ON HOMOGENEOUS SPACES FOR DIAGONAL IND-GROUPS

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ABSTRACT. We study the homogeneous ind-spaces $\mathrm{GL}(\mathbf{s})/\mathbf{P}$ where $\mathrm{GL}(\mathbf{s})$ is a strict diagonal ind-group defined by a supernatural number $\mathbf{s}$ and $\mathbf{P}$ is a parabolic ind-subgroup of $\mathrm{GL}(\mathbf{s})$. We construct an explicit exhaustion of $\mathrm{GL}(\mathbf{s})/\mathbf{P}$ by finite-dimensional partial flag varieties. As an application, we characterize all locally projective $\mathrm{GL}(\infty)$-homogeneous spaces, and some direct products of such spaces, which are $\mathrm{GL}(\mathbf{s})$-homogeneous for a fixed $\mathbf{s}$. The very possibility for a $\mathrm{GL}(\infty)$-homogeneous space to be $\mathrm{GL}(\mathbf{s})$-homogeneous for a strict diagonal ind-group $\mathrm{GL}(\mathbf{s})$ arises from the fact that the automorphism group of a $\mathrm{GL}(\infty)$-homogeneous space is much larger than $\mathrm{GL}(\infty)$.

CONTENTS

1. INTRODUCTION

The ind-group $\mathrm{GL}(\infty) = \varinjlim \mathrm{GL}(n) = \bigcup_{n\geq 1} \mathrm{GL}(n)$ is a most natural direct limit algebraic group, and its locally projective homogeneous spaces are quite well studied by now, see for instance [3], [4], [7], [10]. A larger class of direct limit algebraic groups are the so called diagonal ind-groups. A rather obvious such group, non-isomorphic to $\mathrm{GL}(\infty)$, is the ind-group $\mathrm{GL}(2^\infty) = \varinjlim \mathrm{GL}(2^n)$ where $\mathrm{GL}(2^{n-1})$ is embedded into $\mathrm{GL}(2^n)$ via the

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map

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

A general definition of a diagonal Lie algebra has been given by A. Baranov and A. Zhilinskii in [1], and this definition carries over in a straightforward way to classical Lie groups, producing the class of diagonal Lie groups.

Locally projective homogeneous ind-spaces of diagonal ind-groups have been studied much less extensively than those of $\mathrm{GL}(\infty)$, see [4] and [2]. In this paper, we undertake such a study for a class of diagonal ind-groups which we call strict diagonal ind-groups of type A. These ind-groups are characterized by supernatural numbers $\mathbf{s}$, and are denoted $\mathrm{GL}(\mathbf{s})$. We consider reasonably general parabolic subgroups $\mathbf{P} \subset \mathrm{GL}(\mathbf{s})$ and describe the homogeneous ind-spaces $\mathrm{GL}(\mathbf{s})/\mathbf{P}$ as direct limits of embeddings

$$G_{n-1}/P_{n-1} \to G_n/P_n$$

of usual ind-varieties. Our main theorem is an explicit formula for the so arising embeddings, and this formula is an analogue of the formula for standard extensions introduced in [10] (and used in a particular case in [3]).

This result shows that it is realistic to aim at a future detailed understanding (possibly a classification) of locally projective homogeneous ind-spaces of strict (and of general) diagonal ind-groups. These homogeneous ind-varieties should exhibit interesting properties and each of them should yield a different Borel–Weil–Bott type theory.

In the current paper we restrict ourselves to the following application of the above explicit formula: we determine which locally projective homogeneous ind-spaces of $\mathrm{GL}(\infty)$, i.e., ind-varieties of generalized flags [3], are also $\mathrm{GL}(\mathbf{s})$-homogeneous for a given infinite supernatural number $\mathbf{s}$. Furthermore, we also characterize explicitly direct products of ind-varieties of generalized flags which are $\mathrm{GL}(\mathbf{s})$-homogeneous.

The very possibility of an ind-variety of generalized flags being a homogeneous space for $\mathrm{GL}(\mathbf{s})$, where $\mathbf{s}$ is an infinite supernatural number, is an interesting phenomenon, and can be seen as one possible motivation for our studies of $\mathrm{GL}(\mathbf{s})$-homogeneous ind-spaces. Indeed, recall the following fact for a finite-dimensional algebraic group. If G is a centerless simple algebraic group of classical type and rank at least four and P is a parabolic subgroup, a well-known result of A. Onishchik [9] implies that the connected component of unity of the automorphism group of the homogeneous space G/P coincides with G, except in two special cases when G/P is a projective space and G is a symplectic group, and when G/P is a maximal orthogonal isotropic grassmannian and G is an orthogonal group of type B. Consequently, unless G/P is a projective space or a maximal isotropic grassmannian, G/P cannot be a homogeneous G'-space for a centerless algebraic group $G' \not\cong G$.

The explanation of why the situation is very different if one replaces G by the ind-group $\mathrm{GL}(\infty)$, is that, as shown in [7], the automorphism group of an ind-variety of generalized flags is much larger than $\mathrm{GL}(\infty)$. In this way, our results provide embeddings of $\mathrm{GL}(\mathbf{s})$ into such automorphism groups, with the property that the action of $\mathrm{GL}(\mathbf{s})$ on the respective ind-variety of generalized flags is transitive. As a corollary we obtain that a "generic" ind-variety of generalized flags is $\mathrm{GL}(\mathbf{s})$-homogeneous also for any ind-group $\mathrm{GL}(\mathbf{s})$. This statement should provide a basis for comparison of Borel–Weil–Bott type theories for different locally reductive ind-groups on the same homogeneous ind-variety. In other words, a potential application of our result could be a realization of representations of different ind-groups by means of the same invertible sheaf. Finally a future detailed understanding of the homogeneous ind-varieties of the diagonal ind-groups could lead to an interesting description of their automorphism groups, generalizing the work on the automorphism groups of ind-varieties of generalized flags [7].

The paper is organized as follows. Sections 2, 3, 4 are devoted to preliminaries. We start by introducing the ind-groups $\mathrm{GL}(\mathbf{s})$ where $\mathbf{s}$ is a supernatural number. We then discuss Cartan, Borel, and parabolic ind-subgroups of $\mathrm{GL}(\mathbf{s})$. In Section 3 we review the notions of linear embedding of flag varieties and standard extension of flag varieties, and in Section 4 we recall the necessary results on ind-varieties of generalized flags.

In Section 5 we prove our explicit formula for embeddings of partial flag varieties $\mathrm{GL}(n)/Q \hookrightarrow \mathrm{GL}(dn)/P$ induced by pure diagonal embeddings $\mathrm{GL}(n) \hookrightarrow \mathrm{GL}(dn)$. In Section 6 we use this formula to describe all $\mathrm{GL}(\mathbf{s})$-homogeneous ind-varieties of generalized flags. Finally, in Section 7 we characterize direct products of ind-varieties of generalized flags, which are $\mathrm{GL}(\mathbf{s})$-homogeneous.

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2. The ind-group $\mathrm{GL}(\mathbf{s})$

2.1. **Direct systems associated to a supernatural number.** Throughout this paper we consider a fixed supernatural number $\mathbf{s}$, in other words

$$\mathbf{s} = \prod_{p \in \mathcal{P}} p^{\alpha_p}$$

where $\mathcal{P}$ is a (possibly infinite) set of prime numbers and α_p is either a positive integer or ∞. Moreover, we suppose that $\mathbf{s}$ is infinite, hence at least one of the exponents α_p is infinite or the set $\mathcal{P}$ is infinite. By $\mathcal{D}(\mathbf{s})$ we denote the set of finite divisors of $\mathbf{s}$.

Let $\mathcal{A}$ be a direct system of sets with injective maps. We say that $\mathcal{A}$ is *associated to the supernatural number* $\mathbf{s}$ if the sets in $\mathcal{A}$

$$A(s), \quad s \in \mathcal{D}(\mathbf{s})$$

are parametrized by the finite divisors of $\mathbf{s}$, and the injective maps

$$\delta_{s,s'} : A(s) \hookrightarrow A(s')$$

correspond to pairs of divisors $s, s' \in \mathcal{D}(\mathbf{s})$ such that $s|s'$. Then, if $L(\mathcal{A}) = \varinjlim A(s)$, the resulting map

$$\delta_s : A(s) \hookrightarrow L(A)$$

is injective for every $s \in \mathcal{D}(\mathbf{s})$.

Definition 2.1. We call *exhaustion of* $\mathbf{s}$ any sequence $\{s_n\}_{n\geq 1}$ of integers such that

- $s_n \in \mathcal{D}(\mathbf{s})$ for all n,
- s_n divides s_{n+1} for all n,
- any $s \in \mathcal{D}(\mathbf{s})$ is a multiple of s_n for some n. ■

Lemma 2.2. *Let $\{s_n\}_{n\geq 1}$ be an exhaustion of $\mathbf{s}$. Then $L(\mathcal{A})$ coincides with the limit of the inductive system formed by the sets $A(s_n)$ and the maps $\delta_n = \delta_{s_n,s_{n+1}} : A(s_n) \hookrightarrow A(s_{n+1})$.*

Proof. Straightforward. □

According to the lemma, the limit $L(\mathcal{A})$ can be described in terms of an exhaustion

$$L(\mathcal{A}) = \bigcup_n A(s_n).$$

- In the case where $A(s)$ are vector spaces and the maps $\delta_{s,s'}$ are linear, then $L(\mathcal{A})$ is the direct limit in the category of vector spaces.
- In the case where $A(s)$ are algebraic varieties and the maps $\delta_{s,s'}$ are closed embeddings, the limit $L(\mathcal{A})$ is an ind-variety as defined in [11] and [8].
- In the case where $A(s)$ are algebraic groups and the maps $\delta_{s,s'}$ are group homomorphisms, the limit is both an ind-variety and a group. It is in particular an ind-group[1].

2.2. **Definition of the groups $\mathrm{GL}(\mathbf{s})$ and $\mathrm{SL}(\mathbf{s})$.** Whenever s, s' are two positive integers such that s divides s', we have a diagonal embedding

$$\delta_{s,s'} : \mathrm{GL}(s) \to \mathrm{GL}(s'),\ x \mapsto \mathrm{diag}(\underbrace{x, \ldots, x}_{\frac{s'}{s} \text{ blocks}}).$$

We refer to the embeddings $\delta_{s,s'}$ as *strict diagonal embeddings*. A more general definition of *diagonal embeddings* is given, at the Lie algebra level, in [1].

The groups $\mathrm{GL}(s)$ (for $s \in \mathcal{D}(\mathbf{s})$) and the maps $\delta_{s,s'}$ (for all pairs of integers $s, s' \in \mathcal{D}(\mathbf{s})$ such that s divides s') form a direct system. By definition, the ind-group $\mathrm{GL}(\mathbf{s})$ is the limit of this direct system.

[1] An ind-group is an ind-variety with a group structure such that the multiplication $(x, y) \mapsto xy$ and the inversion $x \mapsto x^{-1}$ are morphisms of ind-varieties.

The group $\mathrm{GL}(\mathbf{s})$ can be viewed as the group of infinite $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$-matrices consisting of one diagonal block of size equal to any (finite) divisor s of $\mathbf{s}$, repeated infinitely many times along the diagonal:

$$\mathrm{GL}(\mathbf{s}) \quad = \quad \left\{ \begin{pmatrix} x & 0 & \cdots \\ 0 & x & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} : x \in \mathrm{GL}(s), \ s \in \mathcal{D}(\mathbf{s}) \right\}. \tag{2.1}$$

Similarly, we define $\mathrm{SL}(\mathbf{s})$ as the limit of the direct system formed by the groups $\mathrm{SL}(s)$ and the same maps $\delta_{s,s'}$. In fact, $\mathrm{SL}(\mathbf{s})$ is the derived group of $\mathrm{GL}(\mathbf{s})$. By $\mathfrak{gl}(\mathbf{s})$ and $\mathfrak{sl}(\mathbf{s})$, we denote the Lie algebras of $\mathrm{GL}(\mathbf{s})$ and $\mathrm{SL}(\mathbf{s})$, respectively. Thus $\mathfrak{sl}(\mathbf{s}) = [\mathfrak{gl}(\mathbf{s}), \mathfrak{gl}(\mathbf{s})]$.

Remark 2.3. Lemma 2.2 shows that the group $\mathrm{GL}(\mathbf{s})$ can be obtained through any exhaustion

$$\mathrm{GL}(\mathbf{s}) = \bigcup_n \mathrm{GL}(s_n)$$

where $\{s_n\}_{n\geq 1}$ is an exhaustion of $\mathbf{s}$ (see Definition 2.1). However, the ind-group $\mathbf{GL}(\mathbf{s})$ has various other exhaustions. If we set

$$\mathbf{K}(n) := \underbrace{\mathrm{GL}(s_n) \times \cdots \times \mathrm{GL}(s_n)}_{\frac{s_{n+1}}{s_n} \text{ factors}}$$

and

$$\psi_n : \mathbf{K}(n) \to \mathbf{K}(n+1), \ (x_1, \ldots, x_{d_n}) \mapsto (\underbrace{\mathrm{diag}(x_1, \ldots, x_{d_n}), \ldots, \mathrm{diag}(x_1, \ldots, x_{d_n})}_{\frac{s_{n+2}}{s_{n+1}} \text{ terms}}),$$

then the direct system $\{\mathbf{K}(n) \xrightarrow{\psi_n} \mathbf{K}(n+1)\}$ intertwines in a natural way with the direct system $\{\mathrm{GL}(s_n) \xrightarrow{\delta_{s_n,s_{n+1}}} \mathrm{GL}(s_{n+1})\}$ considered above. This yields an equality

$$\mathrm{GL}(\mathbf{s}) = \varinjlim \mathrm{GL}(s_n) = \varinjlim \mathbf{K}(n).$$

■

We say that two exhaustions $\mathbf{G} = \bigcup_n G_n = \bigcup_n G'_n$ of a given ind-group are *equivalent* if there are $n_0 \geq 1$ and a commutative diagram

$$\begin{tikzcd}
G_{n_0} \arrow[r] \arrow[d, "\sim"] & \cdots \arrow[r] & G_n \arrow[r] \arrow[d, "\sim"] & G_{n+1} \arrow[r] \arrow[d, "\sim"] & \cdots \\
G'_{n_0} \arrow[r] & \cdots \arrow[r] & G'_n \arrow[r] & G'_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

such that the vertical arrows are isomorphisms of algebraic groups and the horizontal arrows are the embeddings of the exhaustions.

Lemma 2.4. (a) *Any exhaustion of* $\mathrm{SL}(\mathbf{s})$ *by almost simple, simply connected algebraic groups is equivalent to* $\{\mathrm{SL}(s_n), \delta_{s_n,s_{n+1}}\}_{n\geq 1}$ *for an exhaustion* $\{s_n\}_{n\geq 1}$ *of* $\mathbf{s}$.

(b) *Any exhaustion of* $\mathrm{GL}(\mathbf{s})$ *by classical groups (i.e. by groups of the form* $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{SO}(n)$, *or* $\mathrm{Sp}(n)$*) is equivalent to* $\{\mathrm{GL}(s_n), \delta_{s_n,s_{n+1}}\}_{n\geq 1}$ *for an exhaustion* $\{s_n\}_{n\geq 1}$ *of* $\mathbf{s}$.

Proof. (a) It suffices to prove the claim at the level of Lie algebras. Let $\mathfrak{sl}(\mathbf{s}) = \bigcup_n \mathfrak{g}_n$ be an exhaustion by simple Lie algebras, hence of classical type for n large enough. There is a subsequence $\{\mathfrak{g}_{k_n}\}_{n\geq 1}$ and an exhaustion $\{s_n\}_{n\geq 1}$ of $\mathbf{s}$ such that we have a commutative diagram of embeddings

$$\begin{tikzcd}
\mathfrak{sl}(s_n) \arrow[r, "\delta_{s_n,s_{n+1}}"] \arrow[d, "\eta_n"] & \mathfrak{sl}(s_{n+1}) \arrow[r] \arrow[d, "\eta_{n+1}"] & \cdots \\
\mathfrak{g}_{k_n} \arrow[r] \arrow[ur, "\xi_n"] & \mathfrak{g}_{k_{n+1}} \arrow[r] \arrow[ur] & \cdots
\end{tikzcd}$$

By [1, Lemma 2.7], the embeddings η_n and ξ_n are diagonal, in the sense that there is an isomorphism of $\mathfrak{sl}(s_n)$-modules

$$W_n \cong V_n^{\oplus t} \oplus V_n^{*\oplus r} \oplus \mathbb{C}^{\oplus s}$$

and an isomorphism of $\mathfrak{g}_{k_n}$-modules

$$V_{n+1} \cong W_n^{\oplus t'} \oplus W_n^{*\oplus r'} \oplus \mathbb{C}^{\oplus s'}$$

for some triples of nonnegative integers (t, r, s) and (t', r', s'), where V_n and W_n denote the natural representations of $\mathfrak{sl}(s_n)$ and $\mathfrak{g}_{k_n}$, and $\mathbb{C}$ is a trivial representation. Also since $\delta_{s_n,s_{n+1}}$ is strict diagonal, we have an isomorphism of $\mathfrak{sl}(s_n)$-modules

$$V_{n+1} \cong \underbrace{V_n \oplus \ldots \oplus V_n}_{\frac{s_{n+1}}{s_n} \text{ copies}}. \tag{2.2}$$

Arguing by contradiction, assume that $\mathfrak{g}_{k_n}$ is not of type A. Then [1, Proposition 2.3] implies that $t = r$. Moreover, $t' + r' > 0$ since otherwise V_{n+1} would be a trivial representation of $\mathfrak{sl}(s_n)$. Altogether this implies that V_n^* is isomorphic to a direct summand of V_{n+1} considered as an $\mathfrak{sl}(s_n)$-module, which is impossible in view of (2.2). We conclude that $\mathfrak{g}_{k_n}$ is of type A for all n.

Moreover, from (2.2), we obtain $s = s' = 1$ and either $r = r' = 0$ or $t = t' = 0$. Up to replacing $\mathfrak{g}_{k_n} = \mathfrak{sl}(W_n)$ by $\mathfrak{sl}(W_n^*)$, we can assume that $r = r' = 0$, and so $\mathfrak{g}_{k_n} \cong \mathfrak{sl}(s'_{k_n})$ for some integer such that $s_n | s'_{k_n}$, $s'_{k_n} | s_{n+1}$, and the embedding $\mathfrak{g}_{k_n} \hookrightarrow \mathfrak{g}_{k_{n+1}}$ is induced by $\delta_{s'_{k_n}, s'_{k_{n+1}}}$.

If $k := k_n + 1 < k_{n+1}$, we get a commutative diagram

$$\begin{tikzcd}
\mathfrak{sl}(s'_{k_n}) \arrow[rr, "\delta_{s'_{k_n}, s'_{k_{n+1}}}"] \arrow[d, "\sim"] & & \mathfrak{sl}(s'_{k_{n+1}}) \arrow[d, "\sim"] \\
\mathfrak{g}_{k_n} \arrow[r] & \mathfrak{g}_k \arrow[r] & \mathfrak{g}_{k_{n+1}}
\end{tikzcd}$$

where the horizontal arrows are embeddings. Relying as above on [1, Proposition 2.3], we get that $\mathfrak{g}_k$ is necessarily of type A, and up to replacing $\mathfrak{g}_k = \mathfrak{sl}(W)$ by $\mathfrak{sl}(W^*)$, we can assume that $\mathfrak{g}_k \cong \mathfrak{sl}(s'_k)$ for some s'_k with $s'_{k_n}|s'_k$, $s'_k|s'_{k_{n+1}}$, and that the embeddings $\mathfrak{g}_{k_n} \hookrightarrow \mathfrak{g}_k \hookrightarrow \mathfrak{g}_{k_{n+1}}$ are induced by $\delta_{s'_{k_n}, s'_k}$ and $\delta_{s'_k, s'_{k_{n+1}}}$.

By iterating the reasoning, we obtain an exhaustion $\{s'_n\}_{n\geq 1}$ of $\mathbf{s}$ such that the exhaustions $\mathfrak{sl}(\mathbf{s}) = \bigcup_n \mathfrak{g}_n$ and $\mathfrak{sl}(\mathbf{s}) = \bigcup_n \mathfrak{sl}(s'_n)$ are equivalent. This shows (a).

(b) From (a) it follows that for every n, the derived group (G_n, G_n) is isomorphic to $\mathrm{SL}(s_n)$ and, after identifying (G_n, G_n) with $\mathrm{SL}(s_n)$ and (G_{n+1}, G_{n+1}) with $\mathrm{SL}(s_{n+1})$, the map $(G_n, G_n) \hookrightarrow (G_{n+1}, G_{n+1})$ becomes the restriction of $\delta_{s_n, s_{n+1}}$. This implies that G_n is either isomorphic to $\mathrm{SL}(s_n)$ or to $\mathrm{GL}(s_n)$. For $n \geq 1$ large enough, G_n has to contain the center $Z(\mathrm{GL}(\mathbf{s}))$, which is isomorphic to $\mathbb{C}^*$. Since the connected component of the center of $\mathrm{SL}(s_n)$ is trivial, this forces $G_n \cong \mathrm{GL}(s_n)$. Moreover, since $G_n = Z(G_n)(G_n, G_n)$ and the embedding $G_n \hookrightarrow G_{n+1}$ maps $Z(G_n) = Z(\mathrm{GL}(\mathbf{s}))$ into $Z(G_{n+1})$, we deduce that this embedding $G_n \hookrightarrow G_{n+1}$ coincides with $\delta_{s_n, s_{n+1}} : \mathrm{GL}(s_n) \hookrightarrow \mathrm{GL}(s_{n+1})$ after suitably identifying G_n with $\mathrm{GL}(s_n)$ and G_{n+1} with $\mathrm{GL}(s_{n+1})$. ☐

The following statement is a corollary of the classification of general diagonal Lie algebras [1]. We give a proof for the sake of completeness.

Proposition 2.5. (a) *If* $\mathbf{s}$ *and* $\mathbf{s}'$ *are two different infinite supernatural numbers, then the ind-groups* $\mathrm{GL}(\mathbf{s})$ *and* $\mathrm{GL}(\mathbf{s}')$ *(resp.* $\mathrm{SL}(\mathbf{s})$ *and* $\mathrm{SL}(\mathbf{s}')$*) are not isomorphic.*

(b) *If* $\mathbf{s}$ *is an infinite supernatural number, then* $\mathrm{GL}(\mathbf{s})$ *is not isomorphic to* $\mathrm{GL}(\infty)$*, and* $\mathrm{SL}(\mathbf{s})$ *is not isomorphic to* $\mathrm{SL}(\infty)$*.*

Proof. (a) Since $\mathrm{SL}(\cdot)$ is the derived group of $\mathrm{GL}(\cdot)$, it suffices to establish the claim concerning $\mathrm{SL}(\mathbf{s})$ and $\mathrm{SL}(\mathbf{s}')$. Assume there is an isomorphism of ind-groups $\varphi : \mathrm{SL}(\mathbf{s}') \to \mathrm{SL}(\mathbf{s})$. Then any exhaustion $\{s'_n\}$ of $\mathbf{s}'$ yields an exhaustion $\mathrm{SL}(\mathbf{s}) = \bigcup_n \varphi(\mathrm{SL}(s'_n))$ of the group $\mathrm{SL}(\mathbf{s})$, and Lemma 2.4 implies $\mathbf{s} = \mathbf{s}'$, a contradiction.

(b) By definition, $\mathrm{SL}(\infty)$ has an exhaustion by the groups $\mathrm{SL}(n)$ $(n \geq 1)$ via the standard embeddings $\mathrm{SL}(n) \to \mathrm{SL}(n+1)$, $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$. Clearly this exhaustion is not equivalent to $\{\mathrm{SL}(s_n), \delta_{s_n, s_{n+1}}\}_{n\geq 1}$ for any exhaustion $\{s_n\}_{n\geq 1}$ of $\mathbf{s}$. Therefore, the ind-groups $\mathrm{SL}(\mathbf{s})$ and $\mathrm{SL}(\infty)$ are not isomorphic by Lemma 2.4 (a). The same argument shows that $\mathrm{GL}(\mathbf{s})$ and $\mathrm{GL}(\infty)$ are not isomorphic. ☐

2.3. **Parabolic and Borel subgroups.** An ind-subgroup $\mathbf{H} \subset \mathrm{GL}(\mathbf{s})$ is said to be a *(locally splitting) Cartan subgroup* if there is an exhaustion $\mathrm{GL}(\mathbf{s}) = \bigcup_n G_n$ by classical groups such that $G_n \cap \mathbf{H}$ is a Cartan subgroup of G_n for all n. For instance, the subgroup of invertible periodic diagonal matrices in the realization (2.1) is a Cartan subgroup of $\mathrm{GL}(\mathbf{s})$.

If $\mathbf{P}$ is an ind-subgroup of $\mathrm{GL}(\mathbf{s})$, then the quotient $\mathrm{GL}(\mathbf{s})/\mathbf{P}$ is an ind-variety obtained as the direct limit of the quotients $\mathrm{GL}(s)/\mathbf{P}(s)$ for $s \in \mathcal{D}(\mathbf{s})$.

For the purposes of this paper, we say that an ind-subgroup $\mathbf{P} \subset \mathrm{GL}(\mathbf{s})$ is a *parabolic subgroup* if there exists an exhaustion $\mathrm{GL}(\mathbf{s}) = \bigcup_n G_n$ by classical groups such that $G_n \cap \mathbf{P}$ is a parabolic subgroup of G_n for all n (cf. [2]). This implies in particular that the ind-variety $\mathrm{GL}(\mathbf{s})/\mathbf{P}$ is *locally projective* as it has an exhaustion

$$\mathrm{GL}(\mathbf{s})/\mathbf{P} = \bigcup_n G_n/(G_n \cap \mathbf{P})$$

by projective varieties. If, in addition, the unipotent radical of $G_n \cap \mathbf{P}$ is contained in the unipotent radical of $G_{n+1} \cap \mathbf{P}$ for every n, then we say that $\mathbf{P}$ is a *strong parabolic subgroup*.

An ind-subgroup $\mathbf{B} \subset \mathrm{GL}(\mathbf{s})$ is said to be a *Borel subgroup* if it is locally solvable and parabolic. This means equivalently that there is an exhaustion $\mathrm{GL}(\mathbf{s}) = \bigcup_n G_n$ as above for which $G_n \cap \mathbf{B}$ is a Borel subgroup of G_n for all n. Note that a Borel subgroup is necessarily a strong parabolic subgroup.

Lemma 2.6. *A subgroup $\mathbf{G}'$ of $\mathrm{GL}(\mathbf{s})$ is a Cartan (respectively, parabolic or Borel) subgroup of $\mathbf{G}$ if and only if there is an exhaustion $\{s_n\}_{n\geq 1}$ of $\mathbf{s}$ such that for every n the intersection $\mathbf{G}' \cap \mathrm{GL}(s_n)$ is a Cartan (respectively, parabolic or Borel) subgroup of $\mathrm{GL}(s_n)$.*

Proof. This follows from Lemma 2.4. □

The following example shows that for a given parabolic subgroup $\mathbf{P} \subset \mathrm{GL}(\mathbf{s})$, the property that the group $G_n \cap \mathbf{P}$ is a parabolic subgroup of G_n may no longer hold for a refinement of the exhaustion used to define $\mathbf{P}$.

Example 2.7. Let $\mathbf{s} = 2^\infty$, $s_n = 2^{2n-2}$, and $s'_n = 2^{n-1}$. Then both $\{s_n\}_{n\geq 1}$ and $\{s'_n\}_{n\geq 1}$ are exhaustions of $\mathbf{s}$, and $\{s'_n\}_{n\geq 1}$ is a refinement of $\{s_n\}_{n\geq 1}$. Let $H_n \subset \mathrm{GL}(s_n)$ be the subgroup of diagonal matrices. We define a Borel subgroup $B_n \subset \mathrm{GL}(s_n)$ that contains H_n, by induction in the following way: $B_1 := \mathrm{GL}(1)$, and

$$B_{n+1} := \begin{pmatrix} B_n & * & * & * \\ 0 & B_n & * & * \\ 0 & 0 & B_n & 0 \\ 0 & 0 & * & B_n \end{pmatrix}$$

for $n \geq 2$, where all the blocks are square matrices of size s_n. Then $B_{n+1} \cap \mathrm{GL}(s_n) = B_n$ for all n, which implies that $\mathbf{B} = \bigcup_{n\geq 1} B_n$ is a well-defined Borel subgroup of $\mathrm{GL}(\mathbf{s})$

arising from the exhaustion $\{s_n\}_{n\geq 1}$ of $\mathbf{s}$. However, for all n,

$$\mathbf{B}\cap \mathrm{GL}(s'_{2n}) = \begin{pmatrix} B_n & 0 \\ 0 & B_n \end{pmatrix}$$

is not a Borel subgroup (nor a parabolic subgroup) of $\mathrm{GL}(s'_{2n})$. ■

3. On embeddings of flag varieties

In this section we review some preliminaries on finite-dimensional (partial) flag varieties. In particular, we recall the notions of linear embedding and standard extension introduced in [10].

3.1. **Grassmannians and (partial) flag varieties.** Let V be a finite-dimensional vector space. For an integer $0 \leq p \leq \dim V$, we denote by $\mathrm{Gr}(p;V)$ the grassmannian of p-dimensional subspaces in V. This grassmannian can be realized as a projective variety by the Plücker embedding $\mathrm{Gr}(p;V) \hookrightarrow \mathbb{P}(\bigwedge^p V)$. Moreover, the Picard group $\mathrm{Pic}(\mathrm{Gr}(p;V))$ is isomorphic to $\mathbb{Z}$ with generator $\mathcal{O}_{\mathrm{Gr}(p;V)}(1)$, the pull-back of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(\bigwedge^p V)$.

For a sequence of integers $0 < p_1 < \ldots < p_{k-1} < p_k < \dim V$, we denote by $\mathrm{Fl}(p_1,\ldots,p_k;V)$ the variety of (partial) flags

$$\mathrm{Fl}(p_1,\ldots,p_k;V) = \{(V_1,\ldots,V_k) \in \mathrm{Gr}(p_1;V)\times\cdots\times\mathrm{Gr}(p_k;V) : V_1 \subset \ldots \subset V_k\}.$$

We have

$$\mathrm{Pic}(\mathrm{Fl}(p_1,\ldots,p_k;V)) \cong \mathbb{Z}^k.$$

If we let L_i be the pull-back

$$L_i = \mathrm{proj}_i^* \mathcal{O}_{\mathrm{Gr}(p_i;V)}(1)$$

along the projection

$$\mathrm{proj}_i : \mathrm{Fl}(p_1,\ldots,p_k;V) \to \mathrm{Gr}(p_i;V)$$

(for $i = 1,\ldots,k$), then $[L_1],\ldots,[L_k]$ is a set of generators of the Picard group, to which we refer as *preferred generators* of $\mathrm{Pic}\,\mathrm{Fl}(p_1,\ldots,p_k;V)$.

By *embedding of flag varieties* we mean a closed immersion

$$\varphi : X = \mathrm{Fl}(p_1,\ldots,p_k;V) \hookrightarrow Y = \mathrm{Fl}(q_1,\ldots,q_\ell;W).$$

If $\mathcal{F} = \{F_1,\ldots,F_k\} \in X$ is a variable point, we set

$$C_i(\varphi) = \bigcap_{\mathcal{F}\in X} \varphi(\mathcal{F})_i.$$

Then $C_1(\varphi) \subset \ldots \subset C_\ell(\varphi)$ is a chain of subspaces of W with possible repetitions. We define the *support* of φ to be the set of indices $i \in \{1,\ldots,\ell\}$ such that $\dim C_i(\varphi) < q_i$.

3.2. **Linear embedding.** Let

$$Q := \mathrm{Gr}(q_1; W_1) \times \cdots \times \mathrm{Gr}(q_\ell; W_\ell)$$

where $W_1, \ldots, W_\ell$ is a sequence of vector spaces and $0 < q_j < \dim W_j$ for all j. Consider an embedding

$$\psi : X = \mathrm{Fl}(p_1, \ldots, p_k; V) \to Q.$$

We use the notation of the previous section for X. The Picard group of Q is isomorphic to $\mathbb{Z}^\ell$, with generators associated to the line bundles $M_j = \mathrm{proj}_j^* \mathcal{O}_{\mathrm{Gr}(q_j;W_j)}(1)$.

Definition 3.1. We say that the embedding ψ is *linear* if we have

$$[\psi^* M_j] = 0 \quad \text{or} \quad [\psi^* M_j] \in \{[L_1], \ldots, [L_k]\}$$

for all $j \in \{1, \ldots, \ell\}$. ■

Let

$$\varphi : X = \mathrm{Fl}(p_1, \ldots, p_k; V) \hookrightarrow Y = \mathrm{Fl}(q_1, \ldots, q_\ell; W)$$

be an embedding of flag varieties. The following definition is equivalent to [10, Definition 2.1].

Definition 3.2. The embedding φ is said to be *linear* if the composed embedding $\psi = \pi \circ \varphi$ is linear, where $\pi := \prod_{j=1}^{\ell} \mathrm{proj}_j : Y \to \prod_{j=1}^{\ell} \mathrm{Gr}(q_j; W)$. ■

3.3. **Standard extension.**

Definition 3.3 ([10])**.** (a) The embedding $\varphi : \mathrm{Fl}(p_1, \ldots, p_k; V) \hookrightarrow \mathrm{Fl}(q_1, \ldots, q_\ell; W)$ is said to be a *strict standard extension* if there are

- a decomposition of vector spaces $W = V' \oplus Z$ with a linear isomorphism $\varepsilon : V \xrightarrow{\sim} V'$,
- a chain of subspaces $Z_1 \subset \ldots \subset Z_\ell$ of Z (with possible repetitions),
- a nondecreasing map $\kappa : \{1, \ldots, \ell\} \to \{0, 1, \ldots, k, k+1\}$,

such that

$$\varphi\big(\{V_1, \ldots, V_k\}\big) = \{\varepsilon(V_{\kappa(1)}) + Z_1, \ldots, \varepsilon(V_{\kappa(\ell-1)}) + Z_{\ell-1}, \varepsilon(V_{\kappa(\ell)}) + Z_\ell\} \tag{3.1}$$

where $V_0 := 0$ and $V_{k+1} := V$.

(b) More generally, we say that φ is a *standard extension* if φ itself is a strict standard extension or its composition with the duality map $\mathrm{Fl}(q_1, \ldots, q_\ell; W) \to \mathrm{Fl}(\dim W - q_1, \ldots, \dim W - q_\ell; W^*)$ is a strict standard extension. ■

Remark 3.4. Since the map φ of (3.1) is an embedding of flag varieties, the following conditions must hold: $1, \ldots, k$ have preimages by κ, and the map $j \in \{1, \ldots, \ell\} \mapsto (\kappa(j), Z_j)$ is injective and does not contain $(0, 0)$ nor $(k+1, Z)$ in its image. ■

Note that, if φ is a strict standard extension, then $C_i(\varphi) = Z_i$ for all $i \in \{1, \ldots, \ell\}$, and the support of φ is the interval $\kappa^{-1}([1, k])$.

Also, a composition of standard extensions is a standard extension.

Example 3.5. Let $W = V \oplus Z$, where $\dim Z = d$. For $1 \leq k_0 \leq k+1$, we consider the embeddings

$$\begin{array}{rcl} \varphi : \mathrm{Fl}(p_1, \ldots, p_k; V) & \hookrightarrow & \mathrm{Fl}(q_1, \ldots, q_k; W) \\ \{V_1, \ldots, V_k\} & \mapsto & \{V_1, \ldots, V_{k_0-1}, V_{k_0} + Z, \ldots, V_k + Z\} \end{array}$$

and

$$\begin{array}{rcl} \bar\varphi : \mathrm{Fl}(p_1, \ldots, p_k; V) & \hookrightarrow & \mathrm{Fl}(\bar q_1, \ldots, \bar q_{k+1}; W) \\ \{V_1, \ldots, V_k\} & \mapsto & \{V_1, \ldots, V_{k_0-1}, V_{k_0-1} + Z, V_{k_0} + Z, \ldots, V_k + Z\} \end{array}$$

where

$$q_i = \begin{cases} p_i & \text{if } 1 \leq i < k_0, \\ p_i + d & \text{if } k_0 \leq i \leq k \end{cases} \quad \text{and} \quad \bar q_i = \begin{cases} p_i & \text{if } 1 \leq i < k_0, \\ p_{i-1} + d & \text{if } k_0 \leq i \leq k+1. \end{cases}$$

Here, we still use the convention that $V_0 := 0$ and $V_{k+1} := V$, and we set accordingly $p_0 := 0$ and $p_{k+1} := \dim V$. Then φ and $\bar\varphi$ are strict standard extensions, associated with the respective chains of subspaces

$$\underbrace{0 \subset \ldots \subset 0}_{k_0 - 1 \text{ times}} \subset \underbrace{Z \subset \ldots \subset Z}_{k+1-k_0 \text{ times}} \quad \text{and} \quad \underbrace{0 \subset \ldots \subset 0}_{k_0 - 1 \text{ times}} \subset \underbrace{Z \subset \ldots \subset Z}_{k+2-k_0 \text{ times}}$$

and respective maps κ and $\bar\kappa$, where $\kappa(i) = i$ for all i, $\bar\kappa(i) = i$ for $i \leq k_0 - 1$, $\bar\kappa(i) = i - 1$ for $i \geq k_0$. ■

Remark 3.6. Every strict standard extension is the composition of, possibly several, maps φ and $\bar\varphi$ as in Example 3.5. ■

4. A review of generalized flags

4.1. **Generalized flags.** Let V be an infinite-dimensional vector space of countable dimension and let $E = \{e_1, e_2, \ldots\}$ be a basis of V. By $\langle S \rangle$ we denote the linear span of a subset $S \subset V$. Following [3], we call *generalized flag* a collection $\mathcal{F}$ of subspaces of V that satisfies the following conditions:

- $\mathcal{F}$ is totally ordered by inclusion;
- every subspace $F \in \mathcal{F}$ has an immediate predecessor or an immediate successor in $\mathcal{F}$;
- $V \setminus \{0\} = \bigcup_{(F', F'')} (F'' \setminus F')$, where the union is over pairs of consecutive subspaces in $\mathcal{F}$.

Moreover, a generalized flag $\mathcal{F}$ is said to be *E-compatible* if every subspace $F \in \mathcal{F}$ is spanned by elements of E. An E-compatible generalized flag $\mathcal{F}$ can be encoded by a (not order preserving) surjective map $\sigma : \mathbb{Z}_{>0} \to A$ onto a totally ordered set $(A, \leq)$ such that $\mathcal{F} = \{F'_a, F''_a\}_{a \in A}$ where $F'_a = \langle e_k : \sigma(k) < a \rangle$ and $F''_a = \langle e_k : \sigma(j) \leq a \rangle$. More generally, a generalized flag $\mathcal{F}$ is said to be *weakly E-compatible* if it is E'-compatible for some basis E' of V differing from E in finitely many vectors.

Let

$$\mathrm{GL}(E) = \{g \in \mathrm{GL}(V) : g(e_k) = e_k \text{ for all but finitely many } k\}.$$

Then $\mathrm{GL}(E)$ is an ind-group, isomorphic to the finitary classical ind-group $\mathrm{GL}(\infty)$. The group $\mathrm{GL}(E)$ acts on the set of all weakly E-compatible generalized flags. Furthermore, it is established in [3] that weakly E-compatible generalized flags $\mathcal{F}$ of V are in one-to-one correspondence with splitting parabolic subgroups $\mathbf{P} \subset \mathrm{GL}(E)$. More precisely, the map

$$\mathcal{F} \mapsto \mathbf{P} = \mathrm{Stab}_{\mathrm{GL}(E)}(\mathcal{F})$$

is a bijection between these two sets.

We define a *natural representation* of $\mathrm{GL}(\mathbf{s})$ as a direct limit of natural representations of $\mathrm{GL}(s)$ for $s \in \mathcal{D}(\mathbf{s})$, where by the natural representation of $\mathrm{GL}(s)$ we mean the standard representation on column vectors of length s. Two natural representations of $\mathrm{GL}(\mathbf{s})$ do not have to be isomorphic since a direct limit of natural representations can depend on the respective embeddings; see [5].

Assume now that V is a natural representation for $\mathrm{GL}(\mathbf{s})$, $\mathbf{H} \subset \mathrm{GL}(\mathbf{s})$ is a Cartan subgroup such that there is a basis E of V consisting of eigenvectors of $\mathbf{H}$. The group $\mathrm{GL}(\mathbf{s})$ acts in a natural way on the generalized flags in V, and a generalized flag is E-compatible if and only if it is $\mathbf{H}$-stable. However, generalized flags are less suited for describing parabolic subgroups of $\mathrm{GL}(\mathbf{s})$ than for describing parabolic subgroups of $\mathrm{GL}(\infty) \cong \mathrm{GL}(E)$, since the stabilizer of a generalized flag in $\mathrm{GL}(\mathbf{s})$ is not always a parabolic subgroup. Moreover, there are parabolic subgroups of $\mathrm{GL}(\mathbf{s})$ which cannot be realized as stabilizers of generalized flags in a prescribed natural representation. These observations are illustrated by the following two examples.

Example 4.1. For every $n \geq 0$, we define inductively a subset $I_n \subset \{1, \ldots, 2^{n+1}\}$ by setting

$$I_0 := \{1\} \subset \{1, 2\}, \qquad I_n := I_{n-1} \cup \{2^n + i : i \in \{1, \ldots, 2^n\} \setminus I_{n-1}\} \text{ for } n \geq 1.$$

Note that $\{I_n\}_{n\geq 0}$ is a nested sequence of sets, and let $I := \bigcup_{n\geq 0} I_n$. For $V = \langle e_1, e_2, \ldots\rangle$ as above, put

$$W := \langle e_i : i \in I\rangle.$$

Thus $\mathcal{F} := \{0 \subset W \subset V\}$ is a generalized flag.

By Lemma 2.4 (b), any exhaustion of $\mathrm{GL}(2^\infty)$ by classical groups is equivalent to $\{\mathrm{GL}(s_n), \delta_{s_n,s_{n+1}}\}_{n\geq 1}$ for an exhaustion $\{s_n = 2^{k_n}\}_{n\geq 1}$ of $\mathbf{s}$. Every element $g \in \mathrm{GL}(2^{k_n})$ stabilizing $\mathcal{F}$ should be such that the blockwise diagonal matrix

$$\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

stabilizes $\langle e_i : i \in I_{k_n}\rangle = \langle e_i : i \in I_{k_n-1}\rangle \oplus \langle e_{2^{k_n-1}+i} : i \in \{1, \ldots, 2^{k_n}\} \setminus I_{k_n-1}\rangle$, hence g should stabilize both subspaces $\langle e_i : i \in I_{k_n-1}\rangle$ and $\langle e_i : i \in \{1, \ldots, 2^{k_n}\} \setminus I_{k_n-1}\rangle$. This implies that the stabilizer of W in $\mathrm{GL}(2^{k_n})$ is not a parabolic subgroup, for all $n \geq 1$. Therefore, $\mathrm{Stab}_{\mathrm{GL}(2^\infty)}(\mathcal{F})$ is not a parabolic subgroup of $\mathrm{GL}(2^\infty)$. ■

Example 4.2. (a) Let $V = \bigcup_n \mathbb{C}^{2^n}$ be seen as a natural representation of $\mathrm{GL}(2^\infty)$ where the embedding $\mathbb{C}^{2^n} \cong \mathbb{C}^{2^n} \times \{0\}^{2^n} \subset \mathbb{C}^{2^{n+1}}$ is considered. For $n \geq 1$, let $P_n \subset \mathrm{GL}(2^n)$ be the stabilizer of $L_n := \{0\}^{2^n-1} \times \mathbb{C}$, the line spanned by the 2^n-th vector of the standard basis of $\mathbb{C}^{2^n}$. Then $P_{n+1} \cap \mathrm{GL}(2^n) = P_n$ for all $n \geq 1$, hence $\mathbf{P} := \bigcup_{n\geq 1} P_n$ is a parabolic subgroup of $\mathrm{GL}(2^\infty)$. However, $\mathbf{P}$ acts transitively on the nonzero vectors of V, so that there is no nonzero proper subspace of V which is stable by $\mathbf{P}$. Therefore, $\mathbf{P}$ cannot be realized as the stabilizer of a generalized flag in V.

(b) If in part (a) we replace the embeddings defining the structure of natural representation on V by $\mathbb{C}^{2^n} \cong \{0\}^{2^n} \times \mathbb{C}^{2^n} \subset \mathbb{C}^{2^{n+1}}$, then $L_n = L_1$ for all $n \geq 1$ and the parabolic subgroup $\mathbf{P}$ of (a) becomes the stabilizer of the generalized flag $\{0 \subset L_1 \subset V\}$. ■

4.2. **Ind-varieties of generalized flags.**

Definition 4.3. (a) Two generalized flags $\mathcal{F}$ and $\mathcal{G}$ are said to be *E-commensurable* [3] if $\mathcal{F}$ and $\mathcal{G}$ are weakly E-compatible and there is an isomorphism of totally ordered sets $\phi : \mathcal{F} \to \mathcal{G}$ and there is a finite-dimensional subspace $U \subset V$ such that, for all $F \in \mathcal{F}$, $F + U = \phi(F) + U$ and $\dim F \cap U = \dim \phi(F) \cap U$.

(b) Given an E-compatible generalized flag $\mathcal{F}$, we define $\mathrm{Fl}(\mathcal{F}, E)$ as the set of all generalized flags which are E-commensurable with $\mathcal{F}$. ■

Let $\mathcal{F}$ be an E-compatible generalized flag. We now recall the ind-variety structure on $\mathrm{Fl}(\mathcal{F}, E)$ [3]. To do this, we write $E = \{e_k\}_{k\geq 1}$ and, for $n \geq 1$, set $V_n := \langle e_1, \ldots, e_n\rangle$. The collection of subspaces $\{F \cap V_n : F \in \mathcal{F}\}$ determines a flag $F_1^{(n)} \subset \ldots \subset F_{p_n-1}^{(n)}$ in $F_{p_n}^{(n)} := V_n$; furthermore we set $d_i^{(n)} := \dim F_i^{(n)}$ and

$$X_n := \mathrm{Fl}(d_1^{(n)}, \ldots, d_{p_n-1}^{(n)}; V_n).$$

We define an embedding $\eta_n : X_n \to X_{n+1}$ in the following way. Let $i_0 \in \{1, \ldots, p_{n+1}\}$ be minimal such that $e_{n+1} \in F_{i_0}^{(n+1)}$. We have either $p_{n+1} = p_n$ or $p_{n+1} = p_n + 1$. In the former case we set

$$\eta_n : \{M_1, \ldots, M_{p_n-1}\} \mapsto \{M_1, \ldots, M_{i_0-1}, M_{i_0} \oplus \langle e_{n+1}\rangle, \ldots, M_{p_n-1} \oplus \langle e_{n+1}\rangle\}.$$

In the latter case, we define

$$\eta_n : \{M_1, \ldots, M_{p_n-1}\} \mapsto \{M_1, \ldots, M_{i_0-1}, M_{i_0-1} \oplus \langle e_{n+1}\rangle, \ldots, M_{p_n-1} \oplus \langle e_{n+1}\rangle\}.$$

Proposition 4.4 ([3])**.** (a) *The maps $\{\eta_n\}_{n\geq 1}$ are strict standard extensions and they yield an exhaustion $\mathrm{Fl}(\mathcal{F}, E) = \bigcup_{n\geq 1} X_n$. This endows $\mathrm{Fl}(\mathcal{F}, E)$ with a structure of locally projective ind-variety.*

(b) *If $\mathbf{P} = \mathrm{Stab}_{\mathrm{GL}(E)}(\mathcal{F})$, then there is a natural isomorphism of ind-varieties $\mathrm{GL}(E)/\mathbf{P} \xrightarrow{\sim} \mathrm{Fl}(\mathcal{F}, E)$.*

Note also that, up to isomorphism, the ind-variety $\mathrm{Fl}(\mathcal{F}, E)$ only depends on the *type* of $\mathcal{F}$, i.e., on the isomorphism type of the totally ordered set $(\mathcal{F}, \subset)$ and on the dimensions $\dim F''/F'$ of the quotients of consecutive subspaces in $\mathcal{F}$.

5. Embedding of flag varieties arising from diagonal embedding of groups

In this section we study embeddings of flag varieties induced by strictly diagonal embeddings of general linear groups.

Let us fix the following data:

- positive integers $m < n$ such that m divides n, and $d := \frac{n}{m}$;
- $\mathrm{GL}(m)$ seen as a subgroup of $\mathrm{GL}(n)$ through the diagonal embedding
$$x \mapsto \mathrm{diag}(x, \ldots, x);$$
- a decomposition of the natural representation $V := \mathbb{C}^n$ of $\mathrm{GL}(n)$ as
$$V = W^{(1)} \oplus \ldots \oplus W^{(d)}$$
where $W^{(i)} := \{0\}^{(i-1)m} \times \mathbb{C}^m \times \{0\}^{(d-i)m}$; let $\chi_i : W := \mathbb{C}^m \to W^{(i)}$ be the natural isomorphism. For a subspace $M \subset W$, we write $M^{(i)} := \chi_i(M)$.

5.1. **Restriction of parabolic subgroup.** Let $\{e_1, \ldots, e_n\}$ be a basis of V such that $\{e_1, \ldots, e_m\}$ is a basis of $W^{(1)} \cong W$ and $e_{mi+j} = \chi_{i+1}(e_j)$ for all $i = 1, \ldots, d-1$ and all $j = 1, \ldots, m$. By $H = H(n) \subset \mathrm{GL}(n)$ we denote the maximal torus for which $e_1, \ldots, e_n$ are eigenvectors. Then $H' := H \cap \mathrm{GL}(m)$ is a maximal torus of $\mathrm{GL}(m)$.

A parabolic subgroup $P = P(n) \subset \mathrm{GL}(n)$ that contains H is the stabilizer of a flag
$$\mathcal{F}_\alpha = \{\langle e_i : \alpha(i) \leq j \rangle\}_{j=1}^{p-1}$$
for some surjective map $\alpha : \{1, \ldots, n\} \to \{1, \ldots, p\}$. The following statement determines under what condition the intersection $P \cap \mathrm{GL}(m)$ is a parabolic subgroup.

Lemma 5.1. *Consider the map*
$$\beta : \{1, \ldots, m\} \to \{1, \ldots, p\}^d,\ r \mapsto (\alpha(r), \alpha(m+r), \ldots, \alpha((d-1)m+r)),$$
and denote by $\mathcal{I}$ the image of β. Let $\leq$ denote the partial order on $\{1, \ldots, p\}^d$ such that $(x_1, \ldots, x_d) \leq (y_1, \ldots, y_d)$ if $x_i \leq y_i$ for all i.

(a) *The intersection $Q := P \cap \mathrm{GL}(m)$ is a parabolic subgroup of $\mathrm{GL}(m)$ if and only if $\leq$ restricts to a total order on $\mathcal{I}$. Moreover, letting $b_1, \ldots, b_q$ be the elements of $\mathcal{I}$ written in increasing order, we have*
$$Q = \mathrm{Stab}_{\mathrm{GL}(m)}(\mathcal{F}_\beta)$$
where
$$\mathcal{F}_\beta = \{\langle e_i : \beta(i) \leq b_j \rangle\}_{j=1}^{q-1}.$$
In particular, if $d_j = \#\beta^{-1}(\{b_1, \ldots, b_j\})$ then $\mathrm{GL}(m)/Q$ can be identified with the flag variety $\mathrm{Fl}(d_1, \ldots, d_{q-1}; W)$.

(b) *If Q is a parabolic subgroup, the inclusion $U_Q \subset U_P$ of unipotent radicals holds if and only if any two distinct elements $(x_1, \ldots, x_d), (y_1, \ldots, y_d)$ of $\mathcal{I}$ satisfy $x_i \neq y_i$ for all $i \in \{1, \ldots, d\}$.*

Proof. (a) We have a decomposition

$$\mathfrak{gl}(n) = \mathfrak{gl}(V) = \mathfrak{h} \oplus \bigoplus_{1\leq i\neq j\leq n} \mathfrak{g}_{i,j}$$

where $\mathfrak{h} = \operatorname{Lie} H$ and $\mathfrak{g}_{i,j} = \mathbb{C}(e_i \otimes e_j^*)$. With this notation,

$$\mathfrak{p} := \operatorname{Lie} P = \mathfrak{h} \oplus \bigoplus_{\alpha(i)\leq\alpha(j)} \mathfrak{g}_{i,j} \supset \mathfrak{nil}(\mathfrak{p}) = \bigoplus_{\alpha(i)<\alpha(j)} \mathfrak{g}_{i,j}, \tag{5.1}$$

where $\mathfrak{nil}(\mathfrak{p})$ is the nilpotent radical of $\mathfrak{p}$.

There is a similar decomposition

$$\mathfrak{gl}(m) = \mathfrak{gl}(W) = \mathfrak{h}' \oplus \bigoplus_{1\leq i\neq j\leq m} \mathfrak{g}'_{i,j}.$$

Set $\mathfrak{q} := \operatorname{Lie} Q$ where $Q = P \cap \mathrm{GL}(m)$ as before. Since we already know that $\mathfrak{h}' \subset \mathfrak{q}$, the subalgebra $\mathfrak{q}$ is parabolic if and only if

$$1 \leq i \neq j \leq m \quad \Longrightarrow \quad (\mathfrak{g}'_{i,j} \subset \mathfrak{q} \quad \text{or} \quad \mathfrak{g}'_{j,i} \subset \mathfrak{q}). \tag{5.2}$$

In view of (5.1) and the diagonal embedding $\mathfrak{gl}(m) \subset \mathfrak{gl}(n)$, whenever $1 \leq i \neq j \leq m$ we have the equivalence

$$\begin{aligned} \mathfrak{g}'_{i,j} \subset \mathfrak{q} \quad &\Longleftrightarrow \quad \mathfrak{g}_{i+km,j+km} \subset \mathfrak{p} \ \ \forall k = 0, \ldots, d-1 \\ &\Longleftrightarrow \quad \alpha(i+km) \leq \alpha(j+km) \ \forall k = 0, \ldots, d-1 \\ &\Longleftrightarrow \quad \beta(i) \leq \beta(j). \end{aligned}$$

Hence, from (5.2) we obtain that $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{gl}(m)$ if and only if

$$1 \leq i \neq j \leq m \quad \Longrightarrow \quad (\beta(i) \leq \beta(j) \quad \text{or} \quad \beta(j) \leq \beta(i)).$$

The condition means that $\leq$ is a total order set on $\mathcal{I}$. We also have the equality

$$\begin{aligned} \mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\beta(i)\leq\beta(j)} \mathfrak{g}'_{i,j} \quad &= \quad \{X \in \mathfrak{gl}(W) : X(\langle e_i : \beta(i) \leq b_j \rangle) \subset \langle e_i : \beta(i) \leq b_j \rangle \ \forall j\} \\ &= \quad \operatorname{Lie}(\operatorname{Stab}_{\mathrm{GL}(m)}(\mathcal{F}_\beta)) \end{aligned}$$

which implies that $Q = \operatorname{Stab}_{\mathrm{GL}(m)}(\mathcal{F}_\beta)$.

(b) Assume that Q is a parabolic subgroup of $\mathrm{GL}(m)$. The inclusion $U_Q \subset U_P$ holds if and only if the similar inclusion holds for the nilradicals of the Lie algebras. Through the diagonal embedding of $\mathfrak{gl}(m)$ into $\mathfrak{gl}(n)$, the nilradical of $\mathfrak{q}$ can be described as

$$\mathfrak{nil}(\mathfrak{q}) = \bigoplus_{\substack{1\leq i\neq j\leq m \\ \beta(i)<\beta(j)}} (\mathfrak{g}'_{i,j} \oplus \mathfrak{g}'_{i+m,j+m} \oplus \ldots \oplus \mathfrak{g}'_{i+(d-1)m,j+(d-1)m}).$$

Therefore, the desired inclusion $\mathfrak{nil}(\mathfrak{q}) \subset \mathfrak{nil}(\mathfrak{p})$ holds if and only if, for all $i, j \in \{1, \ldots, m\}$,

$$\beta(i) < \beta(j) \quad \Longleftrightarrow \quad \alpha(i+km) < \alpha(j+km) \ \ \forall k \in \{0, \ldots, d-1\}.$$

This condition is equivalent to the one stated in (b) (knowing that the partial order $\leq$ restricts to a total order on $\mathcal{I}$, due to (a)). $\square$

5.2. **Diagonal embedding of flag varieties.** Assuming that the condition of Lemma 5.1 (a) is fulfilled, we now describe the embedding of partial flag varieties

$$\phi : \mathrm{GL}(m)/Q = \mathrm{Fl}(d_1, \ldots, d_{q-1}; W) \to \mathrm{GL}(n)/P \tag{5.3}$$

obtained in this case. We rely on a combinatorial object, introduced in the next definition.

Definition 5.2. (a) We call *E-graph* an unoriented graph with the following features:

- The vertices consist of two sets $\{l_1, \ldots, l_q\}$ ("left vertices") and $\{r_1, \ldots, r_p\}$ ("right vertices"), displayed from top to bottom in two columns, and two vertices are joined by an edge only if they belong to different sets.
- The edges are partitioned into d subsets E_c corresponding to a given colour $c \in \{1, \ldots, d\}$.
- Every vertex is incident with at least one edge, and every vertex is incident with at most one edge of a given colour. The vertex l_q is incident with exactly d edges (one per colour).
- Two edges of the same colour never cross, that is, if (l_i, r_j) and (l_k, r_ℓ), with $i < k$, are joined with two edges of the same colour, then $j < \ell$.

In an E-graph, we call "bounding edges" the edges passing through l_q, and we call "ordinary edges" all other edges.

(b) With the notation of Lemma 5.1, we define the E-graph $\mathcal{G}(\alpha, \beta)$ such that

- we put an edge of colour k between l_i and r_j whenever $b_i = (x_1, \ldots, x_d)$ satisfies $x_k = j$ and i is maximal for this property.

(The conditions given in Lemma 5.1 (a) justify that $\mathcal{G}(\alpha, \beta)$ is a well-defined E-graph.) $\blacksquare$

In the following statement we describe explicitly the embedding ϕ of (5.3) and its properties in terms of the E-graph $\mathcal{G}(\alpha, \beta)$. Recall that for a subspace $M \subset W$, we write $M^{(i)} = \chi_i(M)$, where $\chi_i : W \cong \mathbb{C}^m \to W^{(i)}$ is the natural isomorphism.

Proposition 5.3. (a) *The map* $\phi : Y = \mathrm{GL}(m)/Q = \mathrm{Fl}(d_1, \ldots, d_{q-1}; W) \to X = \mathrm{GL}(n)/P$ *is given by*

$$\phi : \{F_1, \ldots, F_{q-1}\} \mapsto \{V_1, \ldots, V_{p-1}\}$$

where for all $j \in \{1, \ldots, p-1\}$ *we have*

$$V_j = V_{j-1} + F_{i_1}^{(1)} \oplus \ldots \oplus F_{i_d}^{(d)}, \tag{5.4}$$

where $V_0 = F_0 := 0$, $F_q := W$, *and*

$$i_k := \begin{cases} i & \text{if the vertex } r_j \text{ is incident with an edge } (l_i, r_j) \text{ of colour } k \text{ in } \mathcal{G}(\alpha, \beta), \\ 0 & \text{if there is no edge of colours } k \text{ passing through } r_j. \end{cases}$$

We have also

$$V_j = F^{(1)}_{i'_1} \oplus \ldots \oplus F^{(d)}_{i'_d},$$

where i'_k is the index of the left end point of the last edge of colour k arriving at or above r_j in $\mathcal{G}(\alpha, \beta)$, with $i'_k = 0$ if there is no such edge.

(b) *Let $([L_1], \ldots, [L_{p-1}])$ and $([M_1], \ldots, [M_{q-1}])$ denote the sequences of preferred generators of $\operatorname{Pic} X$ and $\operatorname{Pic} Y$, respectively (see Section 3.1). The map $\phi^* : \operatorname{Pic} X \to \operatorname{Pic} Y$ is given by*

$$\phi^*[L_j] = \sum_{k=1}^{d} [M_{i'_k}],$$

where we set by convention $[M_0] = [M_q] = 0$.

(c) *The map ϕ is linear if and only if, whenever $r_j, r_{j'}$ with $j < j'$ are incident with edges of the same colour c in the graph $\mathcal{G}(\alpha, \beta)$, every ordinary edge arriving at $r_{j''}$ for $j \leq j'' < j'$ is also of colour c.*

(d) *The map ϕ is a standard extension if and only if all ordinary edges of $\mathcal{G}(\alpha, \beta)$ are of the same colour. Moreover, in this case, ϕ is a strict standard extension.*

Proof. (a) As in Section 5.1, we write $P = \operatorname{Stab}(\mathcal{F}_\alpha)$ where $\alpha : \{1, \ldots, n\} \to \{1, \ldots, p\}$ is surjective. Then we have $Q = \operatorname{Stab}(\mathcal{F}_\beta)$ where $\beta : \{1, \ldots, m\} \to \mathcal{I} \subset \{1, \ldots, p\}^d$ is described in Lemma 5.1. Let $\hat{\phi} : \mathrm{GL}(m)/Q \to \mathrm{GL}(n)/P$ be the map given by formula (5.4). Thus we have to show that $\hat{\phi} = \phi$. Since the maps ϕ and $\hat{\phi}$ are $\mathrm{GL}(m)$-equivariant, it suffices to show that $\hat{\phi}(\mathcal{F}_\beta) = \mathcal{F}_\alpha$. We write $\mathcal{F}_\alpha = \{F_{\alpha,1}, \ldots, F_{\alpha,p-1}\}$ and $\mathcal{F}_\beta = \{F_{\beta,1}, \ldots, F_{\beta,q-1}\}$. For $j \in \{1, \ldots, p\}$, we have

$$F_{\alpha,j} = \langle e_i : \alpha(i) \leq j \rangle = F_{\alpha,j-1} + \langle e_i : \alpha(i) = j \rangle \tag{5.5}$$

where $F_{\alpha,0} := 0$. Every $i \in \{1, \ldots, n\}$ can be written $i = (k-1)m + r \in \{1, \ldots, n\}$ with $k \in \{1, \ldots, d\}$ and $r \in \{1, \ldots, m\}$, so that $e_i = \chi_k(e_r)$.

Assume that $\alpha(i) = j$. Then there is $b_{i'} = (x_1, \ldots, x_d) \in \mathcal{I}$ with $i' \in \{1, \ldots, q\}$ maximal such that $x_k = j$. Moreover there is $s \in \{r, \ldots, m\}$ such that $x_\ell = \alpha((\ell-1)m + s)$ for all $\ell \in \{1, \ldots, d\}$. This implies that the graph $\mathcal{G}(\alpha, \beta)$ contains an edge of colour k joining $b_{i'}$ and j, and we have

$$e_i = \chi_k(e_r) \in \chi_k(F_{\beta,i'}) = F^{(k)}_{\beta,i'}$$

where $F_{\beta,q} := W$. Conversely, assume that there is an edge of colour k joining $b_{i'}$ and j. The subspace $F^{(k)}_{\beta,i'}$ is spanned by vectors of the form $\chi_k(e_r)$ with $r \in \{1, \ldots, m\}$ such that $\beta(r) = (\alpha((\ell-1)m + r)^d_{\ell=1} \leq b_{i'}$. The latter inequality implies $\alpha((k-1)m + r) \leq j$. Hence $\chi_k(e_r) = e_{(k-1)m+r} \in F_{\alpha,j}$.

Combining these observations with (5.5), we deduce that

$$F_{\alpha,j} = F_{\alpha,j-1} + F^{(1)}_{\beta,i_1} \oplus \ldots \oplus F^{(d)}_{\beta,i_d}$$

where $i_1, \ldots, i_d$ are as defined in the statement of the proposition. Therefore, the claimed equality $\hat{\phi}(\mathcal{F}_\beta) = \mathcal{F}_\alpha$ holds.

The second formula stated in (a) is an immediate consequence of (5.4). The proof of (a) is complete.

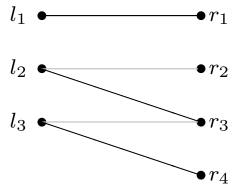
Part (b) is a corollary of the second formula in (a), whereas parts (c) and (d) of the proposition easily follow from parts (a) and (b). The proof of the proposition is complete. □

Remark 5.4. Proposition 5.3 shows how the E-graph $\mathcal{G}(\alpha, \beta)$ describes the embedding $\phi : Y \to X$. Moreover, the chain of constant spaces $(C_j(\phi))$ is expressed in the following way. We enumerate the colours $k_1, \ldots, k_d$ so that $i_1 \leq \ldots \leq i_d$ where r_{i_j} is the right end point of the bounding edge of colour k_j. Then

$$C_j(\phi) = F_q^{(k_1)} \oplus \ldots \oplus F_q^{(k_j)} \quad \text{for } j = 1, \ldots, d.$$

■

Example 5.5. (a) Let us consider for instance the graph



.

It encodes an embedding

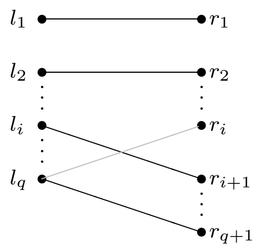
$$\begin{array}{rcl} \phi : X = \mathrm{Fl}(d_1, d_2; \mathbb{C}^n) & \to & Y = \mathrm{Fl}(d_1, d_1 + d_2, d_2 + n; \mathbb{C}^{2n} = \mathbb{C}^n \oplus \overline{\mathbb{C}^n}) \\ \{V_1, V_2\} & \mapsto & \{V_1, V_1 \oplus \overline{V_2}, V_2 \oplus \overline{\mathbb{C}^3}\}. \end{array}$$

If we denote by $([L_1], [L_2])$ and $([M_1], [M_2], [M_3])$ the sets of preferred generators of the Picard groups of X and Y respectively, then the induced map $\phi^* : \mathrm{Pic}\, Y \to \mathrm{Pic}\, X$ is given by

$$[M_1] \mapsto [L_1], \quad [M_2] \mapsto [L_1] + [L_2], \quad [M_3] \mapsto [L_2].$$

Thus ϕ is not linear in this case.

(b) Here we consider the graph



.

There are two colours which means that the embedding is from a flag variety of a space V to the flag variety of a doubled space $W = V \oplus \overline{V}$:

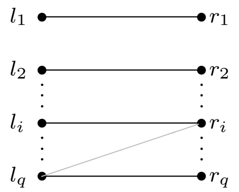
$$\mathrm{Fl}(d_1, \dots, d_{q-1}; V) \hookrightarrow \mathrm{Fl}(d'_1, \dots, d'_q; W = V \oplus \overline{V}).$$

The embedding has the following explicit form

$$\{F_1, \dots, F_{q-1}\} \mapsto \{F_1, \dots, F_{i-1}, F_{i-1} \oplus \overline{V}, \dots, F_{q-1} \oplus \overline{V}\}. \tag{5.6}$$

Note that $\dim F_{i-1} \oplus \overline{V} / \dim F_{i-1} = \dim V$. The dimensions of the other quotients are unchanged.

(c) Now consider



In this case we get an embedding

$$\mathrm{Fl}(d_1, \dots, d_q; V) \hookrightarrow \mathrm{Fl}(d'_1, \dots, d'_q; W = V \oplus \overline{V})$$

given by

$$\{F_1, \dots, F_{q-1}\} \mapsto \{F_1, \dots, F_{i-1}, F_i \oplus \overline{V}, \dots, F_{q-1} \oplus \overline{V}\}. \tag{5.7}$$

The only quotient whose dimension changes is $F_i \oplus \overline{V}/F_{i-1}$ which has dimension $\dim V + \dim F_i/F_{i-1}$.

By Proposition 5.3 (d), the embeddings of parts (a) and (b) of this example are the only possible standard extensions that can come from a diagonal embedding $\mathrm{GL}(n) \hookrightarrow \mathrm{GL}(2n)$.

(d) In the case of a diagonal embedding of the form $\mathrm{GL}(n) \hookrightarrow \mathrm{GL}(dn)$, if the embedding of flag varieties is a standard extension, then it can be described as a composition of embeddings of the previous form, involving a subspace $\overline{V}$ still of dimension n. ■

Remark 5.6. The fact that $U_Q \subset U_P$ is equivalent to the following property of the graph $\mathcal{G}(\alpha, \beta)$: every left vertex is incident with exactly d edges (one per colour). ■

Proposition 5.3 has the following corollary.

Corollary 5.7. *For an embedding $\phi : Y = \mathrm{GL}(n)/Q \to X = \mathrm{GL}(m)/P$ as in Proposition 5.3 and for every $j \in \{1, \dots, q-1\}$, we have $\mathrm{Im}\, \phi^* \not\subset \langle [M_i] : i \in \{1, \dots, q-1\} \setminus \{j\} \rangle$.*

5.3. **Application to ind-varieties.**

Definition 5.8. Let $\{s_n\}_{n \geq 1}$ be an exhaustion of $\mathbf{s}$. We call $\mathbf{s}$-*graph* a graph with infinitely many columns of vertices B_n, with $1 \leq |B_n| \leq s_n$ for all $n \geq 1$, such that the subgraph consisting of B_n, B_{n+1} and the corresponding edges is an E-graph. ■

A parabolic subgroup $\mathbf{P}$ of $\mathrm{GL}(\mathbf{s})$ gives rise to an $\mathbf{s}$-graph. According to the above proposition, this graph encodes the embeddings of flag varieties in an exhaustion of $\mathrm{GL}(\mathbf{s})/\mathbf{P}$. Conversely, any $\mathbf{s}$-graph arises from a parabolic subgroup $\mathbf{P}$ of $\mathrm{GL}(\mathbf{s})$.

6. Ind-varieties of generalized flags as homogeneous spaces of $\mathrm{GL}(\mathbf{s})$

Our purpose in this section is to characterize ind-varieties of generalized flags (introduced in Section 4.2) which can be realized as homogeneous spaces $\mathrm{GL}(\mathbf{s})/\mathbf{P}$ for the given supernatural number $\mathbf{s}$.

6.1. **The case of finitely many finite-dimensional subspaces.** We start with a special situation which is easier to deal with: let $\mathbf{X} = \mathrm{Fl}(\mathcal{F}, E)$ where $\mathcal{F} = \{F'_a, F''_a\}_{a\in A}$ is an E-compatible generalized flag, for an arbitrary totally ordered set $(A, \leq)$, but with the assumption that

$$\dim F''_a/F'_a = +\infty \quad \text{for all but finitely many } a \in A. \tag{6.1}$$

Theorem 6.1. *If condition (6.1) holds, then for every supernatural number* $\mathbf{s}$*, there is an isomorphism of ind-varieties* $\mathrm{Fl}(\mathcal{F}, E) \cong \mathrm{GL}(\mathbf{s})/\mathbf{P}$ *for an appropriate parabolic subgroup* $\mathbf{P} \subset \mathrm{GL}(\mathbf{s})$.

Proof. In the situation of the theorem, the ind-variety $\mathbf{X} = \mathrm{Fl}(\mathcal{F}, E)$ has an exhaustion

$$X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n \overset{\phi_n}{\hookrightarrow} X_{n+1} \hookrightarrow \cdots$$

such that X_n is a finite-dimensional variety of flags in the space $\mathbb{C}^{s_n}$ for some exhaustion $\{s_n\}_{n\geq 1}$ of $\mathbf{s}$ with s_1 sufficiently large, and $\phi_n : X_n \to X_{n+1}$ is one of the two maps from Example 5.5 (b) and (c). Using the maps ϕ_n, one constructs nested parabolic subgroups $P_n \subset \mathrm{GL}(s_n)$ such that $X_n \cong \mathrm{GL}(s_n)/P_n$ and $P_n = \mathrm{GL}(s_n) \cap P_{n+1}$ for all n. The union $\mathbf{P} = \bigcup_{n\geq 1} P_n$ is then a parabolic subgroup of $\mathrm{GL}(\mathbf{s})$ which satisfies the conditions of the theorem. $\square$

6.2. **The general case.** To treat the general case, we need to start with a definition.

Definition 6.2. Let $\mathcal{F} = \{F'_a, F''_a\}_{a\in A}$ be an E-compatible generalized flag and let $A' = \{a \in A : \dim F''_a/\dim F'_a < +\infty\}$. We say that the ind-variety $\mathrm{Fl}(\mathcal{F}, E)$ is $\mathbf{s}$-*admissible* if either A' is finite or A' is infinite and there are a exhaustion $\{s_n\}_{n\geq 1}$ for $\mathbf{s}$ and a numbering $A' = \{k_n\}_{n\geq 1}$ (not necessarily compatible with the total order on A') such that, for all $n \geq 0$:

$$\tfrac{\dim F''_{k_n}/F'_{k_n}}{s_n} \in \{1, \ldots, \tfrac{s_{n+1}}{s_n} - 1\} \text{ and } s_n | \dim F''_a/F'_a \text{ for all } a \in A' \setminus \{k_1, \ldots, k_n\}.$$

■

Theorem 6.3. *The following conditions are equivalent:*

(i) *The ind-variety* $\mathrm{Fl}(\mathcal{F}, E)$ *is* $\mathbf{s}$*-admissible.*

(ii) *There is a parabolic subgroup* $\mathbf{P} \subset \mathrm{GL}(\mathbf{s})$ *and an isomorphism of ind-varieties* $\mathrm{Fl}(\mathcal{F}, E) \cong \mathrm{GL}(\mathbf{s})/\mathbf{P}$.

Proof. (i)⇒(ii): The ind-variety $\mathrm{Fl}(\mathcal{F}, E)$ admits an exhaustion $\mathrm{Fl}(\mathcal{F}, E) = \bigcup_n X_n$ with embeddings of the form

$$\begin{aligned} \phi_n : X_n = \mathrm{Fl}(p_1, \ldots, p_{k_n}; V_n) &\to X_{n+1} = \mathrm{Fl}(q_1, \ldots, q_{\ell_n}; V_n \oplus C_n), \\ \{F_1, \ldots, F_{k_n}\} &\mapsto \{F_{\tau(1)} \oplus C_1^n, \ldots, F_{\tau(\ell_n)} \oplus C_{\ell_n}^n\} \end{aligned} \tag{6.2}$$

(with $F_0 := 0$, $F_{k_n+1} := V_n$) for a nondecreasing surjective map $\tau : \{1, \ldots, \ell_n\} \to \{0, 1, \ldots, k_n, k_n + 1\}$ and a sequence $C_1^n \subset \ldots \subset C_{\ell_n}^n$ (with possible repetitions) of subspaces of C_n.

Assume that there is another exhaustion $\mathrm{Fl}(\mathcal{F}, E) = \bigcup_n Y_n$ for which the embeddings are as described in Proposition 5.3, where $Y_n = \mathrm{Fl}(r_1, \ldots, r_{m_n}; W_n)$ and $\dim W_n = s_n$ for an exhaustion $\{s_n\}$ of $\mathbf{s}$. Then the two exhaustions interlace, and there is no loss of generality in assuming that the interlacing holds for the sequences (X_n) and (Y_n), and not only for subsequences:

$$\begin{tikzcd} X_n \arrow[r, hook, "\phi_n"] \arrow[d, hook, "\xi_n"'] & X_{n+1} \arrow[d, hook, "\xi_{n+1}"] \\ Y_n \arrow[r, hook, "\psi_n"'] \arrow[ur, hook, "\chi_n"] & Y_{n+1}. \end{tikzcd}$$

Claim. The embedding ξ_n is a standard extension.

First we show that ξ_n is linear. Arguing by contradiction, assume that there is a generator $[M_i]$ among the sequence $[M_1], \ldots, [M_{q-1}]$ of preferred generators of $\mathrm{Pic}\, X_n$ such that $\xi_n^*[M_i]$ is neither 0 nor a preferred generator of $\mathrm{Pic}\, Y_n$. Since $\phi_n^* = \chi_n^* \circ \xi_{n+1}^*$, we have the inclusion $\mathrm{Im}\, \phi_n^* \subset \mathrm{Im}\, \chi_n^*$, and due to Corollary 5.7 we get that there is a generator $[L] \in \mathrm{Pic}\, Y_{n+1}$ such that

$$\chi_n^*[L] = \sum_{j=1}^{q-1} \lambda_j [M_j] \quad \text{with } \lambda_i \neq 0.$$

Since the map χ_n is an embedding, we have $\lambda_j \geq 0$ for all j and in particular $\lambda_i \geq 1$. The same argument applied to ξ_n implies that $\xi_n^*[M_j]$ should be a linear combination of the preferred generators of $\mathrm{Pic}\, Y_n$ with nonnegative integer coefficients. This implies that $\psi_n^*[L] = \xi_n^* \chi_n^*[L]$ is neither 0 nor a preferred generator of $\mathrm{Pic}\, Y_n$, contradicting the linearity of the standard extension ψ_n.

Recall that in [10] the notion of an embedding factoring through a direct product is introduced. Note that ξ_n cannot factor through a direct product: otherwise, ψ_n would also factor through a direct product, which is impossible since this is a standard extension. Consequently, ξ_n is a standard extension, and the claim is established.

Now we can assume that ξ_1 is a strict standard extension. Since the maps ϕ_n are strict standard extensions, by using the formula for ψ_n in Proposition 5.3 we derive that ξ_n is a strict standard extension for all $n \geq 1$.

Due to (6.2) and Proposition 5.3, one has $W_n = V_n \oplus Z_n$ and the map ξ_n has the form

$$\xi_n : \{F_1, \ldots, F_{k_n}\} \mapsto \{F_{\sigma(1)} \oplus Z_1^n, \ldots, F_{\sigma(p_n)} \oplus Z_{p_n}^n\}.$$

Since this applies likewise to ξ_{n+1}, and taking into account the form of ϕ_n in (6.2), we see that the map ψ_n has the form

$$\{F_1, \ldots, F_{k_n}, F_1', \ldots, F_{p_n}'\} \mapsto \{F_1, \ldots, F_{k_n}, R_1^n, \ldots, R_{\ell_n}^n, \Gamma_1^n, \ldots, \Gamma_{\delta_n}^n\}$$

for an arbitrary map $\zeta_n : \{F_1', \ldots, F_{p_n}'\} \mapsto \{R_1^n, \ldots, R_{\ell_n}^n\}$ as described in Proposition 5.3, and where Γ_i^n are constant subspaces which are copies of W_n in $W_{n+1} = \bigoplus_{i=1}^{d_n} W_n^{(i)}$. This implies $\dim V_n = d_n' s_n$ for some $d_n' \in \{1, \ldots, d_n = \frac{s_{n+1}}{s_n}\}$.

Since $\{V_n \oplus Z_1^n, \ldots, V_n \oplus Z_{p_n}^n\} = \{\Gamma_1^{n-1}, \ldots, \Gamma_{\delta_{n-1}}^{n-1}\}$, we must have $p_n = \delta_{n-1}$ and the dimension of Z_i^n is a multiple of s_{n-1}. Therefore $\dim R_i^n$ is also a multiple of s_{n-1} for all i. Condition (ii) is established.

(ii)⇒(i): Let $d_n' = \frac{\dim F_{k_n}}{s_n} \in \{1, \ldots, d_n\}$ and set $V_n = F_{k_n}$. The conditions imply that we can choose a decomposition $W_n = V_n \oplus W_{n-1}^{(1)} \oplus \ldots \oplus W_{n-1}^{(d_n - d_n')}$ where the $W_{n-1}^{(i)}$'s are copies of W_{n-1} such that the strict standard extension $\mathrm{Fl}_n(\mathcal{F}, E) \to \mathrm{Fl}_{n+1}(\mathcal{F}, E)$ is given by

$$\phi_n : \{F_1, \ldots, F_{k_n}\} \mapsto \{F_{k_1} + C_1^n, \ldots, F_{k_{n+1}} + C_{k_{n+1}}^n\}$$

with $C_i^n = W_{n-1}^{(1)} \oplus \ldots \oplus W_{n-1}^{(m_i)} = C_i'^n \oplus C_i''^n$ for some nondecreasing sequence $m_1, \ldots, m_{k_{n+1}}$.

Letting $\xi_n : \mathrm{Fl}_n(\mathcal{F}, E) \to \mathrm{Fl}(\mathbf{t}_n; W_n)$ be given by

$$\xi_n : \{F_1, \ldots, F_{k_n}\} \mapsto \{F_{k_1} + C_1'^n, \ldots, F_{k_{n+1}} + C_{k_{n+1}}'^n\},$$

and $\psi_n : \mathrm{Fl}(\mathbf{t}_n; W_n) \to \mathrm{Fl}(\mathbf{t}_{n+1}; W_{n+1})$

$$\psi_n : \{F_1, \ldots, F_{k_n}\} \mapsto \{F_1 + C_1''^{n+1}, \ldots, F_{\ell_n + 1} + C_{\ell_{n+1}}''^{n+1}\}$$

(for suitable types $\mathbf{t}_n$), we get exhaustions of $\mathrm{Fl}(\mathcal{F}, E)$ and a homogeneous space for $\mathrm{GL}(\mathbf{s})$, which interlace. Hence if $\mathcal{F}$ satisfies the condition above, then we can realize $\mathrm{Fl}(\mathcal{F}, E)$ as a homogeneous space for $\mathrm{GL}(\mathbf{s})$. □

Remark 6.4. It is shown in [2, Corollary 5.40] that $\mathrm{GL}(\mathbf{s})/\mathbf{B}$ is never projective when $\mathbf{B}$ is a Borel subgroup. On the other hand, according to [3, Proposition 7.2], an ind-variety of generalized flags is projective if and only if the total order on the flag can be induced by a subset of $(\mathbb{Z}, \leq)$, and Theorem 6.3 shows that in many situations $\mathrm{GL}(\mathbf{s})/\mathbf{P}$ is projective. ■

7. The case of direct products of ind-varieties of generalized flags

In this section, we point out that many direct products of ind-varieties of generalized flags can be homogeneous spaces for the group $\mathrm{GL}(\mathbf{s})$.

7.1. **Direct products of ind-varieties.** Let $\mathbf{X}_i = \bigcup_{n\geq 1} X_{i,n}$ ($i \in I$) be a collection of ind-varieties indexed by $\mathbb{Z}_{>0}$ or a finite subset of it. For each $i \in I$ we pick an element $x_i \in X_{i,1}$ and we set $X_{i,0} = \{x_i\}$. The direct product in the category of pointed ind-varieties is then given by

$$\prod_{i\in I} \mathbf{X}_i := \bigcup_{n\geq 1} \prod_{i\in I} X_{i,\phi_i(n)}$$

for a collection of increasing maps $\phi_i : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ such that for every $n \in \mathbb{Z}_{>0}$ we have $\phi_i(n) = 0$ for all but finitely many $i \in I$ (the definition does not depend essentially on the choice of the maps ϕ_i).

Remark 7.1. (a) As a set, the direct product can be identified with the set of sequences $(y_i)_{i\in I}$ where $y_i \in \mathbf{X}_i$ for all $i \in I$ and $y_i = x_i$ for all but finitely many $i \in I$.

(b) For a finite set of indices I, as a set, $\prod_{i\in I} \mathbf{X}_i$ coincides with the usual cartesian product, and its structure of ind-variety is given by the exhaustion $\prod_{i\in I} \mathbf{X}_i := \bigcup_{n\geq 1} X_{i,n}$.

Fixing an index $i_0 \in I$, there are a canonical projection

$$\mathrm{proj}_{i_0} : \prod_{i\in I} \mathbf{X}_i \to \mathbf{X}_{i_0},\ (y_i) \mapsto y_{i_0}$$

and an embedding

$$\mathrm{emb}_{i_0} : \mathbf{X}_{i_0} \to \prod_{i\in I} \mathbf{X}_i,\ x \mapsto (y_i) \text{ with } y_i = \begin{cases} x_i & \text{if } i \neq i_0, \\ x & \text{if } i = i_0, \end{cases}$$

which are morphisms of ind-varieties.

If the product is endowed with an action of a group G, then each ind-variety $\mathbf{X}_i$ inherits an action of G defined through the maps proj_i and emb_i. Conversely, if every ind-variety $\mathbf{X}_i$ is endowed with an action of a group G, then we obtain an action of G on the product defined diagonally provided that the following condition is fulfilled:

$$\text{every } g \in G \text{ fixes } x_i \text{ for all but finitely many } i \in I. \tag{7.1}$$

(This condition is automatically satisfied in the case where I is finite.) Moreover, in both directions, when $G = \mathbf{G}$ is an ind-group, we have that the obtained action is algebraic provided that the initial one is. The following lemma is an immediate consequence of this discussion.

Lemma 7.2. *Assume that the direct product $\prod_{i\in I} \mathbf{X}_i$ is a homogeneous space for an ind-group $\mathbf{G}$. Then, every ind-variety $\mathbf{X}_i$ is also a homogeneous space for $\mathbf{G}$.*

Note also that a direct product $\prod_{i\in I} \mathbf{X}_i$ is locally projective if and only if it is the case of $\mathbf{X}_i$ for all $i \in I$.

7.2. **The case of ind-varieties of generalized flags.** We start with an example.

Example 7.3. Let $\mathbf{s} = 2^\infty$. We consider the space V of countable dimension, endowed with its fixed basis $E = \{e_k\}_{k\in\mathbb{Z}_{>0}}$, and we set $V_n := \langle e_1, \ldots, e_n\rangle$. We have the exhaustion $\mathrm{GL}(\mathbf{s}) = \bigcup_{n\geq 1} \mathrm{GL}(V_{2^n})$ defined through the diagonal embedding $\mathrm{GL}(V_{2^n}) \hookrightarrow \mathrm{GL}(V_{2^{n+1}})$, $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. Consider the sequence of parabolic subgroups

$$P_n := \mathrm{Stab}_{\mathrm{GL}(V_{2^n})}(V_1, V_{2^n-1}), \quad n \geq 2.$$

In this way, $P_n \cap \mathrm{GL}(V_{2^{n-1}}) = P_{n-1}$ for all $n \geq 3$. Moreover, every quotient $\mathrm{GL}(V_{2^n})/P_n$ is a flag variety formed by flags $(F_1 \subset F_2 \subset V_{2^n})$ of length 2, and we have an embedding of flag varieties

$$\mathrm{GL}(V_{2^{n-1}})/P_{n-1} \to \mathrm{GL}(V_{2^n})/P_n,\ (F_1, F_2) \mapsto (F_1, V_{2^{n-1}} + \overline{F_2})$$

where as before the map

$$V_{2^{n-1}} = \langle e_1, \ldots, e_{2^{n-1}}\rangle \to \overline{V_{2^n-1}} = \langle e_{2^{n-1}+1}, \ldots, e_{2^{n-1}+2^{n-1}}\rangle,\ v \mapsto \overline{v},$$

is the isomorphism from Section 5. For every n, this embedding factors through a direct product of grassmannians

$$\mathrm{GL}(V_{2^{n-1}})/P_{n-1} \to \mathrm{Gr}(1; V_{2^{n-1}}) \times \mathrm{Gr}(2^{n-1} - 1; \overline{V_{2^{n-1}}}) \to \mathrm{GL}(V_{2^n})/P_n,$$

which allows us to chech that the ind-variety $\mathrm{GL}(\mathbf{s})/\mathbf{P}$, where $\mathbf{P} = \bigcup_n P_n$, is isomorphic as an ind-variety to a direct product of two ind-grassmannians.

The following theorem shows that many homogeneous spaces for $\mathrm{GL}(\mathbf{s})$ can be isomorphic to direct products of ind-varieties of generalized flags.

Theorem 7.4. *Let $\mathbf{X} = \mathrm{GL}(\mathbf{s})/\mathbf{P}$ be a homogeneous space, defined by a parabolic subgroup $\mathbf{P}$. Assume that we have an exhaustion $\mathbf{X} = \bigcup_{n\geq 1} \mathrm{GL}(s_n)/P_{s_n}$ determined by an exhaustion $\{s_n\}_{n\geq 1}$ of $\mathbf{s}$, where each embedding $\mathrm{GL}(s_n)/P_{s_n} \hookrightarrow \mathrm{GL}(s_{n+1})/P_{s_{n+1}}$ is linear. Then, $\mathbf{X}$ is isomorphic as an ind-variety to a direct product of ind-varieties of generalized flags $\prod_{i\in I} \mathrm{Fl}(\mathcal{F}^i, E^i)$ where I is either $\mathbb{Z}_{>0}$ or a finite subset of it.*

Proof. Fix $n \geq 1$. Let $d_n = \frac{s_{n+1}}{s_n}$ and fix a decomposition

$$V_n = W_n^{(1)} \oplus \ldots \oplus W_n^{(d_n)} \tag{7.2}$$

of the space $V_n = \mathbb{C}^{s_{n+1}}$ as in Section 5. Thus we are in the setting of Proposition 5.3, and the embedding

$$\phi_n : \mathrm{GL}(s_n)/P_{s_n} \hookrightarrow \mathrm{GL}(s_{n+1})/P_{s_{n+1}}$$

can be encoded by an E-graph with d_n colours in the sense of Proposition 5.3 (a). The formula therein, combined with the characterization of ϕ_n given in Proposition 5.3 (c), yields a commutative diagram

$$\begin{tikzcd}
\mathrm{GL}(s_n)/P_{s_n} \arrow[r, "\phi_n"] \arrow[d, "\prod_i \psi^{(i)}"] & \mathrm{GL}(s_{n+1})/P_{s_{n+1}} \\
\prod_{i=1}^{d_n} \mathrm{Fl}(\underline{\ell}^{(i)}; W_n^{(i)}) \arrow[ur, "\xi_n"] &
\end{tikzcd}$$

where $\psi^{(i)}$ is the embedding corresponding to the subgraph of $\mathcal{G}(\alpha, \beta)$ formed by removing all the ordinary edges which are not of colour i ($\psi^{(i)}$ is a strict standard extension due to Proposition 5.3 (c)–(d)), $\underline{\ell}^{(i)}$ is an appropriate dimension vector, and the embedding ξ_n is induced by the decomposition (7.2). The theorem follows from this construction. □

The above proof yields the following sharpening of Theorem 7.4.

Corollary 7.5. *In the framework of Theorem 7.4, let $\mathcal{G}$ be the $\mathbf{s}$-graph corresponding to $\mathbf{P}$ in the sense of Section 5.3. Let $\mathcal{G} = \bigcup_{i \in \mathcal{I}} \mathcal{G}_i$ be a decomposition into subgraphs so that all ordinary edges of $\mathcal{G}_i$ are of the same colour. Then, the ind-variety $\mathbf{X}$ is isomorphic to a direct product of ind-varieties of generalized flags $\prod_{i \in \mathcal{I}} \mathbf{X}_i$ where $\mathbf{X}_i$ has an exhaustion with embeddings encoded by $\mathcal{G}_i$.*

Outlook

We see the results of this paper as a small first step in the study of locally projective homogeneous ind-spaces of locally reductive ind-groups. One inevitable question for a future such study is, given two non-isomorphic locally reductive ind-groups $\mathbf{G}$ and $\mathbf{G}'$, when are two homogeneous spaces $\mathbf{G}/\mathbf{P}$ and $\mathbf{G}'/\mathbf{Q}$ isomorphic as ind-varieties ? A further natural direction of research could be a comparison of Bott–Borel–Weil type results on $\mathbf{G}/\mathbf{P}$ and $\mathbf{G}'/\mathbf{Q}$. We finish the paper by pointing out that the reader can verify that Theorem 6.1 remains valid if one replaces $\mathrm{GL}(\mathbf{s})$ by any pure diagonal ind-group in the terminology of [1].

* * * * *

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