Math 3A, Section B, Spring 1997

Marcel Oliver

Midterm Exam

1. Let

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & -1 \\ 0 & 0 & 7 & -7 \\ 5 & -5 & 1 & 4 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 7 \\ 6 \end{pmatrix}.$$

   Find:

   (a) The solution set of the equation $A\mathbf{x} = \mathbf{b}$. (Do not use back-substitution. Use the method taught in class.)

   (b) A basis for the row space of A.

   (c) A basis for the null space of A.

   (d) A basis for the column space of A.

   (e) An example of a vector $\mathbf{c}$ such that the equation $A\mathbf{x} = \mathbf{c}$ does not have a solution.

   (f) $\operatorname{rank} A$ and $\operatorname{nullity} A$; verify that for this matrix the "rank equation" holds.

   (g) Extend the basis found in part (d) to a basis of $\mathbb{R}^4$.

   (h) Is A invertible? Give at least *four* different reasons for why or why not.

   (10+5+5+5+5+5+5+5+5)

2. Let V be a vector space with basis $\{v_1, v_2, v_3\}$. Prove that $\{v_1, v_1 - v_2, v_1 - v_2 - v_3\}$ is also a basis for V. (10)

3. Prove that, if W is a subspace of an n-dimensional vector space V and $\dim W = n$, then $W = V$. (10)

4. Let W be a subspace of $\mathbb{R}^n$ and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ a linear transformation. Prove that

$$U = T(W) = \{T(w)\colon w \in W\}$$

is a subspace of $\mathbb{R}^n$. (10)

5. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation.

   (a) Show that $T(0) = 0$.

   (b) Is $S\colon \mathbb{R}^n \to \mathbb{R}^n$, defined through

   $$S(\mathbf{x}) = 3\mathbf{x} + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

   a linear transformation?

   (c) Give an example of two linear transformations S and T where $S \circ T \neq T \circ S$.

(5+5+10)