ON AN INFINITE LIMIT OF BGG CATEGORIES O

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Abstract. We study a version of the BGG category $\mathcal{O}$ for Dynkin Borel subalgebras of root-reductive Lie algebras, such as $\mathfrak{g} = \mathfrak{gl}(\infty)$. We prove results about extension fullness and compute the higher extensions of simple modules by Verma modules. We also show that the category is Ringel self-dual and initiate the study of Koszul duality. An important tool in obtaining these results is an equivalence we establish between appropriate Serre subquotients of category $\mathcal{O}$ for $\mathfrak{g}$ and category $\mathcal{O}$ for finite dimensional reductive subalgebras of $\mathfrak{g}$.

Introduction

After about 20 years of study of the representation theory of the three infinite-dimensional finitary Lie algebras $\mathfrak{sl}(\infty), \mathfrak{so}(\infty), \mathfrak{sp}(\infty)$, there still is no standard analogue of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for there Lie algebras. One reason is that each of these Lie algebras has uncountably many conjugacy classes of Borel subalgebras, so potentially there are many "categories $\mathcal{O}$". Therefore, one is faced with a selection process trying to sort through various options in constructing interesting and relevant analogues of the BGG category $\mathcal{O}$. Existing results concerning integrable $\mathfrak{g}$-modules, as well as primitive ideals in $U(\mathfrak{g})$, for Lie algebras $\mathfrak{g}$ as above, motivate the study of interesting analogues of category $\mathcal{O}$ for two types of Borel subalgebras. These are the *Dynkin Borel subalgebras*, or Borel subalgebras having "enough simple roots", and, on the other hand, the *ideal Borel subalgebras* defined in [PP1] and [PP2]. These nonintersecting classes of Borel subalgebras are "responsible" for different classes of representations, and naturally lead to different "categories $\mathcal{O}$".

The case of Dynkin Borel subalgebras is considered in the recent paper [Na2](see also [Na1]) where a category $\overline{\mathcal{O}}$ is defined. This category consists of all weight modules with finite-dimensional weight spaces which carry a locally finite action of the entire locally nilpotent radical of a fixed Dynkin Borel subalgebra. A direct consequence of the definition of a Dynkin Borel subalgebras is that Verma modules are objects of $\overline{\mathcal{O}}$. Nevertheless $\overline{\mathcal{O}}$ is not a highest weight category due to lack of projective or injective modules. Another result is that the subcategory of $\overline{\mathcal{O}}$ consisting of integrable modules (integrable modules are direct limits of finite-dimensional modules over finite-dimensional subalgebras) is a semisimple category. This makes $\overline{\mathcal{O}}$ somewhat similar to the original BGG category $\mathcal{O}$. A concrete motivation to study versions of category $\mathcal{O}$ for this type of Borel subalgebras comes from the representation theory of finite dimensional Lie superalgebras. Through the concept of "super-duality", the category of finite dimensional modules over the general linear superalgebra $\mathfrak{gl}(m|n)$ is related to modules in (parabolic subcategories) of category $\mathcal{O}$ for $\mathfrak{gl}(\infty)$, see e.g [CLW, CWZ]. Such super-dualities appear also for category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$ and for Lie superalgebras of other types.

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On the other hand, in [PS] an analogue of category $\mathcal{O}$ is defined, for an ideal Borel subalgebra. Verma modules are not objects of this category, but its subcategory of integrable modules coincides with the nonsemisimple category of tensor modules studied in [DPS]. This latter category is itself an interesting highest weight category.

The current paper arose from an attempt to understand better the homological structure of the category $\overline{\mathcal{O}}$ introduced in [Na2]. It turned out that it is convenient to extend Nampaisarn's category to a category $\mathbf{O}$ whose objects are weight modules which are locally finite with respect to the locally nilpotent radical of a Dynkin Borel subalgebra, but do not necessarily have finite-dimensional weight spaces.

Let's give a brief description of the content of the paper. Sections 1 and 2 are of preliminary nature. Here we recall some general facts about abelian categories and about root-reductive Lie algebras. In particular, we go over the notions of Cartan subalgebras, Borel subalgebras and Weyl groups for root-reductive Lie algebras. In Section 3 we collect some basic facts about Verma modules and dual Verma modules. This section reproves some results of [Na1] and [Na2] and explores some of the peculiarities of Verma modules for Borel algebras which are not of Dynkin type.

From Section 4 on, we only consider Dynkin Borel subalgebras and introduce the category $\mathbf{O}$. We demonstrate that the category $\mathbf{O}$ decomposes as the product of indecomposable blocks described by the Weyl group orbits in the dual Cartan subalgebra $\mathfrak{h}^*$. This also reproves Nampaisarn's result about linkage in $\overline{\mathcal{O}}$. Next, we study blocks after truncation to upper finite ideals in $\mathfrak{h}^*$. We prove equivalence of the categories of the truncated blocks with categories of modules over certain locally finite dimensional associative algebras. We then show that any truncated category $\mathbf{O}$ is extension full in $\mathbf{O}$, and also in the category of weight modules. These results allow to transfer certain homological questions in $\mathbf{O}$ to categories which have enough projective objects. It is an open question whether the entire category $\mathbf{O}$ is extension full in the category of weight modules, and whether the category $\overline{\mathcal{O}}$ is extension full in $\mathbf{O}$.

In Section 5, we prove that the Serre quotient category of two appropriately chosen truncations of $\mathbf{O}$ is equivalent to $\mathcal{O}(\mathfrak{g}_n, \mathfrak{b}_n)$ for arbitrarily large n, where $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ for finite dimensional reductive Lie algebras $\mathfrak{g}_n$. For dominant blocks, it suffices to consider a quotient category of $\mathbf{O}$, and for antidominant blocks it suffices to consider a subcategory of $\mathbf{O}$ to establish such equivalences. Using the homological results in Section 4, this shows that the higher extensions of simple modules by Verma modules in $\mathbf{O}$ are governed by Kazhdan-Lusztig-Vogan polynomials. This was conjectured for $\overline{\mathcal{O}}$ in [Na1]. As another application, we show that all blocks of $\mathbf{O}$ corresponding to integral dominant regular weights are equivalent. In this section we also address the Koszulity of blocks of the category $\mathbf{O}$. We prove that truncations of $\mathbf{O}$ admit graded covers, in the sense of [BGS]. In the graded setting, we show that extensions of simple modules by Verma modules (and extensions of dual Verma modules by simple modules) satisfy the Koszulity pattern. For BGG category $\mathcal{O}$, this property actually implies ordinary Koszulity, see [ADL]. Here, we leave open the question of whether extensions of simple modules by simple modules in the graded cover of $\mathbf{O}$ also satisfy the required pattern. This is a nice question for further research.

Section 6 and 7 are devoted to another natural structural question: Ringel duality in the category $\mathbf{O}$. In Section 6 we construct and study the semi-regular $U(\mathfrak{g})$-bimodule. For Kac-Moody algebras corresponding to a finite dimensional Cartan matrix, this bimodule was introduced in [Ar, So], and we extend the procedure to Kac-Moody algebras for infinite dimensional Cartan matrices, such as $\mathfrak{sl}(\infty), \mathfrak{so}(\infty)$ and $\mathfrak{sp}(\infty)$. In Section 7 we show that the category $\mathbf{O}$, as a whole, is Ringel self-dual, by establishing a (covariant) equivalence between the category of modules with a Verma flag and the category of modules with a dual Verma flag. Since this equivalence sends the Verma module $\Delta(\lambda)$ to the dual Verma module $\nabla(-\lambda - 2\rho)$, the blocks of $\mathbf{O}$ are not Ringel self-dual. In particular, dominant blocks are dual to anti-dominant blocks. The Ringel duality functor also implies existence of tilting modules in appropriate Serre quotients and determines their decomposition multiplicities.

The paper is concluded by brief appendices on certain theories to which we refer throughout the text: Serre quotient categories, Ringel duality, graded covers, and quasi-hereditary Koszul algebras.

We conclude the introduction by mentioning some related results, obtained independently at the same time by Chen and Lam in [CL]. There, specific dominant blocks in parabolic subcategories, with respect to specific Levi subalgebras of finite corank, of $\mathcal{O}$ for $\mathfrak{gl}(\infty), \mathfrak{so}(\infty), \mathfrak{sp}(\infty)$ are studied. For $\mathfrak{gl}(\infty)$, this leads to categories where the modules have finite length. In that setting, also in [CL] equivalences with the finite rank case are shown and used to obtain results on Koszulity. It seems that neither the methods of [CL] nor in the current paper extend to the other case immediately.

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1. Preliminaries

We fix an algebraically closed field $\Bbbk$ of characteristic zero. For any Lie algebra $\mathfrak{k}$, the universal enveloping algebra will be denoted by $U(\mathfrak{k})$. The restriction functor from the category of $\mathfrak{k}$-modules, to the category of $\mathfrak{l}$-modules, for a subalgebra $\mathfrak{l} \subset \mathfrak{k}$, is denoted by $\mathrm{Res}^{\mathfrak{k}}_{\mathfrak{l}}$. We set $\mathbb{N} = \{0, 1, 2, \ldots\}$. If A is a set $|A|$ denotes its cardinality.

1.1. **Abelian categories.**
Let $\mathcal{C}$ be an arbitrary abelian category.

1.1.1. *Multiplicities.* We follow [So, Definition 4.1] regarding multiplicities. For $M \in \mathcal{C}$ and simple $L \in \mathcal{C}$, the multiplicity $[M : L] \in \mathbb{N} \cup \{\infty\}$ of L in M is

$$[M : L] \;=\; \sup_{F_\bullet} |\{i \,|\, F_iM/F_{i+1}M \cong L\}|,$$

where $F_\bullet$ ranges over all *finite* filtrations $0 = F_pM \subset \cdots \subset F_{i+1}M \subset F_iM \subset \cdots \subset F_0M = M$.

1.1.2. *Extensions.* For each $i \in \mathbb{N}$, we define the extension functor

$$\mathrm{Ext}^i_{\mathcal{C}}(-,-) \;:\; \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \;\to\; \mathbf{Ab}$$

as in [Ve, III.3], see also [CM1, Section 2.1]. For an abelian subcategory $\iota : \mathcal{B} \hookrightarrow \mathcal{C}$, the exact inclusion functor ι induces group homomorphisms

$$\iota^i_{XY} : \mathrm{Ext}^i_{\mathcal{B}}(X,Y) \;\to\; \mathrm{Ext}^i_{\mathcal{C}}(X,Y), \quad \text{for all } i \in \mathbb{N} \text{ and } X, Y \in \mathcal{B}. \tag{1.1}$$

In general, these are neither epimorphisms nor monomorphisms. When all ι^i_{XY} are isomorphisms, we say that $\mathcal{B}$ is **extension full** in $\mathcal{C}$.

1.1.3. *Coproducts.* We denote the coproduct of a family $\{X_\alpha\}$ of objects in $\mathcal{C}$, if it exists, by $\bigoplus_\alpha X_\alpha$. By definition, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}\left(\bigoplus_\alpha X_\alpha, Y\right) \;\tilde{\to}\; \prod_\alpha \mathrm{Hom}_{\mathcal{C}}(X_\alpha, Y), \quad f \mapsto (f \circ \iota_\alpha)_\alpha. \tag{1.2}$$

The following lemma can be generalised substantially, but it will suffice for our purposes.

Lemma 1.1.4. *If for a family $\{X_\alpha\}_\alpha$ of objects in $\mathcal{C}$ and $Y \in \mathcal{C}$ we have $\mathrm{Ext}^1_{\mathcal{C}}(X_\alpha, Y) = 0$ for all α, then $\mathrm{Ext}^1_{\mathcal{C}}(\bigoplus_\alpha X_\alpha, Y) = 0$.*

Proof. Represent an element of $\mathrm{Ext}^1_{\mathcal{C}}(\bigoplus_\alpha X_\alpha, Y)$ as the upper line of the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & M & \xrightarrow{f} & \bigoplus_\alpha X_\alpha & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle \phi_\beta} & & \uparrow{\scriptstyle \iota_\beta} & & \\ 0 & \longrightarrow & Y & \longrightarrow & M_\beta & \xrightarrow{f_\beta} & X_\beta & \longrightarrow & 0. \end{array}$$

With M_β the pullback of X_β in M, we obtain the above commuting diagram with exact rows, for every β. By assumption, there exist $g_\beta : X_\beta \to M_\beta$ with $f_\beta \circ g_\beta = 1_{X_\beta}$. Equation (1.2) yields a morphism $g : \bigoplus_\alpha X_\alpha \to M$ such that $g \circ \iota_\beta = \phi_\beta \circ g_\beta$. Commutativity of the diagram implies that $f \circ g \circ \iota_\beta = \iota_\beta$. Since β is arbitrary, isomorphism (1.2) implies that $f \circ g$ is the identity of $\bigoplus_\alpha X_\alpha$ and the extension defined by the upper row of the diagram splits. □

1.1.5. *Serre subcategories.* A non-empty full subcategory $\mathcal{B}$ of $\mathcal{C}$ is a **Serre subcategory** ("thick subcategory" in [Ga]) if for every short exact sequence in $\mathcal{C}$

$$0 \to Y_1 \to X \to Y_2 \to 0,$$

we have $X \in \mathcal{B}$ if and only if $Y_1, Y_2 \in \mathcal{B}$. The exact inclusion functor $\imath : \mathcal{B} \to \mathcal{C}$ is fully faithful (meaning that all ι^0_{XY} are isomorphisms) and such that also all ι^1_{XY} are isomorphisms. In addition, it follows immediately that $\mathcal{B}$ is a strictly full (full and replete) subcategory.

1.2. **Locally finite algebras.**

1.2.1. A $\Bbbk$-algebra A is **locally unital**, if there exists a family of mutually orthogonal idempotents $\{e_\alpha \,|\, \alpha \in \Lambda\}$ for which we have

$$A \;=\; \bigoplus_\alpha e_\alpha A \;=\; \bigoplus_\alpha A e_\alpha.$$

We denote by A-Mod the category of left A-modules M which satisfy $M = \bigoplus_\alpha e_\alpha M$.

1.2.2. A locally unital algebra A is **locally finite** if, for all α, β, we have $\dim_\Bbbk e_\alpha A e_\beta < \infty$. Fur such an algebra we have the full subcategory A-mod of A-Mod, of modules M which satisfy $\dim_\Bbbk e_\alpha M < \infty$, for all α. Clearly the projective modules Ae_α are in A-mod, although A-mod will generally not contain *enough* projective objects.

1.3. **Partial orders.** Fix a partially ordered set $(S, \preceq)$. We will denote the induced partial order on any subset of S by the same notation $\preceq$.

1.3.1. Any two elements $\lambda, \mu \in S$ determine an **interval** $\{\nu \in S \,|\, \mu \preceq \nu \preceq \lambda\}$. An **ideal** K is a subset of S with the property that $\lambda \in \mathsf{K}$ and $\mu \preceq \lambda$ implies $\mu \in \mathsf{K}$. An ideal K is **finitely generated** if there are finitely many $\{a_1, \ldots, a_p\}$ such that each $\mu \in \mathsf{K}$ satisfies $\mu \preceq a_i$, for some $1 \leq i \leq p$. A subset J is **upper finite**, resp. **lower finite**, if for any $\mu \in \mathsf{J}$ there are only finitely many $\lambda \in \mathsf{J}$ for which $\lambda \succeq \mu$, resp. $\lambda \preceq \mu$. A subset I is called **complete** if it is a union of intervals, *i.e.* if $\lambda, \mu \in \mathsf{I}$ and $\mu \preceq \nu \preceq \lambda$ implies $\nu \in \mathsf{I}$. A subset C of S is a **coideal** if $\lambda \in \mathsf{C}$ and $\mu \succeq \lambda$ implies $\mu \in \mathsf{C}$. Clearly, the intersection of an ideal and a coideal is a complete subset. Furthermore, the coideals in S are precisely the sets $S \backslash \mathsf{I}$ for ideals I in S.

1.3.2. To any complete subset $\mathsf{l} \subset S$, we associate two ideals

$$\bar{\mathsf{l}} = \{\mu \in S \,|\, \mu \preceq \lambda, \text{ for some } \lambda \in \mathsf{l}\} \quad \text{and} \quad \mathring{\mathsf{l}} \;=\; \bar{\mathsf{l}} \setminus \mathsf{l}.$$

A pair of elements is called **remote** if the interval $\{\nu \in S \,|\, \mu \preceq \nu \preceq \lambda\}$ has infinite cardinality. With this convention, incomparable elements are never remote. A partial order is **interval finite** if every interval is a finite set, or equivalently if no two elements are remote. For a partial order which is interval finite, all finitely generated ideals are upper finite ideals.

2. Root-reductive Lie algebras and triangular decompositions

2.1. Root-reductive Lie algebras.

2.1.1. A Lie algebra $\mathfrak{g}$ over $\Bbbk$ is **locally reductive** if it has a collection of subalgebras $\{\widetilde{\mathfrak{g}}_n \,|\, n \in \mathbb{N}\}$ such that

$$\mathfrak{g} \;=\; \varinjlim \widetilde{\mathfrak{g}}_n,$$

where, for each $n \in \mathbb{N}$, $\widetilde{\mathfrak{g}}_n$ is a finite dimensional reductive Lie algebra which is reductive in $\widetilde{\mathfrak{g}}_{n+1}$. In other words, both the adjoint representation of $\widetilde{\mathfrak{g}}_n$ and its restriction as a $\widetilde{\mathfrak{g}}_{n-1}$-module are semisimple for all $n > 0$.

2.1.2. Consider a locally reductive Lie algebra $\mathfrak{g}$ as above. If, for each $n \in \mathbb{N}$, there exists a Cartan subalgebra $\mathfrak{h}_n \subset \widetilde{\mathfrak{g}}_n$, such that $\mathfrak{h}_n \subset \mathfrak{h}_{n+1}$ and such that each root vector in $\widetilde{\mathfrak{g}}_n$ is also a root vector in $\widetilde{\mathfrak{g}}_{n+1}$, the Lie algebra $\mathfrak{g}$ is called **root-reductive**.

For such $\mathfrak{g}$, we have the corresponding abelian subalgebra $\mathfrak{h} = \varinjlim \mathfrak{h}_n$. Such subalgebras $\mathfrak{h} \subset \mathfrak{g}$ are known as **splitting maximal total subalgebras** of $\mathfrak{g}$. These subalgebras are also **Cartan subalgebras** of $\mathfrak{g}$, according to the definition and results in [DPS, Section 3]. We will simply use the term "Cartan subalgebra" when referring to splitting maximal total subalgebras. We also introduce the subalgebras

$$\mathfrak{g}_n := \widetilde{\mathfrak{g}}_n + \mathfrak{h}.$$

Lemma 2.1.3. [DPS, Theorem 4.1], [DP, Theorem 1] *For any root-reductive Lie algebra* $\mathfrak{g}$*, the derived algebra* $[\mathfrak{g},\mathfrak{g}]$ *is a root-reductive Lie algebra which is a countable direct sum of Lie algebras isomorphic to* $\mathfrak{sl}(\infty)$*,* $\mathfrak{so}(\infty)$*,* $\mathfrak{sp}(\infty)$ *or finite dimensional simple Lie algebras.*

2.1.4. If $\mathfrak{g}$ is a root-reductive Lie algebra with Cartan subalgebra $\mathfrak{h}$, we have a corresponding decomposition into weight spaces

$$\mathfrak{g} \;=\; \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}^{\alpha}, \tag{2.1}$$

for the set of roots $\Phi \subset \mathfrak{h}^*$. By construction, we have $\dim_{\Bbbk} \mathfrak{g}^{\alpha} = 1$, for each $\alpha \in \Phi$, and $0 \notin \Phi$. We denote the subset of roots belonging to $\mathfrak{g}_n$ as Φ_n, for each $n \in \mathbb{N}$.

2.1.5. We introduce the category $\mathbf{C}(\mathfrak{g},\mathfrak{h})$ of $\mathfrak{g}$-modules which are semisimple as $\mathfrak{h}$-modules. For any $\mu \in \mathfrak{h}^*$ and $M \in \mathbf{C}(\mathfrak{g},\mathfrak{h})$, we denote by M_μ the μ-weight space in M. By assumption, we have $M = \bigoplus_\mu M_\mu$ for $M \in \mathbf{C}(\mathfrak{g},\mathfrak{h})$. For any module $M \in \mathbf{C}(\mathfrak{g},\mathfrak{h})$, we consider its support $\mathrm{supp} M \subset \mathfrak{h}^*$, which is the set of all weights μ such that $M_\mu \neq 0$.

The full subcategory of $\mathbf{C}(\mathfrak{g},\mathfrak{h})$ of modules M which satisfy $\dim_{\Bbbk} M_\mu < \infty$ for all $\mu \in \mathfrak{h}^*$, is denoted by $\mathcal{C}(\mathfrak{g},\mathfrak{h})$. This is clearly a Serre subcategory. We have the duality $M \mapsto M^{\circledast}$ on $\mathcal{C}(\mathfrak{g},\mathfrak{h})$ which takes M to its $\mathfrak{h}$-finite dual, *i.e.* to the maximal $\mathfrak{h}$-semisimple submodule of $M^* = \mathrm{Hom}_{\Bbbk}(M,\Bbbk)$. We also have the duality $M \mapsto M^{\vee}$ which twists the action on $M^{\circledast}$ with τ, the anti-involution τ of $\mathfrak{g}$ which maps $\mathfrak{g}^{\alpha}$ to $\mathfrak{g}^{-\alpha}$ for all $\alpha \in \Phi$, and acts as -1 on $\mathfrak{h}^*$. In particular, we have

$$\mathrm{supp} M \;=\; \mathrm{supp} M^{\vee}, \qquad \text{for all } M \in \mathcal{C}(\mathfrak{g},\mathfrak{h}). \tag{2.2}$$

Remark 2.1.6. If we apply the definition of $\mathbf{C}(\mathfrak{g},\mathfrak{h})$ to $\mathfrak{g}_n$, and then only consider modules with support belonging to a fixed coset of $\mathfrak{h}^*/\mathbb{Z}\Phi_n$, we automatically get an equivalence with a correspondingly defined category for $\widetilde{\mathfrak{g}}_n$. We will therefore freely use results for the finite dimensional reductive Lie algebra $\widetilde{\mathfrak{g}}_n$, for instance related to category $\mathcal{O}$, when working over $\mathfrak{g}_n$.

2.2. **Triangular decompositions.** Fix a root-reductive Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$.

2.2.1. Choose a subset $\Phi^+ \subset \Phi$ such that $\Phi = \Phi^+ \amalg \Phi^-$, with $\Phi^- := -\Phi^+$, and $\alpha+\beta \in \Phi^+$ whenever $\alpha, \beta \in \Phi^+$. Then we set

$$\mathfrak{n}^+ := \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}^\alpha, \quad \mathfrak{n}^- := \bigoplus_{\alpha\in\Phi^-} \mathfrak{g}^\alpha.$$

The elements of Φ^+, resp. Φ^-, which cannot be written as a sum of two other elements of Φ^+, resp. Φ^-, are known as **simple roots**. The positive simple roots constitute the subset $\Sigma \subset \Phi^+$.

The **splitting Borel subalgebras** of $\mathfrak{g}$ are by definition precisely the subalgebras $\mathfrak{b} := \mathfrak{h}\oplus\mathfrak{n}^+$ obtained in the above way. (The decomposition $\mathfrak{b} = \mathfrak{h}\oplus\mathfrak{n}^+$ is a direct sum of vector spaces, not of Lie algebras.) Note that $\mathfrak{b}^- = \mathfrak{h}\oplus\mathfrak{n}^-$, the opposite Borel subalgebra to $\mathfrak{b}$, is also a splitting Borel subalgebra. We will simply use the term "Borel subalgebra" when referring to splitting Borel subalgebras. A splitting Borel subalgebra for $\mathfrak{g}$ leads to Borel subalgebras for $\mathfrak{g}_n$ and $\widetilde{\mathfrak{g}}_n$:

$$\mathfrak{b}_n := \mathfrak{g}_n \cap \mathfrak{b}, \qquad \widetilde{\mathfrak{b}}_n := \widetilde{\mathfrak{g}}_n \cap \mathfrak{b}.$$

2.2.2. For each $\lambda \in \mathfrak{h}^*$, we have the corresponding **Verma module**

$$\Delta^{\mathfrak{b}}_{\mathfrak{g}}(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \Bbbk_\lambda, \tag{2.3}$$

where $\Bbbk_\lambda$ is the one-dimensional $\mathfrak{h}$-module of weight λ with trivial $\mathfrak{n}^+$-action. We will leave out the indices $\mathfrak{g}$ and $\mathfrak{b}$ when it is clear which algebras are considered.

We denote by Γ^+ the subset of $\mathfrak{h}^*$, consisting of 0 and finite sums of elements in Φ^+. The partial order $\le$ on $\mathfrak{h}^*$ is defined as

$$\mu \le \lambda \quad\Leftrightarrow\quad \lambda - \mu \in \Gamma^+ \quad\Leftrightarrow\quad \Delta(\lambda)_\mu \neq 0.$$

We use the notation $\le_n$ for the partial order on $\mathfrak{h}^*$ obtained from the above procedure applied to $\Phi_n^+ = \Phi_n \cap \Phi^+$.

2.3. **Parabolic subalgebras.**

2.3.1. For a fixed Borel subalgebra $\mathfrak{b}$, any subalgebra $\mathfrak{p} \subset \mathfrak{g}$ which contains $\mathfrak{b}$ is called a **parabolic** subalgebra. The reductive part $\mathfrak{l} \subset \mathfrak{p}$ is spanned by $\mathfrak{h}$ and all root spaces $\mathfrak{g}^\alpha$ such that both $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ are in $\mathfrak{p}$. We denote by $\Phi(\mathfrak{l}) \subset \Phi$ the set of roots occurring in $\mathfrak{l}$. We have the corresponding parabolic decomposition (of vector spaces)

$$\mathfrak{g} \;=\; \mathfrak{u}^- \oplus \mathfrak{l} \oplus \mathfrak{u}^+, \qquad \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}^+.$$

The following lemma is an easy consequence of the definitions.

Lemma 2.3.2. *For any $\lambda \in \mathfrak{h}^*$ and reductive part $\mathfrak{l} \subset \mathfrak{g}$ of a parabolic subalgebra, the subset $\lambda + \mathbb{Z}\Phi(\mathfrak{l}) \subset \mathfrak{h}^*$ is complete for $\le$.*

2.4. **Induction and restriction.** Fix a Borel subalgebra $\mathfrak{b}$ and a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ with reductive part $\mathfrak{l}$. We have the exact functor

$$\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{l},+} \;:\; \mathfrak{l}\text{-Mod} \;\to\; \mathfrak{g}\text{-Mod},$$

which is given by interpreting $\mathfrak{l}$-modules as $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}^+$-modules with trivial $\mathfrak{u}^+$-action, followed by ordinary induction $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{p}} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} -$. For any $\lambda \in \mathfrak{h}^*$, we also consider

$$\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{l},\lambda} \;:\; \mathbf{C}(\mathfrak{g}, \mathfrak{h}) \;\to\; \mathbf{C}(\mathfrak{l}, \mathfrak{h}),$$

for the ordinary restriction functor followed by taking the maximal direct summand with support in $\lambda + \mathbb{Z}\Phi(\mathfrak{l})$.

2.5. **Dynkin Borel subalgebras.** Consider a root-reductive Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$.

Proposition 2.5.1. *For a splitting Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $\mathfrak{h}$, the following conditions are equivalent:*

(i) The elements of Γ^+ are the finite sums of elements in Σ.
(ii) We can write $\mathfrak{g} = \varinjlim \widetilde{\mathfrak{g}}_n$ as in 2.1.1, with the additional condition that $\widetilde{\mathfrak{g}}_n + \mathfrak{b}$ is a (parabolic) subalgebra of $\mathfrak{g}$, for each $n \in \mathbb{N}$.
(iii) The partial order $\leq$ is interval finite.
(iv) The Lie algebra $\mathfrak{g}$ is generated by $\mathfrak{h}$ and the simple (positive and negative) root spaces.
(v) For each $\lambda \in \mathfrak{h}^*$, the Verma module $\Delta(\lambda)$ is locally $U(\mathfrak{b})$-finite.
(vi) For each $\lambda \in \mathfrak{h}^*$, the Verma module $\Delta(\lambda)$ has finite dimensional weight spaces.

If one of the conditions is satisfied, $\mathfrak{b}$ is called a **Dynkin Borel** *subalgebra.*

Proof. First we show that (ii) and (iv) are equivalent. Choose finite subsets $\Sigma_n \subset \Sigma$ for $n \in \mathbb{N}$, such that $\Sigma = \cup_n \Sigma_n$ and $\Sigma_n \subset \Sigma_{n+1}$. Then we let $\widetilde{\mathfrak{g}}_n$ be the subalgebra of $\mathfrak{g}$ generated by the root vectors corresponding to $\Sigma_n \sqcup -\Sigma_n$. If (iv) is satisfied, it is easy to see that $\{\widetilde{\mathfrak{g}}_n\}$ satisfies all properties in (ii). Now assume that (ii) is satisfied. Since any $X \in \mathfrak{g}$ is contained in $\widetilde{\mathfrak{g}}_n$, for some n, and $\widetilde{\mathfrak{g}}_n$ is generated by $\widetilde{\mathfrak{g}}_n \cap \mathfrak{h}$ and the simple root spaces of $\mathfrak{g}$ which belong to $\widetilde{\mathfrak{g}}_n$, it follows that (iv) is satisfied.

That (i) and (iv) are equivalent is clear.

Now we show that (i) and (iii) are equivalent. If (i) is satisfied, then $\lambda \geq \mu$ implies that $\lambda - \mu$ is a finite sum of simple roots. It follows that the interval between λ and μ is finite. On the other hand, if (i) is not satisfied, we have $\beta \in \Phi^+$ such that we can consecutively subtract elements of Σ and always obtain an element of Φ^+. It follows that the interval between β and 0 has infinite cardinality.

Consider $\gamma \in \Gamma^+$. There are finitely many ways to write γ as a sum of elements in Φ^+ with non-negative coefficients if and only if $\dim_{\Bbbk} \Delta(\lambda)_{\lambda-\gamma} < \infty$, and it is clear that the latter condition is independent of $\lambda \in \mathfrak{h}^*$. It follows that (i) and (vi) are equivalent.

Now assume that (i) is satisfied. By the above, also (vi) is satisfied. We thus have

$$\sum_{\mu \geq \lambda - \gamma} \dim_{\Bbbk} \Delta(\lambda)_\mu < \infty,$$

for an arbitrary $\lambda \in \mathfrak{h}^*$ and $\gamma \in \Gamma^+$. It follows that (v) is also satisfied, so (i) implies (v).

Now assume that (i) is not satisfied. Then there exists $\beta \in \Phi^+$ which is not a finite sum of elements of Σ. It follows from standard $\mathfrak{sl}_2$-arguments that $X_{-\beta}v$, with $X_{-\beta} \in \mathfrak{g}^{-\beta}$ and v the highest weight vector of an arbitrary Verma module, generates an infinite dimensional $U(\mathfrak{b})$-module. □

2.5.2. Consider again an arbitrary Borel subalgebra $\mathfrak{b}$. Following [Na1, Section 6], we define the **$\mathfrak{b}$-finite root-reductive subalgebra** as the subalgebra $\mathfrak{l}_{\mathfrak{b}}$ of $\mathfrak{g}$ generated by $\mathfrak{h}$ and all root spaces for simple roots, with respect to $\mathfrak{b}$. Then $\mathbb{Z}\Phi(\mathfrak{l}_{\mathfrak{b}}) = \mathbb{Z}\Sigma$ and $\mathfrak{l}_{\mathfrak{b}}$ is the reductive part of the parabolic subalgebra $\mathfrak{l}_{\mathfrak{b}} + \mathfrak{b}$.

We have $\mathfrak{l}_{\mathfrak{b}} = \mathfrak{g}$ if and only if $\mathfrak{b}$ is a Dynkin Borel subalgebra. In general, $\mathfrak{b} \cap \mathfrak{l}_{\mathfrak{b}}$ is a Dynkin Borel subalgebra of $\mathfrak{l}_{\mathfrak{b}}$.

2.6. **The Weyl group.** In this section, we consider a Dynkin Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ of $\mathfrak{g}$. By 2.5.1(ii), we can assume that $\mathfrak{g} = \varinjlim \widetilde{\mathfrak{g}}_n$, where each $\widetilde{\mathfrak{g}}_n + \mathfrak{b}$ is a (parabolic) subalgebra.

2.6.1. The Weyl group $W_n := W(\widetilde{\mathfrak{g}}_n : \mathfrak{h}_n)$ is naturally a subgroup of W_{n+1}. Moreover, by assumption, the simple reflections of W_n as a Coxeter group are mapped to simple reflections in W_{n+1}. The infinite Coxeter group

$$W(\mathfrak{g} : \mathfrak{h}) \;=\; W \;:=\; \varinjlim W_n$$

has a natural action on $\mathfrak{h}^*$. For any $\alpha \in \Phi^+$, we denote the corresponding reflection by $r_\alpha \in W$.

2.6.2. It can easily be checked, see e.g. [Na2, Corollary 1.8], that there exists $\rho \in \mathfrak{h}^*$, such that the restriction $\rho|_{\mathfrak{h}_n^*}$ is the half sum of $\widetilde{\mathfrak{b}}_n$-positive roots for $\widetilde{\mathfrak{g}}_n$, with respect to Borel subalgebra. The dot action of W on $\mathfrak{h}^*$ is given by

$$w \cdot \lambda \;=\; w(\lambda + \rho) - \rho.$$

2.6.3. For a weight $\lambda \in \mathfrak{h}^*$, the **integral Weyl group** $W[\lambda]$ is the subgroup of W of elements $w \in W$ for which $w \cdot \lambda - \lambda \in \mathbb{Z}\Phi$. A weight λ is **integral** if $W = W[\lambda]$. A weight λ is **dominant** if $w \cdot \lambda \leq \lambda$, for all $w \in W[\lambda]$, and **antidominant** if $w \cdot \lambda \geq \lambda$, for all $w \in W[\lambda]$. A weight is **regular** if $w \cdot \lambda \neq \lambda$, for all $w \in W[\lambda]$. The **orbit** of a weight λ is denoted by $[\![\lambda]\!] = W[\lambda] \cdot \lambda$.

3. Verma modules

Consider a root-reductive Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$.

3.1. **Simple and (dual) Verma modules.** Recall the Verma module

$$\Delta(\lambda) \;=\; U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \Bbbk_\lambda$$

from equation (2.3). It is easy to see that it has a unique maximal submodule. The corresponding simple quotient of $\Delta(\lambda)$ is denoted by $L(\lambda)$. We will typically use the notation v_λ for a non-zero element in $1 \otimes \Bbbk_\lambda \subset \Delta(\lambda)$.

The following lemma states the universality property of Verma modules.

Lemma 3.1.1. *For $M \in \mathbf{C}(\mathfrak{g}, \mathfrak{h})$ with $M_\nu = 0$ for all $\nu > \mu$, we have an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta(\mu), M) \;\tilde{\to}\; M_\mu, \quad \alpha \mapsto \alpha(v_\mu).$$

Consequently, we have $\dim \mathrm{Hom}_{\mathfrak{g}}(\Delta(\mu), M) \;=\; [M : L(\mu)]$.

Proof. By adjunction, we have

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta(\mu), M) \cong \mathrm{Hom}_{\mathfrak{h}}(\Bbbk_\mu, M^{\mathfrak{n}^+}) \cong \mathrm{Hom}_{\mathfrak{h}}(\Bbbk_\mu, M) \cong M_\mu,$$

where the second isomorphism follows from the assumptions on supp$M$. □

Corollary 3.1.2. *For $M \in \mathbf{C}(\mathfrak{g}, \mathfrak{h})$ with $M_\nu = 0$ for all $\nu > \mu$, we have*

$$\mathrm{Ext}^1_{\mathbf{C}(\mathfrak{g},\mathfrak{h})}(\Delta(\mu), M) = 0.$$

Proof. Consider a short exact sequence

$$0 \to M \to \widetilde{M} \xrightarrow{\beta} \Delta(\mu) \to 0$$

in $\mathbf{C}(\mathfrak{g},\mathfrak{h})$. Take $v \in \widetilde{M}_\mu$ in the preimage $\beta^{-1}(v_\mu)$. By Lemma 3.1.1, we have a morphism $\alpha : \Delta(\mu) \to \widetilde{M}$, such that $\beta \circ \alpha = 1_{\Delta(\mu)}$. This implies that the extension splits. □

3.1.3. If $\mathfrak{b}$ is a Dynkin Borel subalgebra, then $\Delta(\lambda) \in \mathcal{C}(\mathfrak{g},\mathfrak{h})$ by Proposition 2.5.1. In that case, we introduce the **dual Verma module**

$$\nabla(\lambda) \;:=\; \Delta(\lambda)^\vee,$$

where $\vee$ is the duality on $\mathcal{C}(\mathfrak{g},\mathfrak{h})$ of 2.1.5. It follows from equation (2.2) that $L(\lambda) \cong L(\lambda)^\vee$.

3.2. **Reduction to root-reductive subalgebras.** Consider an arbitrary parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}$ with reductive part $\mathfrak{l}$.

Lemma 3.2.1. *For $\lambda, \mu \in \mathfrak{h}^*$ with $\lambda - \mu \in \mathbb{Z}\Phi(\mathfrak{l})$, we have*
(i) $[\Delta(\lambda) : L(\mu)] = [\Delta_{\mathfrak{l}}(\lambda) : L_{\mathfrak{l}}(\mu)]$;
(ii) $\mathrm{Hom}_{\mathfrak{g}}(\Delta(\mu), \Delta(\lambda)) \;\cong\; \mathrm{Hom}_{\mathfrak{l}}(\Delta_{\mathfrak{l}}(\mu), \Delta_{\mathfrak{l}}(\lambda))$.

Proof. Part (i) follows from the observations

$$\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{l},+}\Delta_{\mathfrak{l}}(\lambda) \cong \Delta(\lambda), \qquad \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{l},\lambda}\Delta(\lambda) \cong \Delta_{\mathfrak{l}}(\lambda),$$

$$[\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{l},+}L_{\mathfrak{l}}(\mu) : L(\mu)] = 1 \qquad \text{and} \qquad \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{l},\mu}L(\mu) \cong L_{\mathfrak{l}}(\mu).$$

By adjunction, we have

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta(\mu), \Delta(\lambda)) \;\cong\; \mathrm{Hom}_{\mathfrak{l}}(\Delta_{\mathfrak{l}}(\mu), \Delta(\lambda)^{\mathfrak{u}^+}) \;\cong\; \mathrm{Hom}_{\mathfrak{l}}(\Delta_{\mathfrak{l}}(\mu), \Delta_{\mathfrak{l}_{\mathfrak{b}}}(\lambda)),$$

where the last isomorphism follows from weight considerations. This proves part (ii). □

3.3. **Verma modules for Dynkin Borel subalgebras.** Assume that $\mathfrak{b}$ is a Dynkin Borel subalgebra. By 2.5.1(ii), without loss of generality we may *assume that each* $\widetilde{\mathfrak{g}}_n + \mathfrak{b}$ *is a (parabolic) subalgebra.*

Theorem 3.3.1. *Consider arbitrary $\lambda, \mu \in \mathfrak{h}^*$. For any $n \in \mathbb{N}$ such that $\lambda - \mu \in \mathbb{Z}\Phi_n$, we have*
(i) $[\Delta(\lambda) : L(\mu)] \;=\; [\Delta_n(\lambda) : L_n(\mu)]$;
(ii) $\mathrm{Hom}_{\mathfrak{g}}(\Delta(\mu), \Delta(\lambda)) \;\cong\; \mathrm{Hom}_{\mathfrak{g}_n}(\Delta_n(\mu), \Delta_n(\lambda))$.

Proof. This is a special case of Lemma 3.2.1. □

Remark 3.3.2. For integral regular weights, Theorem 3.3.1(i) was first obtained in [Na2, Proposition 3.6] through different methods. Our results completely determine the decomposition multiplicities of Verma modules for Dynkin Borel subalgebras in terms of the Kazhdan-Lusztig multiplicities for finite dimensional reductive Lie algebras.

Analogues of Theorem 3.3.1 for parabolic Verma modules, where the reductive subalgebra of the parabolic subalgebra has finite rank, can be proved using the same method. Analogues for specific parabolic subalgebras with reductive subalgebra of finite *co*rank have been proved in e.g. [CLW, CWZ].

3.3.3. The **Bruhat order** on $\mathfrak{h}^*$ is the partial order $\uparrow$ generated by the relation

$$\mu \uparrow \lambda \qquad \text{if} \qquad \mu = r_\alpha \cdot \lambda \;\text{ for some }\; \alpha \in \Phi^+ \quad \text{and} \quad \mu \le \lambda.$$

Corollary 3.3.4. *Consider arbitrary $\lambda, \mu \in \mathfrak{h}^*$. For any $n \in \mathbb{N}$ such that $\lambda - \mu \in \mathbb{Z}\Phi_n$, we have*

(i) $[\Delta(\lambda) : L(\mu)] \neq 0$ *if and only if* $\mu \uparrow \lambda$*;*

(ii) $\dim \mathrm{Hom}_{\mathfrak{g}}(\Delta(\mu), \Delta(\lambda)) = \begin{cases} 1 & \text{if } \mu \uparrow \lambda \\ 0 & \text{otherwise.} \end{cases}$

Proof. This follows immediately from Theorem 3.3.1 and the BGG theorem, see [Hu, Theorem 5.1] and [Hu, Theorem 4.2(b)]. □

Proposition 3.3.5. *For all $\lambda, \mu \in \mathfrak{h}^*$, we have*

(i) $\mathrm{Ext}^1_{\mathbf{C}(\mathfrak{g},\mathfrak{h})}(\Delta(\lambda), \nabla(\mu)) = 0$*;*

(ii) $\dim_{\Bbbk} \mathrm{Hom}_{\mathfrak{g}}(\Delta(\lambda), \nabla(\mu)) = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu; \end{cases}$

(iii) $\mathrm{Ext}^1_{\mathbf{C}(\mathfrak{g},\mathfrak{h})}(\Delta(\lambda), \Delta(\mu)) = 0,$ *unless* $\lambda < \mu$*.*

Proof. Part (iii) is a special case of Corollary 3.1.2. If $\lambda \not< \mu$, part (i) follows also from Corollary 3.1.2. If $\lambda < \mu$, part (i) follows from the previous case by applying $\vee$. Similarly if $\lambda \not> \mu$, part (ii) follows from Lemma 3.1.1, and in the remaining cases from $\vee$. □

Remark 3.3.6. Proposition 3.3.5 was first obtained in [Na2, Propositions 3.8 and 3.9].

3.4. **Verma modules for non-Dynkin Borel subalgebras.** First we determine all morphism spaces between Verma modules in terms of those for Dynkin Borel subalgebras (in Theorem 3.3.1).

Proposition 3.4.1. *Consider $\lambda, \mu \in \mathfrak{h}^*$. Let $\mathfrak{l}_{\mathfrak{b}}$ be the $\mathfrak{b}$-finite root-reductive subalgebra of $\mathfrak{g}$.*

(i) If λ and μ are not remote, then

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta(\mu), \Delta(\lambda)) \;\cong\; \mathrm{Hom}_{\mathfrak{l}_{\mathfrak{b}}}(\Delta_{\mathfrak{l}_{\mathfrak{b}}}(\mu), \Delta_{\mathfrak{l}_{\mathfrak{b}}}(\lambda)).$$

(ii) If λ and μ are remote, then $\mathrm{Hom}_{\mathfrak{g}}(\Delta(\mu), \Delta(\lambda)) \;=\; 0.$

Proof. Part (i) is a special case of Lemma 3.2.1(ii).

Now we prove part (ii). We take a basis of $\mathfrak{n}^-$ consisting of root vectors. We extend the partial order $\le$ on Φ^+ to a total order $\preceq$ such that all roots of finite length are smaller than all roots of infinite length. Then we take a PBW basis of $U(\mathfrak{n}^-)$, where each basis element is a product of root vectors, and elements of $\mathfrak{g}^{-\alpha}$ appear to the left of elements of $\mathfrak{g}^{-\beta}$ if $\alpha \succ \beta$. An arbitrary weight vector of $\Delta(\lambda)$ is then of the form

$$w = \sum_{i=1}^{k} u_i \otimes v,$$

with $v \in \Bbbk_\lambda$ and each u_i a PBW basis element of $U(\mathfrak{n}^-)$.

Let $\mu \le \lambda$ be remote from λ and assume that w is of weight μ. By construction, a minimal positive root of infinite length such that $\mathfrak{g}^{-\alpha}$ appears in one of the u_i. Now take $\beta \in \Sigma$ such that $\alpha - \beta \in \Phi^+$. We thus have $[\mathfrak{g}^\beta, \mathfrak{g}^{-\alpha}] \neq 0$, and for a non-zero $X \in \mathfrak{g}^\beta$ we consider

$$Xw = \sum_{i=1}^{k} [X, u_i] \otimes v.$$

Amongst other possible terms, any $[X, u_i]$ such that $\mathfrak{g}^{-\alpha}$ appears in u_i, has a term (in the natural expansion of $[X, u_i]$ based on the action of X on each factor in the product u_i) with a factor in $\mathfrak{g}^{-\alpha+\beta}$ which is by construction a PBW basis element. Moreover, this basis element does not appear in other terms of Xw. It thus follows that $X \in \mathfrak{g}^{\beta} \subset \mathfrak{n}^{+}$ acts non-trivially on w. Consequently, there exists no non-zero morphism from $\Delta(\mu)$ to $\Delta(\lambda)$ □

Remark 3.4.2. Proposition 3.4.1(i) was first obtained in [Na1, Section 6.4].

3.4.3. We will use the labelling set $\mathsf{N} = \mathbb{N} \amalg \{\bar{i} \mid i \in \mathbb{N}\}$, with linear order $\preceq$ such that

$$0 \prec 1 \prec 2 \prec 3 \prec \cdots \prec \overline{3} \prec \overline{2} \prec \overline{1} \prec \overline{0}.$$

We set $\mathfrak{g} = \mathfrak{gl}_\infty$, with

$$\Phi = \{\epsilon_a - \epsilon_b \mid a, b \in \mathsf{N}\},$$

and choose as positive roots

$$\Phi^+ = \{\epsilon_a - \epsilon_b \mid a \prec b\}.$$

The corresponding Borel subalgebra $\mathfrak{b}$ is not of Dynkin type.

Lemma 3.4.4. *With $\mathfrak{g}$ and $\mathfrak{b}$ as in 3.4.3, there exists a simple $\mathfrak{g}$-module L, with $[\Delta(\mathbf{0}) : L] \neq 0$, which does not have a highest weight vector with respect to $\mathfrak{b}$.*

Proof. We consider the Dynkin Borel subalgebra $\mathfrak{b}' \supset \mathfrak{h}$ which corresponds to the choice of positive roots $\{\epsilon_a - \epsilon_b \mid a \prec' b\}$, for the linear order

$$\cdots \prec' 3 \prec' 2 \prec' 1 \prec' 0 \prec' \overline{0} \prec' \overline{1} \prec' \overline{2} \prec' \overline{3} \prec' \cdots .$$

It follows easily that there exists a non-zero homomorphism

$$\Delta^{\mathfrak{b}'}_{\mathfrak{g}}(-\varepsilon_0 + \varepsilon_{\overline{0}}) \;\to\; \Delta^{\mathfrak{b}}_{\mathfrak{g}}(\mathbf{0}), \quad w \mapsto X_{-\epsilon_0 + \epsilon_{\overline{0}}} \otimes v,$$

where w, resp. v, is a vector in $\Bbbk_{-\varepsilon_0+\varepsilon_{\overline{0}}}$, resp. $\Bbbk_{\mathbf{0}}$. This implies that

$$[\Delta^{\mathfrak{b}}_{\mathfrak{g}}(0) : L^{\mathfrak{b}'}_{\mathfrak{g}}(-\varepsilon_0 + \varepsilon_{\overline{0}})] \;\neq\; 0.$$

Since $L := L^{\mathfrak{b}'}_{\mathfrak{g}}(-\varepsilon_0 + \varepsilon_{\overline{0}})$ satisfies

$$L_{-\varepsilon_a + \varepsilon_{\overline{b}}} \neq 0 \quad \text{for all } a, b \in \mathbb{N},$$

and the only weight in $\mathrm{supp}\Delta(0)$ which is higher than all of these is 0, it follows that L does not have a highest weight with respect to $\mathfrak{b}$. □

Lemma 3.4.4 implies that, for $\mathfrak{b}$ the Borel subalgebra of 3.4.3, any Serre subcategory of $\mathfrak{g}$-modules which contains the Verma modules will contain simple modules which are not highest weight modules. *Due to this observation, we will restrict to Dynkin Borel subalgebras for the remainder of the paper.*

3.5. **Modules with Δ-flag or ∇-flag.** Assume that $\mathfrak{b}$ is a Dynkin Borel subalgebra.

3.5.1. Denote by $\mathcal{F}^{\Delta}(\mathfrak{g}, \mathfrak{b})$, resp. $\mathcal{F}^{\nabla}(\mathfrak{g}, \mathfrak{b})$, the full exact subcategory of modules in $\mathbf{C}(\mathfrak{g}, \mathfrak{h})$ which admit a finite Δ-flag, resp. ∇-flag. By a Δ-flag of M, we mean a filtration

$$0 = F_k M \subset F_{k-1} M \subset \cdots F_1 M \subset F_0 M = M, \tag{3.1}$$

with $F_i M / F_{i+1} M \cong \Delta(\mu_i)$ for some $\mu_i \in \mathfrak{h}^*$, for each $0 \leq i < k$.

For $M \in \mathcal{F}^{\Delta}$, we denote by $(M : \Delta(\lambda))$ the number of indices i for which $F_i M / F_{i+1} M$ in (3.1) is isomorphic to $\Delta(\lambda)$. It is easy to see that $(M : \Delta(\lambda))$ is independent of the chosen filtration, for instance by looking at the character of the modules, or from the following lemma.

Lemma 3.5.2. *For $M \in \mathcal{F}^\Delta$ and $\lambda \in \mathfrak{h}^*$, we have*

$$(M : \Delta(\lambda)) \;=\; \dim_{\Bbbk} \mathrm{Hom}_{\mathfrak{g}}(M, \nabla(\lambda)).$$

Proof. This can easily be obtained by induction on the length of the filtration, by applying the properties in Proposition 3.3.5(i) and (ii). □

We have the following alternative characterisation of the categories $\mathcal{F}^\Delta$ and $\mathcal{F}^\nabla$.

Lemma 3.5.3. *The category $\mathcal{F}^\Delta$, resp. $\mathcal{F}^\nabla$, is the full subcategory of $\mathbf{C}(\mathfrak{g},\mathfrak{h})$ consisting of finite direct sums of modules isomorphic to $U(\mathfrak{n}^-)$ when considered as $U(\mathfrak{n}^-)$-modules, resp. finite direct sums of modules isomorphic to $U(\mathfrak{n}^+)^{\circledast}$ when considered as $U(\mathfrak{n}^+)$-modules.*

Proof. It is clear that objects in $\mathcal{F}^\Delta$, resp. $\mathcal{F}^\nabla$, restrict to finite direct sums of modules isomorphic to $U(\mathfrak{n}^-)$ when considered as an $U(\mathfrak{n}^-)$-module, resp. finite direct sums of modules isomorphic to $U(\mathfrak{n}^+)^{\circledast}$ when considered as an $U(\mathfrak{n}^+)$-module.

Now consider $M \in \mathbf{C}(\mathfrak{g},\mathfrak{h})$, such that $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{n}^-} M \cong U(\mathfrak{n}^-)$. Since M must be a weight module, the element $1 \in U(\mathfrak{n}^-)$ corresponds to a one dimensional space of weight λ in M which must be annihilated by $\mathfrak{n}^+$ and generates M as an $\mathfrak{n}$-module. It follows that $M \cong \Delta(\lambda)$.

Now consider $M \in \mathbf{C}(\mathfrak{g},\mathfrak{h})$, such that $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{n}^-} M \cong U(\mathfrak{n}^-)^{\oplus k}$, for some $k > 1$. Since M is a weight module, as an $\mathfrak{h}$-module M is isomorphic to $\oplus_i \Delta(\lambda_i)$, for some weights $\lambda_1, \ldots, \lambda_k$. Without loss of generality we assume that there is no weight in the set higher than λ_1. This shows that there is an injective $\mathfrak{g}$-module morphism $\Delta(\lambda_1) \hookrightarrow M$. We can then proceed by considering $M/\Delta(\lambda_1)$. □

The above lemma has the following three immediate consequences.

Corollary 3.5.4. *For $M, M' \in \mathbf{C}(\mathfrak{g},\mathfrak{h})$, we have that $M \oplus M'$ belongs to $\mathcal{F}^\Delta$, resp. $\mathcal{F}^\nabla$, if and only if both M and M' belong to $\mathcal{F}^\Delta$, resp. $\mathcal{F}^\nabla$.*

Corollary 3.5.5. *Consider a short exact sequence in $\mathbf{C}(\mathfrak{g},\mathfrak{h})$*

$$0 \to A \to B \to C \to 0$$

with $C \in \mathcal{F}^\Delta$. Then we have $A \in \mathcal{F}^\Delta$ if and only if $B \in \mathcal{F}^\Delta$.

Corollary 3.5.6. *The duality functor $\circledast$, resp. $\vee$, on $\mathcal{C}(\mathfrak{g},\mathfrak{h})$ restricts to an exact contravariant equivalence*

$$\circledast : \mathcal{F}^\Delta(\mathfrak{g},\mathfrak{b}) \tilde{\to} \mathcal{F}^\nabla(\mathfrak{g},\mathfrak{b}^-), \quad \textit{resp.} \quad \vee : \mathcal{F}^\Delta \tilde{\to} \mathcal{F}^\nabla.$$

4. The category O

For the rest of the paper, fix a root-reductive Lie algebra $\mathfrak{g} = \varinjlim \widetilde{\mathfrak{g}}_n$ with Dynkin Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. By 2.5.1(ii), without loss of generality we may *assume that each $\widetilde{\mathfrak{g}}_n + \mathfrak{b}$ is a (parabolic) subalgebra.*

4.1. **Definitions.**

The main object of study will be the following abelian category.

Definition 4.1.1. The category $\mathbf{O} = \mathbf{O}(\mathfrak{g},\mathfrak{b})$ is the full subcategory of $\mathbf{C} = \mathbf{C}(\mathfrak{g},\mathfrak{h})$ of modules M on which $\mathfrak{b}$ acts locally finitely.

The simple objects in $\mathbf{O}$ are, up to isomorphism, precisely the simple highest weight modules $L(\lambda)$ for $\lambda \in \mathfrak{h}^*$. The category $\mathcal{F}^\Delta$ is an exact, but not abelian, subcategory of $\mathbf{O}$.

Remark 4.1.2. In case $\mathfrak{g}$ is finite dimensional (so a reductive Lie algebra) the universal enveloping algebra $U(\mathfrak{g})$ is noetherian and the ordinary BGG category $\mathcal{O}(\mathfrak{g},\mathfrak{b})$ of [BGG, Hu] is the full subcategory of $\mathbf{O}(\mathfrak{g},\mathfrak{b})$ of *finitely generated* modules. In this case, the relation between the categories $\mathcal{O}$ and $\mathbf{O}$ is summarised in Proposition 4.4.6 below.

Remark 4.1.3. In [Na2], the abelian category $\bar{\mathcal{O}}(\mathfrak{g},\mathfrak{b})$ is studied, which is the full subcategory of $\mathbf{O}(\mathfrak{g},\mathfrak{b})$ of modules with finite dimensional weight spaces. We thus have Serre subcategories

$$\begin{array}{ccc} \mathcal{C}(\mathfrak{g},\mathfrak{h}) & \hookrightarrow & \mathbf{C}(\mathfrak{g},\mathfrak{h}) \\ \uparrow & & \uparrow \\ \bar{\mathcal{O}}(\mathfrak{g},\mathfrak{b}) & \hookrightarrow & \mathbf{O}(\mathfrak{g},\mathfrak{b}). \end{array}$$

From no on we will leave out the references to $\mathfrak{g}, \mathfrak{b}$ and $\mathfrak{h}$ in $\mathbf{O}(\mathfrak{g},\mathfrak{b})$, $\mathcal{C}(\mathfrak{g},\mathfrak{h})$ etc.

4.1.4. *Serre subcategories by truncation.* Let K be any ideal in $\mathfrak{h}^*$. The Serre subcategory ${}^{\mathsf{K}}\mathbf{O}$ of $\mathbf{O}$ is defined as the full subcategory of modules M with $\mathrm{supp} M \subset \mathsf{K}$. Clearly, we have

$$\Delta(\lambda) \in {}^{\mathsf{K}}\mathbf{O} \quad \Leftrightarrow \quad \lambda \in \mathsf{K} \quad \Leftrightarrow \quad L(\lambda) \in {}^{\mathsf{K}}\mathbf{O}. \tag{4.1}$$

Similarly, ${}^{\mathsf{K}}\bar{\mathcal{O}}$ is the subcategory of $\bar{\mathcal{O}}$ of modules with support in K.

A special role will be played by ideals K which are upper finite. The following lemma is obvious from the fact that simple highest weight modules have finite dimensional weight spaces.

Lemma 4.1.5. *For an upper finite ideal* $\mathsf{K} \subset \mathfrak{h}^*$ *and* $M \in {}^{\mathsf{K}}\mathbf{O}$*, we have*

$$M \in {}^{\mathsf{K}}\bar{\mathcal{O}} \quad \Leftrightarrow \quad [M : L(\mu)] < \infty \text{ for all } \mu \in \mathsf{K}.$$

Remark 4.1.6. When $\mathfrak{g}$ is not finite dimensional, there exist indecomposable modules in $\mathbf{O}$ for which $\mathrm{supp} M$ is not upper finite. For instance, when λ is integral, regular and antidominant we can consider an infinite chain

$$\lambda = \lambda^0 \uparrow \lambda^1 \uparrow \lambda^2 \uparrow \cdots.$$

By Corollary 3.3.4(ii), we have morphisms $\Delta(\lambda^i) \to \Delta(\lambda^{i+1})$, for all $i \in \mathbb{N}$. The $\mathfrak{g}$-module $M := \varinjlim \Delta(\lambda^i)$ belongs to $\mathbf{O}$. However, since M is not in $\bar{\mathcal{O}}$, this does not yet answer the question, raised in [Na1, Section 5.3], of whether there exist indecomposable modules in $\bar{\mathcal{O}}$ whose support is not upper finite.

4.1.7. A special class of ideals in $\mathfrak{h}^*$ is formed by the $\mathbb{Z}\Phi$**-cosets** in $\mathfrak{h}^*$, by which we mean the subsets $\lambda + \mathbb{Z}\Phi$ for arbitrary $\lambda \in \mathfrak{h}^*$. Note that $\mathbb{Z}\Phi$ naturally forms a subgroup of $\mathfrak{h}^*$ for operation given by addition of vectors.

4.2. **Locally projective modules.**

Theorem 4.2.1. *Let* $\mathsf{K} \subset \mathfrak{h}^*$ *be an upper finite ideal.*

- *(i) For each* $\mu \in \mathsf{K}$*, there exists a module* $P_{\mathsf{K}}(\mu) \in {}^{\mathsf{K}}\bar{\mathcal{O}} \subset {}^{\mathsf{K}}\mathbf{O}$ *such that:*
  - *(a)* $\dim_{\Bbbk} \mathrm{Hom}_{{}^{\mathsf{K}}\mathbf{O}}(P_{\mathsf{K}}(\mu), -) = [- : L(\mu)]$.
  - *(b) we have* $P_{\mathsf{K}}(\mu) \in \mathcal{F}^{\Delta}[\mathsf{K}]$*, with*

$$(P_{\mathsf{K}}(\mu) : \Delta(\nu)) = [\Delta(\nu) : L(\mu)], \qquad \text{for all } \nu \in \mathsf{K}.$$

  - *(c) we have* $[P_{\mathsf{K}}(\mu) : L(\nu)] = 0$*, unless* $\nu \in [\![\mu]\!]$.
- *(ii) The category* ${}^{\mathsf{K}}\mathbf{O}$ *has enough projective objects. Each projective object is a direct sum of modules isomorphic to* $P_{\mathsf{K}}(\mu)$*, with* $\mu \in \mathsf{K}$.

We precede the proof with some discussions and a lemma.

Remark 4.2.2.

- (i) The existence of projective objects $P_{\mathsf{K}}(\mu)$ in ${}^{\mathsf{K}}\bar{\mathcal{O}}$ was first established through different methods in [Na2, Section 4].

(ii) Even though $P_{\mathsf{K}}(\mu) \in {}^{\mathsf{K}}\bar{\mathcal{O}}$, that category does generally not have *enough* projective objects. An example is given by considering a regular integral dominant $\lambda \in \mathfrak{h}^*$ and $M := \bigoplus_{\mu\in[\![\lambda]\!]} L(\mu)$. By Corollary 3.3.4(i), we have $\dim M_\nu \le \dim \Delta(\lambda)_\nu$ for all $\nu \in \mathfrak{h}^*$, so $M \in \bar{\mathcal{O}}$. On the other hand, by Theorem 4.2.1(i)(b), a projective cover of M has infinite dimensional weight spaces. This answers [Na2, Open Question 4.15] negatively.

(iii) It will follow *a posteori* that the condition on the ideal $\mathsf{K} \subset \mathfrak{h}^*$ in Theorem 4.2.1 can be weakened to demand that it be upper finite with respect to the Bruhat order $\uparrow$.

(iv) Given an ideal $\mathsf{K} \subset \mathfrak{h}^*$ and *one specific* $\mu \in \mathsf{K}$, the existence of a module $P_{\mathsf{K}}(\mu)$ as in Theorem 4.2.1(i) follows if $\{\nu \in \mathsf{K} \,|\, \nu \ge \mu\}$ is finite (or even just that $\{\nu \in \mathsf{K} \,|\, \mu \uparrow \nu\}$ is finite).

We follow the approach of [BGG, Section 4], see also [CM1]. We fix $\mu \in \mathsf{K}$.

4.2.3. We define a $U(\mathfrak{b})$-module V^{K}_μ with $\mathrm{supp} V^{\mathsf{K}}_\mu \subset \mathsf{K}$ with presentation

$$\bigoplus_{\kappa\in S} U(\mathfrak{b}) \otimes_{U(\mathfrak{h})} \Bbbk_\kappa \to U(\mathfrak{b}) \otimes_{U(\mathfrak{h})} \Bbbk_\mu \to V^{\mathsf{K}}_\mu \to 0, \tag{4.2}$$

where S is a multiset of weights in $\mathfrak{h}^*\backslash\mathsf{K}$ such that each $\kappa \in \mathfrak{h}^*\backslash\mathsf{K}$ appears $\dim U(\mathfrak{b})_{\kappa-\mu}$ times. Since the set $\{\nu \in \mathsf{K} \,|\, \nu \ge \mu\}$ is finite, V^{K}_μ is finite dimensional. We then define

$$Q_{\mathsf{K}}(\mu) \;:=\; U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V^{\mathsf{K}}_\mu.$$

By construction, we have $Q_{\mathsf{K}}(\mu) \in \mathcal{F}^\Delta[\mathsf{K}] \subset {}^{\mathsf{K}}\mathbf{O}$. The module $Q_{\mathsf{K}}(\mu)$ is generated by a vector v_μ, which we take in the image of $\Bbbk_\mu$ under the epimorphism in (4.2).

4.2.4. *Example.* If $\mathsf{K} = \{\nu \in \mathfrak{h}^* \,|\, \nu \le \mu\}$, we have $V^{\mathsf{K}}_\mu = \Bbbk_\mu$ and thus $Q_{\mathsf{K}}(\mu) \cong \Delta(\mu)$.

Lemma 4.2.5. *The module $Q_{\mathsf{K}}(\mu)$ is projective in ${}^{\mathsf{K}}\mathbf{O}$ and for any $M \in {}^{\mathsf{K}}\mathbf{O}$, we have an isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(Q_{\mathsf{K}}(\mu), M) \;\tilde{\to}\; M_\mu, \qquad \alpha \mapsto \alpha(v_\mu).$$

Proof. Apply the exact induction functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} -$ to the exact sequence (4.2), followed by application of the left exact contravariant functor $\mathrm{Hom}_{\mathfrak{g}}(-, M)$. This yields an exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(Q_{\mathsf{K}}(\mu), M) \to \mathrm{Hom}_{\mathfrak{h}}(\Bbbk_\mu, M) \to \prod_{\kappa\in S} \mathrm{Hom}_{\mathfrak{h}}(\Bbbk_\kappa, M),$$

where we have used adjunction and equation (1.2). The right term is zero since $\mathrm{supp} M \subset \mathsf{K}$, which concludes the proof. □

Proof of Theorem 4.2.1. First assume that there are no simple modules L in ${}^{\mathsf{K}}\mathbf{O}$, other than $L = L(\mu)$, such that $L_\mu \neq 0$. It then follows from Lemma 4.2.5 that $Q_{\mathsf{K}}(\mu)$ satisfies the property of $P_{\mathsf{K}}(\mu)$ in (i)(a). If there are other $\nu \in \mathsf{K}$ such that $L(\nu)_\mu \neq 0$, then there are only finitely many. By induction we can assume that we already constructed $P_{\mathsf{K}}(\nu)$ for all of them. It follows that all these are direct summands (appearing $\dim L(\nu)_\mu$ times each) of $Q_{\mathsf{K}}(\mu)$. The remaining direct summand of $Q_{\mathsf{K}}(\mu)$ satisfies the properties of $P_{\mathsf{K}}(\mu)$ in (i)(a).

By Corollary 3.5.4 and induction, we find that $P_{\mathsf{K}}(\mu)$ is in $\mathcal{F}^\Delta$. Lemma 3.5.2 implies that for any $\nu \in \mathfrak{h}^*$

$$(P_{\mathsf{K}}(\mu) : \Delta(\nu)) \;=\; \dim \mathrm{Hom}_{\mathfrak{g}}(P_{\mathsf{K}}(\mu), \nabla(\nu)).$$

If $\nu \in \mathsf{K}$, then part (i)(a) implies we also have

$$\dim \mathrm{Hom}_{\mathfrak{g}}(P_{\mathsf{K}}(\mu), \nabla(\nu)) \;=\; [\nabla(\nu) : L(\mu)] \;=\; [\Delta(\nu) : L(\mu)],$$

where the latter equality follows from the duality $\vee$. This concludes the proof of part (i)(b).

Part (i)(c) follows from part (i)(b) and Corollary 3.3.4(i).

We consider an arbitrary module $M \in {}^{\mathsf{K}}\mathbf{O}$. It has a set of generating elements $\{v_\alpha\} \subset M$, which we can choose to be weight vectors. Since $M \in {}^{\mathsf{K}}\mathbf{O}$, it follows $U(\mathfrak{g})v_\alpha$ is a quotient of $Q_{\mathsf{K}}(\mu)$, for $\mu \in \mathfrak{h}^*$ the weight of v_α. Hence, we have an epimorphism $\bigoplus_\mu Q_{\mathsf{K}}(\mu) \twoheadrightarrow M$. From the universality property of coproducts, or alternatively from Lemma 1.1.4, it follows that $\bigoplus_\mu Q_{\mathsf{K}}(\mu)$ is projective. This proves part (ii). □

4.3. **Blocks.** For $\lambda \in \mathfrak{h}^*$, let $\mathbf{O}_{[\![\lambda]\!]}$ denote the full subcategory of modules M in $\mathbf{O}$ such that $[M : L(\mu)] = 0$ whenever $\mu \notin [\![\lambda]\!]$. We use similar notation for $\bar{\mathcal{O}}$ and $\mathcal{O}$.

Proposition 4.3.1. *We have an equivalence of categories*

$$\prod_{[\![\lambda]\!]} \mathbf{O}_{[\![\lambda]\!]} \xrightarrow{\sim} \mathbf{O}, \qquad (M_{[\![\lambda]\!]})_{[\![\lambda]\!]} \mapsto \bigoplus_{[\![\lambda]\!]} M_{[\![\lambda]\!]}.$$

Proof. By definition, we have

$$\mathrm{Hom}_{\prod_{[\![\lambda]\!]}\mathbf{O}_{[\![\lambda]\!]}}\left((M_{[\![\nu]\!]})_{[\![\nu]\!]}, (N_{[\![\mu]\!]})_{[\![\mu]\!]}\right) \;=\; \prod_{[\![\lambda]\!]} \mathrm{Hom}_{\mathbf{O}_{[\![\lambda]\!]}}(M_{[\![\lambda]\!]}, N_{[\![\lambda]\!]}).$$

On the other hand, by (1.2), we have

$$\mathrm{Hom}_{\mathbf{O}}(\bigoplus_{[\![\lambda]\!]} M_{[\![\lambda]\!]}, \bigoplus_{[\![\mu]\!]} N_{[\![\mu]\!]}) \;\cong\; \prod_{[\![\lambda]\!]} \mathrm{Hom}_{\mathbf{O}}(M_{[\![\lambda]\!]}, \bigoplus_{[\![\mu]\!]} N_{[\![\mu]\!]}) \;\cong\; \prod_{[\![\lambda]\!]} \mathrm{Hom}_{\mathbf{O}}(M_{[\![\lambda]\!]}, N_{[\![\lambda]\!]}).$$

Hence, the functor $\prod_{[\![\lambda]\!]} \mathbf{O}_{[\![\lambda]\!]} \to \mathbf{O}$ is fully faithful. We denote the isomorphism closure of its image by $\mathbf{O}'$, which is a subcategory closed under taking quotients. By Theorem 4.2.1, $\mathbf{O}$ has enough projective objects and they are all contained in $\mathbf{O}'$. This proves that $\mathbf{O}' = \mathbf{O}$. □

Remark 4.3.2. For any ideal $\mathsf{K} \subset \mathfrak{h}^*$ and $\lambda \in \mathsf{K}$, we use the notation ${}^{\mathsf{K}}\mathbf{O}_{[\![\lambda]\!]}$ for the full subcategory of $\mathbf{O}$ consisting of modules which are both in ${}^{\mathsf{K}}\mathbf{O}$ and $\mathbf{O}_{[\![\lambda]\!]}$. It is clear, by equation (4.1), that we have ${}^{\mathsf{K}'}\mathbf{O}_{[\![\lambda]\!]} = {}^{\mathsf{K}}\mathbf{O}_{[\![\lambda]\!]}$, for two ideals K and K' such that $\mathsf{K} \cap [\![\lambda]\!] = \mathsf{K}' \cap [\![\lambda]\!]$.

Remark 4.3.3. Proposition 4.3.1 implies in particular [Na2, Theorem 3.4], which was obtained through a different approach. Note however that we have proper inclusions of categories

$$\bigoplus_{[\![\lambda]\!]} \bar{\mathcal{O}}_{[\![\lambda]\!]} \;\subsetneq\; \bar{\mathcal{O}} \;\subsetneq\; \prod_{[\![\lambda]\!]} \bar{\mathcal{O}}_{[\![\lambda]\!]}.$$

4.4. **Describing algebras.** Fix an upper finite ideal $\mathsf{K} \subset \mathfrak{h}^*$ and $\lambda \in \mathsf{K}$.

4.4.1. We set ${}^{\mathsf{K}}[\![\lambda]\!] = \mathsf{K} \cap [\![\lambda]\!]$. We define the vector space

$$A^{\mathsf{K}}_{[\![\lambda]\!]}(\mathfrak{g}, \mathfrak{b}) = A^{\mathsf{K}}_{[\![\lambda]\!]} \;:=\; \bigoplus_{\mu,\nu \in {}^{\mathsf{K}}[\![\lambda]\!]} \mathrm{Hom}_{\mathfrak{g}}(P_{\mathsf{K}}(\mu), P_{\mathsf{K}}(\nu)),$$

which is an algebra with multiplication $fg = g \circ f$. The algebra $A^{\mathsf{K}}_{[\![\lambda]\!]}$ is then locally finite, with idempotents e_ν given by the identity of $P_{\mathsf{K}}(\nu)$, for all $\nu \in \mathsf{K}$.

Theorem 4.4.2. *We have an equivalence of categories*

$${}^{\mathsf{K}}\mathbf{O}_{[\![\lambda]\!]} \;\xrightarrow{\sim}\; A^{\mathsf{K}}_{[\![\lambda]\!]}\text{-Mod}, \quad M \mapsto \bigoplus_{\mu \in {}^{\mathsf{K}}[\![\lambda]\!]} \mathrm{Hom}_{\mathfrak{g}}(P_{\mathsf{K}}(\mu), M).$$

Proof. We write $A = A^{\mathsf{K}}_{[\![\lambda]\!]}$ and $\mathcal{F} = \bigoplus_\mu \mathrm{Hom}_{\mathfrak{g}}(P_{\mathsf{K}}(\mu), -)$. We observe that

$$\mathcal{F}(P_{\mathsf{K}}(\nu)) \;\cong\; Ae_\nu, \quad \text{for all } \nu \in \mathsf{K}.$$

Furthermore, $\mathcal{F}$ induces an isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(P_{\mathsf{K}}(\nu), P_{\mathsf{K}}(\kappa)) \xrightarrow{\sim} \mathrm{Hom}_A(Ae_\nu, Ae_\kappa), \quad \text{for all } \nu, \kappa \in \mathsf{K}.$$

It is clear that the functor $\mathcal{F}$ preserves arbitrary coproducts. Hence, $\mathcal{F}$ restricts to an equivalence between the categories of projective objects in ${}^{\mathsf{K}}\mathbf{O}$ and A-Mod. The fact that the functor $\mathcal{F}$ is exact then implies that $\mathcal{F}$ is an equivalence of categories between ${}^{\mathsf{K}}\mathbf{O}$ and A-Mod. □

Remark 4.4.3. In case $\lambda \in \mathfrak{h}^*$ is dominant, we can take $\mathsf{K} := \{\mu \in \mathfrak{h}^* \,|\, \mu \leq \lambda\}$, in which case we have ${}^{\mathsf{K}}\mathbf{O}_{[\![\lambda]\!]} = \mathbf{O}_{[\![\lambda]\!]}$. It thus follows that $\mathbf{O}_{[\![\lambda]\!]}$ is equivalent to the category of modules over a locally finite associative algebra.

Corollary 4.4.4. *We have an equivalence of categories*

$${}^{\mathsf{K}}\bar{\mathcal{O}}_{[\![\lambda]\!]} \xrightarrow{\sim} A^{\mathsf{K}}_{[\![\lambda]\!]}\text{-mod}, \quad M \mapsto \bigoplus_{\mu \in {}^{\mathsf{K}}[\![\lambda]\!]} \mathrm{Hom}_{\mathfrak{g}}(P_{\mathsf{K}}(\mu), M).$$

Proof. This is the restriction of the equivalence of Theorem 4.4.2 to the Serre subcategories of modules with finite multiplicities, by Lemma 4.1.5. □

Lemma 4.4.5. *For an upper finite ideal $\mathsf{K}' \subset \mathfrak{h}^*$ containing K, we have an isomorphism*

$$A^{\mathsf{K}'}_{[\![\lambda]\!]}/I \;\cong\; A^{\mathsf{K}}_{[\![\lambda]\!]}, \quad \textit{with} \quad I := \sum_{\mu \in {}^{\mathsf{K}'}[\![\lambda]\!] \backslash {}^{\mathsf{K}}[\![\lambda]\!]} A^{\mathsf{K}'}_{[\![\lambda]\!]} e_\mu A^{\mathsf{K}'}_{[\![\lambda]\!]}.$$

Proof. Consider the short exact sequence

$$0 \to N(\nu) \to P_{\mathsf{K}'}(\nu) \overset{p_\nu}{\to} P_{\mathsf{K}}(\nu) \to 0,$$

for any $\nu \in \mathsf{K}$. For all $\nu, \kappa \in \mathsf{K}$, we have

$$\mathrm{Hom}_{\mathbf{C}}(N(\nu), P_{\mathsf{K}}(\kappa)) = 0 \quad \text{and} \quad \mathrm{Ext}^1_{\mathbf{C}}(P_{\mathsf{K}'}(\nu), N(\kappa)) = 0.$$

Studying the long exact sequences coming from the bifunctor $\mathrm{Hom}_{\mathbf{C}}(-,-)$ acting on short exact sequences as above then yields an epimorphism and isomorphism

$$\mathrm{Hom}_{\mathbf{C}}(P_{\mathsf{K}'}(\nu), P_{\mathsf{K}'}(\kappa)) \xrightarrow{p_\kappa \circ -} \mathrm{Hom}_{\mathcal{C}}(P_{\mathsf{K}'}(\nu), P_{\mathsf{K}}(\kappa)) \xleftarrow[-\circ p_\nu]{\sim} \mathrm{Hom}_{\mathcal{C}}(P_{\mathsf{K}}(\nu), P_{\mathsf{K}}(\kappa)).$$

Composing the epimorphism and the inverse of the isomorphism yields the epimorphism

$$\mathrm{Hom}_{\mathbf{C}}(P_{\mathsf{K}'}(\nu), P_{\mathsf{K}'}(\kappa)) \twoheadrightarrow \mathrm{Hom}_{\mathcal{C}}(P_{\mathsf{K}}(\nu), P_{\mathsf{K}}(\kappa)), \quad \alpha \mapsto \phi, \quad \text{when } p_\kappa \circ \alpha = \phi \circ p_\nu.$$

This clearly yields an algebra morphism $A^{\mathsf{K}'}_{[\![\lambda]\!]} \twoheadrightarrow A^{\mathsf{K}}_{[\![\lambda]\!]}$.

The projective cover of $N(\kappa)$ is a direct sum of modules $P_{\mathsf{K}'}(\mu)$, with $\mu \in {}^{\mathsf{K}'}[\![\lambda]\!] \backslash {}^{\mathsf{K}}[\![\lambda]\!]$. Any $\alpha : P_{\mathsf{K}'}(\nu) \to P_{\mathsf{K}'}(\kappa)$ with $p_\kappa \circ \alpha = 0$ thus factors through such a projective module. This shows that the ideal I is the kernel of the above epimorphism. □

Proposition 4.4.6. *Assume that $\mathfrak{g}$ is finite dimensional, and hence a reductive Lie algebra.*

(i) We have equivalences of categories

$$\mathbf{O} \;\cong\; \prod_{[\![\lambda]\!]} \mathbf{O}_{[\![\lambda]\!]} \quad \textit{and} \quad \mathcal{O} \;\cong\; \bigoplus_{[\![\lambda]\!]} \mathcal{O}_{[\![\lambda]\!]}.$$

(ii) For each $\lambda \in \mathfrak{h}^*$, there exist finite dimensional algebra A such that

$$\mathcal{O}_{[\![\lambda]\!]} \;\cong\; A\text{-mod} \quad \textit{and} \quad \mathbf{O}_{[\![\lambda]\!]} \;\cong\; A\text{-Mod}.$$

(iii) The subcategory $\mathcal{O}$ is extension full in $\mathbf{O}$.

Proof. Part (i) is a special case of Proposition 4.3.1, see also [BGG]. Part (ii) follows from Theorem 4.4.2 and Corollary 4.4.4, by observing that $\bar{\mathcal{O}}_{[\![\lambda]\!]} = \mathcal{O}_{[\![\lambda]\!]}$, see e.g. [Na1, Section 5.1]. Part (iii) follows immediately from observing that a minimal projective resolution in $\mathbf{O}$ of $M \in \mathcal{O}$, is actually in $\mathcal{O}$. □

4.5. **Extension fullness.**

Theorem 4.5.1. *Let $\mathsf{K} \subset \mathfrak{h}^*$ be an upper finite ideal.*

(i) The category ${}^{\mathsf{K}}\mathbf{O}$ is extension full in $\mathbf{O}$.
(ii) The category ${}^{\mathsf{K}}\mathbf{O}$ is extension full in $\mathbf{C}$.
(iii) For the inclusion $\iota : {}^{\mathsf{K}}\bar{\mathcal{O}} \hookrightarrow {}^{\mathsf{K}}\mathbf{O}$, the morphism ι^i_{MN} in (1.1) is an isomorphism for all $i \in \mathbb{N}$, if $M \in \mathcal{F}^\Delta$ or $N \in \mathcal{F}^\nabla$.

Before proving the theorem, we note the following special case.

Corollary 4.5.2. *For all $\mu, \nu \in \mathfrak{h}^*$, we have*

$$\mathrm{Ext}^i_{\mathbf{O}}(\Delta(\mu), \nabla(\nu)) = 0, \quad \textit{for all } i > 0.$$

Proof. Assume first that $\nu \not> \mu$ and let K be the ideal generated by μ and ν. Then $\Delta(\mu) = P_{\mathsf{K}}(\mu)$, so $\mathrm{Ext}^i_{{}^{\mathsf{K}}\mathbf{O}}(\Delta(\mu), \nabla(\nu)) = 0$ and the conclusion follows from Theorem 4.5.1(i). If $\nu > \mu$, we can reduce to the previous case by Theorem 4.5.1(iii) and application of the duality $\vee$ on $\bar{\mathcal{O}}$. □

We start the proof of the theorem by proving the following proposition.

Proposition 4.5.3. *Consider an upper finite ideal K and a family $\{M_\alpha\}$ of objects in $\mathcal{F}^\Delta[\mathfrak{h}^*\backslash\mathsf{K}]$, and set $M = \bigoplus_\alpha M_\alpha$. We have*

$$\mathrm{Ext}^k_{\mathbf{O}}(M, N) = 0, \quad \textit{for all } N \in {}^{\mathsf{K}}\mathbf{O} \textit{ and } k \in \mathbb{N}.$$

Proof. The case $k = 0$ is obvious by equation (1.2). The case $k = 1$ follows from Lemma 1.1.4 and Corollary 3.1.2. Now we take $k > 1$, and assume the proposition is proved for $k - 1$. An element $\mathrm{Ext}^k_{\mathbf{O}}(M, N)$ can be represented by the upper exact sequence in the diagram

$$\begin{array}{ccccccccccccccc}
0 & \to & N & \to & E_k & \to & E_{k-1} & \to & \cdots & \to & E_2 & \to & E_1 & \to & M & \to & 0 \\
 & & \| & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \| & & \\
0 & \to & N & \to & E'_k & \to & E'_{k-1} & \to & \cdots & \to & E'_2 & \to & \bigoplus_\alpha P^\alpha & \to & M & \to & 0.
\end{array}$$

Now $M_\alpha \in \mathcal{F}^\Delta$ is generated by finitely many weight vectors, say of weights $\{\mu^\alpha_j\}_j$ in $\mathfrak{h}^*\backslash\mathsf{K}$, and we take a weight vector in E_1 in the preimage of each such generating weight vector. Since $E_1 \in \mathbf{O}$ is locally $U(\mathfrak{b})$ finite, it follows that the submodule of E_1 generated by those weight vectors is a quotient of a module of the form $P^\alpha := \bigoplus_j P_{\mathsf{K}^\alpha_j}(\mu^\alpha_j)$ for upper finite ideals $\mathsf{K}^\alpha_j \ni \mu^\alpha_j$, where the direct sum is finite. Using pull-backs we thus arrive at the above commutative diagram with exact rows. By Corollary 3.5.5, the kernel K^α of $P^\alpha \twoheadrightarrow M_\alpha$ is in $\mathcal{F}^\Delta$. By construction, K^α is even in $\mathcal{F}^\Delta[\mathfrak{h}^*\backslash\mathsf{K}]$. By induction, the extension

$$0 \to N \to E'_k \to E'_{k-1} \to \cdots \to E'_2 \to \bigoplus_\alpha K^\alpha \to 0$$

is zero. It follows that the original extension is also zero. □

Lemma 4.5.4. *Let $\mathsf{K} \subset \mathfrak{h}^*$ be an upper finite ideal and S a coideal in K (for instance $\mathsf{S} = \mathsf{K}$). For any $M \in \mathcal{F}^\Delta[\mathsf{S}]$, we have a short exact sequence in $\mathcal{F}^\Delta[\mathsf{S}]$*

$$0 \to X \to \bigoplus_\mu P_{\mathsf{K}}(\mu) \to M \to 0.$$

Proof. This is an immediate consequence of the structure of projective objects in ${}^{\mathsf{K}}\mathbf{O}$ and Corollary 3.5.5. □

Proof of Theorem 4.5.1. For part (i), it suffices to prove that $\mathrm{Ext}^i_{\mathbf{O}}(P,N) = 0$, for all $i > 0$, $N \in {}^{\mathsf{K}}\mathbf{O}$ and P a projective object in ${}^{\mathsf{K}}\mathbf{O}$, by [CM1, Corollary 5]. Using the main method in the proof of Proposition 4.5.3, one shows that $\mathrm{Ext}^i_{\mathbf{O}}(P,N) \neq 0$ implies $\mathrm{Ext}^{i-1}_{\mathbf{O}}(M,N) \neq 0$ for M some direct sum of modules in $\mathcal{F}^{\Delta}[\mathfrak{h}^*\backslash\mathsf{K}]$. This contradicts Proposition 4.5.3.

Part (ii) can also be proved by an application of [CM1, Corollary 5]. It follows from the construction in Section 4.2 that each projective object P in ${}^{\mathsf{K}}\mathbf{O}$ has a resolution by direct sums of modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \Bbbk_\kappa$, with $\kappa \notin \mathsf{K}$ outside of position 0 of the resolution. This implies that $\mathrm{Ext}^i_{\mathbf{C}}(P,N) = 0$ for all $i > 0$ and $N \in {}^{\mathsf{K}}\mathbf{O}$.

By Lemma 4.5.4, any module in $\mathcal{F}^{\Delta}[\mathsf{K}]$ has a projective resolution in ${}^{\mathsf{K}}\mathbf{O}$ which is actually contained in ${}^{\mathsf{K}}\bar{\mathcal{O}}$. Part (iii) follows from this observation and by applying the duality $\vee$ on $\mathcal{C}$. □

Corollary 4.5.5. *For arbitrary $i \in \mathbb{N}$, $\mu \in \mathfrak{h}^*$ and $M \in \mathbf{O}$ with upper finite* supp$M$*, we have*

$$\mathrm{Ext}^i_{\mathbf{O}}(\Delta(\mu), M) \;\cong\; \mathrm{Hom}_{\mathfrak{h}}(\Bbbk_\mu, \mathrm{H}^i(\mathfrak{n}^+, M)).$$

Proof. Let K be the ideal generated by supp$M$ and μ. Theorem 4.5.1(i) and (ii) imply

$$\mathrm{Ext}^i_{\mathbf{O}}(\Delta(\mu), M) \;\cong\; \mathrm{Ext}^i_{{}^{\mathsf{K}}\mathbf{O}}(\Delta(\mu), M) \;\cong\; \mathrm{Ext}^i_{\mathbf{C}}(\Delta(\mu), M).$$

The reformulation in terms of $\mathfrak{n}^+$-Lie algebra cohomology then follows as in the finite dimensional case, see e.g. [Hu, Theorem 6.15(b)] or [CM1, Corollary 14]. □

Question 4.5.6. *Fix a root-reductive Lie algebra $\mathfrak{g}$ with Dynkin Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ and an upper finite ideal $\mathsf{K} \subset \mathfrak{h}^*$.*

(i) Is $\mathbf{O}$ extension full in $\mathbf{C}$?
(ii) Is $\bar{\mathcal{O}}$ extension full in $\mathbf{O}$?

Note that Question 4.5.6(i) has a positive answer when restricting to the block $\mathbf{O}_{[\![\lambda]\!]}$, for a dominant λ, by Remark 4.4.3 and Theorem 4.5.1(ii).

4.6. **Extensions in Serre quotients.** For any ideal L, we denote the Serre quotient $\mathbf{O}/{}_{\mathsf{L}}\mathbf{O}$, see Appendix A, by ${}_{\mathsf{L}}\mathbf{O}$.

Proposition 4.6.1. *Let $\mathsf{L} \subset \mathsf{K} \subset \mathfrak{h}^*$ be ideals and let K be finite. For $M \in \mathcal{F}^{\Delta}[\mathsf{K}\backslash\mathsf{L}]$ and $N \in {}^{\mathsf{K}}\mathbf{O}$, we have isomorphisms*

$$\pi : \mathrm{Ext}^i_{{}^{\mathsf{K}}\mathbf{O}}(M,N) \;\tilde{\to}\; \mathrm{Ext}^i_{{}^{\mathsf{K}}_{\mathsf{L}}\mathbf{O}}(M,N), \qquad \text{for all } i \in \mathbb{N}.$$

Proof. First we consider the case $i = 0$. Clearly $M \in \mathcal{F}^{\Delta}[\mathsf{K}\backslash\mathsf{L}]$ has no proper submodule M' such that M/M' is in ${}^{\mathsf{L}}\mathbf{O}$, hence

$$\mathrm{Hom}_{{}_{\mathsf{L}}\mathbf{O}}(M,N) \;=\; \varinjlim \mathrm{Hom}_{\mathbf{O}}(M, N/N'),$$

where N' runs over all submodules of N which are in ${}^{\mathsf{L}}\mathbf{O}$. For such N', the exact sequence

$$\mathrm{Hom}_{\mathbf{O}}(M,N') \to \mathrm{Hom}_{\mathbf{O}}(M,N) \to \mathrm{Hom}_{\mathbf{O}}(M,N/N') \to \mathrm{Ext}^1_{\mathbf{O}}(M,N')$$

has first and last term equal to zero, see Lemma 3.1.2. Consequently all the maps in the direct limit are isomorphisms. Hence, we find an isomorphism

$$\pi : \mathrm{Hom}_{\mathbf{O}}(M,N) \;\tilde{\to}\; \mathrm{Hom}_{{}_{\mathsf{L}}\mathbf{O}}(M,N).$$

Lemma 4.5.4 implies that, inside ${}^{\mathsf{K}}\mathbf{O}$, the module M has a projective resolution $P_\bullet$, with $P_i \in \mathcal{F}^{\Delta}(\mathsf{K}\backslash\mathsf{L})$, for all $i \in \mathbb{N}$. By Lemma A.1.3, $\pi(P_\bullet)$ is a projective resolution of M in ${}^{\mathsf{K}}_{\mathsf{L}}\mathbf{O}$. Since the extension groups are then calculated as $\mathrm{H}^i(\mathrm{Hom}(P_\bullet, N))$ in the respective categories, the conclusion follows from the above paragraph. □

Remark 4.6.2. For an upper finite ideal $\mathsf{K} \subset \mathfrak{h}^*$ and an arbitrary ideal $\mathsf{L} \subset \mathsf{K}$, the category ${}^{\mathsf{K}}_{\mathsf{L}}\mathbf{O}$ has enough projective objects by Theorem 4.2.1 and Lemma A.1.3.

This observation extends to the case of ideals $\mathsf{L} \subset \mathsf{K} \subset \mathfrak{h}^*$ such that $\mathsf{K}\backslash\mathsf{L}$ is upper finite in $\mathfrak{h}^*\backslash\mathsf{L}$, using Remark 4.2.2(ii).

4.7. **Example: $\mathfrak{gl}_\infty$.** For $\mathfrak{gl}_\infty$ we have precisely two Dynkin Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$, up to conjugacy. We consider a Cartan subalgebra $\mathfrak{h}$ contained in both Borel subalgebras.

4.7.1. For $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$, we choose a realisation where

$$\Phi = \{\epsilon_i - \epsilon_j \,|\, i \neq j \in \mathbb{N}\} \quad \text{and} \quad \Phi^+ = \{\epsilon_i - \epsilon_j \,|\, i < j\}.$$

For $\mathfrak{g} \supset \mathfrak{b}' \supset \mathfrak{h}$, we can choose a realisation where

$$\Phi = \{\epsilon'_i - \epsilon'_j \,|\, i \neq j \in \mathbb{Z}\} \quad \text{and} \quad \Phi^+ = \{\epsilon'_i - \epsilon'_j \,|\, i < j\}.$$

We show that these lead to different theories in the following sense.

Lemma 4.7.2. *There exists no equivalence of categories $\mathbf{O}_{[\![0]\!]}(\mathfrak{g},\mathfrak{b}) \to \mathbf{O}_{[\![0]\!]}(\mathfrak{g},\mathfrak{b}')$ which exchanges Verma modules.*

Proof. We set $W = W(\mathfrak{g} : \mathfrak{h})$. We denote the simple reflections with respect to $\mathfrak{b}$ by $s_i = r_{\epsilon_i - \epsilon_{i+1}} \in W$, for $i \in \mathbb{N}$ and with respect to $\mathfrak{b}'$ by $s'_i = r_{\epsilon'_i - \epsilon'_{i+1}} \in W$, for $i \in \mathbb{Z}$. We denote the Bruhat order corresponding to $\mathfrak{b}$ by $\uparrow$ and the one corresponding to $\mathfrak{b}'$ by $\uparrow'$. By Corollary 3.3.4(ii), it suffices to prove that the partially ordered sets $(W \cdot 0, \uparrow)$ and $(W \cdot 0, \uparrow')$ are not isomorphic.

To look for a contradiction, we assume that we have an isomorphism of posets $\phi : (W \cdot 0, \uparrow) \to (W \cdot 0, \uparrow')$. Both posets have a unique maximal element, 0, which must be exchanged by ϕ. The sets of elements μ which are covered by 0 (which means $\mu \uparrow 0$, but there exists no $\lambda \notin \{0, \mu\}$ such that $\mu \uparrow \lambda \uparrow 0$) must also be exchanged by ϕ. We must thus have a bijection

$$\phi : \{s_i \cdot 0, i \in \mathbb{N}\} \ \to \ \{s'_i \cdot 0, i \in \mathbb{Z}\}.$$

Now we define $c(\mu) \in \mathbb{N}$ for any of the weights μ in $\{s_i \cdot 0, i \in \mathbb{N}\}$ as the number of other elements ν in that set such that there exist (at least) 2 elements in $W \cdot 0$ which are covered by both μ and ν. We find that $c(s_0 \cdot 0) = 1$, because we only have $s_1 \cdot 0$ such that these two weights cover two weights, namely $s_0 s_1 \cdot 0$ and $s_1 s_0 \cdot 0$. We have $c(s_i \cdot 0) = 2$ for $i > 0$, coming from $s_{i-1} \cdot 0$ and $s_{i+1} \cdot 0$. Similarly, the same definition leads to $c(s'_i \cdot 0) = 2$ for all $i \in \mathbb{Z}$. This contradicts the existence of ϕ. □

We conclude this section with an explicit example of an infinite dimensional extension space in $\mathbf{O}$ for simple modules.

Lemma 4.7.3. *For $\mathfrak{g} = \mathfrak{gl}_\infty$ and both of the Dynkin Borel subalgebras, we have*

$$\dim_{\Bbbk} \mathrm{Ext}^2_{\mathbf{O}}(\Bbbk, \Bbbk) \ = \ \infty.$$

Proof. Theorem 4.5.1(i) shows that we can equivalently calculate the extension in ${}^{\mathsf{K}}\mathbf{O}$ for some upper finite ideal $\mathsf{K} \ni 0$. Theorem 4.5.1(ii) then shows we can equivalently calculate the extension in $\mathbf{C}(\mathfrak{g},\mathfrak{h})$. Taking the standard projective resolution of $\Bbbk$ in $\mathbf{C}(\mathfrak{g},\mathfrak{h})$ shows that $\mathrm{Ext}^\bullet_{\mathbf{C}}(\Bbbk,\Bbbk)$ is the cohomology of the complex

$$0 \to \Bbbk \to \mathrm{Hom}_{\mathfrak{h}}(\mathfrak{g}/\mathfrak{h}, \Bbbk) \to \mathrm{Hom}_{\mathfrak{h}}(\Lambda^2\mathfrak{g}/\mathfrak{h}, \Bbbk) \to \mathrm{Hom}_{\mathfrak{h}}(\Lambda^3\mathfrak{g}/\mathfrak{h}, \Bbbk) \to \cdots .$$

Since $\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{g}/\mathfrak{h}, \Bbbk) = 0$, we find the well-known properties $\mathrm{Hom}_{\mathbf{C}}(\Bbbk,\Bbbk) = \Bbbk$, $\mathrm{Ext}^1_{\mathbf{C}}(\Bbbk,\Bbbk) = 0$ and also that $\mathrm{Ext}^2_{\mathcal{C}}(\Bbbk,\Bbbk)$ is the kernel of $\mathrm{Hom}_{\mathfrak{h}}(\Lambda^2\mathfrak{g}/\mathfrak{h}, \Bbbk) \to \mathrm{Hom}_{\mathfrak{h}}(\Lambda^3\mathfrak{g}/\mathfrak{h}, \Bbbk)$. That kernel is easily seen to be infinite dimensional. □

5. Link with finite case

5.1. **Induction and restriction functors.**

5.1.1. For each $n \in \mathbb{N}$ and $\lambda \in \mathfrak{h}^*$, we have the complete set $\Lambda_n := [\lambda]_n = \lambda + \mathbb{Z}\Phi_n$ in $\mathfrak{h}^*$. We have the corresponding ideals $\mathring{\Lambda}_n$ and $\overline{\Lambda}_n$ as in 1.3.2. The exact functors of Section 2.4 restrict to exact functors

$$\mathrm{Ind}^n_+ = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_n,+} : {}^{\Lambda_n}\mathbf{O}(\mathfrak{g}_n,\mathfrak{b}_n) \to {}^{\overline{\Lambda}_n}\mathbf{O}, \qquad \mathrm{Res}^n_\lambda = \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}_n,\lambda} : {}^{\overline{\Lambda}_n}\mathbf{O} \to {}^{\Lambda_n}\mathbf{O}(\mathfrak{g}_n,\mathfrak{b}_n).$$

The following is an infinite rank version of [CMZ, Theorem 32]. We provide an alternative proof.

Theorem 5.1.2. *For $n \in \mathbb{N}$ and $\lambda \in \mathfrak{h}^*$, we have mutually inverse equivalences of abelian categories Ψ and Φ, admitting a commuting diagram*

$$\begin{array}{ccccc}
{}^{\Lambda_n}\mathbf{O}(\mathfrak{g}_n,\mathfrak{b}_n) & \xrightarrow{\mathrm{Ind}^n_+} & {}^{\overline{\Lambda}_n}\mathbf{O} & \xrightarrow{\mathrm{Res}^n_\lambda} & {}^{\Lambda_n}\mathbf{O}(\mathfrak{g}_n,\mathfrak{b}_n) \\
& {\scriptstyle\Psi}\searrow & \downarrow{\scriptstyle\pi} & \nearrow{\scriptstyle\Phi} & \\
& & {}^{\overline{\Lambda}_n}_{\mathring{\Lambda}_n}\mathbf{O} & &
\end{array}$$

Moreover, for any $\mu \in \Lambda_n$, we have $\Psi(\Delta_n(\mu)) \cong \Delta(\mu)$ in ${}^{\overline{\Lambda}_n}_{\mathring{\Lambda}_n}\mathbf{O}$.

Proof. We define $\Psi := \pi \circ \mathrm{Ind}^n_+$. The existence of a functor Φ which completes the commuting diagram follows from Lemma A.1.2 in Appendix A. By construction, we have $\mathrm{Res}^n_\lambda \circ \mathrm{Ind}^n_+ \cong \mathrm{id}_{{}^{\Lambda_n}\mathbf{O}(\mathfrak{g}_n,\mathfrak{b}_n)}$. By commutativity of the diagram, we then have $\Phi \circ \Psi \cong \mathrm{id}_{\mathbf{O}_{\Lambda_n}(\mathfrak{g}_n,\mathfrak{b}_n)}$. To conclude the proof it suffices to show that Φ is faithful. We will write $\mathcal{A} := {}^{\overline{\Lambda}_n}_{\mathring{\Lambda}_n}\mathbf{O}$.

Consider $M, N \in {}^{\overline{\Lambda}_n}\mathbf{O}$ and $f \in \mathrm{Hom}_{\mathcal{A}}(M,N)$. By Lemma A.1.2, we have $\Phi(f) = \mathrm{Res}^{\Lambda_n}_n(g)$, for any representative $g \in \mathrm{Hom}_{\mathfrak{g}}(M', N/N')$ of f, where $M' \subset M$ with $\mathrm{supp}(M/M') \subset \mathring{\Lambda}_n$ and $N' \subset N$ with $\mathrm{supp}N' \subset \mathring{\Lambda}_n$.

Now assume $\Phi(f) = 0$, which thus implies that g restricted to the weight spaces of M' for weights in Λ_n is zero. Since g is in particular a morphism of $\mathfrak{h}$-modules this means that the image of g is of the form N''/N' for some $N'' \supset N'$ with $\mathrm{supp}N'' \subset \mathring{\Lambda}_n$. Thus the morphism

$$\mathrm{Hom}_{\mathfrak{g}}(M', N/N') \to \mathrm{Hom}_{\mathfrak{g}}(M', N/N''),$$

in the direct limit defining $\mathrm{Hom}_{\mathcal{A}}(M,N)$, satisfies $g \mapsto 0$. This implies $g \sim 0$ or $f = 0$. We hence find that Φ is indeed faithful. □

Corollary 5.1.3. *We use the notation of Theorem 5.1.2.*

(i) For dominant $\lambda \in \mathfrak{h}^*$, the functor Ψ restricts to an equivalence between $\mathbf{O}_{[\![\lambda]\!]_n}(\mathfrak{g}_n,\mathfrak{b}_n)$ and the Serre quotient of $\mathbf{O}_{[\![\lambda]\!]}$ with respect to the subcategory with only non-zero multiplicities for simple modules $L(w\cdot\lambda)$ with $w \notin W_n$.

(ii) For antidominant $\lambda \in \mathfrak{h}^*$, the functor Ψ restricts to an equivalence between $\mathbf{O}_{[\![\lambda]\!]_n}(\mathfrak{g}_n,\mathfrak{b}_n)$ and the Serre subcategory of $\mathbf{O}_{[\![\lambda]\!]}$ with only non-zero multiplicities for simple modules $L(w\cdot\lambda)$ with $w \in W_n$.

5.1.4. Consider an arbitrary set $\{\lambda_1, \lambda_2, \cdots, \lambda_k\} \subset \mathfrak{h}^*$ and $n \in \mathbb{N}$ large enough such that $\lambda_i - \lambda_j \in \mathbb{Z}\Phi_n$ for all $1 \le i,j \le k$. Denote by K, resp. K_n, the ideal in $(\mathfrak{h}^*, \le)$, resp. $(\mathfrak{h}^*, \le_n)$, generated by $\{\lambda_1, \lambda_2, \cdots, \lambda_k\}$. The set K_n is complete in $(\mathfrak{h}^*, \le)$, so $\mathring{\mathsf{K}}_n := \mathsf{K}\backslash\mathsf{K}_n$ is also an ideal in $(\mathfrak{h}^*, \le)$.

By restricting the equivalence in Theorem 5.1.2, we obtain the following corollary.

Corollary 5.1.5. *With notation and assumptions as in 5.1.4, we have mutually inverse equivalences of abelian categories Ψ and Φ, admitting a commuting diagram*

$$\begin{array}{ccccc} {}^{\mathsf{K}_n}\mathbf{O}(\mathfrak{g}_n,\mathfrak{b}_n) & \xrightarrow{\mathrm{Ind}^n_+} & {}^{\mathsf{K}}\mathbf{O} & \xrightarrow{\mathrm{Res}^n_\lambda} & {}^{\mathsf{K}_n}\mathbf{O}(\mathfrak{g}_n,\mathfrak{b}_n) \\ & \searrow^{\Psi} & \downarrow{\scriptstyle\pi} & \nearrow_{\Phi} & \\ & & {}^{\mathsf{K}}_{\mathring{\mathsf{K}}_n}\mathbf{O} & & \end{array}$$

Moreover, we have $\Psi(\Delta_n(\mu)) \cong \Delta(\mu)$ in ${}^{\mathsf{K}}_{\mathring{\mathsf{K}}_n}\mathbf{O}$ for all $\mu \in \mathsf{K}_n$.

Theorem 5.1.6. *With notation and assumptions as in 5.1.4 and $\lambda \in \mathsf{K}_n$, consider the algebras $A := A^{\mathsf{K}}_{[\![\lambda]\!]}$ and $A_n := A^{\mathsf{K}_n}_{[\![\lambda]\!]_n}(\mathfrak{g}_n,\mathfrak{b}_n)$ as in 4.4.1. For the idempotent $\varepsilon_n = \sum_\mu e_\mu \in A$, with μ ranging over $\mathsf{K}_n \cap [\![\lambda]\!]_n$, we have an algebra isomorphism $\varepsilon_n A \varepsilon_n \cong A_n$.*

Proof. By Theorem 4.4.2, we have an equivalence

$${}^{\mathsf{K}_n}\mathbf{O}_{[\![\lambda]\!]_n}(\mathfrak{g}_n,\mathfrak{b}_n) \;\cong\; A_n\text{-Mod}.$$

By Theorem 4.4.2 and Lemma A.2.1, we have equivalences

$${}^{\mathsf{K}_n}\mathbf{O}_{[\![\lambda]\!]_n}(\mathfrak{g}_n,\mathfrak{b}_n) \;\cong\; {}^{\mathsf{K}}_{\mathring{\mathsf{K}}_n}\mathbf{O} \;\cong\; \varepsilon_n A \varepsilon_n\text{-Mod}.$$

By construction, both A_n and $\varepsilon_n A\varepsilon_n$ are the endomorphism algebra of a direct sum of all indecomposable projective objects (without repeating isomorphism classes) in ${}^{\mathsf{K}_n}\mathbf{O}_{[\![\lambda]\!]_n}(\mathfrak{g}_n,\mathfrak{b}_n)$, implying that they are isomorphic. □

Corollary 5.1.7. *For two integral dominant regular weights λ, λ', we have an equivalence of categories*

$$\mathbf{O}_{[\![\lambda]\!]} \;\tilde{\to}\; \mathbf{O}_{[\![\lambda']\!]} \quad \textit{with} \quad L(w\cdot\lambda) \mapsto L(w\cdot\lambda'), \quad \textit{for all } w \in W.$$

Proof. We denote by K, resp. K', the ideal in $(\mathfrak{h}^*, \leq)$ generated by λ, resp. λ'. Set $A := A^{\mathsf{K}}_{[\![\lambda]\!]}$ and $B := A^{\mathsf{K}'}_{[\![\lambda']\!]}$. It follows from Theorem 5.1.6 and [Hu, Proposition 7.8] that, for all n, we have a commuting square of algebra morphisms

$$\begin{array}{ccc} \varepsilon_n A \varepsilon_n & \hookrightarrow & \varepsilon_{n+1} A \varepsilon_{n+1} \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \varepsilon_n B \varepsilon_n & \hookrightarrow & \varepsilon_{n+1} B \varepsilon_{n+1}. \end{array}$$

We thus have $A \cong \varinjlim \varepsilon_n A \varepsilon_n \cong B$ and the equivalence follows. □

5.2. **Extensions of Verma modules.**

Theorem 5.2.1. *Consider arbitrary $\lambda, \mu \in \mathfrak{h}^*$ and $i \in \mathbb{N}$. For any $n \in \mathbb{N}$ such that $\lambda - \mu \in \mathbb{Z}\Phi_n$, we have*

$$\mathrm{Ext}^i_{\mathbf{O}}(\Delta(\mu), L(\lambda)) \;\cong\; \mathrm{Ext}^i_{\mathcal{O}(\mathfrak{g}_n,\mathfrak{b}_n)}(\Delta_n(\mu), L_n(\lambda)).$$

Proof. Let K be the ideal in $(\mathfrak{h}^*, \leq)$ generated by μ, and λ and K_n be the ideal in $(\mathfrak{h}^*, \leq_n)$ generated by μ and λ. By Theorem 4.5.1, it suffices to prove

$$\mathrm{Ext}^i_{{}^{\mathsf{K}}\mathbf{O}}(\Delta(\mu), L(\lambda)) \;\cong\; \mathrm{Ext}^i_{{}^{\mathsf{K}_n}\mathbf{O}(\mathfrak{g}_n,\mathfrak{b}_n)}(\Delta_n(\mu), L_n(\lambda)).$$

By Proposition 4.6.1, the left-hand side is isomorphic to $\mathrm{Ext}^i_{{}^{\mathsf{K}}_{\mathring{\mathsf{K}}_n}\mathbf{O}}(\Delta(\mu), L(\lambda))$. The theorem then follows from Corollary 5.1.5. □

In BGG category $\mathcal{O}(\mathfrak{g}_n, \mathfrak{b}_n)$, the dimensions of the extension spaces $\mathrm{Ext}^i(\Delta_n(\mu), L_n(\lambda))$ are determined by the KLV polynomials. Theorem 5.2.1 thus shows that the same is true in $\mathbf{O}$. For instance, let $\mu \in \mathfrak{h}^*$ be integral, regular and anti-dominant. With any unexplained notation taken from [Hu, Section 8], the combination of Theorem 5.2.1 and [Hu, Theorem 8.11(b)] yields

$$P_{x,w}(q) \; = \; \sum_{i\geq 0} q^i \dim \mathrm{Ext}^{\ell(x,w)-2i}_{\mathbf{O}}(\Delta(x\cdot\mu), L(w\cdot\mu)), \qquad \text{for all } x, w \in W$$

with $P_{x,w}$ the KLV polynomial corresponding to the Weyl group W_n, with n big enough so that $x, w \in W_n$. In [Na1, Conjecture 8.17], this formula was conjectured for extensions in $\overline{\mathcal{O}}$.

Corollary 5.2.2. *Conjecture 8.17 in* [Na1] *is true for* $\mathbf{O}$*.*

The original question in [Na1] therefore becomes a special case of Question 4.5.6(ii).

Lemma 5.2.3. *Consider arbitrary $\lambda, \mu \in \mathfrak{h}^*$ and $i \in \mathbb{N}$. For any $n \in \mathbb{N}$ such that $\lambda - \mu \in \mathbb{Z}\Phi_n$, we have*

$$\mathrm{Ext}^i_{\mathbf{O}}(\Delta(\mu), \Delta(\lambda)) \; \cong \; \mathrm{Ext}^i_{\mathcal{O}(\mathfrak{g}_n,\mathfrak{b}_n)}(\Delta_n(\mu), \Delta_n(\lambda)).$$

Proof. Mutatis mutandis Theorem 5.2.1. □

5.3. **Standard Koszulity.** We use the notion of a graded cover of an abelian category as in Definition C.1.1 and refer to Appendix D for the justification of the use of the term "standard Koszulity". We will frequently refer to results in those appendices in this section.

Theorem 5.3.1 (Standard Koszulity)**.** *Let K be a finitely generated ideal in $(\mathfrak{h}^*, \leq)$. The category ${}^{\mathsf{K}}\mathbf{O}$ admits a graded cover ${}^{\mathsf{K}}\mathbf{O}^{\mathbb{Z}}$ such that simple and (dual) Verma modules admit graded lifts. We use the same symbol for the graded lifts and can choose the normalisation such that, for any $\mu \in \mathsf{K}$, we have non-zero morphisms $\Delta(\mu) \to L(\mu) \to \nabla(\mu)$ (without applying shifts $\langle k\rangle$) in ${}^{\mathsf{K}}\mathbf{O}^{\mathbb{Z}}$. Then, for all $\mu, \nu \in \mathsf{K}$, we have*

$$\mathrm{Ext}^i_{{}^{\mathsf{K}}\mathbf{O}^{\mathbb{Z}}}(\Delta(\mu), L(\nu)\langle j\rangle) \; = \; 0 \; = \; \mathrm{Ext}^i_{{}^{\mathsf{K}}\mathbf{O}^{\mathbb{Z}}}(L(\mu), \nabla(\nu)\langle j\rangle), \quad \text{if } i \neq j.$$

Proof. It suffices to take an arbitrary $\lambda \in \mathsf{K}$ and restrict to ${}^{\mathsf{K}}\mathbf{O}_{[\![\lambda]\!]}$. Set $A := A^{\mathsf{K}}_{[\![\lambda]\!]}$ and consider the equivalence

$$\mathcal{F} : \; {}^{\mathsf{K}}\mathbf{O}_{[\![\lambda]\!]} \; \xrightarrow{\sim} \; A\text{-Mod}$$

from Theorem 4.4.2. For each $n \in \mathbb{N}$ large enough we define the ideal $\mathring{\mathsf{K}}_n$ in $(\mathfrak{h}^*, \leq)$ as in 5.1.4. We have the idempotents $\varepsilon_n \in A$ from Theorem 5.1.6, with $A_n = \varepsilon_n A \varepsilon_n$ and $A \cong \varinjlim_n A_n$. By Proposition D.2.1, the algebras A_n have a Koszul grading. By Theorem D.1.4(ii), the grading on A_n inherited from the one on A_{n+1} from the relation $\varepsilon_n A_{n+1} \varepsilon_n = A_n$ is also Koszul. By uniqueness of Koszul gradings, see e.g. [BGS, Corollary 2.5.2], the gradings on the algebras $\{A_n\}$ are thus consistent and induce a grading on $A \cong \varinjlim_n A_n$. By Example C.2.2, the category

$${}^{\mathsf{K}}\mathbf{O}^{\mathbb{Z}}_{[\![\lambda]\!]} := A\text{-gMod}$$

is thus a graded cover of ${}^{\mathsf{K}}\mathbf{O}_{[\![\lambda]\!]}$.

Now, for $\mu \in \mathsf{K}$, we consider the A-module $M := \mathcal{F}(\Delta(\mu))$. We then have a short exact sequence

$$0 \to \sum_{\kappa \not\leq \mu} A e_\kappa A e_\mu \to A e_\mu \to M \to 0,$$

as follows from Theorem 4.2.1(i)(b). It follows that the A-module M admits an $\mathbb{N}$-grading. We thus have a projective resolution of M in A-gMod. For n large enough that $\mu \in \mathsf{K}_n$, it follows from Lemma 4.5.4 (or Corollary 4.5.5) that all terms in the complex are direct sums of modules $P_{\mathsf{K}}(\kappa)$, with $\kappa \in \mathsf{K}_n$. The exact full functor

$$\varepsilon_n : A\text{-gMod} \to A_n\text{-gMod} \tag{5.1}$$

shows via the standard Koszulity of A_n that $\mathrm{Ext}^i_{{}^{\mathsf{K}}\mathbf{O}^{\mathbb{Z}}_{[\![\lambda]\!]}}(\Delta(\mu), L(\nu)\langle j\rangle) = 0$, if $i \neq j$. The statement for dual Verma modules follows similarly. □

The following proposition suggests that any complete theory of Koszul *duality* for $\mathbf{O}$ would restrict to a duality between dominant and antidominant blocks. For $\mu \in \mathsf{K}$, $j \in \mathbb{Z}$ and $M \in {}^{\mathsf{K}}\mathbf{O}^{\mathbb{Z}}$, we set

$$[M : L(\mu)\langle j\rangle] = \dim \mathrm{Hom}_{{}^{\mathsf{K}}\mathbf{O}^{\mathbb{Z}}}(P_{\mathsf{K}}(\mu)\langle j\rangle, M),$$

with $P_{\mathsf{K}}(\mu)$ the projective cover of $L(\mu)\langle 0\rangle$ in ${}^{\mathsf{K}}\mathbf{O}^{\mathbb{Z}}$.

Proposition 5.3.2. *Let $\lambda, \mu \in \mathfrak{h}^*$ be integral and regular, with λ dominant and μ antidominant. For all $w, x \in W$ and $j \in \mathbb{N}$, we have*

(i) $\dim \mathrm{Ext}^j_{\mathbf{O}}(\Delta(w\cdot\lambda), L(x\cdot\lambda)) \;=\; [\Delta(w^{-1}\cdot\mu) : L(x^{-1}\cdot\mu)\langle j\rangle]$,

(ii) $\dim \mathrm{Ext}^j_{\mathbf{O}}(\Delta(w\cdot\mu), L(x\cdot\mu)) \;=\; [\Delta(w^{-1}\cdot\lambda) : L(x^{-1}\cdot\lambda\langle j\rangle)]$.

Proof. Take $n \in \mathbb{N}$ big enough such that $w\cdot\lambda - x\cdot\lambda \in \mathbb{Z}\Phi_n$ and the corresponding conditions for μ and x^{-1}, w^{-1} are also satisfied. By Theorem 5.2.1, the left-hand side corresponds to the corresponding dimensions in $\mathcal{O}(\mathfrak{g}_n, \mathfrak{b}_n)$. Choosing an appropriate finitely generated ideal $\mathsf{K} \subset \mathfrak{h}^*$ and using equation (5.1) shows that the right-hand side can be computed in $\mathcal{O}^{\mathbb{Z}}(\mathfrak{g}_n, \mathfrak{b}_n)$. The result thus follows from [BGS, Proposition 1.3.1]. □

Despite the fact that the property in Theorem 5.3.1 implies ordinary Koszulity in the case of finite dimensional (quasi-hereditary) algebras, we still have the following open question.

Question 5.3.3. *Do we have $\mathrm{Ext}^i_{{}^{\mathsf{K}}\mathbf{O}^{\mathbb{Z}}_{[\![\lambda]\!]}}(L(\mu), L(\nu)\langle j\rangle) \;=\; 0$, if $i \neq j$?*

The difficulty in answering this question lies in the fact that the indecomposable projective modules appearing in a fixed position in the projective resolution of a simple module in ${}^{\mathsf{K}}\mathbf{O}_{[\![\lambda]\!]}$ will generally form a set $\{P(\mu)\,|\,\mu \in S\}$, for some multiset of weights S which is not lower finite. This already happens for instance in the projective cover of the kernel of $P_{\mathsf{K}}(0) \twoheadrightarrow L(0)$.

Another open question is whether we can construct a cover without taking a Serre subcategory of $\mathbf{O}$ via truncation.

Question 5.3.4. *Is it possible to construct a graded cover of* $\mathbf{O}$*?*

6. The semiregular bimodule

6.1. **Definitions.**

6.1.1. *The group* Γ. Let S be a countable set. We consider the free abelian group $\Gamma_S \in \mathbf{Ab}$ with basis S,

$$\Gamma_S := \bigoplus_{s\in S}\mathbb{Z}, \qquad \text{with group homomorphism} \quad \mathsf{ht} : \Gamma_S \to \mathbb{Z}; \quad (a_s)_{s\in S} \mapsto \sum_{s\in S} a_s.$$

Hence Γ_S is isomorphic either to $\mathbb{Z}^{\oplus k}$, for some $k \in \mathbb{N}$, or to $\mathbb{Z}^{\oplus\aleph_0}$. In the following we leave out the reference to S.

For any two Γ-graded vector spaces $V = \bigoplus_a V^a$ and $W = \bigoplus_a W^a$, we define the Γ-graded vector space $\mathsf{Hom}_{\Bbbk}(V, W)$ by

$$\mathsf{Hom}_{\Bbbk}(V, W)^a \;=\; \{f \in \mathrm{Hom}_{\Bbbk}(V, W) \mid \text{with } \; f(V^b) \subset W^{b+a} \text{ for all } b \in \Gamma\}.$$

We equip the one dimensional vector space $\Bbbk$ with the trivial Γ-grading. Then we define $V^{\circledast} = \mathsf{Hom}_{\Bbbk}(V, \Bbbk)$ as the subspace of V^* of functionals which vanish on all but finitely many degrees. We will interpret $(-)^{\circledast}$ as a duality functor on the category of Γ-graded vector spaces and relevant subcategories.

6.1.2. We will work with Γ-graded Lie algebras over $\Bbbk$, denoted by $\mathfrak{k} = \bigoplus_{a\in\Gamma} \mathfrak{k}^a$. Any Γ-graded Lie algebra has an **associated $\mathbb{Z}$-grading** through the homomorphism ht:

$$\mathfrak{k} \;=\; \bigoplus_{i\in\mathbb{Z}} \mathfrak{k}^{(i)}, \qquad \mathfrak{k}^{(i)} = \bigoplus_{\mathsf{ht}(a)=i} \mathfrak{k}^a.$$

Definition 6.1.3. We say that a Γ-grading on a Lie algebra $\mathfrak{k}$ is **triangular** if

(i) $\mathfrak{k}^a = 0$, whenever $a = (a_s) \in \Gamma$ contains both positive and negative integers;
(ii) $\dim_{\Bbbk} \mathfrak{k}^a < \infty$ if $\mathsf{ht}(a) < 0$;
(iii) $\mathfrak{k}$ is generated by the subspace $\mathfrak{k}^{(1)} \oplus \mathfrak{k}^{(0)} \oplus \mathfrak{k}^{(-1)}$.

Condition (i) implies in particular that $\mathfrak{k}^{(0)} = \mathfrak{k}^0$.

6.1.4. *Example.* Definition 6.1.3 is tailored to cover Kac-Moody algebras for arbitrary (possibly infinite) generalised Cartan matrices. The group Γ is then to be identified with the root lattice. When the Cartan matrix is finite dimensional (and hence Γ is finitely generated) the spaces $\mathfrak{k}^{(i)}$ are already finite dimensional. In this case, one might as well work with the associated $\mathbb{Z}$-grading.

6.1.5. For a triangularly Γ-graded Lie algebra $\mathfrak{k}$, we set

$$\mathfrak{k}_< = \bigoplus_{i<0} \mathfrak{k}^{(i)}, \;\; \mathfrak{k}_\geq = \bigoplus_{i\geq 0} \mathfrak{k}^{(i)}, \;\; N = U(\mathfrak{k}_<), \;\; B = U(\mathfrak{k}_\geq), \text{ and } \; U = U(\mathfrak{k}).$$

All these algebras are naturally Γ-graded.

6.1.6. *Semi-infinite characters.* Consider a triangularly Γ-graded Lie algebra $\mathfrak{k}$. Following [So, Definition 1.1], see also [Ar], we call a character $\gamma : \mathfrak{k}^0 \to \Bbbk$ **semi-infinite for $\mathfrak{k}$** if

$$\gamma([X,Y]) \;=\; \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y : \mathfrak{k}^0 \to \mathfrak{k}^0) \quad \text{for all } X \in \mathfrak{k}^{(1)} \text{ and } Y \in \mathfrak{k}^{(-1)}.$$

6.2. **Some bimodules.** Keeping notation as above, we consider a triangularly Γ-graded Lie algebra $\mathfrak{k}$.

6.2.1. *The bimodule $N^{\circledast}$.* We have the natural N-bimodule structure on $N^* = \mathrm{Hom}_{\Bbbk}(N, \Bbbk)$, with $(fn)(u) = f(nu)$ and $(nf)(u) = f(un)$, for $f \in N^*$ and $u, n \in N$. The subspace

$$N^{\circledast} := \mathsf{Hom}_{\Bbbk}(N, \Bbbk),$$

clearly constitutes a sub-bimodule of N^*.

6.2.2. The (N, B)-bimodule structure on $N^{\circledast} \otimes_{\Bbbk} B$ is induced from the left N-module structure on $N^{\circledast}$ and the right module structure on B. The N-bimodule structure on $N^{\circledast}$ and U viewed as an (N, U)-bimodule, yield an (N, U)-bimodule structure on $N^{\circledast} \otimes_N U$.

6.2.3. Now fix an arbitrary character $\gamma : \mathfrak{k}^{(0)} \to \Bbbk$ and define the one-dimensional left B-module $\Bbbk_\gamma$ via the character $\gamma : \mathfrak{k}^{(0)} \to \Bbbk$ and the surjection $\mathfrak{k}_\geq \twoheadrightarrow \mathfrak{k}^{(0)}$. Then we have the B-bimodule $\Bbbk_\gamma \otimes_{\Bbbk} B$, which as a left module is the tensor product of $\Bbbk_\gamma$ and the left regular module. The right B-action is only on B. Then we consider U as a (B, U)-bimodule, which allows to introduce the (U, B)-bimodule $\mathrm{Hom}_B(U, \Bbbk_\gamma \otimes B)$.

Lemma 6.2.4. *We consider some arbitrary elements $n \in N$, $b, b' \in B$ and $f \in N^{\circledast}$.*

(i) *The (N,B)-bimodule morphism*

$$\psi : N^{\circledast}\otimes_{\Bbbk} B \;\to\; N^{\circledast}\otimes_N U,\qquad \psi(f\otimes b)=f\otimes b,$$

is an isomorphism.

(ii) *The (N,B)-bimodule morphism*

$$\phi : N^{\circledast}\otimes_{\Bbbk} B \;\to\; \mathsf{Hom}_B(U,\Bbbk_\gamma\otimes B),\quad \phi(f\otimes b)(b'n)=b'(f(n)\otimes b),$$

is an isomorphism.

Proof. These are immediate applications of the PBW theorem. □

6.3. **The semi-regular bimodule.** We continue with assumptions and notation as above and now also *assume that $\gamma:\mathfrak{k}^{(0)}\to\Bbbk$ is a semi-infinite character for $\mathfrak{k}$.* On the space $N^{\circledast}\otimes_{\Bbbk} B$, we can define a right U-action through the isomorphism ψ in Lemma 6.2.4(i) and a left U-action through the isomorphism ϕ in Lemma 6.2.4(ii).

Proposition 6.3.1. *The left and right U-action on $N^{\circledast}\otimes_{\Bbbk} B$ commute if γ is a semi-infinite character.*

Proof. This results from the same reasoning as in the proof of [So, Theorem 1.3]. By construction, we only need to prove that the left B-action commutes with the right N-action.

For the left B-action it suffices to consider the action of $\mathfrak{k}^{(0)}\oplus\mathfrak{k}^{(1)}$, by 6.1.3(iii). For $H\in\mathfrak{k}^{(0)}$, $f\in N^{\circledast}$ and $b\in B$, a direct computation shows that

$$H(\phi(f\otimes b)) \;=\; -\phi(f\circ \mathrm{ad}_H\otimes b)+\gamma(H)\phi(f\otimes b)+\phi(f\otimes Hb). \tag{6.1}$$

Note that $\mathrm{ad}_H\in \mathrm{End}_{\Bbbk}(N)^0\subset \mathsf{End}_{\Bbbk}(N)$, so that $f\circ\mathrm{ad}_H\in N^{\circledast}$ is well-defined. That this left action commutes with the right N-action follows as in [So, Theorem 1.3]. Now we consider the left action of $\mathfrak{k}^{(1)}$. By 6.1.3(ii), it suffices to consider $X\in\mathfrak{k}^{\gamma}$ for basis elements $\gamma\in S\subset\Gamma$. By 6.1.3(i) we then find that the dimension of $[X,\mathfrak{n}]=[X,\mathfrak{k}^{-\gamma}]$ is finite. We take a basis $\{H_i\}$ of this space, which allows to define $H^i,F\in\mathsf{End}_{\Bbbk}(N)$ by

$$nX \;=\; Xn+\sum_i H_iH^i(n)+F(n),\qquad \text{for all } n\in N.$$

A direct computation shows that

$$X\phi(f\otimes b) \;=\; \phi(f\otimes Xb)+\phi(f\circ F\otimes b)+\sum_i\gamma(H_i)\phi(f\circ H^i\otimes b)+\sum_i\phi(f\circ H^i\otimes H_ib). \tag{6.2}$$

That this action commutes with the left N-action follows again from the same computation as in [So, Theorem 1.3]. □

The resulting bimodule in Proposition 6.3.1 will be denoted by S_γ, and referred to as the **semi-regular** bimodule.

Corollary 6.3.2. *Consider the inclusion of N-bimodules $\iota : N^{\circledast}\hookrightarrow S_\gamma$, corresponding to $N^{\circledast}\hookrightarrow N^{\circledast}\otimes_{\Bbbk} B=S_\gamma$.*

(i) *The adjoint action of H on the bimodule S_γ satisfies*

$$\mathrm{ad}_H(\iota(f)) \;=\; \gamma(H)\iota(f)-\iota(f\circ\mathrm{ad}_H),\qquad \textit{for } H\in\mathfrak{k}^{(0)} \textit{ and } f\in N^{\circledast}.$$

(ii) *The (U,N)-bimodule morphism*

$$\xi : U\otimes_N N^{\circledast} \;\to\; S_\gamma;\quad u\otimes f\mapsto u\iota(f),$$

is an isomorphism.

Proof. Part (i) is essentially equation (6.1).

For part (ii), it suffices to interpret ξ as a vector space morphism

$$\xi:\ B\otimes_{\Bbbk} N^{\circledast}\to N^{\circledast}\otimes_{\Bbbk} B,\quad b\otimes f\mapsto b(f\otimes 1),$$

where the right-hand side is to be interpreted as $\phi^{-1}(b\phi(f\otimes 1))$. Now consider arbitrary $X_1,\dots,X_k\in\mathfrak{k}^{(0)}\cup\mathfrak{k}^{(1)}$. Equations (6.1) and (6.2) imply that, for $f\in N^{\circledast}$, we have

$$\xi(X_1\cdots X_k\otimes f)\ =\ f\otimes X_1\cdots X_k\ +\ \sum g\otimes u,$$

where $\sum g\otimes u$ stands for a finite sum of elements $g\otimes u$, where $g\in N^{\circledast}$ and $u\in B$ such that u is

- a product of strictly fewer than k elements of $\mathfrak{k}^{(0)}\cup\mathfrak{k}^{(1)}$; or
- a product of exactly k elements of $\mathfrak{k}^{(0)}\cup\mathfrak{k}^{(1)}$, but strictly more elements belonging to $\mathfrak{k}^{(0)}$ than in $X_1\cdots X_k$.

From this, it is now easy to show that ξ must be an isomorphism. □

7. Ringel duality

Now we return to the root-reductive Lie algebra $\mathfrak{g}$ as in the beginning of Section 4.

7.1. Triangular Γ-grading and semi-infinite characters.

7.1.1. Using the notation of 6.1.1, we set

$$\Gamma:=\Gamma_{\Sigma}\cong\mathbb{Z}\Sigma\cong\mathbb{Z}\Phi.$$

The root decomposition (2.1), where we set $\mathfrak{h}=\mathfrak{g}^0$, is thus a Γ-grading. It is easily checked that this makes $\mathfrak{g}$ a triangularly Γ-graded Lie algebra. We then have $\mathfrak{g}_{\geq}=\mathfrak{b}$ and $\mathfrak{g}_{<}=\mathfrak{n}^-$, and thus $B=U(\mathfrak{b})$ and $N=U(\mathfrak{n}^-)$, for the algebras introduced in 6.1.5.

7.1.2. By the above, we can view Γ as a subgroup of $\mathfrak{h}^*$, which we denote by $\sigma:\Gamma\hookrightarrow\mathfrak{h}^*$. In particular, this equips any Γ-graded vector space V with the structure of a semisimple $\mathfrak{h}$-module, by setting $H(v)=\sigma(\gamma)(H)v$, for any $v\in V_\gamma$ and $H\in\mathfrak{h}$. The dual $V^{\circledast}$ of 6.1.1 then corresponds to the finite dual of V as a semisimple $\mathfrak{h}$-module as in 2.1.5. In particular, we can interpret $N^{\circledast}$ as in 6.2.1 in this way by using the adjoint $\mathfrak{h}$-action.

Lemma 7.1.3. *The semi-infinite characters $\gamma\in\mathfrak{h}^*$ are those characters $\gamma:\mathfrak{h}\to\Bbbk$, for which $\gamma(H)=2\rho(H)$ for all $H\in\mathfrak{h}\cap[\mathfrak{g},\mathfrak{g}]$ and with ρ the global half sum of positive roots, as in 2.6.2.*

Proof. We consider the Chevalley generator $\{E_\alpha\}$ and $\{F_\alpha\}$, and set $H_\alpha:=[E_\alpha,F_\alpha]$. By applying the definition in 6.1.6, we find

$$\gamma([E_\alpha,F_\alpha])=\alpha(H_\alpha).$$

Since the latter number is $2\rho(H_\alpha)$. The conclusion then follows, since $H\in\mathfrak{h}\cap[\mathfrak{g},\mathfrak{g}]$ is spanned by vectors H_α as above, for a Dynkin Borel subalgebra. □

This determines all semi-infinite characters for Dynkin Borel subalgebras in case $\mathfrak{g}$ is simple.

Corollary 7.1.4. *For $\mathfrak{g}$ equal to $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$ or $\mathfrak{sp}_\infty$, the unique semi-infinite character is the global half sum of $\mathfrak{b}$-positive roots.*

7.2. **The AS duality functor.** In this subsection, we consider the analogue of the duality functor constructed by Arkhipov and Soergel for (affine) Kac-Moody algebras in [Ar, So].

We set $\gamma = 2\rho$, which is a semi-infinite character by Lemma 7.1.3, and consider $S := S_{2\rho}$, the corresponding semiregular bimodule.

Lemma 7.2.1. *For any $\lambda \in \mathfrak{h}^*$, we have an isomorphism $S \otimes_U \Delta(\lambda) \;\cong\; \Delta(-\lambda - 2\rho)^{\circledast}$.*

Proof. Using the notation of Corollary 6.3.2 we find that $S \otimes_U \Delta(\lambda)$ is equal to its subspace $\iota(N^{\circledast}) \otimes \Bbbk_\lambda$, with $\Bbbk_\lambda$ the one dimensional subspace of $\Delta(\lambda)$ of weight λ. By Corollary 6.3.2(i) we have for any $H \in \mathfrak{h}$, $f \in N^{\circledast}$ and $v \in \Bbbk_\lambda$

$$H(\iota(f) \otimes v) \;=\; 2\rho(H)(\iota(f \otimes v)) + \iota(f) \otimes Hv - \iota(f \circ \mathrm{ad}_H) \otimes v.$$

Hence, we have an isomorphism

$$\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{h}} S \otimes_U \Delta(\lambda) \;\cong\; N^{\circledast} \otimes_{\Bbbk} \Bbbk_{\lambda + 2\rho},$$

for the canonical adjoint $\mathfrak{h}$-action on $N^{\circledast}$. In particular, $S \otimes_U \Delta(\lambda)$ is a weight module, so Lemma 3.5.3 implies

$$(S \otimes_U \Delta(\lambda))^{\circledast} \cong \Delta(\mu),$$

for some $\mu \in \mathfrak{h}^*$. Comparison of

$$\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{h}} \Delta(\mu)^{\circledast} \;\cong\; N^{\circledast} \otimes_{\Bbbk} \Bbbk_{-\mu}.$$

with $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{h}} S \otimes_U \Delta(\lambda)$ implies that $\mu = -\lambda - 2\rho$. ☐

Lemma 7.2.2. *The functor $\mathcal{F} : \mathcal{F}^{\Delta}(\mathfrak{g}, \mathfrak{b}) \to \mathcal{F}^{\nabla}(\mathfrak{g}, \mathfrak{b}^-)$, obtained by the restriction of $S \otimes_U -$, is an equivalence of exact categories.*

Proof. We start by considering the functor $S \otimes_U - : \mathcal{F}^{\Delta}(\mathfrak{g}, \mathfrak{b}) \to U\text{-Mod}$. Since

$$\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{n}^-} S \otimes_U - \;\cong\; N^{\circledast} \otimes_N \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{n}^-} -,$$

Lemma 3.5.3 implies that $S \otimes_U -$ is exact. Lemma 7.2.1 then implies that the image of objects in $\mathcal{F}^{\Delta}(\mathfrak{g}, \mathfrak{b})$ are contained in $\mathcal{F}^{\nabla}(\mathfrak{g}, \mathfrak{b}^-)$. We denote the corresponding exact functor by $\mathcal{F}$.

By tensor-hom adjunction, we have the right adjoint functor $\mathrm{Hom}_U(S, -)$. By Corollary 6.3.2(ii), we have an isomorphism of functors

$$\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{n}^-} \circ \mathrm{Hom}_U(S, -) \;\cong\; \mathrm{Hom}_N(N^{\circledast}, \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{n}^-} -).$$

Hence, this yields an exact functor $\mathcal{G} : \mathcal{F}^{\nabla}(\mathfrak{g}, \mathfrak{b}^-) \to \mathcal{F}^{\Delta}(\mathfrak{g}, \mathfrak{b})$. That the adjoint pair $(\mathcal{F}, \mathcal{G})$ are mutually inverse functors follows as in [So, Theorem 2.1]. ☐

We can compose the functor $\mathcal{F}$ with the duality functor $(-)^{\circledast}$ on $\mathcal{C}(\mathfrak{g}, \mathfrak{h})$, we denote the corresponding functor by $\mathcal{D}$.

Corollary 7.2.3. *The functor $\mathcal{D}$ yields an exact contravariant equivalence $\mathcal{F}^{\Delta} \xrightarrow{\sim} \mathcal{F}^{\Delta}$, mapping $\Delta(\lambda)$ to $\Delta(-\lambda - 2\rho)$.*

Proof. This is immediate from Lemmata 7.2.2, 7.2.1 and Corollary 3.5.6. ☐

7.3. **Ringel duality and tilting modules.** We can also compose the functor $\mathcal{F}$ with the twist by the automorphism τ, or equivalently, the functor $\mathcal{D}$ with the duality functor $\vee$ on $\mathcal{C}(\mathfrak{g}, \mathfrak{h})$ of 2.1.5. By comparing the following proposition with Theorem B.2.1(iii), we can interpret

$$\mathcal{R} := {}_{\tau}S \otimes_U \cong (-)^{\vee} \circ \mathcal{D}$$

as the Ringel duality functor.

Proposition 7.3.1 (Ringel self-duality of **O**)**.** *The functor $\mathcal{R}$ yields an exact equivalence $\mathcal{F}^{\Delta} \xrightarrow{\sim} \mathcal{F}^{\nabla}$, mapping $\Delta(\lambda)$ to $\nabla(-\lambda - 2\rho)$.*

Proof. This is immediate from Lemmata 7.2.2, 7.2.1 and Corollary 3.5.6. □

Remark 7.3.2. By Theorem B.2.1(iii), we can thus state that $\mathbf{O}_{[\![\lambda]\!]}$ is *Ringel dual* to $\mathbf{O}_{[\![-\lambda-2\rho]\!]}$.

The following Proposition represents the combinatorial shadow of the Ringel duality between $\mathbf{O}_{[\![\lambda]\!]}$ and $\mathbf{O}_{[\![-\lambda-2\rho]\!]}$, see Theorem B.2.1(iv).

Proposition 7.3.3. *Let $\mathsf{C}\subset\mathfrak{h}^*$ be a lower finite coideal and $\nu\in\mathsf{C}$. There exists a module $T_{\mathsf{C}}(\nu)\in\mathcal{F}^{\nabla}[\mathsf{C}]$ such that, for all $\kappa\in\mathsf{C}$,*

$$(T_{\mathsf{C}}(\nu):\nabla(\kappa)) \;=\; [\Delta(-\kappa-2\rho):L(-\nu-2\rho)] \qquad \textit{and} \qquad \mathrm{Ext}^1_{\mathbf{O}}(T_{\mathsf{C}}(\nu),\nabla(\kappa)) \;=\; 0.$$

Proof. We define the upper finite ideal

$$\mathsf{K} \;:=\; \{-\mu-2\rho\,|\,\mu\in\mathsf{C}\}$$

and the module $N:=\mathcal{R}(P_{\mathsf{K}}(\lambda))$, with $\lambda:=-\nu-2\rho$, and use freely the results of Theorem 4.2.1. By Proposition 7.3.1, we have

$$(N:\nabla(\kappa)) \;=\; (P_{\mathsf{K}}(\lambda):\Delta(-\kappa-2\rho)) \;=\; [\Delta(-\kappa-2\rho):\Delta(\lambda)].$$

By Proposition 7.3.1, we also have

$$\mathrm{Ext}^1_{\mathbf{O}}(N,\nabla(\mu)) \;=\; \mathrm{Ext}^1_{\mathbf{O}}(P_{\mathsf{K}}(\lambda),\Delta(-\mu-2\rho)) \;=\; 0, \quad \text{for all } \mu\in\mathsf{C}.$$

This concludes the proof. □

7.3.4. *Example.* For $\nu\in\mathfrak{h}^*$, we set $\mathsf{C}:=\{\lambda\in\mathfrak{h}^*\,|\,\lambda\ge\mu\}$. Then we have $T_{\mathsf{C}}(\nu)=\nabla(\nu)$.

Remark 7.3.5. The vanishing of extensions in Proposition 7.3.3 implies that inside the quotient ${}_{\mathsf{L}}\mathbf{O}$, for $\mathsf{L}=\mathfrak{h}^*\backslash\mathsf{C}$, the module $T_{\mathsf{C}}(\nu)$ becomes a tilting module, that is a module with Verma and dual Verma flag. This follows from [Ri, Theorem 4] and the results in Section 4.

As a special case of the above remark, we have the following corollary, which also follows from Corollary 5.1.3(ii).

Corollary 7.3.6. *If $\lambda\in\mathfrak{h}^*$ is antidominant, we have a $\mathfrak{g}$-module $T(\mu)$ for each $\nu\in[\![\lambda]\!]$ which is in $\mathcal{F}^{\Delta}\cap\mathcal{F}^{\nabla}$ and satisfies*

$$(T(\nu):\nabla(\kappa)) \;=\; [\Delta(-\kappa-2\rho):L(-\nu-2\rho)], \qquad \textit{for all } \kappa\in[\![\lambda]\!].$$

Appendix A. Homological algebra in Serre quotient categories

A.1. **Serre quotient categories.** We recall some results from [Ga, Chapitre III]. We fix an abelian category $\mathcal{C}$ with Serre subcategory $\mathcal{B}\subset\mathcal{C}$ for the entire subsection.

A.1.1. The Serre quotient category $\mathcal{C}/\mathcal{B}$ is defined by setting $\mathrm{Ob}\,(\mathcal{C}/\mathcal{B})=\mathrm{Ob}\,\mathcal{C}$ and for $X,Y\in\mathrm{Ob}\,\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{B}}(X,Y) \;:=\; \varinjlim \mathrm{Hom}_{\mathcal{C}}(X',Y/Y'),$$

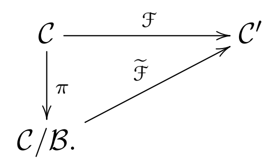
where X', resp. Y', runs over all subobjects in $\mathcal{C}$ (ordered by inclusion) of X, resp. Y, such that $X/X'\in\mathcal{B}\ni Y'$. For the precise definition of the composition of two morphisms in $\mathcal{C}/\mathcal{B}$ we refer to [Ga].

By [Ga, Proposition III.1.1], the category $\mathcal{C}/\mathcal{B}$ is abelian and we have an exact functor $\pi:\mathcal{C}\to\mathcal{C}/\mathcal{B}$, which is the identity on objects and is given on morphisms by the composition

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \;\hookrightarrow\; \bigoplus_{X',Y'}\mathrm{Hom}_{\mathcal{C}}(X',Y/Y') \;\twoheadrightarrow\; \varinjlim \mathrm{Hom}_{\mathcal{C}}(X',Y/Y').$$

The following is the universality property of Serre quotient categories.

Lemma A.1.2. [Ga, Corollaires III.1.2 and III.1.3] *Assume that for an abelian category* $\mathcal{C}'$ *and an exact functor* $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$*, we have* $\mathcal{F}(X) = 0$*, for all* $X \in \mathcal{B}$*. There exists a unique functor* $\widetilde{\mathcal{F}}$*, making the following diagram commute*



Furthermore, the functor $\widetilde{\mathcal{F}}$ *is exact and we have* $\widetilde{\mathcal{F}}(f) = \mathcal{F}(g)$ *for any* $f \in \mathrm{Hom}_{\mathcal{C}/\mathcal{B}}(X, Y)$ *represented by some* $g \in \mathrm{Hom}_{\mathcal{C}}(X', Y/Y')$*.*

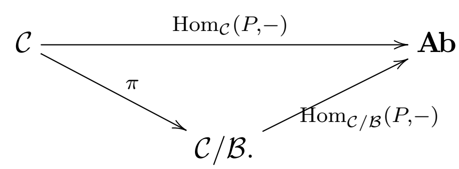
Lemma A.1.3. *Consider a projective object* P *in* $\mathcal{C}$ *with a unique maximal subobject* X*, and assume* P/X *is not an object of* $\mathcal{B}$*. Then* P *is also projective in* $\mathcal{C}/\mathcal{B}$*, and we have*

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{B}}(\pi P, \pi -) \;=\; \mathrm{Hom}_{\mathcal{C}}(P, -), \qquad \textit{as functors } \mathcal{C} \to \mathbf{Ab}.$$

Proof. By assumption, there is no subobject $P' \subset P$ with $P/P' \in \mathcal{B}$. This implies

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{B}}(P, M) \;=\; \varinjlim \mathrm{Hom}_{\mathcal{C}}(P, M/M') \quad \text{for all } M \in \mathcal{C},$$

where the limit is taken over all $\mathcal{B} \ni M' \subset M$. By assumption, we have $\mathrm{Hom}_{\mathcal{C}}(P, M') = 0$ and we find a commuting diagram of functors



That $\mathrm{Hom}_{\mathcal{C}/\mathcal{B}}(P, -)$ is exact thus follows from Lemma A.1.2. □

A.2. **Example: Locally unital algebras.** Let A be a locally unital algebra for the orthogonal idempotents $\{e_\alpha \,|\, \alpha \in \Lambda\}$. For any subset $\Lambda' \subset \Lambda$, we have the locally unital algebra

$$A' \;=\; \bigoplus_{\alpha,\beta \in \Lambda'} e_\alpha A e_\beta.$$

Lemma A.2.1. *Set* $\mathcal{C} = A$-Mod *and* $\mathcal{C}' = A'$-Mod*. With* $\mathcal{B}$ *the Serre subcategory of* A*-modules which satisfy* $e_\alpha M = 0$ *for all* $\alpha \in \Lambda'$*, we have* $\mathcal{C}/\mathcal{B} \cong \mathcal{C}'$*.*

Proof. We have the exact functor

$$\mathcal{F} : \mathcal{C} \to \mathcal{C}', \quad M \mapsto \bigoplus_{\alpha \in \Lambda'} e_\alpha M$$

and the right exact functor

$$\mathcal{K} = X \otimes_{A'} - : \mathcal{C}' \to \mathcal{C}, \quad \text{with } X = \textstyle\bigoplus_{\alpha \in \Lambda'} A e_\alpha.$$

Clearly, the composition $\mathcal{F} \circ \mathcal{K}$ is the identity on $\mathcal{C}'$. It then follows from Lemma A.1.2 that we have a dense and full functor $\widetilde{\mathcal{F}} : \mathcal{C}/\mathcal{B} \to \mathcal{C}'$.

Now we prove that $\widetilde{\mathcal{F}}$ is faithful. Assume that $\widetilde{\mathcal{F}}(f) = 0$ for $f \in \mathrm{Hom}_{\mathcal{C}/\mathcal{B}}(M, N)$. We take a representative $g \in \mathrm{Hom}_A(M', N/N')$ of f, with $M/M' \in \mathcal{B} \ni N'$. Lemma A.1.2 implies $\mathcal{F}(g) = 0$. This means that the restriction of g to $\bigoplus_{\alpha \in \Lambda'} e_\alpha M'$ is zero. It follows that $\mathrm{im}\, g \in \mathcal{B}$. So we can define $N'' \in \mathcal{B}$ with $N' \subset N'' \subset N$ and $N''/N' = \mathrm{im}\, g$. Hence, the map $\mathrm{Hom}_{\mathcal{C}}(M', N/N') \to \mathrm{Hom}_{\mathcal{C}}(M', N/N'')$ in the direct limit maps g to zero, which shows that $f = 0$. □

Appendix B. Ringel duality

B.1. **Quasi-hereditary algebras.** For a *finite dimensional, unital and basic* algebra A, fix an orthogonal decomposition of the identity element $1 = e_1 + e_2 + \cdots + e_n$ into primitive idempotents. We write (A, e), when we consider the algebra A together with the above *ordered* choice of primitive idempotents. We set

$$\varepsilon_i = e_i + \cdots + e_n, \quad \text{for all } 1 \le i \le n, \text{ and } \quad \varepsilon_{n+1} = 0.$$

B.1.1. The **standard modules** are given by

$$\Delta(i) \;=\; Ae_i/(A\varepsilon_{i+1}Ae_i), \qquad 1 \le i \le n.$$

We also have the projective cover $P(i) = Ae_i$ of the simple module $L(i)$. With our conventions we always have $\Delta(n) = P(n)$ and, if A has finite global dimension, also $\Delta(1) = L(1)$.

We denote the category of finite dimensional modules with Δ-flag by $\mathcal{F}_A^\Delta$. We can dually define the costandard modules $\nabla(i)$ and $\mathcal{F}_A^\nabla$.

Theorem B.1.2. [DR, Theorem 2] *The algebra (A, e) is* **quasi-hereditary***, see* [CPS]*, if $[\Delta(i) : L(i)] = 1$ for all $1 \le i \le n$ and one of the following equivalent conditions is satisfied*

(i) ${}_AA \in \mathcal{F}_A^\Delta$;
(ii) $\mathrm{Ext}_A^k(\Delta(i), \nabla(j))$*, for all $k > 0$ and $1 \le i, j \le n$.*

Lemma B.1.3. *If (A, e) is quasi-hereditary, then both $\varepsilon_i A \varepsilon_i$ and $A/(A\varepsilon_{i+1}A)$ are quasi-hereditary, for any $1 \le i \le n$. The order on the simple modules is induced from the one for A.*

B.2. **Ringel duality.** The following is [Ri, Section 6], using [DR, Theorem 3].

Theorem B.2.1. *Let (A, e) denote a quasi-hereditary algebra.*

(i) The category $\mathcal{F}_A^\Delta \cap \mathcal{F}_A^\nabla$ contains precisely n indecomposable non-isomorphic modules $\{T(i) \,|\, 1 \le i \le n\}$. For every $1 \le i \le n$, there exists a short exact sequence

$$0 \to N(i) \to T(i) \to \nabla(i) \to 0, \qquad \text{with } N(i) \in \mathcal{F}_A^\nabla.$$

(ii) The **Ringel dual** *algebra (A', f) of (A, e), defined by $A' := \mathrm{End}_A(\oplus_i T(i))^{\mathrm{op}}$ and $f_i := 1_{T(n+1-i)}$, is quasi-hereditary with and satisfies $A'' \cong A$.*
(iii) The algebra (A', f) is the unique (basic) quasi-hereditary algebra for which there exists an exact equivalence

$$\mathcal{R} :\; \mathcal{F}_A^\nabla \;\tilde{\to}\; \mathcal{F}_{A'}^\Delta.$$

(iv) For all $1 \le i, j \le n$, we have

$$(T_A(i) : \nabla_A(j)) \;=\; (P_{A'}(n+1-i) : \Delta_{A'}(n+1-j)).$$

Note that we can take $\mathrm{Hom}_A(\oplus_i T_A(i), -)$ for the functor $\mathcal{R}$ in (iii), yielding in particular $\mathcal{R}(T_A(i)) \cong P_{A'}(n+1-i)$.

Appendix C. Graded covers

We introduce 'graded covers' of abelian categories $\mathcal{C}$, similarly to [BGS, Section 4.3].

C.1. **Definition.** By an **abelian $\mathbb{Z}$-category** $\mathcal{G}$, we mean an abelian category with a strict $\mathbb{Z}$-action. A strict $\mathbb{Z}$-action is a collection of exact functors $\{\langle j\rangle \,|\, j \in \mathbb{Z}\}$ on $\mathcal{G}$, which satisfy $\langle i\rangle\langle j\rangle = \langle i+j\rangle$ and $\langle 0\rangle = \mathbf{Id}_{\mathcal{G}}$.

Definition C.1.1. A **graded cover** of $\mathcal{C}$ is an abelian $\mathbb{Z}$-category $\mathcal{C}^{\mathbb{Z}}$, with an exact functor $\mathcal{G}: \mathcal{C}^{\mathbb{Z}} \to \mathcal{C}$, such that

(i) $\mathcal{G}\langle i\rangle = \mathcal{G}$, for all $i \in \mathbb{Z}$;
(ii) for all $M, N \in \mathcal{C}^{\mathbb{Z}}$, the functor $\mathcal{G}$ induces group isomorphisms

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^l_{\mathcal{C}^{\mathbb{Z}}}(M, N\langle i\rangle) \xrightarrow{\sim} \mathrm{Ext}^l_{\mathcal{C}}(\mathcal{G}M, \mathcal{G}N), \quad \text{for all } l \in \mathbb{N};$$

(iii) all simple objects in $\mathcal{C}$ are isomorphic to objects in the image of $\mathcal{G}$.

C.2. **Positively graded algebras.**

C.2.1. We say that a locally unital algebra A is $\mathbb{Z}$-graded if $A = \bigoplus_{j\in\mathbb{Z}} A_j$ with $A_jA_k = A_{j+k}$ and $e_\alpha \in A_0$, for all α. The category of $\mathbb{Z}$-graded locally unital A-modules with morphism preserving the grading is denoted by A-gMod. If A is locally finite, we denote by A-gmod the full subcategory of A-gMod of locally finite dimensional modules.

For $M \in A$-gMod and $i \in \mathbb{Z}$, the shifted module $M\langle i\rangle$ is identical to M as an ungraded module, but with grading

$$M\langle i\rangle_j \;=\; M_{j-i}, \quad \text{for all } j \in \mathbb{Z}.$$

We denote the exact functor forgetting the grading by

$$\mathcal{G}: A\text{-gMod} \to A\text{-Mod}.$$

A module $\widetilde{M} \in A$-gMod satisfying $\mathcal{G}\widetilde{M} \cong M$ is a **graded lift** of M.

A $\mathbb{Z}$-graded algebra A is **positively graded** if $A_i = 0$ for $i < 0$ and if A_0 is semisimple. Clearly, for such an algebra, the simple modules admit graded lifts which are contained in one degree.

C.2.2. *Example.* Let A be a locally unital $\mathbb{Z}$-graded algebra such that every simple module has a graded lift, for instance A is positively graded. Then A-gMod is a graded cover of A-Mod for the forgetful functor.

Appendix D. Standard Koszul algebras

We review some results about Koszul quasi-hereditary algebras, based on [ADL, BGS]. The algebra A is assumed to be associative, unital, finite dimensional and basic.

D.1. **Algebras.**

D.1.1. Assume that A is $\mathbb{Z}$-graded. The homomorphism spaces in A-gmod are denoted by $\hom_A$ and the extension functors by ext^k_A. We take the convention that, unless otherwise specified, a graded lift of a simple, standard or projective module is normalised (using $\langle j\rangle$) such that the top is in degree zero. Similarly, graded lifts of costandard an injective modules are chosen to have their socle in degree zero.

D.1.2. Now assume that A is positively graded. The subcategory of A-gmod of modules M which satisfy

$$\mathrm{ext}^j_A(M, L\langle i\rangle) = 0, \qquad \text{if } i \neq j,$$

for all simple modules L (contained in degree 0), is denoted by $\mathcal{PL}_A$. Similarly, the subcategory of A-gmod of modules M which satisfy

$$\mathrm{ext}^j_A(L, M\langle i\rangle) = 0, \qquad \text{if } i \neq j,$$

for all simple modules L, is denoted by $\mathcal{IL}_A$. The positively graded algebra A is **Koszul** if all simple modules are in $\mathcal{CL}_A$, or equivalently in $\mathcal{IL}_A$, see [BGS, Proposition 2.1.3].

D.1.3. Consider a quasi-hereditary algebra (A, e) as in Section B.1, which is positively graded. Then A is **standard Koszul** if each standard module $\Delta(i)$ is in $\mathcal{PL}_A$ and each costandard module is in $\mathcal{IL}_A$. Note that, by construction, $\Delta(i)$ and $\nabla(i)$ admit graded lifts. By definition, standard Koszul algebra are thus assumed to be positively graded quasi-hereditary algebras.

If A is positively graded, then so are $A/(AeA)$ and eAe for any idempotent $e \in A_0$.

Theorem D.1.4. [ADL, Theorem 1.4 and Proposition 3.9] *Let (A, e) be a standard Koszul algebra.*

(i) The positively graded algebra A is Koszul.
(ii) The quasi-hereditary algebras $A/(A\varepsilon_{i+1}A)$ and $\varepsilon_i A \varepsilon_i$, from Lemma B.1.3, are standard Koszul, for all $1 \leq i \leq n$.

D.2. **Category $\mathcal{O}$.**

Proposition D.2.1. [ADL] *Let $\mathfrak{g}$ be a (finite dimensional) reductive Lie algebra, with $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ a Cartan and Borel subalgebra and a weight $\lambda \in \mathfrak{h}^*$.*

(i) The basic algebra A for which A-mod $\cong \mathcal{O}_{[\![\lambda]\!]}$, has a positive grading for which it is standard Koszul.
(ii) For K an ideal in $(\mathfrak{h}^*, \leq)$, the basic algebra A^{K} for which A^{K}-mod $\cong {}^{\mathsf{K}}\mathcal{O}_{[\![\lambda]\!]}$, which is of the form eAe, for some idempotent $e \in A_0$ is standard Koszul for the grading inherited from A.

Proof. Part (i) is [ADL, Corollary 3.8], which is based on results in [BGS]. Part (ii) is an application of Theorem D.1.4(ii). Note that for this we should complete $\leq$ on $W \cdot \lambda$ to an arbitrary total order such that $\mathsf{K} \cap W \cdot \lambda$ is still an ideal. □

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