HW problem on absorbing Markov chains:

#2

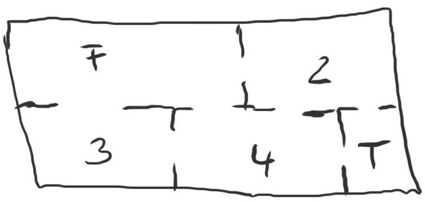
$$
\begin{array}{c|cc|ccc}
 & 0 & 4 & 1\ (\text{Vijay}) & 2 & 3 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 \\
4 & 0 & 1 & 0 & 0 & 0 \\
\hline
1\ (\text{Andrej}) & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
2 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
3 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0
\end{array}
\qquad
\left(\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right)
$$

$$
I - Q = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \Rightarrow (I-Q)^{-1} = \begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ \frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix}
$$

$$
(I-Q)^{-1} R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 4 \end{array} \\ \underbrace{\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}}_{\text{matrix of absorbing probabilities}} \end{array}
$$

$\frac{3}{4}$ is the probab. that Andrej, having \$1, will lose it all.

#5:



$$
\begin{array}{c|cc|ccc}
 & F & T & 2 & 3 & 4 \\
\hline
F & 1 & 0 & 0 & 0 & 0 \\
T & 0 & 1 & 0 & 0 & 0 \\
\hline
2 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\
3 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\
4 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\
 & \multicolumn{2}{c}{R} & \multicolumn{3}{c}{Q}
\end{array}
$$

$$
I - Q = \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}
$$

$$
(I-Q)^{-1} = \frac{1}{19} \begin{pmatrix} 21 & 2 & 8 \\ 3 & 22 & 12 \\ 6 & 6 & 24 \end{pmatrix}
\qquad
(I-Q)^{-1} R = \frac{1}{19} \begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} F & T \end{array} \\ \begin{pmatrix} 10 & 9 \\ 15 & 4 \\ 11 & 8 \end{pmatrix} \end{array}
$$

So the prob. that rat will end up in the trap if initially placed in room 3 is $\frac{4}{19}$.

Back to two-player zero-sum games:

Recall: strictly determined games: payoff matrix has a saddle point

row minima *
column maxima □
simultaneously

$$\begin{array}{cc} & \text{Player C} \\ \text{Player R} & \begin{matrix} -25^* & 10 \\ \boxed{10^*} & \boxed{50} \end{matrix} \end{array}$$

saddle point

(made-up example)

"Value of this game is 10" (expected payoff for the row player if both follow optimal strategy)

Mixed strategies:

In the absence of a saddle point, optimal strategies are random choices of options with certain probabilities.

$$\begin{array}{ccc} & & \text{C} \\ & & \begin{matrix} \text{dime} & \text{quarter} \end{matrix} \\ \text{R} & \begin{matrix} \text{dime} \\ \text{quarter} \end{matrix} & \underbrace{\begin{matrix} 10 & -10 \\ -25 & 25 \end{matrix}}_{A} \end{array}$$

A mixed strategy for Robert is a probability vector $(p \quad 1-p)$

" " " " Claire " " " " $\begin{pmatrix} q \\ 1-q \end{pmatrix}$

$$A \begin{pmatrix} q \\ 1-q \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where c_i is the expected payoff for Robert, given Claire's strategy, if Robert has chosen row i

$(p \quad 1-p)\, A = (r_1 \quad r_2)$ r_i is the expected payoff for Robert given his strategy if Claire has chosen column i

Given both strategies,

$$(p \quad 1-p)\, A \begin{pmatrix} q \\ 1-q \end{pmatrix}$$ is the expected payoff (for Robert) of the game.

E.g.: $p = \frac{2}{10}$ $q = \frac{7}{10}$

$$\begin{pmatrix} \frac{2}{10} & \frac{8}{10} \end{pmatrix} \begin{pmatrix} 10 & -10 \\ -25 & 25 \end{pmatrix} \begin{pmatrix} \frac{7}{10} \\ \frac{3}{10} \end{pmatrix} = \begin{pmatrix} \frac{2}{10} & \frac{8}{10} \end{pmatrix} \begin{pmatrix} 4 \\ -10 \end{pmatrix} = -\frac{72}{10} = -7.2$$

With these given mixed strategies, Robert loses -7.2 ct. on average per game.

Robert should use the mixed strategy where $r_1 = r_2$, otherwise Claire could observe and prefer that choice of column where Robert is worse off.

$$(p \quad 1-p) \begin{pmatrix} 10 & -10 \\ -25 & 25 \end{pmatrix} = \Big(\underbrace{(10p - 25(1-p))}_{\substack{r_1 \\ = 35p - 25}} \quad \underbrace{(-10p + 25(1-p))}_{\substack{r_2 \\ = 25 - 35p}} \Big)$$

$r_1 = r_2 \implies 35p - 25 = 25 - 35p$

$\implies 70p = 50$

$\implies p = \frac{5}{7}$