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Ind-Varieties of Generalized Flags as Homogeneous Spaces for Classical Ind-Groups

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To Joseph A. Wolf with admiration

1 Introduction

Flag varieties play a fundamental role both in representation theory and in algebraic geometry. There are two standard approaches to flag varieties, the group-theoretic one, where a flag variety is defined as G/P for a classical algebraic group G and a parabolic subgroup P, and the geometric one, where a flag variety is defined as the set of all chains of subspaces of fixed dimensions in a finite-dimensional vector space, which in addition are assumed isotropic in the presence of a bilinear form. The very existence of these two approaches is in the heart of the interplay between representation theory and geometry.

The main topic of this paper is a purely geometric construction of homogeneous spaces for the classical ind-groups $SL(\infty)$, $SO(\infty)$, and $Sp(\infty)$. Despite the fact that many phenomena related to inductive limits of classical groups have been studied (see, for instance, [6, 7, 9, 11, 12]), many natural questions remain unanswered. In particular, the only approach to homogeneous spaces of classical ind-groups discussed in the literature is a representation-theoretic one and has been introduced by Wolf and his collaborators [3, 5].

The difficulty in the purely geometric approach is that the consideration of flags, that is, chains of subspaces enumerated by integers, is no longer sufficient. To illustrate

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the problem, let, more specifically, G denote the ind-group $SL(\infty)$ over a field of characteristic 0, and let P be a parabolic subgroup of G. By definition, G is the union of a standard system of nested algebraic groups $SL(n)$ and P is the union of parabolic subgroups. If V is the natural representation of G, all P-invariant subspaces in V form a chain $\mathcal{C}$ of subspaces of V. In general, the chain $\mathcal{C}$ has a rather complicated structure and is not necessarily a flag, that is, it cannot be indexed by integers. We show, however, that $\mathcal{C}$ always contains a canonical subchain $\mathcal{F}$ of subspaces of V with the property that every element of $\mathcal{F}$ is either the immediate predecessor F' of a subspace $F'' \in \mathcal{F}$ or the immediate successor F'' of a subspace $F' \in \mathcal{F}$, and, in addition, each nonzero vector $v \in V$ belongs to a difference $F''\backslash F'$. These two properties define the generalized flags. (Maximal generalized flags already appeared in [2] in a related but somewhat different context.) If, in addition, the vector space V is equipped with a nondegenerate bilinear (symmetric or antisymmetric) form, we introduce the notion of an isotropic generalized flag.

Informally, we think of two (possibly isotropic) generalized flags being commensurable if they only differ in a finite-dimensional subspace of V in which they reduce to flags of the same type. The precise definition is given in Section 5. The main result of this paper is the construction of the ind-varieties of commensurable generalized flags and their identification with homogeneous ind-spaces G/P for classical locally linear ind-groups G isomorphic to $SL(\infty)$, $SO(\infty)$, or $Sp(\infty)$, and corresponding parabolic subgroups P.

The paper is concluded by providing two applications, an explicit computation of the Picard group of any ind-variety of commensurable generalized flags X and a criterion for projectivity of X. We show that the Picard group of X admits a description very similar to the classical one; however, X is projective if and only if it is an ind-variety of usual flags.

The "flag realization" of the ind-varieties G/P given in the present paper opens the way for a detailed and explicit study of the geometry of G/P, which should play a role as prominent as the geometric representation theory of the classical algebraic groups.

Conventions. $\mathbb{N}$ stands for $\{1, 2, \dots\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The ground field is a field k of characteristic 0 which will be assumed algebraically closed only when explicitly indicated in the text. As usual, $k^\times$ is the multiplicative group of k. The superscript * denotes dual vector space. The signs $\varinjlim$ and $\varprojlim$ stand, respectively, for direct and inverse limit over a direct or inverse system of morphisms parametrized by $\mathbb{N}$ or $\mathbb{Z}_+$. $\Gamma(X, \mathcal{L})$ denotes the global sections of a sheaf $\mathcal{L}$ on a topological space X. All orders $\prec$ are assumed linear and strict (i.e., $\alpha \prec \alpha$ never holds), and all partial orders $\prec$ are assumed to have the additional property that the relation "x is not comparable with y" (i.e., "neither $x \prec y$ nor $y \prec x$") is an equivalence relation. Such partial orders have the property that they induce a linear order on the set of equivalence classes of noncomparable elements.

2 Preliminaries

An *ind-variety* (over k) is a set X with a filtration

$$X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots \tag{2.1}$$

such that $X = \cup_{n\in\mathbb{Z}_+} X_n$, each X_n is a Noetherian algebraic variety, and the inclusions $X_n \subset X_{n+1}$ are closed immersions of algebraic varieties. An ind-variety X is automatically a topological space: a subset $U \subset X$ is *open* in X if and only if, for each n, $U \cap X_n$ is an open subvariety of X_n. The *sheaf of regular functions on* X, or *the structure sheaf* $\mathcal{O}_X$ of X, is the inverse limit $\mathcal{O}_X = \varprojlim \mathcal{O}_{X_n}$ of the sheaves of regular functions $\mathcal{O}_{X_n}$ on X_n. An ind-variety $X = \cup_{n\in\mathbb{Z}_+} X_n = \varinjlim X_n$ is *proper* if and only if all the varieties X_n are proper and is *affine* if and only if all the X_n are affine. A *morphism* from an ind-variety X to an ind-variety Y is a map $\varphi : X \to Y$ such that, for every $n \geq 0$, the restriction $\varphi|_{X_n}$ is a morphism of X_n into Y_m for some $m = m(n)$. An *isomorphism* of ind-varieties is a morphism which admits an inverse morphism. An *ind-subvariety* Z of X is a subset $Z \subset X$ such that $Z \cap X_n$ is a subvariety of X_n for each n. Finally, an *ind-group* is by definition a group object in the category of ind-varieties. In this paper, we consider only ind-groups G which are *locally linear*, that is, ind-varieties G with an ind-variety filtration $G_0 \subset G_1 \subset \cdots \subset G_n \subset G_{n+1} \subset \cdots$, such that all G_n are linear algebraic groups and the inclusions are group morphisms.

Let V be a vector space of countable dimension. Fix an integer $l \geq 1$. The set $\mathrm{Gr}(l; V)$ of all l-dimensional subspaces of V has a canonical structure of proper ind-variety: any filtration $0 \subset V_l \subset V_{l+1} \subset \cdots \subset V = \cup_{r\geq 0} V_{l+r}$, $\dim V_{l+r} = l + r$, induces a filtration

$$\mathrm{Gr}\big(l; V_l\big) \subset \mathrm{Gr}\big(l; V_{l+1}\big) \subset \cdots \subset \mathrm{Gr}(l; V), \tag{2.2}$$

and the associated ind-variety structure on $\mathrm{Gr}(l; V)$ is independent of the choice of filtration on V. For $l = 1$, $\mathbb{P}(V) := \mathrm{Gr}(1; V)$ is by definition the *projective ind-space associated to* V.

An *invertible sheaf* on an ind-variety X is a sheaf of $\mathcal{O}_X$-modules locally isomorphic to $\mathcal{O}_X$. The set of isomorphism classes of invertible sheaves on X is an abelian group (the group structure being induced by the operation of tensor product over $\mathcal{O}_X$ of invertible sheaves). By definition, the latter is the *Picard group* $\mathrm{Pic}\, X$ *of* X. It is an easy exercise to show that $\operatorname{Pic} X = \varprojlim \operatorname{Pic} X_n$ for any filtration (2.1). If $X = \mathbb{P}(V)$, then $\operatorname{Pic} X \cong \mathbb{Z}$. The preimage of 1 under this isomorphism is the class of the standard sheaf $\mathcal{O}_{\mathbb{P}(V)}(1)$, where, by definition, $\mathcal{O}_{\mathbb{P}(V)}(1) := \varprojlim \mathcal{O}_{\mathbb{P}(V_n)}(1)$.

An invertible sheaf $\mathcal{L}$ on a proper ind-variety X is *very ample* if, for some filtration (2.1), its restrictions $\mathcal{L}_n$ on X_n are very ample for all n, and all restriction maps $\Gamma(X_n; \mathcal{L}_n) \to \Gamma(X_{n-1}, \mathcal{L}_{n-1})$ are surjective. A very ample invertible sheaf defines a closed immersion of X into $\mathbb{P}(\varinjlim \Gamma(X_n, \mathcal{L}_n)^*)$ as for each n the restrictions $\mathcal{L}_n$ and $\mathcal{L}_{n-1}$ define a commutative diagram of closed immersions

$$\begin{tikzcd} X_{n-1} \arrow[r, hook] \arrow[d, phantom, "\cap" description] & \mathbb{P}\big(\Gamma(X_{n-1}, \mathcal{L}_{n-1})^*\big) \arrow[d, phantom, "\cap" description] \\ X_n \arrow[r, hook] & \mathbb{P}\big(\Gamma(X_n, \mathcal{L}_n)^*\big). \end{tikzcd} \tag{2.3}$$

Conversely, given a closed immersion $X \hookrightarrow \mathbb{P}(V)$, the inverse image of $\mathcal{O}_{\mathbb{P}(V)}(1)$ on X is a very ample invertible sheaf on X. Therefore, a proper ind-variety X is projective, that is, X admits a closed immersion into a projective ind-space, if and only if it admits a very ample invertible sheaf.

3 Generalized flags: definition and first properties

Let V be a vector space over k. A *chain of subspaces in* V is a set $\mathcal{C}$ of pairwise distinct subspaces of V such that for any pair $F', F'' \in \mathcal{C}$, either $F' \subset F''$ or $F'' \subset F'$. Every chain of subspaces $\mathcal{C}$ is ordered by proper inclusion. Given $\mathcal{C}$, we denote by $\mathcal{C}'$ (resp., by $\mathcal{C}''$) the subchain of $\mathcal{C}$ which consists of all $C \in \mathcal{C}$ with an immediate successor (resp., an immediate predecessor). A *generalized flag in* V is a chain of subspaces $\mathcal{F}$ which satisfies the following conditions:

(i) each $F \in \mathcal{F}$ has an immediate successor or an immediate predecessor, that is, $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$;

(ii) $V \backslash \{0\} = \cup_{F' \in \mathcal{F}'} F'' \backslash F'$, where $F'' \in \mathcal{F}''$ is the immediate successor of $F' \in \mathcal{F}'$.

Given a generalized flag $\mathcal{F}$ and a subspace $F'' \in \mathcal{F}''$ (resp., $F' \in \mathcal{F}'$), unless the contrary is stated explicitly, we will denote by F' (resp., by F'') its immediate predecessor (resp., immediate successor). Furthermore, condition (ii) implies that each nonzero vector $v \in V$ determines a unique pair $F'_v \subset F''_v$ of subspaces in $\mathcal{F}$ with $v \in F''_v \backslash F'_v$.

Example 3.1. (i) We define a *flag in* V to be a chain of subspaces $\mathcal{F}$ satisfying (ii) and which is isomorphic as an ordered set to a subset of $\mathbb{Z}$. A flag can be equivalently defined as a chain of subspaces $\mathcal{F}$ for which there exists a strictly monotonic map of ordered sets

$\varphi : \mathcal{F} \to \mathbb{Z}$ and, in addition, $\cap_{F\in\mathcal{F}} F = 0$ and $\cup_{F\in\mathcal{F}} = V$. There are four different kinds of flags: a finite flag of length k $\mathcal{F} = \{0 = F_1 \subset F_2 \subset \cdots \subset F_{k-1} \subset F_k = V\}$; an infinite ascending flag $\mathcal{F} = \{0 = F_1 \subset F_2 \subset F_3 \subset \cdots\}$, where $\cup_{i\geq 1} F_i = V$; an infinite descending flag $\mathcal{F} = \{\cdots \subset F_{-3} \subset F_{-2} \subset F_{-1} = V\}$, where $\cap_{i\leq -1} F_i = 0$; and a two-sided infinite flag $\mathcal{F} = \{\cdots \subset F_{-1} \subset F_0 \subset F_1 \subset \cdots\}$, where $\cap_{i\in\mathbb{Z}} F_i = 0$ and $\cup_{i\in\mathbb{Z}} F_i = V$.

(ii) A simple example of a generalized flag V, which is not a flag, is a generalized flag with both an infinite ascending part and an infinite descending part, that is, $\mathcal{F} = \{0 = F_1 \subset F_2 \subset F_3 \subset \cdots \subset F_{-3} \subset F_{-2} \subset F_{-1} = V\}$, where $\cup_{i\geq 1} F_i = \cap_{j\leq -1} F_j$.

(iii) Let V be a countable-dimensional vector space with basis $\{e_q\}_{q\in\mathbb{Q}}$. Set $F'(q) := \operatorname{span}\{e_r \mid r < q\}$ and $F''(q) := \operatorname{span}\{e_r \mid r \leq q\}$. Then $\mathcal{F} := \cup_{q\in\mathbb{Q}}\{F'(q), F''(q)\}$ is a generalized flag in V. $\mathcal{F}$ is not a flag, and moreover, no $F \in \mathcal{F}$ has both an immediate predecessor and an immediate successor.

The following proposition shows that each of the subchains $\mathcal{F}'$ and $\mathcal{F}''$ reconstructs $\mathcal{F}$.

Proposition 3.2. Let $\mathcal{F}$ be a generalized flag in V. Then

(i) for every $F' \in \mathcal{F}'$, $F' = \cup_{G''\in\mathcal{F}'', G''\subset F'', G''\neq F''} G''$;

(ii) for every $F'' \in \mathcal{F}''$, $F'' = \cap_{G'\in\mathcal{F}', G'\supset F', G'\neq F'} G'$. □

Proof. (i) The inclusion $F' \supset \cup_{G''\in\mathcal{F}'', G''\subset F'', G''\neq F''} G''$ is obvious. Assume now that $v \in F'$. Let $v \in H''\backslash H'$ for some $H' \in \mathcal{F}'$ and its immediate successor $H'' \in \mathcal{F}''$. Then $H' \subset F'$ and hence $H'' \subset F'$, that is, $v \in \cup_{G''\in\mathcal{F}'', G''\subset F'', G''\neq F''} G''$ which proves that $F' \subset \cup_{G''\in\mathcal{F}'', G''\subset F'', G''\neq F''} G''$. Assertion (ii) is proved in a similar way. ■

Any chain $\mathcal{C}$ of subspaces in V determines the following partition of V:

$$V = \sqcup_{v\in V} [v]_{\mathcal{C}}, \tag{3.1}$$

where $[v]_{\mathcal{C}} := \{w \in V \mid w \in F \Leftrightarrow v \in F, \forall F \in \mathcal{C}\}$. Consider this correspondence as a map π from the set of chains of subspaces in V into the set of partitions of V. This map is not injective, for $\pi(\mathcal{C}') = \pi(\mathcal{C})$ if $\mathcal{C}'$ is obtained from $\mathcal{C}$ by adding arbitrary intersections and unions of elements of $\mathcal{C}$. As we show in Proposition 3.3, the notion of a generalized flag provides us with a natural right inverse of π, that is, with a map γ (defined on the image of π) such that $\pi \circ \gamma = \mathrm{id}$. This explains the special role of generalized flags among arbitrary chains of subspaces in V. Namely, every generalized flag in V is a natural representative of the class of chains of subspaces in V which yield the same partition of V.

Proposition 3.3. Given a chain $\mathcal{C}$ of subspaces in V, there exists a unique generalized flag $\mathcal{F}$ in V for which $\pi(\mathcal{C}) = \pi(\mathcal{F})$. □

Proof. To prove the existence, set $F'_v := \cup_{W\in\mathcal{C}, v\notin W} W$ and $F''_v := \cap_{W\in\mathcal{C}, v\in W} W$, and put $\mathcal{F} := \cup_{v\in V\setminus\{0\}}\{F'_v, F''_v\}$. It is obvious from the definition of $\mathcal{F}$ that $\pi(\mathcal{C}) = \pi(\mathcal{F})$. To show that $\mathcal{F}$ is a generalized flag, notice that, for any pair of nonzero vectors $u, v \in V$, exactly one of the following three possibilities holds:

(i) $F'_u = F'_v$, and hence $F''_u = F''_v$;

(ii) $F'_u \subset F'_v$, and hence $F''_u \subset F'_v$;

(iii) $F'_u \supset F'_v$, and hence $F'_u \supset F''_v$.

Indeed, if, for every $W \in \mathcal{C}$, $u \in W$ if and only if $v \in W$, then $F'_u = F'_v$ and $F''_u = F''_v$. Assume now that there exists $W \in \mathcal{C}$ such that $u \in W$ but $v \notin W$. Then $F''_u \subset W \subset F'_v$. Similarly, if there exists $W \in \mathcal{C}$ such that $u \notin W$ but $v \in W$, we have $F''_v \subset W \subset F'_u$. The existence of $\mathcal{F}$ is now established.

The uniqueness follows from the fact that $[v]_{\mathcal{C}} = (\cap_{W\in\mathcal{C}, v\in W} W)\setminus(\cup_{W\in\mathcal{C}, v\notin W} W)$, while, for a generalized flag $\mathcal{F}$, $[v]_{\mathcal{F}} = F''_v\setminus F'_v$. ■

We now define the map γ by setting $\gamma(\pi(\mathcal{C})) := \mathcal{F}$, and put $\mathrm{fl} := \gamma \circ \pi$. Note that, for any generalized flag $\mathcal{F}$ in V, $\mathcal{F} = \mathrm{fl}(\mathcal{F}') = \mathrm{fl}(\mathcal{F}'')$.

In the example below we determine the preimages under fl of the generalized flags introduced in Example 3.1. The computation is based on the following simple fact: if $\bar{\mathcal{C}}$ is any chain in $\mathrm{fl}^{-1}(\mathcal{F})$, then every nonzero subspace $\bar{C} \in \bar{\mathcal{C}}$ is the union of spaces from $\mathcal{F}$.

Example 3.4. The cases (i), (ii), and (iii) below refer to the corresponding cases in Example 3.1.

(i) If $\mathcal{F}$ is a flag in V, then $\mathrm{fl}^{-1}(\mathcal{F})$ consists of $\mathcal{F}$ and the chains obtained from $\mathcal{F}$ by adding 0, V, or both, in case 0 and/or V, do not belong to $\mathcal{F}$.

(ii) In this case, $\mathrm{fl}^{-1}(\mathcal{F})$ consists of two chains: $\mathcal{F}$ itself and the chain obtained by adding $\cup_{i\geq 1} F_i = \cap_{j\leq -1} F_j$ to $\mathcal{F}$.

(iii) In this case, there are infinitely many chains $\bar{\mathcal{C}}$ with $\mathrm{fl}(\bar{\mathcal{C}}) = \mathcal{F}$. Set $F'(x) := \mathrm{span}\{e_r \mid r < x\}$ for any $x \in \mathbb{R}$, and let $\mathcal{C}$ denote the chain $\{F'(x) \mid x \in \mathbb{R}\} \cup \{F''(q) \mid q \in \mathbb{Q}\} \cup \{0, V\}$. It is easy to check that $\mathrm{fl}(\mathcal{C}) = \mathcal{F}$ and that any chain in $\mathrm{fl}^{-1}(\mathcal{F})$ is a subchain of $\mathcal{C}$. To characterize explicitly all chains in $\mathrm{fl}^{-1}(\mathcal{F})$, for any subchain $\bar{\mathcal{C}} \subset \mathcal{C}$, set $\mathbb{R}_{\bar{\mathcal{C}}} := \{x \in \mathbb{R} \mid F'(x) \in \bar{\mathcal{C}}\}$ and $\mathbb{Q}_{\bar{\mathcal{C}}} := \{q \in \mathbb{Q} \mid F''(q) \in \bar{\mathcal{C}}\}$. Then $\mathrm{fl}(\bar{\mathcal{C}}) = \mathcal{F}$ if and only if, for any $r \in \mathbb{Q}$, we have $r \in \mathbb{Q}_{\bar{\mathcal{C}}}$ or $r = \inf\{x \in \mathbb{R}_{\bar{\mathcal{C}}} \cup \mathbb{Q}_{\bar{\mathcal{C}}} \mid r < x\}$, and $r \in \mathbb{R}_{\bar{\mathcal{C}}}$ or $r = \sup\{x \in \mathbb{R}_{\bar{\mathcal{C}}} \cup \mathbb{Q}_{\bar{\mathcal{C}}} \mid x < r\}$.

A generalized flag $\mathcal{F}$ in V is *maximal* if it is not properly contained in another generalized flag in V. It is easy to see that the generalized flags introduced in Example 3.1(ii) and (iii) are maximal. More generally, a generalized flag $\mathcal{F}$ is maximal if and only if $\dim(F''_v/F'_v) = 1$ for every nonzero $v \in V$. Indeed, assume $\dim(F''_{v_0}/F'_{v_0}) > 1$ for some v_0.

Let $F \subset V$ be a subspace with proper inclusions $F'_{v_0} \subset F \subset F''_{v_0}$. Then the generalized flag $\mathcal{F} \cup \{F\}$ properly contains $\mathcal{F}$. Conversely, if $\dim(F''_v/F'_v) = 1$ for every nonzero $v \in V$, and if $\mathcal{G}$ is a generalized flag which contains $\mathcal{F}$, then $F'_v \subset G'_v \subset G''_v \subset F''_v$. Hence $F'_v = G'_v$, $F''_v = G''_v$, that is, $\mathcal{F} = \mathcal{G}$.

The map fl establishes a bijection between maximal chains of subspaces in V and maximal generalized flags in V. More precisely, if $\mathcal{C}$ is a maximal chain, $\mathrm{fl}(\mathcal{C})$ is the unique maximal generalized flag which is a subchain of $\mathcal{C}$. Conversely, $\mathcal{C}$ is the unique maximal chain containing $\mathrm{fl}(C)$. These latter statements are essentially equivalent to [2, Theorem 9]. For example, if $\mathcal{F}$ is the maximal flag from Example 3.1(iii), then its corresponding maximal chain $\mathcal{C}$ is described in Example 3.4(iii).

We conclude this section by introducing isotropic generalized flags. Let $w : V \times V \to k$ be a nondegenerate symmetric or skew-symmetric bilinear form on V. Denote by $U^{\perp}$ the w-orthogonal complement of a subspace $U \subset V$. A generalized flag $\mathcal{F}$ in V is w-*isotropic* if $F^{\perp} \in \mathcal{F}$ for every $F \in \mathcal{F}$, and if, furthermore, the map $F \mapsto F^{\perp}$ is an involution of $\mathcal{F}$. If $\mathcal{F}$ is a w-isotropic flag in V, the involution $F \mapsto F^{\perp}$ induces an involution τ on $\mathcal{F}'$ defined as follows. If $F' \in \mathcal{F}'$, then $(F'')^{\perp}$ is the immediate predecessor of $(F')^{\perp}$, and we set $\tau(F') := (F'')^{\perp}$. We also introduce the subspaces $\mathcal{T}' := \cup_{F \in \mathcal{F}, F \subset F^{\perp}} F$ and $\mathcal{T}'' := \cap_{F \in \mathcal{F}, F \supset F^{\perp}} F$ of V. Clearly, $\mathcal{T}' \subset \mathcal{T}''$. Since $(\sum W_\alpha)^{\perp} = \cap(W_\alpha^{\perp})$ for any family of subspaces $\{W_\alpha\}$ of V, we have $(\mathcal{T}')^{\perp} = \mathcal{T}''$. If $\mathcal{T}' \neq \mathcal{T}''$, then $\mathcal{T}' \in \mathcal{F}'$ and $\mathcal{T}'' \in \mathcal{F}''$ is the immediate successor of $\mathcal{T}'$. As $(\mathcal{T}')^{\perp} = \mathcal{T}''$, and $F \mapsto F^{\perp}$ is an involution of $\mathcal{F}$, we conclude that $(\mathcal{T}'')^{\perp} = \mathcal{T}'$ and hence $\mathcal{T}'$ is the unique fixed point of τ. If $\mathcal{T}' = \mathcal{T}''$, then τ has no fixed point. Moreover, in this case $\mathcal{T}' = \mathcal{T}''$ may or may not belong to $\mathcal{F}$. If $\mathcal{T}' = \mathcal{T}''$ belongs to $\mathcal{F}$, it has both an immediate successor and an immediate predecessor, but as an only exception of our use of the superscripts $'$ and $''$, $\mathcal{T}''$ is clearly not the immediate successor of $\mathcal{T}'$.

4 Compatible bases

If V is finite-dimensional, any ordered basis determines a maximal flag in V. Conversely, a maximal flag in V determines a set of compatible bases in V. More generally, if V is any vector space, $\mathcal{F}$ is a generalized flag in V and $\{e_\alpha\}_{\alpha \in A}$ is a basis of V, we say that $\mathcal{F}$ and $\{e_\alpha\}_{\alpha \in A}$ are *compatible* if there exists a strict partial order $\prec$ on A (satisfying the condition stated in the Conventions) such that $F'_{e_\alpha} = \mathrm{span}\{e_\beta \mid \beta \prec \alpha\}$ and $\mathcal{F} = \mathrm{fl}(\{F'_{e_\alpha}\}_{\alpha \in A})$. Then, as it is easy to see, each $F' \in \mathcal{F}'$ equals F'_{e_α} for some $\alpha \in A$, and $F''_{e_\alpha} = \mathrm{span}\{e_\beta \mid \beta \prec \alpha$ or β is not comparable with $\alpha\}$.

Not every generalized flag admits a compatible basis. Indeed, let $V := \mathbb{C}[[x]]$ be the space of formal power series in the indeterminate x and let $\mathcal{F}$ denote the flag $\cdots \subset F_n \subset F_{n-1} \subset \cdots \subset F_1 \subset F_0 = V$, where $F_n := x^n V$. Clearly, $\mathcal{F}$ is a maximal flag in V as

$\dim(F_{n-1}/F_n) = 1$ for all $n > 0$. However, as V is uncountable-dimensional, no basis of V can be compatible with the countable flag $\mathcal{F}$.

The following proposition shows that the uncountability of $\dim V$ is crucial in the above example.

Proposition 4.1. If V is countable-dimensional, every generalized flag $\mathcal{F}$ in V admits a compatible basis. □

Proof. First assume that $\mathcal{F}$ is a maximal generalized flag in V. Let $\{l_i\}_{i\in\mathbb{N}}$ be a basis of V. Define inductively a basis $\{e_i\}_{i\in\mathbb{N}}$ of V as follows. Put $e_1 := l_1$. Assuming that $e_1, \dots, e_n$ have been constructed, choose e_{n+1} of the form $l_{n+1} + c_1e_1 + \cdots + c_ne_n$ so that $F'_{e_{n+1}}$ is not among $F'_{e_1}, \dots, F'_{e_n}$. Then, obviously,

$$\operatorname{span}\{l_1, \dots, l_n\} = \operatorname{span}\{e_1, \dots, e_n\} \tag{4.1}$$

for every n and the subspaces F'_{e_n} are pairwise distinct. Furthermore, as it is not difficult to check, for every $F' \in \mathcal{F}'$, the set $F''\backslash F'$ contains exactly one element of the basis $\{e_i\}_{i\in\mathbb{N}}$, and hence $\mathbb{N}$ is linearly ordered in the following way: $i \prec j$ if and only if F'_{e_i} is a proper subset of F'_{e_j}. Now it is clear that $F'_{e_n} = \operatorname{span}\{e_i\}_{i\prec n}$ and, since $\mathcal{F} = \mathrm{fl}(\mathcal{F}')$, $\mathcal{F}$ is compatible with $\{e_i\}_{i\in\mathbb{N}}$.

For a not necessarily maximal generalized flag $\mathcal{F}$, it is enough to consider a basis compatible with a maximal generalized flag $\mathcal{G}$ containing $\mathcal{F}$. Such a basis is automatically compatible with $\mathcal{F}$. ■

Let V be a finite- or countable-dimensional vector space and w be a nondegenerate symmetric or skew-symmetric bilinear form on V. Let, furthermore, n take values in $\mathbb{Z}_+$. Define a basis of V of the form $\{e_n, e^n\}$ to be *of type* C if $w(e_i, e_j) = w(e^i, e^j) = 0$ and $w(e_i, e^j) = \delta_{i,j}$ for a skew-symmetric w. A basis of V of the form $\{e_0 = e^0, e_n, e^n\}$ (resp., $\{e_n, e^n\}$) is *of type* B (resp., *of type* D) if $w(e_i, e_j) = w(e^i, e^j) = 0$ and $w(e_i, e^j) = \delta_{i,j}$ for a symmetric w. For uniformity we will always label a basis of type B, C, or D simply as $\{e_n, e^n\}$, where we assume that in the case of B, $e_0 = e^0$ and n runs over $\mathbb{Z}_+$ when V is countable-dimensional, or over a finite subset of $\mathbb{Z}_+$ when V is finite-dimensional, while in the cases of C and D, n runs over $\mathbb{N}$ or over a finite subset of $\mathbb{N}$. A *w-isotropic basis of* V is by definition a basis of V admitting an order which makes it a basis of type B, C, or D. If V is finite-dimensional, then V admits a basis of type B, C, or D if and only if, respectively, w is symmetric and V is odd-dimensional, w is skew-symmetric and then V is necessarily even-dimensional, or w is symmetric and V is even-dimensional. If V

is infinite-dimensional, the following infinite-dimensional analog of the Gram-Schmidt orthogonalization process holds.

Lemma 4.2. Let V be a countable-dimensional vector space, and w a nondegenerate bilinear form on V. If w is skew-symmetric, then V admits a basis of type C. If w is symmetric and k is algebraically closed, then V admits bases both of type B and of type D. □

Proof. Consider first the case when w is symmetric. We start by constructing an orthonormal basis $\{f_n\}_{n\in\mathbb{N}}$ of V.

Fix a basis $\{v_n\}_{n\in\mathbb{N}}$ of V. Next we construct inductively finite subsets $J_0, J_1, J_2, \ldots$ of $\mathbb{N}$ and an orthonormal basis $\{f_n\}_{n\in\mathbb{N}}$ of V satisfying the following three conditions:

(i) $\{1, 2, \ldots, n\} \subset J_n$;

(ii) $J(n) \subset J(n+1)$;

(iii) $\mathrm{span}\{f_i\}_{i\in J_n} = \mathrm{span}\{v_i\}_{i\in J_n}$.

For $n = 0$ we set $J_0 := \varnothing$. Assume J_n together with $\{f_i\}_{i\in J_n}$ have been constructed. Let k be the smallest positive integer not contained in J_n. Set $\widetilde{f_k} := v_k - \sum_{i\in J_n} w(v_k, e_i)e_i$. If $w(\widetilde{f_k}, \widetilde{f_k}) \neq 0$, we put $J_{n+1} := J_n \cup \{k\}$ and $f_k := (1/\sqrt{w(\widetilde{f_k}, \widetilde{f_k})})\widetilde{f_k}$, where the choice of the square root is arbitrary. If $w(\widetilde{f_k}, \widetilde{f_k}) = 0$, then there exists $s \notin (J_n \cup \{k\})$ with $w(\widetilde{f_k}, v_s) \neq 0$. Put $\widetilde{f_s} := v_s - \sum_{i\in J_n} w(v_s, e_i)e_i$. Then the restriction of w on the two-dimensional space $\mathrm{span}\{\widetilde{f_k}, \widetilde{f_s}\}$ is nondegenerate, and hence there is an orthonormal basis $\{f_k, f_s\}$ of $\mathrm{span}\{\widetilde{f_k}, \widetilde{f_s}\}$. Since $\mathrm{span}\{\widetilde{f_k}, \widetilde{f_s}\}$ is orthogonal to $\mathrm{span}\{f_i\}_{i\in J_n}$, the set $J_{n+1} := J_n \cup \{k, s\}$ satisfies conditions (i), (ii), and (iii). This completes the inductive construction of an orthonormal basis $\{f_n\}_{n\in\mathbb{N}}$ of V.

A basis of type B in V is given by $e_0 = e^0 := f_1$, $e_n := (1/\sqrt{2})(f_{2n} + \sqrt{-1}f_{2n+1})$, and $e^n := (1/\sqrt{2})(f_{2n} - \sqrt{-1}f_{2n+1})$ for $n \in \mathbb{N}$. A basis of type D in V is given by $e_n := (1/\sqrt{2})(f_{2n-1} + \sqrt{-1}f_{2n})$ and $e^n := (1/\sqrt{2})(f_{2n-1} - \sqrt{-1}f_{2n})$ for $n \in \mathbb{N}$.

The case when w is skew-symmetric is simpler. Indeed, it is possible to modify the construction of an orthonormal basis in the symmetric case so that the restriction of w on $\mathrm{span}\{f_i\}_{i\in J_n} = \mathrm{span}\{v_i\}_{i\in J_n}$ is nondegenerate and some relabeling $\{e_i, e^i\}_{i\in\{1,\ldots,n\}}$ of $\{f_i\}_{i\in J_n}$ is a basis of type C in $\mathrm{span}\{v_i\}_{i\in J_n}$ for every n. ■

In the rest of the paper, we assume that the dimension of V is countable. We show next that every w-isotropic generalized flag admits a compatible w-isotropic basis.

Proposition 4.3. Let $\mathcal{F}$ be a w-isotropic generalized flag in V.

(i) Assume that w is skew-symmetric. Then V admits a basis of type C compatible with $\mathcal{F}$. In particular, the vector space $\mathcal{T}''/\mathcal{T}'$, is even-dimensional or infinite-dimensional.

(ii) Assume that k is algebraically closed and w is symmetric. If the vector space $\mathcal{T}''/\mathcal{T}'$ is odd-dimensional or infinite-dimensional, then V admits a basis of type D compatible with $\mathcal{F}$. If $\mathcal{T}''/\mathcal{T}'$ is even-dimensional or infinite-dimensional, then V admits a basis of type B compatible with $\mathcal{F}$. □

Proof. Let $\{l_n\}_{n\in\mathbb{N}}$ be a basis of V compatible with $\mathcal{F}$. Set $U_{F'} := \operatorname{span}\{l_i \mid F'_{l_i} = F'\}$ for $F' \in \mathcal{F}'$. Then $F' = \oplus_{G'\in\mathcal{F}', G'\subset F', G'\neq F'} U_{G'}$ and $F'' = F' \oplus U_{F'}$ for every $F' \in \mathcal{F}'$. It is clear therefore that the restriction of w on $U_{F'} \times U_{\tau(F')}$ is a nondegenerate bilinear form for every $F' \in \mathcal{F}'$. Furthermore, if τ has a fixed point, the restriction of w on $U_{\mathcal{T}'} \times U_{\mathcal{T}'}$ is a nondegenerate skew-symmetric (resp., symmetric) bilinear form. If w is skew-symmetric, this implies, in particular, that $U_{\mathcal{T}'}$, and hence $\mathcal{T}''/\mathcal{T}'$, is even-dimensional or infinite-dimensional. Then, by Lemma 4.2, $U_{\mathcal{T}'}$ admits a basis of type C, B, or D depending on whether w is skew-symmetric or symmetric and on the dimension of $\mathcal{T}''/\mathcal{T}'$. Denote such a basis by $\{l'_i, l'^i\}$. If τ does not have a fixed point, then $\mathcal{T}''/\mathcal{T}' = 0$, the corresponding basis is empty, and, hence, of type C or D depending on whether w is skew-symmetric or symmetric. Let, furthermore, $\{l''_i\}$ (resp., $\{l''^i\}$) be the subset of $\{l_n\}$ consisting of all l_n for which F'_{l_n} is properly contained in $\mathcal{T}'$ (resp., F''_{l_n} properly contains $\mathcal{T}''$). Finally, relabel the sets $\{l'_i\}\cup\{l''_i\}$ and $\{l'^i\}\cup\{l''^i\}$ and denote the respective resulting sets by $\{g_n\}$ and $\{g^n\}$, so that $g^n = l'^i$ if and only if $g_n = l'_i$.

We are now ready to construct inductively the desired w-isotropic basis $\{e_n, e^n\}$. Assume that e_i, e^i have been constructed for $i \le n$. Put $e_{n+1} := g_{n+1} - \sum_{i=1}^{n}(w(e_i, g_{n+1})e^i + w(g_{n+1}, e^i)e_i)$. If $g_{n+1} \in \{l''_i\}$, let k be the smallest integer for which $g^k \in U_{\tau(F'_{g_{n+1}})}$ and $w(e_{n+1}, g^k) = 1$. If $g_{n+1} \in \{l'_i\}$, we set $k := n+1$. Set then $e^{n+1} := g^k - \sum_{i=1}^{n}(w(e_i, g^k)e^i + w(g^k, e^i)e_i)$. The construction ensures that $\{e_n, e^n\}$ is a basis of V of the same type as $\{l'_i, l'^i\}$ which is compatible with $\mathcal{F}$. ■

5 Ind-varieties of generalized flags

For a finite-dimensional V, two flags belong to the same connected component of the variety of all flags in V if and only if their types coincide, that is, if the dimensions of the subspaces in the flags coincide. If V is infinite-dimensional, then the notion of type is in general not defined, and flags, or generalized flags, can be compared using a notion of commensurability. Such notions are well known in the special case of subspaces of V, that is, of flags of the form $0 \subset W \subset V$, see [10] and [8, Chapter 7]. Below we introduce a notion of commensurability for generalized flags which in the case of subspaces reduces to a refinement of Tate's notion of commensurability, see [10].

In the rest of the paper we fix a basis $E = \{e_n\}$ of V. In the presence of a bilinear form w on V, we fix a w-isotropic basis $E = \{e_n, e^n\}$, and whenever other bases of V or generalized flags in V are considered they are automatically assumed to be w-isotropic. We call a generalized flag $\mathcal{F}$ *weakly compatible with* E if $\mathcal{F}$ is compatible with a basis L of V such that $E \backslash (E \cap L)$ is a finite set. Furthermore, we define two generalized flags $\mathcal{F}$ and $\mathcal{G}$ in V to be E-*commensurable* if both $\mathcal{F}$ and $\mathcal{G}$ are weakly compatible with E and there exists an inclusion preserving bijection $\varphi : \mathcal{F} \to \mathcal{G}$ and a finite-dimensional subspace $U \subset V$ such that for every $F \in \mathcal{F}$

(i) $F \subset \varphi(F) + U$ and $\varphi(F) \subset F + U$;

(ii) $\dim(F \cap U) = \dim(\varphi(F) \cap U)$.

It follows immediately from the definition that any two E-commensurable generalized flags are isomorphic as ordered sets, and that two flags in a finite-dimensional space are E-commensurable if and only if their types coincide. (In the latter case the condition of weak compatibility with E is empty.) Furthermore, E-commensurability is an equivalence relation. Indeed, it is obviously reflexive and symmetric. It is also transitive. To see this, note first that, in the definition of E-commensurability, one can replace (ii) by

(ii′) $\dim(F/(F \cap \varphi(F))) = \dim(\varphi(F)/(F \cap \varphi(F)))$.

Consider now $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ such that $\mathcal{F}$ is E-commensurable with $\mathcal{G}$ and $\mathcal{G}$ is E-commensurable with $\mathcal{H}$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ be the respective bijections and let U and W be the finite-dimensional subspaces of V corresponding to φ and ψ, respectively. Then $\mathcal{F}$ and $\mathcal{H}$ satisfy (i) and (ii′) with $\psi \circ \varphi : \mathcal{F} \to \mathcal{H}$ and $U + W$.

Example 5.1. (i) Let $\mathcal{F} = \{0 \subset F \subset V\}$ and $\mathcal{G} = \{0 \subset G \subset V\}$. If F and G are finite-dimensional, then $\mathcal{F}$ and $\mathcal{G}$ are automatically weakly compatible with E. Furthermore, $\mathcal{F}$ and $\mathcal{G}$ are E-commensurable if and only if $\dim F = \dim G$. If, however, F and G are infinite-dimensional, then the condition that $\mathcal{F}$ and $\mathcal{G}$ are weakly compatible with E is not automatic. For example, if $F = \operatorname{span}\{e_2, e_3, \dots\}$ and $G = \operatorname{span}\{e_2 - e_1, e_3 - e_1, \dots\}$, then $\mathcal{F}$ is weakly compatible with E but $\mathcal{G}$ is not, and consequently, $\mathcal{F}$ and $\mathcal{G}$ are not E-commensurable. Finally, if F and G are both of finite codimension in V, and $\mathcal{F}$ and $\mathcal{G}$ are weakly E-compatible, then $\mathcal{F}$ and $\mathcal{G}$ are E-commensurable if and only if $\operatorname{codim}_V F = \operatorname{codim}_V G$.

(ii) Let $\mathcal{F} = \{0 = F_1 \subset F_2 \subset F_3 \subset \cdots\}$ and $\mathcal{G} = \{0 = G_1 \subset G_2 \subset G_3 \subset \cdots\}$ be two finite or infinite ascending flags in V compatible with E. If all subspaces F_i and G_i are finite-dimensional, then $\mathcal{F}$ and $\mathcal{G}$ are E-commensurable if and only if $\dim F_i = \dim G_i$ for every i, and $F_n = G_n$ for large enough n. If, however, there are infinite-dimensional spaces among F_i and G_i, the above conditions are still necessary for $\mathcal{F}$ and $\mathcal{G}$ to be E-commensurable but they are not always sufficient. The exact sufficient conditions can be derived as a consequence of the proof of Proposition 5.2 below.

Given a generalized flag $\mathcal{F}$ weakly compatible with E, we denote by $\mathcal{F}\ell(\mathcal{F}, E)$ the set of all generalized flags in V E-commensurable with $\mathcal{F}$. For the rest of the paper we fix the following notations: $E_n = \{e_i\}_{i \leq n}$, $V_n = \operatorname{span} E_n$, $E_n^c := \{e_i\}_{i>n}$, and $V_n^c := \operatorname{span} E_n^c$. If $\mathcal{F}$ is a w-isotropic generalized flag in V, then $\mathcal{F}\ell(\mathcal{F}, w, E)$ stands for the set of all w-isotropic generalized flags E-commensurable with $\mathcal{F}$. If $\mathcal{G}$ is an isotropic generalized flag in V, then the involution τ is an order-reversing involution on $\mathcal{G}'$ considered as an ordered set. In this case, $E_n = \{e_i, e^i\}_{i \leq n}$, $V_n = \operatorname{span} E_n$, $E_n^c := \{e_i, e^i\}_{i>n}$, and $V_n^c := \operatorname{span} E_n^c$. Since all generalized flags in $\mathcal{F}\ell(\mathcal{F}, w, E)$ are isomorphic as ordered sets, we will use the same letter τ to denote the involution on $\mathcal{G}'$ for any $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, w, E)$.

Proposition 5.2. $\mathcal{F}\ell(\mathcal{F}, E)$, as well as $\mathcal{F}\ell(\mathcal{F}, w, E)$, has a natural structure of an ind-variety. $\square$

Proof. We present the proof in the case of $\mathcal{F}\ell(\mathcal{F}, E)$ only. The reader will supply a similar proof for $\mathcal{F}\ell(\mathcal{F}, w, E)$. For any $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, E)$ choose a positive integer $n_{\mathcal{G}}$ such that $\mathcal{F}$ and $\mathcal{G}$ are compatible with bases containing $E^c_{n_{\mathcal{G}}}$, and $V_{n_{\mathcal{G}}}$ contains a finite-dimensional subspace U which (together with the corresponding φ) makes $\mathcal{F}$ and $\mathcal{G}$ E-commensurable. Obviously we can pick $n_{\mathcal{F}}$ so that $n_{\mathcal{F}} \leq n_{\mathcal{G}}$ for every $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, E)$. Set also

$$\mathcal{G}_n := \{G \cap V_n \mid G \in \mathcal{G}\} \tag{5.1}$$

for $n \geq n_{\mathcal{G}}$.

The type of the flag $\mathcal{F}_n$ yields a sequence of integers $0 = d_{n,0} < d_{n,1} < \cdots < d_{n,s_{n-1}} < d_{n,s_n} = n$, and $\mathcal{F}\ell(\mathcal{F}_n, E_n)$ is the usual flag variety $\mathcal{F}\ell(d_n; V_n)$ of type $d_n = (d_{n,1}, \ldots, d_{n,s_n-1})$ in V_n. Notice that $s_{n+1} = s_n$ or $s_{n+1} = s_n + 1$. Furthermore, in both cases an integer j_n is determined as follows: in the former case, $d_{n+1,i} = d_{n,i}$ for $0 \leq i < j_n$ and $d_{n+1,i} = d_{n,i} + 1$ for $j_n \leq i < s_n$, and in the latter case $d_{n+1,i} = d_{n,i}$ for $0 \leq i < j_n$ and $d_{n+1,i} = d_{n,i-1} + 1$ for $j_n \leq i < s_n$.

Now we define a map $\iota_n : \mathcal{F}\ell(d_n; V_n) \to \mathcal{F}\ell(d_{n+1}; V_{n+1})$ for every $n \geq n_{\mathcal{F}}$. Given $\mathcal{G}_n = \{0 = G_0^n \subset G_1^n \subset \cdots \subset G_{s_n}^n = V_n\} \in \mathcal{F}\ell(d_n; V_n)$, put $\iota_n(\mathcal{G}_n) = \mathcal{G}_{n+1} := \{0 = G_0^{n+1} \subset G_1^{n+1} \subset \cdots \subset G_{s_{n+1}}^{n+1} = V_{n+1}\}$, where

$$G_i^{n+1} := \begin{cases} G_i^n & \text{if } 0 \leq i < j_n, \\ G_i^n \oplus k e_{n+1} & \text{if } j_n \leq i \leq s_{n+1} \text{ and } s_{n+1} = s_n, \\ G_{i-1}^n \oplus k e_{n+1} & \text{if } j_n \leq i \leq s_{n+1} \text{ and } s_{n+1} = s_n + 1. \end{cases} \tag{5.2}$$

It is clear that ι_n is a closed immersion of algebraic varieties, and hence $\varinjlim \mathcal{F}\ell(d_n; V_n)$ is an ind-variety. Let $\psi_n : \mathcal{F}\ell(d_n; V_n) \to \varinjlim \mathcal{F}\ell(d_n; V_n)$ denote the canonical embedding corresponding to the direct system $\{\iota_n\}$.

To endow $\mathcal{F}\ell(\mathcal{F}, E)$ with an ind-variety structure, we construct a bijection $\mathcal{F}\ell(\mathcal{F}, E) \to \varinjlim \mathcal{F}\ell(d_n; V_n)$. Set

$$\theta : \mathcal{F}\ell(\mathcal{F}, E) \longrightarrow \varinjlim \mathcal{F}\ell\big(d_n; V_n\big), \qquad \theta(\mathcal{G}) := \varinjlim \mathcal{G}_n, \tag{5.3}$$

see (5.1). Checking that θ is injective is straightforward. To check that θ is surjective, fix $\widetilde{\mathcal{G}} = \varinjlim \widetilde{\mathcal{G}_n} \in \varinjlim \mathcal{F}\ell(d_n; V_n)$, an integer $\tilde{n}$, and a flag $\widetilde{\mathcal{G}_{\tilde{n}}} \in \mathcal{F}\ell(d_{\tilde{n}}; V_{\tilde{n}})$ with $\psi_{\tilde{n}}(\widetilde{\mathcal{G}_{\tilde{n}}}) = \widetilde{\mathcal{G}}$. Denote by $\varphi_{\tilde{n}}$ the inclusion preserving bijection $\varphi_{\tilde{n}} : \mathcal{F}_{\tilde{n}} \to \widetilde{\mathcal{G}_{\tilde{n}}}$. For every $F \in \mathcal{F}$, put $\varphi(F) := \varphi_{\tilde{n}}(F \cap V_{\tilde{n}}) \oplus (F \cap V_{\tilde{n}}^c)$. It is clear that $\mathcal{G} := \{\varphi(F)\}_{F\in\mathcal{F}}$ is a generalized flag in V E-commensurable with $\mathcal{F}$ via φ and $V_{\tilde{n}}$. Furthermore, using (5.2), one verifies that $\theta(\mathcal{G}) = \widetilde{\mathcal{G}}$. Hence θ is surjective. ■

Example 5.3. Let $\mathcal{F} = \{0 \subset F \subset V\}$, see Example 5.1(i). If F is a finite-dimensional subspace of V of dimension l, then, regardless of E, $\mathcal{F}\ell(\mathcal{F}, E)$ is nothing but the ind-variety $\mathrm{Gr}(l; V)$ introduced in Section 2. If F is an infinite-dimensional subspace of V of codimension l, then as a set $\mathcal{F}\ell(\mathcal{F}, E)$ depends on the choice of E. However, the isomorphisms between $\mathrm{Gr}(l; V_n)$ and $\mathrm{Gr}(n - l; V_n)$ extend to an ind-variety isomorphism between $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathrm{Gr}(l; V)$ which depends on E. The latter isomorphism is a particular case of the following general duality. Let $\mathcal{F}$ be an arbitrary generalized flag in V. Assume that E is compatible with $\mathcal{F}$, and for every $F \in \mathcal{F}$ set $F^c := \mathrm{span}\{e \in E \mid e \notin F\}$. Then $\mathcal{F}^c := \{F^c \mid F \in \mathcal{F}\}$ is a generalized flag in V compatible with E and, moreover, $\mathcal{F}\ell(\mathcal{F}^c, E)$ is isomorphic to $\mathcal{F}\ell(\mathcal{F}, E)$.

We complete this section by defining big cells in $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{F}, w, E)$. Let $L = \{l_n\}_{n\in\mathbb{N}}$ be a basis of V compatible with $\mathcal{F}$ and such that $E\backslash(E \cap L)$ is a finite set, and let $U_{F'} = \mathrm{span}\{l \in L \mid F'_l = F'\}$ for any $F' \in \mathcal{F}'$. Denote by $\Phi = \{\Phi_{F'}\}_{F'\in\mathcal{F}'}$ a set of linear maps of finite rank $\Phi_{F'} : F' \to U_{F'}$, such that $\Phi_{F'} \neq 0$ for finitely many subspaces $F'_1 \subset \cdots \subset F'_p$ only. Given Φ, define

$$\begin{aligned} &\Gamma_{F'} : F' \longrightarrow F'', \quad \Gamma_{F'}(v) := v + \Phi_{F'}(v),\\ &\Gamma : V \longrightarrow V, \quad \Gamma(v) := \Gamma_{F'_p} \circ \cdots \circ \Gamma_{F'_{i+1}}(v), \end{aligned} \tag{5.4}$$

where i is the largest integer with $v \notin F'_i$. Put $\Phi(\mathcal{F}) := \mathrm{fl}(\{\Gamma(F')\}_{F'\in\mathcal{F}'})$. Then define the *big cell* $C(\mathcal{F}, E; L)$ *of* $\mathcal{F}\ell(\mathcal{F}, E)$ *corresponding to the basis* L by setting

$$C(\mathcal{F}, E; L) := \big\{\Phi(\mathcal{F}) \mid \text{ for all possible } \Phi\big\}. \tag{5.5}$$

To define the big cell $C(\mathcal{F}, w, E; L)$ in $\mathcal{F}\ell(\mathcal{F}, w, E)$, we start with a w-isotropic basis $L = \{l_n, l^n\}$ of V compatible with $\mathcal{F}$ and such that $E\backslash(E \cap L)$ is a finite set, and repeat the above construction of $\Gamma(F')$ for all $F' \in \mathcal{F}'$ with $F' \subset \tau(F')$. As a result, we obtain subspaces

$\Gamma(F')$ for $F' \subset \tau(F')$ and set $\Phi(\mathcal{F}) := \mathrm{fl}(\{\Gamma(F'), (\Gamma(F'))^{\perp}\}_{F' \in \mathcal{F}, F' \subset \tau(F')})$. Then we set

$$C(\mathcal{F}, w, E; L) := \{\Phi(\mathcal{F}) \mid \text{ for all possible } \Phi\}. \tag{5.6}$$

Note that the role of $\mathcal{F}$ in defining big cells is not special and that big cells $\mathcal{C}(\mathcal{G}, E; L)$ or $\mathcal{C}(\mathcal{G}, w, E; L)$ are well defined for every $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, E)$, or, respectively, $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, w, E)$.

Proposition 5.4. (i) The big cell $C(\mathcal{F}, E; L)$ (resp., $C(\mathcal{F}, w, E; L)$) is an affine open ind-subvariety of $\mathcal{F}\ell(\mathcal{F}, E)$ (resp., of $\mathcal{F}\ell(\mathcal{F}, w, E)$).

(ii) The following equalities hold:

$$\mathcal{F}\ell(\mathcal{F}, E) = \cup_L C(\mathcal{F}, E; L), \tag{5.7}$$

$$\mathcal{F}\ell(\mathcal{F}, w, E) = \cup_L C(\mathcal{F}, w, E; L), \tag{5.8}$$

where the unions run over all bases (resp., w-isotropic bases) L of V compatible with $\mathcal{F}$ and such that $E\backslash(E \cap L)$ is a finite set. □

Proof. We discuss the case of $\mathcal{F}\ell(\mathcal{F}, E)$ only. The argument for the case of $\mathcal{F}\ell(\mathcal{F}, w, E)$ is similar. Put $L_n := \{l_i\}_{i \leq n}$ and $W_n := \operatorname{span} L_n$. Let $\mathcal{F}_n$ and $\mathcal{F}\ell(d_n; W_n)$ be as in the proof of Proposition 5.2. Set $\mathcal{C}(d_n; W_n; L_n) := \{\Phi(\mathcal{F})_n \mid \forall \Phi$ such that for every $F' \in \mathcal{F}', \Phi_{F'}(W_n) \subset W_n$ and $\Phi_{F'}(l_i) = 0$ for $i > n\}$. Obviously, $\mathcal{C}(d_n; W_n; L_n)$ is a big cell in $\mathcal{F}\ell(d_n; W_n)$, and hence is an affine open subset. Therefore, the inclusion $\iota_n(\mathcal{C}(d_n; W_n; L_n)) \subset \mathcal{C}(d_{n+1}; W_{n+1}; L_{n+1})$ and the equality $\varinjlim \mathcal{C}(d_n; W_n; L_n) = C(\mathcal{F}, E; L)$ show that $C(\mathcal{F}, E; L)$ is an affine open ind-subvariety of $\mathcal{F}\ell(\mathcal{F}, E)$. The fact that the set of cells $\{C(\mathcal{F}, E; L) \mid L$ is a basis of V compatible with $\mathcal{F}$ such that $L\backslash(E \cap L)$ is a finite set$\}$ is a covering of $\mathcal{F}\ell(\mathcal{F}, E)$ is an easy consequence of the definition of E-commensurability. ■

6 Ind-varieties of generalized flags as homogeneous ind-spaces

Let $G(E)$ be the group of automorphisms g of V such that $g(e) = e$ for all but finitely many $e \in E$ and in addition $\det g = 1$. Recall that $E_n = \{e_i\}_{i \leq n}$ and $V_n = \operatorname{span} E_n$. The natural inclusion

$$G(E_n) \subset G(E_{n+1}), \qquad g \longmapsto \kappa_n(g), \tag{6.1}$$

where $\kappa_n(g)_{|V_n} = g$ and $\kappa_n(g)(e) = e$ for $e \in E_{n+1}\backslash E_n$, is a closed immersion of algebraic groups. Furthermore, $G(E) = \cup_{n \in \mathbb{N}} G(E_n)$. In particular $G(E)$ is a locally linear ind-group, and $G(E) = G(L)$ for any basis L of V such that $E\backslash(E \cap L)$ is a finite set.

Similarly, when E is a w-isotropic basis of V, let $G^w(E) := \{g \in G(E) \mid w(g(u), g(v)) = w(u,v)$ for any $u, v \in V\}$. There are natural closed immersions $G^w(E_n) \subset G^w(E_{n+1})$, and $G^w(E) = \cup_{n\in\mathbb{N}} G^w(E_n)$, where in this case $E_n := \{e_i, e^i\}_{i\leq n}$.

The ind-group $G(E)$ (resp., $G^w(E)$) is immediately seen to be isomorphic to the classical ind-group $A(\infty)$ (resp., $B(\infty)$, $C(\infty)$, or $D(\infty)$ if E is a w-isotropic basis of type B, C, or D). The ind-groups $A(\infty)$, $B(\infty)$, $C(\infty)$, and $D(\infty)$ are discussed in detail in [3]. An alternative notation for $A(\infty)$ is $SL(\infty)$, and $B(\infty) \cong D(\infty)$ and $C(\infty)$ are also denoted, respectively, by $SO(\infty)$ and $Sp(\infty)$.

In the rest of the paper, the letter G will denote one of the groups $G(E)$ or $G^w(E)$, and G_n will denote, respectively, $G(E_n)$ or $G^w(E_n)$. The basis E equips G with a subgroup H, consisting of all diagonal automorphisms of V in G, that is, of the elements $g \in G$ such that $g(e) \in ke$ for every $e \in E$. We call H a *splitting Cartan subgroup* (in the terminology of [3], H is a Cartan subgroup of G). Following [3], for the purposes of the present paper, we define a *parabolic* (resp., a *Borel*) *subgroup of* G to be an ind-subgroup P (resp., B) of G such that its intersection with G_n for every n is a parabolic (resp., a Borel) subgroup of G_n for some, or equivalently any, order on E.

If $\mathcal{F}$ is a generalized flag in V compatible with E (and w-isotropic, whenever E is w-isotropic), we denote by $P_{\mathcal{F}}$ the stabilizer of $\mathcal{F}$ in G.

Proposition 6.1. (i) $P_{\mathcal{F}}$ is a parabolic subgroup of G containing H;

(ii) the map $\mathcal{F} \mapsto P_{\mathcal{F}}$ establishes a bijection between generalized flags in V compatible with E and parabolic subgroups of G containing H. □

Proof. The inclusion $H \subset P_{\mathcal{F}}$ follows directly from the definition of H and $P_{\mathcal{F}}$. Furthermore, $P_{\mathcal{F}} \cap G_n$ is a parabolic subgroup of G_n as it is the stabilizer of $\mathcal{F}_n$ in G_n. Hence $P_{\mathcal{F}}$ is a parabolic subgroup of G. If, conversely, $P = \cup_n P_n$ is a parabolic subgroup of G containing H, denote by $\mathcal{F}(n)$ the flag in V_n whose stabilizer is P_n. Note that $\mathcal{F}(n)$ maps into $\mathcal{F}(n+1)$. More precisely, for $G = G(E)$, $\mathcal{F}(n+1) = \iota_n(\mathcal{F}(n))$, see (5.2); and for $G = G^w(E)$, the corresponding map is the w-isotropic analog of ι_n which we leave to the reader to reconstruct. In both cases, we define $\mathcal{F}$ as $\theta^{-1}(\varinjlim \mathcal{F}(n))$. A direct checking shows that $P = P_{\mathcal{F}}$. ■

Proposition 6.1 further justifies our consideration of generalized flags, see the discussion before Proposition 3.3. Indeed, it is clear that if $P \subset G$ is the stabilizer of a chain $\mathcal{C}$ of subspaces in V, then P depends only on the partition $\pi(\mathcal{C})$, see (3.1), and not on $\mathcal{C}$ itself. Therefore, the generalized flag $\mathcal{F}$ emerges as a representative of the class of all chains $\mathcal{C}$ which have P as a stabilizer in G. Moreover, Proposition 4.1 together with Proposition 6.1 (resp., Propositions 4.3 and 6.1 for w-isotropic flags) imply that the stabilizer in G of any generalized flag (resp., isotropic generalized flag) compatible with E is a parabolic subgroup of G. Finally, maximal generalized flags in V correspond to Borel subgroups under the above bijection.

Note that for any order on E and for any generalized flag $\mathcal{F}$ compatible with E, $G/P_{\mathcal{F}} = \cup_n (G_n/P_n)$, where $P_n := P_{\mathcal{F}} \cap G_n$. In particular, $G/P_{\mathcal{F}}$ is an ind-variety. Moreover, any other order on E, for which E is isomorphic to $\mathbb{N}$ as an ordered set, defines an isomorphic ind-variety. We are now ready to exhibit the homogeneous ind-space structure on $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{F}, w, E)$.

Theorem 6.2. For any E and $\mathcal{F}$ as above there is a respective isomorphism of ind-varieties $\mathcal{F}\ell(\mathcal{F}, E) \cong G/P_{\mathcal{F}}$ or $\mathcal{F}\ell(\mathcal{F}, w, E) \cong G/P_{\mathcal{F}}$. □

Proof. Given $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, E)$ (or, resp., $\mathcal{G} \in \mathcal{F}\ell(\mathcal{F}, w, E)$), let $U \subset V$ be the finite-dimensional subspace whose existence is ensured by the E-commensurability of $\mathcal{F}$ and $\mathcal{G}$. We may assume that $U = V_n = \operatorname{span} E_n$ for some n. Since $\mathcal{F}_n$ and $\mathcal{G}_n$ are flags of the same type in the finite-dimensional space V_n, there exists $g_n \in G_n$, so that $g(\mathcal{F}_n) = \mathcal{G}_n$. We extend g_n to an element $g \in G$ by setting $g(e) = e$ for $e \in E \backslash E_n$. Now

$$f : \mathcal{F}\ell(\mathcal{F}, E) \longrightarrow G/P_{\mathcal{F}} \ \big(\text{or, resp., } f : \mathcal{F}\ell(\mathcal{F}, w, E) \longrightarrow G/P_{\mathcal{F}}\big), \qquad f(\mathcal{G}) := gP, \tag{6.2}$$

is a well-defined map and it is easy to check that it is an isomorphism of ind-varieties. ■

7 Picard group and projectivity

The interpretation of $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{F}, w, E)$ as homogeneous ind-spaces $G/P_{\mathcal{F}}$ provides us with a representation-theoretic description of the Picard groups of $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{F}, w, E)$. Namely, $\operatorname{Pic} \mathcal{F}\ell(\mathcal{F}, E)$, as well as $\operatorname{Pic} \mathcal{F}\ell(\mathcal{F}, w, E)$, is naturally isomorphic to the group of integral characters of the Lie algebra of the ind-group $P_{\mathcal{F}}$.

Consider $\mathcal{F}\ell(\mathcal{F}, E)$. There is a canonical isomorphism of abelian groups $\operatorname{Pic} \mathcal{F}\ell(\mathcal{F}, E) = \operatorname{Hom}(P_{\mathcal{F}}, k^\times)$. To see this, notice that $\operatorname{Pic} \mathcal{F}\ell(\mathcal{F}, E) = \varprojlim \operatorname{Pic} \mathcal{F}\ell(d_n; V_n) = \varprojlim \operatorname{Pic} G_n/(P_{\mathcal{F}})_n$. It is a classical fact that $\operatorname{Pic} G_n/(P_{\mathcal{F}})_n = \operatorname{Hom}((P_{\mathcal{F}})_n, k^\times)$ for every n, and an immediate verification shows that the diagram

$$\begin{array}{ccc} \operatorname{Pic}\left(G_{n+1}/\left(P_{\mathcal{F}}\right)_{n+1}\right) & \cong & \operatorname{Hom}\left(\left(P_{\mathcal{F}}\right)_{n+1}, k^\times\right) \\ \downarrow & & \downarrow \\ \operatorname{Pic}\left(G_n/\left(P_{\mathcal{F}}\right)_n\right) & \cong & \operatorname{Hom}\left(\left(P_{\mathcal{F}}\right)_n, k^\times\right) \end{array} \tag{7.1}$$

is commutative. Hence $\operatorname{Pic}\mathcal{F}\ell(\mathcal{F},E) \cong \operatorname{Hom}(P_{\mathcal{F}},k^{\times})$, and $\operatorname{Hom}(P_{\mathcal{F}},k^{\times})$ is nothing but the group of integral characters of the Lie algebra of $P_{\mathcal{F}}$. In the case of $\mathcal{F}\ell(\mathcal{F},w,E)$, the desired isomorphism is established by replacing $\operatorname{Hom}((P_{\mathcal{F}})_n,k^{\times})$ and $\operatorname{Hom}((P_{\mathcal{F}})_{n+1},k^{\times})$ in diagram (7.1) with the groups of integral characters of the Lie algebras of $P_{\mathcal{F}_n}$ and $P_{\mathcal{F}_{n+1}}$, respectively.

In the rest of this section, we give a purely geometric description of $\operatorname{Pic}\mathcal{F}\ell(\mathcal{F},E)$ and $\operatorname{Pic}\mathcal{F}\ell(\mathcal{F},w,E)$. Consider the corresponding covering (5.7) or (5.8). Let L and M be two bases compatible with $\mathcal{F}$ for which $E\backslash(E\cap L)$ and $E\backslash(E\cap M)$ are finite sets. Denote by $g_{L,M}$ the automorphism of V such that $g_{L,M}(l_i) = m_i$ for $\mathcal{F}\ell(\mathcal{F},E)$, and $g_{L,M}(l_i) = m_i$, $g_{L,M}(l^i) = m^i$ for $\mathcal{F}\ell(\mathcal{F},w,E)$. It has a well-defined determinant and, moreover, it induces an automorphism of F''/F'. Denote the determinant of this latter automorphism by $\det_{L,M}(F''/F')$. In this way, we obtain an invertible sheaf $\mathcal{L}_{F'}$ with transition functions $\det_{L,M}(F''/F')$ on $C(\mathcal{F},E;L)\cap C(\mathcal{F},E;M)$ or $C(\mathcal{F},w,E;L)\cap C(\mathcal{F},w,E;M)$, respectively. Finally, let $\gamma_{F'} \in \operatorname{Pic}\mathcal{F}\ell(d_n;V_n)$, respectively, $\gamma_{F'} \in \operatorname{Pic}\mathcal{F}\ell(d_n,w;V_n)$, denote the class of $\mathcal{L}_{F'}$.

Proposition 7.1. There are canonical isomorphisms of abelian groups $\operatorname{Pic}\mathcal{F}\ell(\mathcal{F},E) \cong (\Pi_{F'\in\mathcal{F}'}(\mathbb{Z}\gamma_{F'}))/(\mathbb{Z}\Pi_{F'\in\mathcal{F}'}\gamma_{F'})$ and $\operatorname{Pic}\mathcal{F}\ell(\mathcal{F},w,E) \cong \Pi_{F'\in\mathcal{F}',F'\subset\tau(F'),F'\neq\mathcal{T}'}(\mathbb{Z}\gamma_{F'})$. □

Proof. Consider the case of $\mathcal{F}\ell(\mathcal{F},E)$ first. Let $\gamma_{F',n}$ be the class of the restriction $(\mathcal{L}_{F'})_n$ of $\mathcal{L}_{F'}$ to $\mathcal{F}\ell(d_n;V_n)$. Then $\gamma_{F',n} = 0$ unless $F''\cap V_n \neq F'\cap V_n$. Define the group homomorphism $\varphi_n : \Pi_{F'\in\mathcal{F}'}(\mathbb{Z}\gamma_{F'}) \to \operatorname{Pic}\mathcal{F}\ell(d_n;V_n)$ via $\varphi_n(\Pi_{F'\in\mathcal{F}'}m_{F'}\gamma_{F'}) := \sum_{F'\in\mathcal{F}'} m_{F'}\gamma_{F',n}$. The sum $\sum_{F'\in\mathcal{F}'} m_{F'}\gamma_{F',n}$ makes sense because $\gamma_{F',n} = 0$ for all but finitely many $F' \in \mathcal{F}'$. Clearly $\varphi_n = r_n \circ \varphi_{n+1}$, where $r_n : \operatorname{Pic}\mathcal{F}\ell(d_{n+1};V_{n+1}) \to \operatorname{Pic}\mathcal{F}\ell(d_n;V_n)$ is the restriction map. Therefore, by the universality property of $\varprojlim$, there is a homomorphism

$$\varphi : \Pi_{F'\in\mathcal{F}'}\big(\mathbb{Z}\gamma_{F'}\big) \longrightarrow \operatorname{Pic}\mathcal{F}\ell(\mathcal{F},E) = \varprojlim \operatorname{Pic}\mathcal{F}\ell\big(d_n;V_n\big). \tag{7.2}$$

Furthermore, φ is surjective as φ_n is surjective for each n.

To compute $\ker\varphi$, note that $\ker\varphi = \cap\ker\varphi_n$. We have $\ker\varphi_n = (\mathbb{Z}(\Pi_{F'\in\mathcal{F}'}\gamma_{F'})) \times \Pi_{F'\in\mathcal{F}',F'\cap V_n=F''\cap V_n}(\mathbb{Z}\gamma_{F'})$ and therefore $\ker\varphi = \mathbb{Z}(\Pi_{F'\in\mathcal{F}'}\gamma_{F'})$, that is, $\operatorname{Pic}\mathcal{F}\ell(\mathcal{F},E) \cong (\Pi_{F'\in\mathcal{F}'}(\mathbb{Z}\gamma_{F'}))/(\mathbb{Z}\Pi_{F'\in\mathcal{F}'}\gamma_{F'})$.

In the case of $\mathcal{F}\ell(\mathcal{F},w,E)$ homomorphisms, $\varphi_n : \Pi_{F'\in\mathcal{F}'}(\mathbb{Z}\gamma_{F'}) \to \operatorname{Pic}\mathcal{F}\ell(d_n,w;V_n)$ and $\varphi : \Pi_{F'\in\mathcal{F}'}(\mathbb{Z}\gamma_{F'}) \to \operatorname{Pic}\mathcal{F}\ell(\mathcal{F},w,E)$ are defined in a similar way. Here $\ker\varphi_n = \Pi_{F'\in\mathcal{F}',F'\subset\tau(F')}(\mathbb{Z}(\gamma_{F'}+\gamma_{\tau(F')'}))\times\Pi_{F'\in\mathcal{F}',F'\subset\tau(F'),F'\cap V_n=F''\cap V_n}(\mathbb{Z}\gamma_{F'})$, and consequently $\ker\varphi = \Pi_{F'\in\mathcal{F}',F'\subset\tau(F')}(\mathbb{Z}(\gamma_{F'} + \gamma_{\tau(F')'}))$, that is, $\operatorname{Pic}\mathcal{F}\ell(\mathcal{F},w,E) \cong \Pi_{F'\in\mathcal{F}',F'\subset\tau(F'),F'\neq\mathcal{T}'}(\mathbb{Z}\gamma_{F'})$. ■

We complete this paper by an explicit criterion for the projectivity of $\mathcal{F}\ell(\mathcal{F},E)$ and $\mathcal{F}\ell(\mathcal{F},w,E)$. The following proposition is a translation of [3, Proposition 15.1] into the language of generalized flags.

Proposition 7.2. $\mathcal{F}\ell(\mathcal{F}, E)$ or $\mathcal{F}\ell(\mathcal{F}, w, E)$ is projective if and only if $\mathcal{F}$ is a flag. □

Proof. Consider the case of $\mathcal{F}\ell(\mathcal{F}, E)$ (the case of $\mathcal{F}\ell(\mathcal{F}, w, E)$ is similar). $\mathcal{F}\ell(\mathcal{F}, E)$ is projective if and only if it admits a very ample invertible sheaf. An immediate verification shows that an invertible sheaf $\mathcal{L}$, whose class in $\operatorname{Pic} \mathcal{F}\ell(\mathcal{F}, E)$ is the image of $\prod_{F' \in \mathcal{F}'} m_{F'} \gamma_{F'}$, is very ample if and only if the map $c : \mathcal{F}' \to \mathbb{Z}$, $F' \mapsto m_{F'}$ is strictly increasing. Indeed, $\mathcal{F}\ell(\mathcal{F}, E) = \varinjlim \mathcal{F}\ell(d_n; V_n)$, and $\mathcal{L}$ is very ample if and only if its restrictions $\mathcal{L}_n$ onto $\mathcal{F}\ell(d_n; V_n)$ are very ample for all n. Consider the map $c_n : \mathcal{F}'_n \to \mathbb{Z}$, defined via $c_n((\mathcal{F}_n)'_v) := c((\psi_n(\mathcal{F}_n))'_v)$ for every nonzero $v \in V_n$, where ψ_n is defined in the proof of Proposition 5.2 above. (As the reader will check, c_n is well defined, that is, if $(\mathcal{F}_n)'_{v_1} = (\mathcal{F}_n)'_{v_2}$, then $(\psi_n(\mathcal{F}_n))'_{v_1} = (\psi_n(\mathcal{F}_n))'_{v_2}$.) A direct comparison with the classical Bott-Borel-Weil theorem for the group $G(E_n)$ (see, e.g., [1]) shows that $\mathcal{L}_n$ is very ample if and only if the map c_n is strictly increasing. Hence $\mathcal{L}$ is very ample if and only if c is strictly increasing. This enables us to conclude that $\mathcal{F}\ell(\mathcal{F}, E)$ is projective if and only if there exists a strictly increasing map $\mathcal{F}' \to \mathbb{Z}$, that is, if and only if $\mathcal{F}$ is a flag. ■

Propositions 7.1 and 7.2 allow us to make some initial remarks concerning the isomorphism classes of the ind-varieties $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{F}, w, E)$. For example, if $\mathcal{F}$ is a flag of finite length in V, and $\mathcal{G}$ is a flag (or generalized flag) in V of length different from the length of $\mathcal{F}$ (finite or infinite), then $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{G}, L)$ are not isomorphic because their Picard groups are not isomorphic. Furthermore, if $\mathcal{F}$ is a flag in V but $\mathcal{G}$ is not, then $\mathcal{F}\ell(\mathcal{F}, E)$ and $\mathcal{F}\ell(\mathcal{G}, L)$ are not isomorphic because the former ind-variety is projective and the latter is not. Finally, a recent result of Donin and the second-named author, [4], implies that if $\mathcal{F} = \{0 \subset F \subset V\}$ with F both infinite-dimensional and of infinite codimension in V, then the "ind-grassmannian" $\mathcal{F}\ell(\mathcal{F}, E)$ is not isomorphic to $\operatorname{Gr}(l; V)$ for any l (cf. Example 5.3).

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