① HW 4:

Given $v_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ $v_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ $v_3 = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$ $V_4 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

Are $v_1, \dots, v_4$ l.i., if not, find a linear relation among them.

columns with pivots

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & 3 \\ 2 & 1 \\ -2 & 0 \end{matrix}} & \begin{matrix} 4 \\ 3 \\ -2 \end{matrix} & \boxed{\begin{matrix} 2 \\ -1 \\ 1 \end{matrix}} \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 4 & 2 \\ 0 & -5 & -5 & -5 \\ 0 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} \boxed{\begin{matrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{matrix}} & \begin{matrix} 4 \\ 1 \\ 0 \end{matrix} & \boxed{\begin{matrix} 2 \\ 0 \\ 1 \end{matrix}} \end{pmatrix}$$

$\Rightarrow$ V_1, V_2, V_4 l.i.; V_3 linearly depends on the others: $x_1 \boxed{V_1} + x_2 \boxed{V_2} + x_3 \boxed{V_4} = V_3$

To find x_1, x_2, x_3, need to solve system with augmented matrix

$$\left(\boxed{\begin{matrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}} \,\middle|\, \begin{matrix} 4 \\ 1 \\ 0 \end{matrix}\right) \to \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \,\middle|\, \begin{matrix} 1 \\ 1 \\ 0 \end{matrix}\right)$$

$\Rightarrow$ we have shown that $1 \cdot v_1 + 1 \cdot v_2 + 0 \cdot V_4 = V_3$

② Show that if columns of A are l.i. $\Rightarrow$ A^TA is invertible.

$A = \begin{pmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{pmatrix}$ typically tall

$c_1, \dots, c_n$ l.i. $\Leftrightarrow$ $x_1 c_1 + \dots + x_n c_n = 0$ implies $x_1 = \dots = x_n = 0$

$\Leftrightarrow$ $Ax = 0$ implies $x = 0$

$\underbrace{A^TA}_{\text{square}}$ invertible $\Leftrightarrow$ A^TA has full rank $\Leftrightarrow$ $A^TAx = 0$ has only the trivial solution $x = 0$

$\Leftrightarrow$ $A^TAx = 0$ implies $x = 0$

Given $A^TAx = 0$, want to prove that this implies $x = 0$.

$\Downarrow$

$\underbrace{x^TA^TAx}_{\|Ax\|^2} = x^T 0 = 0 \Longrightarrow Ax = 0 \Rightarrow x = 0$

by property of norm

③ Finish last example from the Monday class:

After reduction by dominance, payoff matrix is

$$G_{red} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}$$

Expected payoff for row player with mixed strategy $r^T = (p, (1-p))$

$$r^T G_{red} = (p, \ 1-p) \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} = \left(-2p + 1\cdot(1-p), \ 4p - 2(1-p)\right)$$

$$= (-3p+1, \ 6p-2)$$

Optimal strategy has $-3p+1 = 6p-2 \Rightarrow 3 = 9p \Rightarrow p = \frac{1}{3}$

Value of the game: $-3 \cdot \frac{1}{3} + 1 = 0$ "the game is fair"

④ In how many ways can a 10 question multiple-choice test with 4 answers per question be answered?

Interpretation 1: exactly one of four answers is correct.

$$4^{10}$$

Interpretation 2: any number of answers may be correct

$$2^{4\cdot 10} = 2^{40} > 4^{10} \qquad 2^{4\cdot 10} = (2^4)^{10} = 16^{10}$$

Interpretation 3: One or more answers may be correct

$$\underbrace{\binom{4}{1} + \binom{4}{2}} + \binom{4}{3} + \binom{4}{4} = 15$$ possibilities per question, so 15^{10} possibilities in total.

up to 2 correct answers per question: 10 per question, 10^{10} in total.

(5) In how many ways can 3 books be selected from 4 English and 2 History books, if at least one English book must be chosen?

$$C^4_3 C^2_0 + C^4_2 C^2_1 + C^4_1 C^2_2 = 4 \cdot 1 + 6 \cdot 2 + 4 \cdot 1 = 20$$

choose 3 E, 0 H

WRONG computation:

$$4 \cdot C^6_2 = \dots = 60 \neq 20$$

choose 1 English book

from the remaining 6 books, choose 2.

(6) In how many ways can the word OUTSIDE be re-arranged s.t. it has a consonant in first and last place.

$$3 \cdot 2 \cdot 5! = 720$$

choose first

choose last

# of permutations of the remaining letters