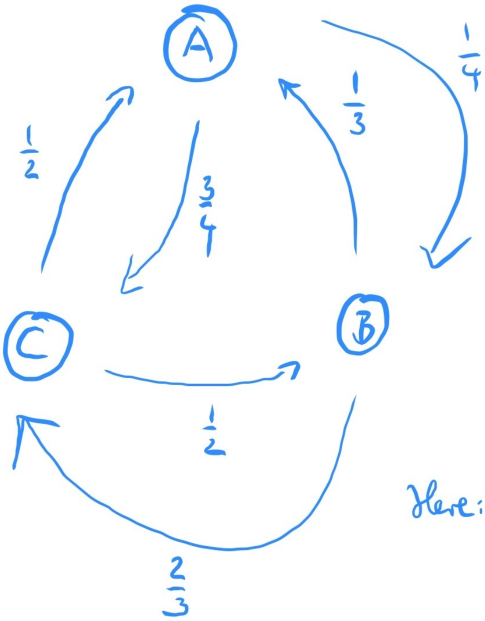
Final check:

$$(16 \quad 15 \quad 22)\begin{pmatrix} 0 & \frac14 & \frac34 \\ \frac13 & 0 & \frac23 \\ \frac12 & \frac12 & 0 \end{pmatrix}$$

$$= (16 \quad 15 \quad 22) \checkmark$$



$$P = \begin{pmatrix} 0 & \frac14 & \frac34 \\ \frac13 & 0 & \frac23 \\ \frac12 & \frac12 & 0 \end{pmatrix}$$

Note: P^2 has non-zero entries everywhere, so chain is regular.

Here: $P^T - I = \begin{pmatrix} -1 & \frac13 & \frac12 \\ \frac14 & -1 & \frac12 \\ \frac34 & \frac23 & -1 \end{pmatrix}$

$$(x^T P)^T = (x^T)^T$$

or $P^T x = x$ or $P^T x - Ix = 0$ or $(P^T - I)x = 0$

$$\begin{pmatrix} -1 & \frac13 & \frac12 \\ \frac14 & -1 & \frac12 \\ \frac34 & \frac23 & -1 \end{pmatrix} \xrightarrow[\frac34 R1 + R3 \to R3]{-R1 \to R1,\ \frac14 R1 + R2 \to R2} \begin{pmatrix} 1 & -\frac13 & -\frac12 \\ 0 & -\frac{11}{12} & \frac58 \\ 0 & \frac{11}{12} & -\frac58 \end{pmatrix} \xrightarrow[R2+R3 \to R3]{-\frac{12}{11} R2 \to R2} \begin{pmatrix} 1 & -\frac13 & -\frac12 \\ 0 & 1 & -\frac{15}{22} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac23 + \frac{\cancel{3}}{4}\frac{1}{\cancel{3}} = \frac{8}{12} + \frac{3}{12}$$

$$-1 + \frac34 \frac12 = -\frac88 + \frac38 = -\frac58$$

$$-\frac{\cancel{12}^{\,3}}{11}\frac{5}{\cancel{8}_{\,2}} = -\frac{15}{22}$$

$$-\frac12 + \frac13\left(-\frac{15}{22}\right)$$

$$= -\frac{33}{66} - \frac{15}{66}$$

$$= -\frac{48}{66} = -\frac{24}{33} = -\frac{8}{11}$$

$$\xrightarrow{\frac13 R2 + R1 \to R1} \begin{pmatrix} 1 & 0 & -\frac{8}{11} \\ 0 & 1 & -\frac{15}{22} \\ 0 & 0 & 0 \end{pmatrix} \quad -1$$

$$x = \begin{pmatrix} \frac{8}{11} \\ \frac{15}{22} \\ 1 \end{pmatrix} \text{ or } x = \begin{pmatrix} 16 \\ 15 \\ 22 \end{pmatrix}$$

To get probabilities: normalise: $p = \frac{1}{16+15+22}\begin{pmatrix} 16 \\ 15 \\ 22 \end{pmatrix} = \frac{1}{53}\begin{pmatrix} 16 \\ 15 \\ 22 \end{pmatrix}$

Inconsistent vs. underdetermined:

$Ax = b$ — Inconsistent $\downarrow$ least-square; underdetermined $\downarrow$ least-norm

$$\left(\begin{array}{c|c} A & b \end{array}\right) \longrightarrow \cdots \longrightarrow \left(\begin{array}{cccccc|c} 1 & 0 & * & * & * & & * \\ 0 & 1 & * & * & * & & * \\ 0 & 0 & \cdots & 0 & & & \boxed{} \end{array}\right)$$

If in red block: all zeros $\Longrightarrow$ system is consistent, but underdetermined

at least one non-zero: $\Rightarrow$ system is inconsistent.

A is a tall matrix $\longrightarrow$ typically, need least squares

$$\begin{pmatrix} \\ A \\ \\ \end{pmatrix} \begin{pmatrix} x \end{pmatrix}_m = \begin{pmatrix} \\ b \\ \\ \end{pmatrix}_n \qquad m < n$$

$$Ax = b$$

$$A^T A x = A^T b$$

(abstract solution: $x = (A^T A)^{-1} A^T b$)

$\begin{pmatrix} A^T \end{pmatrix}\begin{pmatrix} A \end{pmatrix}$ is a small $m \times m$ matrix

Example

$$\underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} x = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{b}$$

$$A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solve $A^T A x = A^T b$, with augmented matrix $\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right)$

$$\to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -3 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & -\frac{1}{3} \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{1}{3} \end{array}\right)$$

$x = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$ is the least-square solution.