ON THE EXISTENCE OF INFINITE-DIMENSIONAL GENERALIZED HARISH-CHANDRA MODULES

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To our friend Joe

Abstract. We prove a general existence result for infinite-dimensional admissible $(\mathfrak{g}, \mathfrak{k})$-modules, where $\mathfrak{g}$ is a reductive finite-dimensional complex Lie algebra and $\mathfrak{k}$ is a reductive in $\mathfrak{g}$ algebraic subalgebra.

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In this note, we draw a corollary of our earlier work [4]. In the subsequent works [5], [6], [7], [8] we have built foundations of an algebraic theory of generalized Harish-Chandra modules.

The base field is $\mathbb{C}$. Let $\mathfrak{g}$ be a finite-dimensional (complex) reductive Lie algebra and let $\mathfrak{k} \in \mathfrak{g}$ be a reductive in $\mathfrak{g}$ algebraic subalgebra. A $(\mathfrak{g}, \mathfrak{k})$-*module* M is a $\mathfrak{g}$-module M on which $\mathfrak{k}$ acts locally finitely, i.e. dim $(U(\mathfrak{k}) \cdot m) < \infty$ for any $m \in M$. Under the assumption that M is a simple $\mathfrak{g}$-module, the requirement that M be a $(\mathfrak{g}, \mathfrak{k})$-module is equivalent to the requirement that as a $\mathfrak{k}$-module M decomposes into a direct sum of simple finite-dimensional $\mathfrak{k}$-modules. An *admissible* $(\mathfrak{g}, \mathfrak{k})$-module M is a $(\mathfrak{g}, \mathfrak{k})$-module which, after restriction to $\mathfrak{k}$, is isomorphic to a direct sum of simple finite-dimensional $\mathfrak{k}$-modules with finite multiplicities.

Both of these notions go back to the 1960's. By a *generalized Harish-Chandra module*, we understand a $\mathfrak{g}$-module M for which there exists a reductive subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ such that M is an admissible $(\mathfrak{g}, \mathfrak{k})$-module. The case of *Harish-Chandra modules* corresponds to the case where $\mathfrak{k}$ is a symmetric subalgebra of $\mathfrak{g}$. Under this latter assumption, there is an extensive literature on $(\mathfrak{g}, \mathfrak{k})$-modules, and here we just direct the reader to [9] and [2] and references therein.

The question of interest in the present note is the following:

What is a necessary and sufficient condition on an algebraic, reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$ for the existence of a simple infinite-dimensional admissible $(\mathfrak{g}, \mathfrak{k})$-module?

We know of no published answer to this question. However, we have observed that the answer is actually implicit in our work. To make it explicit, we prove the following.

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Theorem. *For an algebraic reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$, there exists a simple infinite-dimensional admissible $(\mathfrak{g}, \mathfrak{k})$-module if and only if $\mathfrak{k}$ is not an ideal of $\mathfrak{g}$.*

Proof. If $\mathfrak{k}$ is an ideal of $\mathfrak{g}$, then any simple $(\mathfrak{g}, \mathfrak{k})$-module M is isomorphic to an outer tensor product $M_\mathfrak{k} \boxtimes M'$, where $M_\mathfrak{k}$ is a simple finite-dimensional $\mathfrak{k}$-module and M' is a simple module over a direct complement $\mathfrak{g}'$ of $\mathfrak{k}$ in $\mathfrak{g}$. Indeed, fix a simple finite-dimensional $\mathfrak{k}$-submodule $M_\mathfrak{k}$ of M (which exists because of the locally finite action of $\mathfrak{k}$ on M). Then the isotypic component of $M_\mathfrak{k}$ in M is a $\mathfrak{g}$-submodule since $\mathfrak{k}$ is an ideal in $\mathfrak{g}$. Therefore,

$$M = M_\mathfrak{k} \otimes \mathrm{Hom}_\mathfrak{k}(M_\mathfrak{k}, M).$$

Setting $M' := \mathrm{Hom}_\mathfrak{k}(M_\mathfrak{k}, M)$, we see that the simplicity of M as a $\mathfrak{g}$-module implies the simplicity of M' as a $\mathfrak{g}'$-module. Consequently, if M is admissible then M is finite dimensional.

Assume now that $\mathfrak{k}$ is not an ideal in $\mathfrak{g}$. Without loss of generality we can assume that $\mathfrak{g}$ is semisimple and that $\mathfrak{k}$ does not contain an ideal of $\mathfrak{g}$. We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$ where $^\perp$ indicates orthogonal space with respect to the Killing form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. We fix a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and an element $h \in \mathfrak{t}$ which is regular in $\mathfrak{k}$ and has real eigenvalues in $\mathfrak{g}$. By $\mathfrak{g}^\alpha$ we denote the eigenspaces of h in $\mathfrak{g}$. Then

$$\mathfrak{p} := C_\mathfrak{g}(h) \oplus (\bigoplus_{\alpha>0} \mathfrak{g}^\alpha)$$

is a *minimal* $\mathfrak{t}$*-compatible parabolic subalgebra* of $\mathfrak{g}$. Here $C_\mathfrak{g}(h)$ is the centralizer of h in $\mathfrak{g}$. The notions of $\mathfrak{t}$-compatible and minimal $\mathfrak{t}$-compatible parabolic subalgebra are discussed in [4]. In what follows, we set $\mathfrak{m} := C_\mathfrak{g}(h)$ and $\mathfrak{n} := \bigoplus_{\alpha>0} \mathfrak{g}^\alpha$ and note that in the semidirect sum $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$, $\mathfrak{m}$ is the reductive part of $\mathfrak{p}$ and $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$. Furthermore, $\mathfrak{k}^\perp = (\mathfrak{n} \cap \mathfrak{k}^\perp) \oplus (\mathfrak{m} \cap \mathfrak{k}^\perp) \oplus (\bar{\mathfrak{n}} \cap \mathfrak{k}^\perp)$ where $\bar{\mathfrak{n}} := \bigoplus_{\alpha<0} \mathfrak{g}^\alpha$. The assumption that $\mathfrak{k}$ does not contain an ideal of $\mathfrak{g}$ implies that h does not commute with $\mathfrak{k}^\perp$, and hence

$$\mathfrak{n} \cap \mathfrak{k}^\perp \neq 0.$$

In particular,

$$r := \dim(\mathfrak{n} \cap \mathfrak{k}^\perp) > 0.$$

Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{b} \subset \mathfrak{p}$ and $\mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$. Recall from [4] the construction of the fundamental series $(\mathfrak{g}, \mathfrak{k})$-module $F^s(\mathfrak{p}, E)$ where E is a finite-dimensional simple $\mathfrak{p}$-module which, as an $\mathfrak{m}$-module, has highest weight ν with respect to the Borel subalgebra $\mathfrak{b}_\mathfrak{m} = \mathfrak{b} \cap \mathfrak{m}$ of $\mathfrak{m}$. Note that rk$\mathfrak{m}$=rk$\mathfrak{g}$, and assume that $\nu \in \mathfrak{h}^*$ for a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ lying in $\mathfrak{m}$ and containing $\mathfrak{t}$. Let $\omega \in \mathfrak{t}^*$ be the restriction of ν to $\mathfrak{t}$. By μ we denote the $\mathfrak{t}$-weight $\omega + 2\rho_\mathfrak{n}^\perp$, where $\rho_\mathfrak{n}^\perp$ is the half-sum of the $\mathfrak{t}$-weights of $\mathfrak{n} \cap \mathfrak{k}^\perp$ with multiplicities, that is, the half-sum of the multiset of $\mathfrak{t}$-weights of $\mathfrak{n} \cap \mathfrak{k}^\perp$.

In what follows, we assume that μ is an integral weight of $\mathfrak{k}$, dominant with respect to the Borel subalgebra $\mathfrak{b} \cap \mathfrak{k}$ of $\mathfrak{k}$. We need one further assumption on μ.

Following [4], we call μ *generic* if the following two conditions are satisfied:

(1) $\langle \mathrm{Re}\mu + 2\rho - \rho_{\mathfrak{n}}, \beta\rangle \geq 0$ for every $\mathfrak{t}$-weight β of $\mathfrak{n} \cap \mathfrak{k}$,
(2) $\langle \mathrm{Re}\mu + 2\rho - \rho_S, \rho_S\rangle > 0$ for every submultiset of the multiset S of $\mathfrak{t}$-weights of $\mathfrak{n}$,

where ρ is the half-sum of the $\mathfrak{t}$-roots of $\mathfrak{k}$ and $\rho_{\mathfrak{n}}$ is the half-sum of the multiset of $\mathfrak{t}$-weights of $\mathfrak{n}$.

Theorem 2 of [4] implies that, under the additional assumption of genericity of μ (which is ultimately a condition on ν), the $(\mathfrak{g}, \mathfrak{k})$-module $F^s(\mathfrak{p}, E)$ is a nonzero admissible $(\mathfrak{g}, \mathfrak{k})$-module with a unique simple submodule $\bar{F}^s(\mathfrak{p}, E)$. Here $s = \dim(\mathfrak{k} \cap \mathfrak{n})$. Moreover, Proposition 6 in [4] claims that there is an isomorphism of vector spaces

$$\mathrm{Hom}_{\mathfrak{g}}(M, \bar{F}^s(\mathfrak{p}, E)) \simeq \mathrm{Hom}_{\mathfrak{m}}(H^r(\mathfrak{n}, M), E),$$

for any simple admissible $(\mathfrak{g}, \mathfrak{k})$-module M. We know that $\bar{F}^s(\mathfrak{p}, E)$ is a simple admissible $(\mathfrak{g}, \mathfrak{k})$-module, so it only remains to show the existence of a weight ν which satisfies all above assumptions and such that $\dim \bar{F}^s(\mathfrak{p}, E) = \infty$.

We consider two possibilities. Either there exists a ν as above such that the central character of the $(\mathfrak{g}, \mathfrak{k})$-module $\bar{F}^s(\mathfrak{p}, E)$ is not integral, or the central character of $\bar{F}^s(\mathfrak{p}, E)$ is necessarily integral (as a consequence of all our assumptions on ν). In the former case, we are done as then necessarily $\dim \bar{F}^s(\mathfrak{p}, E) = \infty$. In the latter case we will further assume that ν is integral $\mathfrak{b}$-dominant for $\mathfrak{g}$. Lemma 2.3 in [6] shows that this additional assumption is compatible with all previous assumptions on ν. Then, by Theorem 2 c) in [4], the simple finite-dimensional W with $\mathfrak{b}$-highest weight ν is the only (up to isomorphism) simple finite-dimensional module whose central character coincides with that of $\bar{F}^s(\mathfrak{p}, E)$.

Therefore it suffices to show that

$$\mathrm{Hom}_{\mathfrak{g}}(W, \bar{F}^s(\mathfrak{p}, E)) = \mathrm{Hom}_{\mathfrak{m}}(H^r(\mathfrak{n}, W), E) = 0.$$

For this, recall that Kostant's Theorem [1] asserts that there is an isomorphism of $\mathfrak{m}$-modules

$$H^r(\mathfrak{n}, W) \simeq \bigoplus_{w} E(w(\nu + \tilde{\rho}) - \tilde{\rho}).$$

Here $\tilde{\rho}$ is the half-sum of roots of $\mathfrak{b}$, $E(\gamma)$ is a simple $\mathfrak{m}$-module with highest weight γ, and the sum is taken over all elements w of the Weyl group of $\mathfrak{g}$ of length r for which the weights $w(\nu + \tilde{\rho}) - \tilde{\rho}$ are $\mathfrak{b}_{\mathfrak{m}}$-dominant. Since $r > 0$, we infer that

$$\mathrm{Hom}_{\mathfrak{m}}(H^r(\mathfrak{n}, W), E) = 0,$$

and the theorem is proved. □

In conclusion, we would like to make two brief comments on how the modules, whose existence is claimed in the above theorem, fit into the panorama of well-studied (and not so well-studied) $\mathfrak{g}$-modules. Our first remark is that $\bar{F}^s(\mathfrak{p}, E)$ does not have to be a $(\mathfrak{g}, \mathfrak{k}')$-module for any reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}'$ which contains $\mathfrak{k}$

properly. Indeed, let $\mathfrak{g} = \mathfrak{sl}(n)$ for $n \geq 4$ and let $\mathfrak{k}$ be a principal $\mathfrak{sl}(2)$-subalgebra. By the same argument as in our expository paper [3], using the work of Willenbring and the second author [11], one can show that the $\mathfrak{g}$-module $\bar{F}^1(\mathfrak{p}, E)$ (here $s = 1$) is not a $(\mathfrak{g}, \mathfrak{k}')$-module for any $\mathfrak{k}'$ as above. In particular, $\bar{F}^1(\mathfrak{p}, E)$ is not a Harish-Chandra module for the pair $(\mathfrak{g}, \mathfrak{so}(n))$ if $n = 2k+1$, or the pair $(\mathfrak{g}, \mathfrak{sp}(n))$ if $n = 2k$.

Our second comment is that $\mathfrak{k}$ does not have to be a symmetric subalgebra of $\mathfrak{g}$ for the module $\bar{F}^s(\mathfrak{p}, E)$ to be a Harish-Chandra module. For instance, let $\mathfrak{g}$ be simple and $\mathfrak{k}$ be an ideal in a symmetric subalgebra $\mathfrak{k}'$ of $\mathfrak{g}$. Then $\bar{F}^s(\mathfrak{p}, E)$ is an admissible $(\mathfrak{g}, \mathfrak{k})$-module which is also a $(\mathfrak{g}, \mathfrak{k}')$-module, hence a Harish-Chandra module. The property of a Harish-Chandra module to be admissible over an ideal of the relevant symmetric subalgebra has been studied in the literature. This applies in particular to the work of Orsted and Wolf [10], where certain ideals of symmetric subalgebras are singled out and discrete series modules --admissible over these ideals-- are investigated. Our approach in [4] provides an alternative construction which applies to any ideal of a symmetric subalgebra but, even in the case of Orsted and Wolf, the range of Harish-Chandra modules arising through this construction requires further study.

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