Classification of primitive ideals of $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$

by

Aleksandr Fadeev

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Approved, Thesis Committee

Prof. Dr. Ivan Penkov

Chair, Jacobs University Bremen

Prof. Dr. Alan Huckleberry

Jacobs University Bremen / Ruhr-Universität Bochum

Dr. Alexey Petukhov

Institute for Information Transmission Problems

Date of Defense: *Wednesday, December 04, 2019*

Department of Mathematics and Logistics

Abstract

The purpose of this Ph.D. thesis is to study and classify primitive ideals of the enveloping algebras $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$. Let $\mathfrak{g}(\infty)$ denote any of the Lie algebras $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$. Then $\mathfrak{g}(\infty) = \bigcup_{n\geq 2} \mathfrak{g}(2n)$ for $\mathfrak{g}(2n) = \mathfrak{o}(2n)$ or $\mathfrak{g}(2n) = \mathfrak{sp}(2n)$, respectively. We show that each primitive ideal I of $U(\mathfrak{g}(\infty))$ is weakly bounded, i.e., $I \cap U(\mathfrak{g}(2n))$ equals the intersection of annihilators of bounded weight $\mathfrak{g}(2n)$-modules. To every primitive ideal I of $\mathfrak{g}(\infty)$ we attach a unique irreducible coherent local system of bounded ideals, which is an analog of a coherent local system of finite-dimensional modules, as introduced earlier by A. Zhilinskii. As a result, primitive ideals of $U(\mathfrak{g}(\infty))$ are parametrized by triples (x, y, Z) where x is a nonnegative integer, y is a nonnegative integer or half-integer, and Z is a Young diagram. In the case of $\mathfrak{o}(\infty)$, each primitive ideal is integrable, and our classification reduces to a classification of integrable ideals going back to A. Zhilinskii, A. Penkov and I. Petukhov. In the case of $\mathfrak{sp}(\infty)$, only 'half' of the primitive ideals are integrable, and nonintegrable primitive ideals correspond to triples (x, y, Z) where y is a half-integer.

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Declaration

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Signature: ____________________ Name: Aleksandr Fadeev
Date: December 10, 2019

Contents

1. Introduction

In this work, we classify the primitive ideals of the universal enveloping algebras of the infinite-dimensional Lie algebras $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$.

Let $\mathfrak{g}(\infty)$ denote one of the Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$, and let $\mathfrak{g}(n)$ denote the respective finite-dimensional Lie algebra $\mathfrak{sl}(n)$, $\mathfrak{o}(2n)$, $\mathfrak{o}(2n+1)$ or $\mathfrak{sp}(2n)$. Then the Lie algebra $\mathfrak{g}(\infty)$ is isomorphic to the direct limit $\varinjlim \mathfrak{g}(n)$ for certain natural embeddings $\mathfrak{g}(n) \hookrightarrow \mathfrak{g}(n+1)$.

Classifying the primitive ideals of the universal enveloping algebra $U(\mathfrak{g}(\infty))$ is an important structural problem for $U(\mathfrak{g}(\infty))$ as an associative algebra, and is also a fundamental problem in the representation theory of $\mathfrak{g}(\infty)$. Indeed, for any Lie algebra $\mathfrak{g}$, the annihilator in $U(\mathfrak{g})$ of a simple $\mathfrak{g}$-module is a primitive ideal, but experience shows that even for a simple finite-dimensional Lie algebra $\mathfrak{g}$ the problem of classifying primitive ideals in $U(\mathfrak{g})$ is tractable, while the problem of classifying simple $\mathfrak{g}$-modules when rank of $\mathfrak{g}$ the large enough, is untractable or 'wild'. The simple objects of several important categories of $\mathfrak{g}$-modules have been classified. This concerns category $\mathcal{O}$, the category of Harish–Chandra modules, the category of weight modules of finite type, and some other categories, but there is no known approach to a classification of arbitrary simple $\mathfrak{g}$-modules. On the other hand, the classification of primitive ideals for a simple Lie algebra $\mathfrak{g}$ is now one of the cornerstones of the representation theory of simple, or semisimple, finite-dimensional Lie algebras.

Let me discuss this classification in more detail. The starting point is Duflo's Theorem, see [D], which claims that, given a simple $\mathfrak{g}$-module, there exists a simple highest weight $\mathfrak{g}$-module with the same annihilator. Hence, for the classification of primitive ideals, it is enough to classify the annihilators of all simple highest weight $\mathfrak{g}$-modules. After this, it remains to understand when two different highest weight modules have the same annihilators. A sufficient condition for this was given by A. Joseph [J1], and for $\mathfrak{g} = \mathfrak{sl}(n)$ Joseph was able to solve the problem completely. It turned out, that for integral weights λ and μ lying in one Weyl group orbit, the corresponding primitive ideals coincide precisely when the insertion tableaux in the outputs of the Robinson–Schensted algorithm applied to λ and μ coincide. We recall that this is a combinatorial algorithm which attaches to each permutation two Young tableaux: the insertion tableau and the recording tableau.

For example, let

$$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 5 & 1 & 2 & 8 & 3 & 7 & 4 & 9 \end{pmatrix} \in S_9$$

Then the output of the Robinson–Schensted algorithm applied to δ is

$$Y = \begin{array}{|c|c|c|} \hline 9 & 7 & 4 \\ \hline 8 & 5 & 2 \\ \hline 6 \\ \cline{1-1} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}\ , \quad Y' = \begin{array}{|c|c|c|} \hline 9 & 8 & 7 \\ \hline 6 & 4 & 2 \\ \hline 5 \\ \cline{1-1} 3 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}$$

where Y is called the insertion tableau and Y' is called the recording tableau. For a detailed description of this algorithm and more examples, see Subsection 2.12.

Another important result of Joseph, closely related to the classification of primitive ideals, is that the associated variety of a primitive ideal coincides with the closure of a nilpotent coadjoint orbit.

The classification of primitive ideals of $U(\mathfrak{g})$ for $\mathfrak{g}(n)$ equal to $\mathfrak{o}(2n)$, $\mathfrak{o}(2n+1)$ or $\mathfrak{sp}(2n)$ was completed by D. Barbash and D. Vogan in their work [BV]. As we pointed out above, two annihilators of simple $\mathfrak{sl}(n)$-modules $L(\lambda)$ and $L(\mu)$, with respective highest weights λ and μ, coincide if and only if the insertion tableaux in the outputs of Robinson–Schensted algorithm applied λ and μ coincide. This is no longer true for $\mathfrak{o}(2n)$, $\mathfrak{o}(2n+1)$ or $\mathfrak{sp}(2n)$. In this case, consider a primitive ideal $I = \mathrm{Ann}(L(\lambda))$ for a simple module $L(\lambda)$ with highest weight λ. Then Barbash and Vogan find a weight γ such that $I = \mathrm{Ann}(L(\gamma))$ and the insertion tableau Y in the output of Robinson–Schensted algorithm applied to γ has the property that the lengths of the rows of Y are equal to the sizes of Jordan cells for a nilpotent matrix from the associated variety of I. In this way, primitive ideals of $U(\mathfrak{o}(2n))$, $U(\mathfrak{o}(2n+1))$ and $U(\mathfrak{sp}(2n))$ can also be parameterized by (certain) Young tableaux.

Now, let us turn to the infinite-dimensional setting. First, we will briefly recall the classification of primitive ideals of the universal enveloping algebra for the infinite-dimensional Lie algebra $\mathfrak{sl}(\infty)$, due to I. Penkov and A. Petukhov. In the series of works [PP1], [PP2], [PP3], [PP4], they provided such a classification, and in the paper [PP5] this classification was related to an infinite-dimensional analogue of the Robinson–Schensted algorithm.

We should note that in the case of $\mathfrak{sl}(\infty)$ there are some important differences to the finite-dimensional case. First, it easy to prove that the center of the universal enveloping algebra of the Lie algebra $\mathfrak{sl}(\infty)$ consists of constants only. Second, a 'generic' simple $\mathfrak{sl}(\infty)$-module has zero annihilator, and in [PP2] a criterion for a simple highest weight $\mathfrak{sl}(\infty)$-module to have nonzero annihilator is established. Next, every primitive ideal for $\mathfrak{sl}(\infty)$ is integrable, which is wrong in the finite-dimensional case. Recall that an integrable ideal $I \subset U(\mathfrak{g}(\infty))$ is by definition the annihilator of an integrable $\mathfrak{g}(\infty)$-module M, i.e., of a $\mathfrak{g}(\infty)$-module M which, when restricted to $\mathfrak{g}(n)$ for any n, is a sum of finite-dimensional $\mathfrak{g}(n)$-modules. Finally, the integrability of a primitive ideal $I \subset U(\mathfrak{sl}(\infty))$ does not imply that any simple $\mathfrak{sl}(\infty)$-module M, whose annihilator equals I, is integrable.

The integrability of a primitive ideal $I \subset U(\mathfrak{sl}(\infty))$ was proved by Penkov and Petukhov in the work [PP4]. This reduced the classification of primitive ideals to a classification of annihilators of simple integrable $\mathfrak{sl}(\infty)$-modules. This latter classification had already been carried out in the papers [PP1], [PP2], [PP3] (without classifying simple integrable $\mathfrak{sl}(\infty)$-modules!) and relies essentially on work of A. Zhilinskii.

In a series of works [Zh1], [Zh2], [Zh3], Zhilinskii introduces and classifies certain combinatorial data which he called coherent local systems, c.l.s.. In the work [PP3], Penkov and Petukhov establish a bijection between some irreducible c.l.s. and integrable primitive ideals. In the cases of $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$, integrable primitive ideals of $U(\mathfrak{sl}(\infty))$ are in one-to-one correspondence with all irreducible c.l.s. Next, in the paper [PP4] Penkov and Petukhov introduce the notion of a precoherent local system (p.l.s.) and prove that every primitive ideal of $U(\mathfrak{sl}(\infty))$ is integrable.

More precisely, let $V(n)$ be the natural $\mathfrak{sl}(n)$-module, and let $V = \varinjlim V(n)$ be the natural $\mathfrak{sl}(\infty)$-module. We denote by $S^\bullet(V)$ and $\Lambda^\bullet(V)$ the symmetric algebra and the exterior algebra of V respectively. It turns out that every primitive ideal of $U(\mathfrak{sl}(\infty))$ has the form

$$I(x, y, Y_l, Y_r) := \mathrm{Ann}_{U(\mathfrak{sl}(\infty))}(V_{Y_l} \otimes (S^\bullet(V))^{\otimes x} \otimes (\Lambda^\bullet(V))^{\otimes y} \otimes (V_{Y_r})_*),$$

where $x, y \in \mathbb{Z}_{\geq 0}$, Y_l and Y_r are Young diagrams, and the modules V_{Y_l} and $(V_{Y_r})_*$ are constructed as follows. Let Y be a Young diagram with row lengths

$$l_1 \geq l_2 \geq \cdots \geq l_s > 0.$$

Then for $n \geq s$ we denote by $V_Y(n)$ the $\mathfrak{sl}(n)$-module with highest weight

$$\underbrace{(l_1,\ l_2,\ \ldots,\ l_s,\ 0,\ 0,\ \ldots,\ 0)}_{n \text{ numbers}},$$

and note that the modules $V_Y(n)$ are nested: $V_Y(n) \hookrightarrow V_Y(n+1)$. This allows us to define the $\mathfrak{sl}(\infty)$-module V_Y as the direct limit $\varinjlim V_Y(n)$. Finally, we put

$$(V_Y)_* = \varinjlim (V_Y(n))^*,$$

where $(V_Y(n))^*$ is the module dual to $V_Y(n)$.

As stated above, the goal in the present work is to classify the primitive ideals of the universal enveloping algebras of $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$. For $\mathfrak{o}(\infty)$, Penkov and Petukhov conjectured that every primitive ideal is integrable. Proving this conjecture, reduces the classification of primitive ideals of $U(\mathfrak{o}(\infty))$ to the known classification [PP3] of integrable primitive ideals of $U(\mathfrak{o}(\infty))$. On the other hand, in the case of $\mathfrak{sp}(\infty)$ not every primitive ideal is integrable. Indeed, consider the simple $\mathfrak{sp}(2n)$-modules $SW^+(2n)$ and $SW^-(2n)$ (the Shale–Weil modules) with respective highest weights $(-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, -\frac{3}{2})$. One can check that the direct limits $\varinjlim SW^+(2n)$ and $\varinjlim SW^-(2n)$ are well-defined $\mathfrak{sp}(\infty)$-modules. In the work [PP3] Penkov and Petukhov prove that the annihilators in $U(\mathfrak{sp}(\infty))$ of these modules coincide, and constitute a nonintegrable primitive ideal.

For the $\mathfrak{sp}(\infty)$-case Penkov and Petukhov provided me with a conjectural construction of all primitive ideal of $U(\mathfrak{sp}(\infty))$ by using a generalization of the notion of c.l.s. In this work, I prove this conjecture, as well as the conjecture that all primitive ideals of $U(\mathfrak{o}(\infty))$ are integrable.

In what follows, I describe the contents of this dissertation.

In Section 2 we give most necessary definitions, as well as known statements which are used later in this work. Section 3 is devoted to the proof of the fact that every primitive ideal of $U(\mathfrak{o}(\infty))$ or $U(\mathfrak{sp}(\infty))$ is weakly bounded. In addition, it turns out that all primitive ideals of $U(\mathfrak{o}(\infty))$ and some primitive ideals of $U(\mathfrak{sp}(\infty))$ are locally integrable. In Section 4 we prove that every locally integrable ideal of $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$ is integrable.

In Section 5 we introduce the notions of a coherent local system of bounded ideals (c.l.s.b.) and a precoherent local system of bounded ideals (p.l.s.b.), which generalize the notions of a coherent local system and a precoherent local system of finite-dimensional representations, respectively.

A new combinatorial tool appearing in the case of $\mathfrak{sp}(\infty)$ are the Kazhdan–

Lusztig polynomials. Each Kazhdan–Lusztig polynomial is defined by two elements of the Weyl group of $\mathfrak{sp}(2n)$ for some n (in general, of a Coxeter group); for a more precise definition, see [H] or Subsection 5.1. It is a remarkable fact that the Kazhdan–Lusztig polynomials corresponding to a bounded simple highest weight $\mathfrak{sp}(2n)$-module $L(\lambda)$ are equal to the respective Kazhdan–Lusztig polynomials corresponding to the simple finite-dimensional $\mathfrak{o}(2n)$-module $L(\lambda')$ where $\lambda' = \lambda + \sum_{i=1}^{n} \varepsilon_i$. Using this, we establish a one-to-one correspondence between the set of c.l.s.b. for $\mathfrak{sp}(\infty)$ and the set of c.l.s.b for $\mathfrak{o}(\infty)$, the latter set being equal to the set of c.l.s. for $\mathfrak{o}(\infty)$.

Finally, in Section 6 it is proved that each nonzero primitive ideal $I \subsetneq \mathrm{U}(\mathfrak{sp}(\infty))$ is the annihilator of a unique $\mathfrak{sp}(\infty)$-module of the form

$$(\mathrm{S}^{\bullet}(V))^{\otimes x} \otimes (\Lambda^{\bullet}(V))^{\otimes y} \otimes V_Z \text{ or } (\mathrm{S}^{\bullet}(V))^{\otimes x} \otimes (\Lambda^{\bullet}(V))^{\otimes y} \otimes V_Z \otimes R$$

where $x, y \in \mathbb{Z}_{\geq 0}$, V is the natural $\mathfrak{sp}(\infty)$-module, V_Z is the $\mathfrak{sp}(\infty)$-module defined analogously to the $\mathfrak{sl}(\infty)$ case (for Z is arbitrary Young diagram), and R is the Shale–Weil $\mathfrak{sp}(\infty)$-module which is equal to the direct limit $\varinjlim SW^+(2n)$.

One may note that the notion of a bounded primitive ideal of $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$ is well-defined. This can be deduced for instance from work of I. Penkov and V. Serganova [PS]. Furthermore, the work [GP] of D. Grantcharov and I. Penkov shows that the only nonintegrable bounded ideal of $U(\mathfrak{sp}(\infty))$ is the annihilator of module $\varinjlim SW^+(2n)$. By analogy with the finite-dimensional case, this ideal should be called Joseph ideal. In this way, all nonintegrable primitive ideals of $U(\mathfrak{sp}(\infty))$ are weakly bounded but all of them, with one exception, are not bounded.

2. Preliminaries

All Lie algebras and vector spaces are defined over $\mathbb{C}$. Here we outline some of the preliminaries needed in the sequel. Let $\mathfrak{g}$ be a finite- or countable-dimensional Lie algebra, and $g, g_0 \subset \mathfrak{g}$. Then

$$\mathrm{ad}_g \colon \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{ad}_g(g_0) = [g, g_0]$$

is the adjoint representation of $\mathfrak{g}$. *The universal enveloping algebra* $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the quotient algebra

$$U(\mathfrak{g}) := T(\mathfrak{g})/I,$$

where

$$T(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \dots$$

is the tensor algebra of $\mathfrak{g}$, and I is the two-sided ideal of $T(\mathfrak{g})$ generated by all elements of the form

$$a \otimes b - b \otimes a - [a, b]$$

for $a, b \in \mathfrak{g}$.

For a vector space V, we define the *dual space*

$$V^* := \mathrm{Hom}(V, \mathbb{C}).$$

Let V be a vector space and $S \subset V$ be a subset of V. Then *linear span* $\operatorname{span} S$ *of* S in V defined as

$$\operatorname{span} S = \bigcap V',$$

where the intersection is taken over all subspaces $V' \subset V$ such that $V' \supset S$. Let F be a finite set. Then $\#F$ denotes the number of elements of F.

Next, we introduce the associative algebra $\mathrm{Mat}_n = \mathrm{Mat}_n(\mathbb{C})$ of all $(n \times n)$-matrices over $\mathbb{C}$. Also we fix the special basis of the vector space Mat_n, where e_{ij} is the elementary matrix

$$e_{ij} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & ⋰ & \vdots \\ 0 & \dots & 1_{ij} & \dots & 0 \\ \vdots & ⋰ & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}.$$

The (i, j)-th entry of $e_{i,j}$ equals 1, while all other entries are zero.

Let $X = \{x_{ij}\}$ be a $(n \times n)$-matrix. Then we put

$$\operatorname{tr} X = \sum_{i=1}^{n} x_{ii} \text{ and } X^t = \{a_{ij} \mid a_{ij} = x_{ji}\}.$$

There is a symmetric bilinear form $(\cdot, \cdot)$ on the Lie algebra $\mathfrak{g}$, defined by

$$(x, y) = \operatorname{tr}(\operatorname{ad}_x \operatorname{ad}_y)$$

for $x, y \in \mathfrak{g}$. This symmetric bilinear form is called the *Killing form on* $\mathfrak{g}$.

2.1. Lie algebras with root decomposition

Here we introduce some basic definitions and facts from the structure theory of the Lie algebras.

Let $\mathfrak{g}$ be a finite- or countable-dimensional Lie algebra, and let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$.

Definition 2.1. *A Lie subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *is called a* toral subalgebra of $\mathfrak{g}$ *if, for every nonzero element* $h \in \mathfrak{h}$*, the linear operator* $\operatorname{ad}_h \colon \mathfrak{g} \to \mathfrak{g}$ *is diagonalizable.*

Lemma 2.1. *Each toral subalgebra* $\mathfrak{h}$ *of a finite-dimensional Lie algebra* $\mathfrak{g}$ *is abelian.*

Proof. Consider two nonzero elements $h_1, h_2 \in \mathfrak{h}$ and let $h_2 = \sum_{s=1}^{n} h_2^s$ be the decomposition of h_2 as a linear combination of ad_{h_1}-eigenvectors with distinct eigenvalues λ_s. Then, $\operatorname{ad}_{h_1}(h_2) = \sum_{s=1}^{n} \lambda_s h_2^s$. As $\mathfrak{h}$ is a subspace of $\mathfrak{g}$, every vector of the form $h_i' = \sum_{s=1}^{n} (\lambda_s - \lambda_i) h_2^s$ belongs to $\mathfrak{h}$. The vector h_i' decomposes as a sum of $n-1$ ad_{h_1}-eigenvectors with distinct eigenvalues, and induction on n shows that in fact every vector h_2^s belongs to $\mathfrak{h}$. Next, note that $[h_2^s, [h_2^s, h_1]] = [h_2^s, -\lambda_s h_2^s] = 0$. Since $\operatorname{ad}_{h_2^s}$ is diagonalizable for each s and $\ker_{\operatorname{ad}_{h_2^s}} = \ker_{(\operatorname{ad}_{h_2^s})^2}$, we conclude that $[h_2^s, h_1] = 0$ for all s. Hence, $[h_2, h_1] = 0$. □

We note that, as a corollary of Lemma 2.1 and Definition 2.1, every toral subalgebra of a finite-dimensional Lie algebra is diagonalizable, i.e. all operators in it are simultaneously diagonalizable.

Definition 2.2. *Let* $\mathfrak{h}$ *be a toral subalgebra of* $\mathfrak{g}$*. Then* $\mathfrak{h}$ *is a* maximal toral subalgebra *if there is no proper inclusion* $\mathfrak{h} \subset \mathfrak{h}'$ *for a toral subalgebra* $\mathfrak{h}' \in \mathfrak{g}$.

Definition 2.3. *A maximal toral subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *is* a splitting Cartan subalgebra *if* $\mathfrak{g}$ *has the following* $\mathfrak{h}$-module decomposition*:*

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^\alpha \tag{1}$$

where $\mathfrak{g}^\alpha$ *is the eigenspace* $\{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x : \ \forall h \in \mathfrak{h}\}$ *and* $\mathfrak{g}^0 = \mathfrak{h}$.

For a splitting Cartan subalgebra, the set of nonzero elements α appearing in the decomposition (1) is called the *root system* of $\mathfrak{g}$, or simply *the set of roots of* $\mathfrak{g}$ with respect to $\mathfrak{h}$. We denote the set of roots by Δ.

Let now Δ^+, $\Delta^- \subset \Delta$ be two subsets satisfying the conditions

$$\Delta = \Delta^+ \sqcup \Delta^-, \quad -\Delta^+ = \Delta^-, \quad \alpha, \beta \in \Delta^+, \alpha + \beta \in \Delta \Rightarrow \alpha + \beta \in \Delta^+.$$

Given such subsets, we call Δ^+ the set of *positive roots*, and Δ^- the set of *negative roots*. Then the $\mathbb{Z}$-submodule Λ_Δ of $\mathfrak{h}^*$ generated by Δ is called the *root lattice* of the root system Δ.

If the Lie algebra $\mathfrak{g}$ is finite dimensional, we introduce the notation

$$\rho_{\mathfrak{g}} := \sum_{\alpha \in \Delta^+} \alpha/2.$$

Definition 2.4. *Let* $\mathfrak{g}$ *be a Lie algebra. A Lie subalgebra* $\mathfrak{b}$ *of* $\mathfrak{g}$ *is a* splitting Borel subalgebra *if*

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$$

for some splitting Cartan subalgebra $\mathfrak{h}$*, and some subset* Δ^+ *of positive roots.*

Definition 2.5. *Let* $\mathfrak{b}$ *be a Borel subalgebra defining* Δ^+*. Then an element* $\alpha \in \Delta^+$ *is said to be a* simple $\mathfrak{b}$-positive root*, or a* simple root *with respect to* $\mathfrak{b}$*, if it cannot be decomposed as a (finite) sum of two or more* $\mathfrak{b}$*-positive roots. We usually use the symbol* Σ^+ *or* Σ *for the set of all simple* $\mathfrak{b}$*-positive roots. Similarly, we say that* $\alpha \in \Delta^-$ *is a* simple negative root *with respect to* $\mathfrak{b}$ *if it cannot be decomposed as a sum of two or more* $\mathfrak{b}$*-negative roots. The symbol* Σ^- *denotes the set of simple* $\mathfrak{b}$*-negative roots.*

2.2. Some finite-dimensional Lie algebras

Here we review some important examples of finite-dimensional Lie algebras.

1) The Lie algebra $\mathfrak{g} = \mathfrak{gl}(n) = \mathfrak{gl}(n, \mathbb{C})$ is the Lie algebra of the algebra Mat_n, where $[X, Y] = XY - YX$ for $X, Y \in \mathrm{Mat}_n$.

   The *general linear Lie algebra* is the Lie algebra obtained from the associative algebra $\mathrm{Mat_n}$.

   We can choose a splitting Cartan subalgebra $\mathfrak{h}_{\mathfrak{gl}(n)}$ as the algebra of all diagonal matrices in Mat_n. Indeed, $\mathfrak{h}_{\mathfrak{gl}(n)}$ is clearly a maximal toral subalgebra of $\mathfrak{g}$. We have the following $\mathfrak{h}_{\mathfrak{gl}(n)}$-root decomposition:

$$\mathfrak{g} = \mathfrak{h}_{\mathfrak{gl}(n)} \oplus \bigoplus_{1 \le i,j \le n, i \neq j} \mathrm{span}\{e_{ij}\}.$$

   Consider the basis $b = \{e_{11}, e_{22}, \dots, e_{n\,n}\}$ of the Lie algebra $\mathfrak{h}_{\mathfrak{gl}(n)}$. The dual basis

$$b^* = \{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$$

   of $\mathfrak{h}^*_{\mathfrak{gl}(n)}$ satisfies

$$\varepsilon_i(e_{jj}) = \delta^i_j$$

   for $1 \le i, j \le n$. Here δ^i_j is Kronecker's delta.

   Thus the root system $\Delta_{\mathfrak{gl}(n)}$ of $\mathfrak{gl}(n)$ with respect to $\mathfrak{h}_{\mathfrak{gl}(n)}$ is

$$\Delta_{\mathfrak{gl}(n)} = \{\varepsilon_i - \varepsilon_j \mid i, j \in \mathbb{Z}_{>0}, 1 \le i \neq j \le n\}.$$

   We can choose

$$\Delta^+_{\mathfrak{gl}(n)} = \{\varepsilon_i - \varepsilon_j \mid i, j \in \mathbb{Z}_{>0}, 1 \le i < j \le n\} \qquad (2)$$

   as the set of positive roots, with simple roots

$$\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{n-1} - \varepsilon_n\}.$$

   Then the splitting Borel subalgebra $\mathfrak{b}_{\mathfrak{gl}(n)}$ of $\mathfrak{gl}(n)$ containing $\mathfrak{h}_{\mathfrak{gl}(n)}$ and corresponding to $\Delta^+_{\mathfrak{gl}(n)}$ consists of all upper-triangular matrices.

   The Lie algebra $\mathfrak{gl}(n)$ is not a simple Lie algebra, and we can split it as $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \{\text{scalar matrices}\}$.

   Let's describe the Lie algebra $\mathfrak{sl}(n)$.

2) The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n) = \mathfrak{sl}(n, \mathbb{C})$ is the Lie subalgebra of $\mathfrak{gl}(n)$ consisting of all g such that $\operatorname{tr} g = 0$.

   The Lie algebra $\mathfrak{sl}(n)$ is called *the special linear Lie algebra.*

   We can choose as a Cartan subalgebra the algebra

$$\mathfrak{h}_{\mathfrak{sl}(n)} = \operatorname{span}\{e_{ii} - e_{i+1\,i+1} \mid 1 \leq i \leq n\}$$

   of all diagonal matrices of $\mathfrak{sl}(n)$. Then we have the root decomposition

$$\mathfrak{g} = \mathfrak{h}_{\mathfrak{sl}(n)} \oplus \bigoplus_{1 \leq i,j \leq n, i \neq j} \operatorname{span}\{e_{ij}\}.$$

   Next, we define ε_i as in the previous case. Note that

$$\{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n\}$$

   is the basis of $\mathfrak{h}^*_{\mathfrak{sl}(n)}$ dual to the basis

$$\{\frac{e_{ii}}{2} - \frac{e_{i+1\,i+1}}{2} \mid 1 \leq i \leq n\}.$$

   The root system $\Delta_{\mathfrak{sl}(n)}$ of $\mathfrak{sl}(n)$ with respect to $\mathfrak{h}_{\mathfrak{sl}(n)}$ is equal to the root system of $\mathfrak{gl}(n)$ with respect to $\mathfrak{h}_{\mathfrak{gl}(n)}$.

   Let the positive and simple roots be as in the previous example. The Weyl group $W_{\mathfrak{sl}(n)}$ is isomorphic to the permutation group in n letters $W_{\mathfrak{sl}(n)} \simeq S_n$. It acts on a weight $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$ by permuting its coordinates:

$$w(\lambda) = \sum_{i=1}^{n} \lambda_i \varepsilon_{w(i)},\ w \in W_{\mathfrak{sl}(n)}.$$

   The Lie subalgebra of upper triangular matrices in $\mathfrak{sl}(n)$ is a splitting Borel subalgebra $\mathfrak{b}_{\mathfrak{sl}(n)}$, and $\Delta^+_{\mathfrak{sl}(n)}$ is given by the right-hand side of the formula (2).

3) The Lie algebra $\mathfrak{o}(2n) = \mathfrak{o}(2n, \mathbb{C})$ is a Lie subalgebra of the Lie algebra $\mathfrak{gl}(2n)$. Fix the matrix

$$F = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then the Lie algebra $\mathfrak{o}(2n)$ is

$$\mathfrak{o}(2n) = \{X \in \mathfrak{gl}(2n) \mid XF + FX^t = 0\}.$$

It is a subalgebra because

$$(X+Y)F + F(X+Y)^t = (XF + FX^t) + (YF + FY^t) = 0,$$

and

$$\begin{aligned}[X,Y]F + F[X,Y]^t &= XYF - YXF + F(XY)^t - F(YX)^t = \\ &= -XFY^t + YFX^t + FY^tX^t - FX^tY^t = \\ &= (YF + FY^t)X^t - (XF + FX^t)Y^t = 0,\end{aligned}$$

where $X, Y \in \mathfrak{o}(2n)$.

Throughout the rest of this work, we index the columns and rows of matrices in the ambient Lie algebra $\mathfrak{gl}(2n)$ by

$$(-n, -n+1, \dots, -1, 1, \dots, n-1, n).$$

The subalgebra

$$\mathfrak{h}_{\mathfrak{o}(2n)} = \operatorname{span}\{v_i = e_{ii} - e_{-i-i} \mid 1 \le i \le n\}$$

of all diagonal matrices in $\mathfrak{o}(2n)$ is a splitting Cartan subalgebra of $\mathfrak{o}(2n)$.

Let $\{\varepsilon_i\}$ be the basis of $\mathfrak{h}^*_{\mathfrak{o}(2n)}$ dual to the basis $\{v_i\}$. The root system $\Delta_{\mathfrak{o}(2n)}$ of $\mathfrak{o}(2n)$ with respect to $\mathfrak{h}_{\mathfrak{o}(2n)}$ is

$$\Delta_{\mathfrak{o}(2n)} = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid i, j \in \mathbb{Z}_{>0}, 1 \le i \ne j \le n\}.$$

Let

$$\Delta^+_{\mathfrak{o}(2n)} = \{\varepsilon_i \pm \varepsilon_j \mid i, j \in \mathbb{Z}_{>0}, 1 \le i < j \le n\}$$

be the set of positive roots, with the set of simple roots

$$\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}. \tag{3}$$

The splitting Borel subalgebra $\mathfrak{b}_{\mathfrak{o}(2n)}$ of $\mathfrak{o}(2n)$ with these positive roots is the subalgebra of all upper-triangular matrices in $\mathfrak{o}(2n)$.

In the sequel we fix an inclusion $\mathfrak{h}^*_{\mathfrak{o}(2n)} \hookrightarrow \mathfrak{h}^*_{\mathfrak{sl}(2n)}$ defined by $\varepsilon_i \mapsto \bar{\varepsilon}_i - \bar{\varepsilon}_{-i}$, where $\{\bar{\varepsilon}_1, \bar{\varepsilon}_2 \dots, \bar{\varepsilon}_n, \bar{\varepsilon}_{-n}, \dots, \bar{\varepsilon}_{-2}, \bar{\varepsilon}_{-1}\}$ is the basis of $\mathfrak{h}^*_{\mathfrak{gl}(2n)}$ dual to the basis $\{2e_{ii}\}$.

Under this inclusion, a weight $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i$ is mapped to

$$\sum_{i=1}^{n} \lambda_i \bar{\varepsilon}_i + \sum_{j=-1}^{-n} \lambda_j \bar{\varepsilon}_j$$

where $\lambda_j = -\lambda_{-j}$ for $j < 0$.

Let S_{2n} denote the symmetric group on the $2n$ letters

$$-n, \ldots, -1, 1, \ldots, n.$$

The Weyl group $W_{\mathfrak{o}(2n)}$ of the Lie algebra $\mathfrak{o}(2n)$ is isomorphic to the subgroup of S_{2n} consisting of permutations $w \in S_{2n}$ such that $w(-i) = -w(i)$, $1 \le i \le n$, for which the number $\#\{i > 0 \mid w(i) < 0\}$ is even. 00

We will identify $W_{\mathfrak{o}(2n)}$ with this subgroup, and will use the usual two-line notation

$$g = \left(\begin{array}{cccc|cccc} -n & -n+1 & \ldots & -1 & 1 & \ldots & n-1 & n \\ g(-n) & g(-n+1) & \ldots & g(-1) & g(1) & \ldots & g(n-1) & g(n) \end{array} \right)$$

for an element $g \in W$. Note that, if we identify a weight $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i$ with the sequence of integers

$$(\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{-n}, \ldots, \lambda_{-2}, \lambda_{-1}),$$

then g sends this sequence to

$$(\lambda_{g^{-1}(1)}, \lambda_{g^{-1}(2)}, \ldots, \lambda_{g_{-1}(n)}, \lambda_{g_{-1}(-n)}, \ldots, \lambda_{g^{-1}(-2)}, \lambda_{g^{-1}(-1)}).$$

4) The Lie algebra $\mathfrak{sp}(2n) = \mathfrak{sp}(2n, \mathbb{C})$ is a Lie subalgebra of $\mathfrak{gl}(2n)$. Fix the matrix

$$F = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & -1 & \ldots & 0 & 0 \\ -1 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

with n 1s and (-1)s on the antidiagonal. Then the Lie algebra $\mathfrak{sp}(2n)$ is

$$\mathfrak{sp}(2n) = \{X \in \mathfrak{gl}(2n) \mid XF + FX^t = 0\}.$$

One can check that it is a Lie subalgebra indeed.

As in the case of $\mathfrak{o}(2n)$, we use the set of indices $\{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}$. The subalgebra

$$\mathfrak{h}_{\mathfrak{sp}(2n)} = \operatorname{span}\{e_{ii} - e_{-i-i} \mid 1 \leq i \leq n\},$$

of all diagonal matrices in $\mathfrak{sp}(2n)$ is a splitting Cartan sublagebra of $\mathfrak{sp}(2n)$.

We denote by ε_i the same weights as for $\mathfrak{o}(2n)$. Then the root system $\Delta_{\mathfrak{sp}(2n)}$ of $\mathfrak{sp}(2n)$ with respect to $\mathfrak{h}_{\mathfrak{sp}(2n)}$ is

$$\Delta_{\mathfrak{sp}(2n)} = \{\pm(\varepsilon_i \pm \varepsilon_j), \pm 2\varepsilon_i \mid i, j \in \mathbb{Z}_{>0}, 1 \leq i \neq j \leq n\}.$$

Let

$$\Delta^+_{\mathfrak{sp}(2n)} = \{\varepsilon_i \pm \varepsilon_j, 2\varepsilon_i \mid i, j \in \mathbb{Z}_{>0}, 1 \leq i < j \leq n\}$$

be the set of positive roots with the set of simple roots

$$\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}. \tag{4}$$

Then the splitting Borel subalgebra $\mathfrak{b}_{\mathfrak{sp}(2n)}$ of $\mathfrak{sp}(2n)$ with positive roots $\Delta^+_{\mathfrak{sp}(2n)}$ is the subalgebra of all upper-triangular matrices in $\mathfrak{sp}(2n)$.

As in the case of $\mathfrak{o}(2n)$, we can rewrite any weight $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$ as

$$\lambda = \sum_{i=1}^{n} \lambda_i \bar{\varepsilon}_i + \sum_{j=-1}^{-n} \lambda_j \bar{\varepsilon}_j$$

where $\lambda_j = -\lambda_{-j}$ for $j < 0$. The Weyl group $W_{\mathfrak{sp}(2n)}$ of $\mathfrak{sp}(2n)$ is isomorphic to the subgroup of S_{2n} consisting of permutations $w \in S_{2n}$ such that
$w(-i) = -w(i)$, $1 \leq i \leq n$. As for $\mathfrak{o}(2n)$, we will identify $W_{\mathfrak{sp}(2n)}$ with this subgroup and will use usual two-line notation.

In the sequel, we will refer to the Cartan and Borel subalgebras of $\mathfrak{sl}(n)$, $\mathfrak{o}(2n)$ or $\mathfrak{sp}(2n)$ introduced above, as fixed Cartan and Borel sibalgebras.

2.3. Some countable-dimensional Lie algebras

Here we review some important examples of infinite-dimensional Lie algebras.

1) The Lie algebra $\mathfrak{sl}(\infty) = \mathfrak{sl}(\infty, \mathbb{C})$ is a countable-dimensional Lie algebra. Consider the embeddings

$$\mathfrak{sl}(i) \to \mathfrak{sl}(i+1),$$

$$\mathbf{X} \longmapsto \begin{pmatrix} \mathbf{X} & 0_{i\times 1} \\ 0_{1\times i} & 0 \end{pmatrix}.$$

We set

$$\mathfrak{sl}(\infty) := \varinjlim \mathfrak{sl}(i).$$

Next, we choose a splitting Cartan subalgebra $\mathfrak{h}_{\mathfrak{sl}(\infty)}$ as the direct limit

$$\mathfrak{h}_{\mathfrak{sl}(\infty)} := \varinjlim \mathfrak{h}_i,$$

where $\mathfrak{h}_i$ is the diagonal splitting Cartan subalgebra of $\mathfrak{sl}(i)$. The Lie algebra $\mathfrak{h}_{\mathfrak{sl}(\infty)}$ is obviously a toral subalgebra. It is also maximal toral, as any larger subalgebra contains an elementary nondiagonal matrix, and the latter is a nilpotent element of $\mathfrak{sl}(\infty)$.

The root decomposition of the Lie algebra $\mathfrak{sl}(\infty)$ is

$$\mathfrak{sl}(\infty) = \mathfrak{h}_{\mathfrak{sl}(\infty)} \oplus \bigoplus_{i,j\in\mathbb{Z}_{>0}} \operatorname{span}\{e_{ij}\}.$$

The root system Δ of $\mathfrak{sl}(\infty)$ with respect to $\mathfrak{h}_{\mathfrak{sl}(\infty)}$ is

$$\Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j,\ i,j \in \mathbb{Z}_{>0}\},$$

where

$$\varepsilon_i(e_{jj}) = \delta^i_j.$$

Let

$$\Delta^+ = \{\varepsilon_i - \varepsilon_j \mid i < j,\ i,j \in \mathbb{Z}_{>0}\}$$

be the set of positive roots with the simple roots

$$\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4 \dots\}.$$

Then the splitting Borel subalgebra $\mathfrak{b}_{\mathfrak{sl}(\infty)} \supset \mathfrak{h}_{\mathfrak{sl}(\infty)}$ with positive roots Δ^+ is the direct limit of $\varinjlim \mathfrak{b}_{\mathfrak{sl}(n)}$.

2) The Lie algebra $\mathfrak{o}(\infty) = \mathfrak{o}(\infty, \mathbb{C})$ is a countable-dimensional Lie algebra. Consider the embeddings

$$\phi_{2i} \colon \mathfrak{o}(2i) \to \mathfrak{o}(2i+2),$$

$$\mathbf{X} \longmapsto \left(\begin{array}{ccc} 0 & 0_{1\times i} & 0 \\ 0_{i\times 1} & \mathbf{X} & 0_{i\times 1} \\ 0 & 0_{1\times i} & 0 \end{array} \right).$$

We set

$$\mathfrak{o}(\infty) := \varinjlim \mathfrak{o}(2i).$$

We use $\mathbb{Z}\backslash\{0\}$ as the set of indices for matrices of $\mathfrak{o}(\infty)$. We can choose a splitting Cartan subalgebra $\mathfrak{h}_{\mathfrak{o}(\infty)}$ as the direct limit

$$\mathfrak{h}_{\mathfrak{o}(\infty)} := \varinjlim \mathfrak{h}_{\mathfrak{o}(2i)},$$

where

$$\mathfrak{h}_{\mathfrak{o}(\infty)} = \mathrm{span}\{v_i = e_{ii} - e_{-i\,-i} \mid\ 1 \leq i\}.$$

By

$$\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots\}$$

we denote set dual to the basis $\{v_i\}$ of $\mathfrak{h}_{\mathfrak{o}(\infty)}$, i.e.

$$\varepsilon_i(v_j) = \delta^i_j$$

for $1 \leq i, j$ and $i, j \in \mathbb{Z}_{>0}$.

The root system Δ of $\mathfrak{o}(\infty)$ with respect to $\mathfrak{h}_{\mathfrak{o}}$ is

$$\Delta = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid\ i \neq j,\ i, j \in \mathbb{Z}_{>0}\}.$$

Let

$$\Delta^+ = \{-\varepsilon_i \pm \varepsilon_j \mid\ i > j,\ i, j \in \mathbb{Z}_{>0}\}$$

be the set of positive roots; the corresponding simple roots are

$$\{-\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4 \dots\}.$$

3) The Lie algebra $\mathfrak{sp}(\infty) = \mathfrak{sp}(\infty, \mathbb{C})$ is a countable-dimensional Lie algebra. We consider the embedding

$$\psi_{2i} \colon \mathfrak{sp}(2i) \to \mathfrak{sp}(2i+2),$$

$$\mathbf{X} \longmapsto \begin{pmatrix} 0 & 0_{1\times i} & 0 \\ 0_{i\times 1} & \mathbf{X} & 0_{i\times 1} \\ 0 & 0_{1\times i} & 0 \end{pmatrix},$$

and put

$$\mathfrak{sp}(\infty) := \varinjlim \mathfrak{sp}(2i).$$

We use the set $\mathbb{Z}\backslash\{0\}$ as a set of indexes for matrices of $\mathfrak{sp}(\infty)$. One can choose a splitting Cartan subalgebra $\mathfrak{h}_{\mathfrak{sp}(\infty)}$ as the direct limit

$$\mathfrak{h}_{\mathfrak{sp}(\infty)} := \varinjlim \mathfrak{h}_{\mathfrak{sp}(2i)}.$$

By construction,

$$\mathfrak{h}_{\mathfrak{sp}(\infty)} = \operatorname{span}\{v_i = e_{ii} - e_{-i\,-i} \mid 1 \leq i\}. \tag{5}$$

The vectors ε_i are defined in the same way as for $\mathfrak{o}(\infty)$.

The root system Δ of $\mathfrak{sp}(\infty)$ with respect to $\mathfrak{h}_{\mathfrak{sp}(\infty)}$ is

$$\Delta = \{\pm(\varepsilon_i \pm \varepsilon_j), \pm 2\varepsilon_i \mid i \neq j,\ i, j \in \mathbb{Z}_{>0}\}.$$

We set

$$\Delta^+ = \{-\varepsilon_i \pm \varepsilon_j, -2\varepsilon_i \mid i > j,\ i, j \in \mathbb{Z}_{>0}\}$$

as the set of positive roots; the corresponding simple roots are

$$\{-2\varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4 \ldots\}.$$

2.4. Highest weight modules

In this subsection we introduce the notion of highest weight module, and give some related definitions.

Definition 2.6. *A Lie algebra $\mathfrak{g}$ is* simple *if every ideal $I \subset \mathfrak{g}$ is equal to zero or to $\mathfrak{g}$.*

For the purposes of this paper, we call a Lie algebra semisimple if it is isomorphic to a direct of a simple Lie algebras. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ be a splitting Cartan subalgebra, $\mathfrak{b} \subset \mathfrak{g}$ be a splitting Borel subalgebra with $\mathfrak{b} \supset \mathfrak{h}$, and let $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$. Fix a $\mathfrak{g}$-module M. For each $\lambda \in \mathfrak{h}^*$, define M^λ to be the subspace

$$\{v \in M \mid h \cdot v = \lambda(h)v : \ \forall h \in \mathfrak{h}\} \subset M.$$

Definition 2.7. *If*

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$$

then M is a weight module *over $\mathfrak{g}$. If $M^\lambda \neq 0$, then $\lambda \neq 0$ is said to be a* weight *of M, M^λ is called a* weight subspace *of M, and the elements of M^λ are called* weight vectors of weight λ.

Definition 2.8. *The set* $\operatorname{supp} M$ *of weights λ for which* $\dim M_\lambda > 0$ *is called* the support *of the weight module M.*

One can prove that submodules, quotients and direct sums of weight $\mathfrak{g}$-modules are weight modules as well.

Definition 2.9. *A $\mathfrak{g}$-module M is said to be a* cyclic *over $\mathfrak{g}$ if M is generated as an $U(\mathfrak{g})$-module by a single nonzero vector.*

Definition 2.10. *A $\mathfrak{g}$-module M is called a* highest weight module *with respect to a splitting Borel subalgebra $\mathfrak{b}$ if it is generated by a vector $v \neq 0$ satisfying*

$$\mathfrak{n} \cdot v = 0,$$

and there exists a weight λ (which we will call the highest weight *of M) such that*

$$h \cdot v = \lambda(h)v$$

for each $h \in \mathfrak{g}$. The vector v is called a highest weight vector *of M.*

Each highest weight $\mathfrak{g}$-module is a weight $\mathfrak{g}$-module since $\mathfrak{g}$ is an $\mathrm{ad}_{\mathfrak{h}}$-weight module.

Definition 2.11. *For a fixed finite-dimensional semisimple Lie algebra $\mathfrak{g}$, a splitting Borel sublagebra $\mathfrak{b} \subset \mathfrak{g}$, a splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$, and for every $\lambda \in \mathfrak{h}^*$, we define the highest weight module $M(\lambda) = M(\lambda; \mathfrak{g}, \mathfrak{b}, \mathfrak{h})$ by setting*

$$M(\lambda; \mathfrak{g}, \mathfrak{b}, \mathfrak{h}) := U(\mathfrak{g})/I,$$

where I is the left ideal in $U(\mathfrak{g})$ generated by $\mathfrak{n}$ and by $h - \lambda(h)1_{U(\mathfrak{g})}$ for all $h \in \mathfrak{h}$. The modules $M(\lambda; \mathfrak{g}, \mathfrak{b}, \mathfrak{h})$ are known as Verma modules.

Each Verma module $M(\lambda)$ has a unique maximal proper $U(\mathfrak{g})$-submodule N, which is the sum of all proper submodules of $M(\lambda)$, [DP]. Accordingly,

$$L(\lambda) := M(\lambda)/N$$

is the unique simple quotient of $M(\lambda)$.

We denote by v_λ the image of $1_{U(\mathfrak{g})}$ under the canonical projection $U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})/I$. Clearly, v_λ is a highest weight vector of $M(\lambda)$ of weight λ. Moreover, for any highest weight module M over $\mathfrak{g}$ with highest weight λ, there is a unique surjective homomorphism $\phi\colon M(\lambda) \twoheadrightarrow M$. Hence, up to isomorpsim, $L(\lambda)$ is the unique simple highest weight module over $\mathfrak{g}$ with highest weight λ.

2.5. BGG category $\mathcal{O}$

Here we recall some basics concerning the BGG category $\mathcal{O}$, which was introduced in the early 1970s by Joseph Bernstein, Israel Gelfand, and Sergei Gelfand.

Definition 2.12. *The* BGG category $\mathcal{O}$ *is defined to be the full subcategory of $U(\mathfrak{g})$-modules whose objects are the modules satisfying the following three conditions:*

1) M is a finitely generated $U(\mathfrak{g})$-module.

2) M is $\mathfrak{h}$-semisimple, that is, M is a weight module.

3) M is a locally $\mathfrak{n}$-finite: for each $v \in M$, the subspace $U(\mathfrak{n}) \cdot v$ of M is finite dimensional.

We will call a weight λ *dominant* if $\frac{2(\alpha,\lambda)}{(\alpha,\alpha)}$ is not a negative integer for any $\alpha \in \Delta^+$. A weight λ is *regular* if $\frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \neq 0$ for any $\alpha \in \Delta^+$.

Definition 2.13. *The* dot action *of W on $\mathfrak{h}^*$ is defined by letting*

$$w \cdot \lambda = w(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{g}}.$$

Definition 2.14. The reflection group corresponding to a linear function λ *is the subgroup $W_{[\lambda]}$ of W which consists of the elements*

$$w \in W \text{ such that } w \cdot \lambda - \lambda \in \Lambda_\Delta,$$

where Λ_Δ is the root lattice of $\mathfrak{g}$.

By definition, two weights γ and κ are *linked by* $W_{[\lambda]}$ whenever $\gamma = w \cdot \kappa$ for $w \in W_{[\lambda]}$.

The blocks of the category $\mathcal{O}$ are precisely the subcategories consisting of modules whose all composition factors have highest weights linked by $W_{[\lambda]}$ to a weight λ such that $-\lambda$ is dominant. Thus the blocks are in natural bijection with the dominant weights.

2.6. Duflo's Theorem

In this subsection we recall the important Duflo Theorem, which allows us to reduce the set of primitive ideals to the set of annihilators of simple highest weight modules.

Let A be an associative algebra with identity, and I be an ideal of A.

Definition 2.15. *The ideal $I \subset A$ is* primitive *if I is the annihilator of a simple left A-module.*

This following statement is well known as Duflo's Theorem.

Theorem 2.2. [D] *Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra, I be a primitive ideal of $U(\mathfrak{g})$, and $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. Then there exists an irreducible $\mathfrak{b}$-highest weight $\mathfrak{g}$-module whose annihilator is I.*

2.7. Associated variety

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The following theorem is known as the Poincaré–Birkhoff–Witt Theorem, and plays an important role in this thesis.

Theorem 2.3. *Let $\phi\colon \mathfrak{g} \to U(\mathfrak{g})$ be the canonical map. Denote by $\{g_1, g_2, g_3, \dots, g_n\}$ a basis of $\mathfrak{g}$. Then the monomials $g_1^{v_1} g_2^{v_2} g_3^{v_3} \dots g_n^{v_n}$, where $v_1, v_2, v_3, \dots, v_n \in \mathbb{Z}_{\geq 0}$, constitute a basis of $U(\mathfrak{g})$.*

Define U^i to be the vector subspace of $U(\mathfrak{g})$ spanned by all monomials $g_1^{v_1} g_2^{v_2} g_3^{v_3} \dots g_n^{v_n}$ with $\sum v_j \leq i$. The chain of subspaces $\{U^i\}_{i \in \mathbb{Z}_{\geq 0}}$ is a *filtration* of $U(\mathfrak{g})$.

Definition 2.16. *Put $U^{-1} = \{0\}$. Define* the associated graded algebra of $U(\mathfrak{g})$*:*

$$grU(\mathfrak{g}) := \bigoplus_{d \in \mathbb{Z}_{\geq 0}} (U^d / U^{d-1})$$

As a consequence of the Poincaré–Birkhoff–Witt Theorem, we conclude that this algebra is isomorphic to the symmetric algebra $S^\bullet(\mathfrak{g})$.

Definition 2.17. *In the same way we define the* associated graded ideal *of an ideal $I \subset U(\mathfrak{g})$:*

$$grI := \bigoplus_{d \in \mathbb{Z}_{\geq 0}} (U^d \cap I)/(U^{d-1} \cap I) \subset grU(\mathfrak{g}) \simeq S^\bullet(\mathfrak{g}).$$

Definition 2.18. W*e denote by $Var(I)$ the algebraic variety corresponding to grI; by definition this is the set of common zeros of grI in $\mathfrak{g}^*$ (Here we identify $S^\bullet(\mathfrak{g})$ with the algebra of polynomial functions of $\mathfrak{g}^*$). By identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form, we can assume that $Var(I) \subset \mathfrak{g}$.*

2.8. Weakly bounded ideals

In this subsection we give some definitions which are needed to state the results of this work.

Here $\mathfrak{g}(\infty)$ is one of the Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$.

Definition 2.19. *A $\mathfrak{g}(\infty)$-module M is* integrable *if, for any finitely generated subalgebra $U' \subset U(\mathfrak{g}(\infty))$ and any $m \in M$, we have $\dim(U' \cdot m) < \infty$.*

Definition 2.20. *A two-sided ideal $I \subset U(\mathfrak{g}(\infty))$ is* integrable*, if I is the annihilator of an integrable $\mathfrak{g}(\infty)$-module.*

Definition 2.21. *An ideal $I \subset U(\mathfrak{g}(\infty))$ is* locally integrable *if, for any finitely generated subalgebra $A' \subset U(\mathfrak{g}(\infty))$, the ideal $I \cap A'$ is an integrable ideal of A'.*

One can check that an ideal $I \subset U(\mathfrak{g}(\infty))$ is locally integrable if and only if, for every $n \in \mathbb{Z}_{\geq 0}$, $I \cap U(\mathfrak{g}(2n))$ is an intersection of ideals of finite codimension in $U(\mathfrak{g}(2n))$.

Definition 2.22. *Let $\mathfrak{g}$ be a (possibly infinite-dimensional) Lie algebra. For every weight $\mathfrak{g}$-module M we define* the degree of module

$$\deg(M) := \sup_{\lambda \in \operatorname{supp} M}(\dim M_\lambda).$$

Definition 2.23. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then a weight $\mathfrak{g}$-module M is called* bounded *if* $\deg(M) < \infty$*.*

Definition 2.24. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with a splitting Cartan subalgebra $\mathfrak{h}$. An ideal $I \subset U(\mathfrak{g})$ is an* $\mathfrak{h}$-bounded ideal *if $I = \operatorname{Ann}_{U(\mathfrak{g})}(M)$ where M is a bounded $\mathfrak{h}$-weight $\mathfrak{g}$-module.*

Further, we will simply say *bounded* instead of $\mathfrak{h}$-bounded since the subalgebra $\mathfrak{h}$ will be fixed.

Let I be a bounded ideal such that $I = \operatorname{Ann}_{U(\mathfrak{g})}(M)$ for some simple weight $\mathfrak{g}$-module M. Then M is a bounded module (see [PS]).

Note that there are no infinite-dimensional bounded $\mathfrak{o}(2n)$-modules [F]. There is a classification of bounded infinite-dimensional highest weight simple $\mathfrak{g}$-modules, where $\mathfrak{g}$ is one of Lie algebras $\mathfrak{sl}(n)$, $\mathfrak{o}(2n)$, $\mathfrak{o}(2n+1)$ or $\mathfrak{sp}(2n)$ (actually, such modules exist only for $\mathfrak{sl}(n)$ and $\mathfrak{sp}(2n)$). In particular, for $\mathfrak{sp}(2n)$ we have

Lemma 2.4. [M] *Let $L(\lambda)$ be a simple infinite-dimensional highest weight $\mathfrak{sp}(2n)$-module with highest weight λ. It is bounded if and only if*

1) $\lambda(v_i - v_{i+1}) \in \mathbb{Z}_{>0}$ *for any* $0 < i < n$*,*

2) $\lambda(v_n) \in 1/2 + \mathbb{Z}$*,*

3) $\lambda(v_{n-1} + v_n) \in \mathbb{Z}_{\geq -2}$*,*

where v_i *is defined in Subsection 2.2 in formula (5).*

The following bounded $\mathfrak{sp}(2n)$-modules will play an important role in this thesis.

Example. *Consider the ring* $\mathbb{C}[x_1, x_2, \ldots, x_n]$ *of polynomials in* n *variables. The Lie algebra* $\mathfrak{g}(2n) = \mathfrak{sp}(2n)$ *can be realized as follows:*

$$\mathfrak{g}(2n) = \bigoplus_{1\leq i,j\leq n} \operatorname{span}\{\frac{\partial^2}{\partial x_i \partial x_j}\} \oplus \bigoplus_{1\leq k,l\leq n} \operatorname{span}\{x_k \frac{\partial}{\partial x_l} + \frac{\delta_l^k}{2}\} \oplus \bigoplus_{1\leq m,n\leq n} \operatorname{span}\{x_m x_n\},$$

where δ_l^m *is the Kronecker symbol. The space* $\bigoplus_i \operatorname{span}\{x_i \frac{\partial}{\partial x_i} + \frac{1}{2}\}$ *is a splitting Cartan sublagebra* $\mathfrak{h} \subset \mathfrak{sp}(2n)$*, with simple coroots* $h_i = -x_i \frac{\partial}{\partial x_i} + x_{i+1} \frac{\partial}{\partial x_{i+1}}$ *for* $i \leq i \leq n-1$ *and* $h_n = -x_n \frac{\partial}{\partial x_n} - \frac{1}{2}$*. As an* $\mathfrak{sp}(2n)$*-module,* $\mathbb{C}[x_1, x_2, \ldots, x_n]$ *equals* $SW^+(2n) \oplus SW^-(2n)$*, where* $SW^+(2n)$ *is the subspace of homogeneous polynomials of even degree, and* $SW^-(2n)$ *is the subspace of homogeneous polynomials of odd degree. These two subspaces are simple bounded highest weight modules with respective highest weights* $-\frac{1}{2}\sum_{i=1}^n \varepsilon_i$ *and* $-\frac{1}{2}\sum_{i=1}^{n-1} \varepsilon_i - \frac{3}{2}\varepsilon_n$*. They are known as* Shale–Weil *(or* oscillator) representations*.*

Definition 2.25. *An ideal* $I \subset U(\mathfrak{g}(\infty))$ *is* weakly bounded *if* $I \cap U_n$ *is an intersection of annihilators of bounded weight modules of* U_n *for every* $n \geq 2$*, where* $U_n = U(\mathfrak{sl}(n))$ *for* $\mathfrak{g}(\infty) = \mathfrak{sl}(\infty)$*,* $U_n = U(\mathfrak{o}(2n))$ *for* $\mathfrak{g}(\infty) = \mathfrak{o}(\infty)$*, and* $U_n = U(\mathfrak{sp}(2n))$ *for* $\mathfrak{g}(\infty) = \mathfrak{sp}(\infty)$*.*

The next theorem is one of the main results of the present work.

Theorem 2.5. *Every primitive ideal of* $U(\mathfrak{o}(\infty))$ *or* $U(\mathfrak{sp}(\infty))$ *is weakly bounded. Moreover, each primitive ideal of* $U(\mathfrak{o}(\infty))$ *is integrable.*

Note that this theorem is an analogue of the result for $\mathfrak{sl}(\infty)$ proved by I. Penkov and A. Petukhov in [PP4].

2.9. Integrable ideals and coherent local systems

As before, $\mathfrak{g}(\infty) = \mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$. Also, in this subsection and the next subsection, $\mathfrak{g}(n)$ denotes one of the Lie algebras $\mathfrak{sl}(n)$, $\mathfrak{o}(2n)$ or $\mathfrak{sp}(2n)$. Since we express $\mathfrak{o}(\infty)$ as $\varinjlim \mathfrak{o}(2n)$, we do not need the $\mathfrak{o}(2n+1)$-series of Lie algebras. Let Irr_n denote the set of isomorphism classes of simple finite-dimensional $\mathfrak{g}(n)$-modules.

Definition 2.26. *A* coherent local system of modules *(further c.l.s.) for* $\mathfrak{g}(\infty)$ *is a collection of sets*

$$\{Q_n\}_{n\in\mathbb{Z}_{\geq 1}} \subset \Pi_{n\in\mathbb{Z}_{\geq 1}} Irr_n$$

such that $Q_m = \langle Q_n\rangle_m$ *for any* $n > m$, *where* $\langle Q_n\rangle_m$ *denotes the set of isomorphism classes of all simple* $\mathfrak{g}(m)$*-constituents of the* $\mathfrak{g}(n)$*-modules from* Q_n.

Definition 2.27. *A c.l.s.* Q *is* irreducible *if* $Q \neq Q' \cup Q''$ *with* $Q' \not\subset Q''$ *and* $Q' \not\supset Q''$, *where* Q' *and* Q'' *are nonempty coherent local systems of modules for* $\mathfrak{gl}(\infty)$.

Each c.l.s. Q can be represented uniquely as a finite union $\cup_i Q(i)$ [Zh1] of some maximal (by inclusion within Q) irreducible c.l.s. $Q(i)$; we call $Q(i)$ *the irreducible components* of Q.

Each integrable $\mathfrak{g}(\infty)$-module M determines a c.l.s. $Q := \{Q_n\}_{n\in\mathbb{Z}_{\geq 1}}$, where

$$Q_n := \{z \in Irr_n \mid \mathrm{Hom}_{\mathfrak{g}(n)}(z, M) \neq \{0\}\}$$

We denote this c.l.s. by $Q(M)$.

Definition 2.28. *We say that a c.l.s.* Q *is of* finite type *if the set* Q_n *is finite for all* $n \geq 1$.

Definition 2.29. *An integrable* $\mathfrak{g}(\infty)$*-module* M *is called* locally simple *if* $M = \varinjlim M_n$ *for a chain*

$$M_3 \subset M_n \subset M_{n+1} \subset \dots$$

of simple finite-dimensional $\mathfrak{g}(n)$*-submodules* M_n *of* M.

For every c.l.s. $Q = \{Q_n\}_{n\in\mathbb{Z}_{\geq 1}}$ we can define the following ideal

$$I(Q) := \cup_m(\cap_{z\in Q_m} \mathrm{Ann}_{U(\mathfrak{g}(m))}\, z) \subset U(\mathfrak{g}(\infty)).$$

We say that $I(Q)$ is the *annihilator* of Q.

Proposition 2.6. [Zh1, Lemma 1.1.2] *If Q is an irreducible c.l.s., then $I(Q)$ is the annihilator of some locally simple integrable $\mathfrak{g}(\infty)$-module. In particular, the ideal $I(Q)$ is primitive.*

Corollary 2.7. *The ideal $I(Q)$ is integrable for each c.l.s. Q.*

Proof. Let $Q = \cup_i Q(i)$, where $Q(i)$ are the irreducible components of Q. By Proposition 2.6, the ideal $I(Q(i))$ is the annihilator of a simple integrable $\mathfrak{g}(\infty)$-module $M(i)$. Therefore $I(Q)$ is the annihilator of the integrable $\mathfrak{g}(\infty)$-module $\oplus_i M_i$. □

Every isomorphism class $z \in Irr_n$ of simple $\mathfrak{g}(n)$-modules corresponds to an integral dominant weight λ of $\mathfrak{g}(n)$. Let z_1, z_2 be isomorphism classes of simple $\mathfrak{g}(n)$-modules with respective highest weights λ_1, λ_2. We denote by $z_1 z_2$ the isomorphism class of a simple module with highest weight $\lambda_1 + \lambda_2$. If $S_1, S_2 \subset Irr_n$ we set

$$S_1 S_2 := \{z \in Irr_n \mid z = z_1 z_2 \text{ for some } z_1 \in S_1 \text{ and } z_2 \in S_2\}.$$

Let Q' and Q'' be c.l.s. We denote by $Q'Q''$ the smallest c.l.s. such that $(Q')_n (Q'')_n \subset (Q'Q'')_n$. By definition, $Q'Q''$ is the *product* of Q' and Q''. The definition implies that the operation of product is associative and commutative.

2.10. Zhilinskii's classification of c.l.s.

We set $V = \varinjlim V(n)$, where $V(n)$ is the natural $\mathfrak{g}(n)$-module, and we set $(V)_* = \varinjlim V(n)^*$ where $V(n)^*$ is the conatural $\mathfrak{g}(n)$-module. We denote by $S^\bullet(K)$ and $\Lambda^\bullet(K)$ the symmetric algebra and the exterior algebra of a module K respectively. Also $S^p(K)$ and $\Lambda^p(K)$ denote respectively the pth symmetric power and the pth exterior power of a module K. A simple highest weight $\mathfrak{o}(2n)$-module with highest weight $(\frac{1}{2}\sum_1^{n-1}\varepsilon_i) \pm \frac{1}{2}\varepsilon_n$ is called a *spinor* module.

If K is a $\mathfrak{g}(\infty)$-module, we define the c.l.s. $Q(K)$ for which $Q(K)_n$ is precisely the set of isomorphism classes of all simple constituents of K considered as a $\mathfrak{g}(n)$-module.

For simplicity we will use the following notations:

$$E := Q(\Lambda^\bullet V),\ L_p := Q(\Lambda^p V),\ L_p^\infty := Q(S^\bullet(V \otimes \mathbb{C}^p)),$$

$$R_p := Q(\Lambda^p (V)_*),\ R_p^\infty := Q(S^\bullet((V)_*) \otimes \mathbb{C}^p), R := \{\text{spinor modules}\},$$

$E^\infty := \{$all irreducible modules, which highest weight consists integral entries $\}$

where $p, q \in \mathbb{Z}_{\geq 1}$. Moreover, the following table defines the *basic c.l.s.* for the Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$.

Lie algebra	Basic c.l.s.
$\mathfrak{sl}(\infty)$	E, L_p , L_p^∞, R_p, R_p^∞, E^∞
$\mathfrak{o}(\infty)$	E, L_p , L_p^∞, E^∞, R
$\mathfrak{sp}(\infty)$	E, L_p , L_p^∞, E^∞

By definition the *trivial c.l.s.* is the *c.l.s.* Q such that $Q_n = \{\mathbb{C}\}$, where $g \cdot \mathbb{C} = 0$ for any $g \in \mathfrak{g}(n)$.

Proposition 2.8. [Zh1] *Any irreducible c.l.s. can be uniquely expressed as a product of basic c.l.s. as follows:*

$$(L_v^\infty L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}}) E^m (R_w^\infty R_{w+1}^{z_{w+1}} R_{w+2}^{z_{w+2}} \dots R_{w+t}^{z_{w+t}})$$

for $\mathfrak{g}(\infty) = \mathfrak{sl}(\infty)$,

$$(L_v^\infty L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}}) E^m \text{ or } (L_v^\infty L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}}) E^m R$$

for $\mathfrak{g}(\infty) = \mathfrak{o}(\infty)$,

$$(L_v^\infty L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}}) E^m$$

for $\mathfrak{g}(\infty) = \mathfrak{sp}(\infty)$, *where*

$$m, r, v, w \in \mathbb{Z}_{\geq 0},$$

$$x_i, z_j \in \mathbb{Z}_{\geq 0} \text{ for } v+1 \leq i \leq n \text{ and } w+1 \leq j \leq t.$$

Here, for $v = 0$, L_v^∞ *is assumed to be trivial c.l.s., and, similarly,* R_w^∞ *is assumed to be trivial c.l.s. for* $w = 0$.

2.11. Tensor product of c.l.s.

Here we reformulate the expression of Proposition 2.8 in terms of tensor products.

Definition 2.30. *Let* $S_1, S_2 \subset Irr_n$ *then*

$$S_1 \otimes S_2 := \{z \in Irr_n \mid \mathrm{Hom}_{\mathfrak{g}(n)}(z, z_1 \otimes z_2) \neq \{0\} \text{for some} \, z_1 \in S_1 \, \text{and} \, z_2 \in S_2\}.$$

Given two c.l.s. Q' *and* Q'', *their* tensor product *is the c.l.s. defined by* $(Q' \otimes Q'')_i = Q'_i \otimes Q''_i$.

One can check that

$$(L_v^\infty L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}}) E^m (R_w^\infty R_{w+1}^{z_{w+1}} R_{w+2}^{z_{w+2}} \dots R_{w+t}^{z_{w+t}}) =$$

$$(L_1^\infty)^{\otimes v} \otimes (R_1^\infty)^{\otimes w} \otimes ((L_1^{x_{v+1}} L_2^{x_{v+2}} \dots L_r^{x_{v+r}}) E^m (R_1^{z_{w+1}} R_2^{z_{w+2}} \dots R_t^{z_{w+t}}))$$

for $\mathfrak{g}(\infty) = \mathfrak{sl}(\infty)$,

$$(L_v^\infty L_{v+1}^{x_{v+1}} L_{v+2}^{x_v+2} \dots L_{v+r}^{x_{v+r}}) E^m = (L_1^\infty)^{\otimes v} \otimes (L_1^{x_{v+1}} L_2^{x_{v+2}} \dots L_r^{x_{v+r}}) E^m$$

for $\mathfrak{g}(\infty) = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$, and

$$(L_v^\infty L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}}) E^m R = (L_1^\infty)^{\otimes v} \otimes (L_1^{x_{v+1}} L_2^{x_{v+2}} \dots L_r^{x_{v+r}}) E^m R$$

for $\mathfrak{g}(\infty) = \mathfrak{o}(\infty)$.

We will call an irreducible c.l.s. Q for $\mathfrak{sl}(\infty)$ a *left irreducible c.l.s.* if

$$Q = (L_1^\infty)^{\otimes v} \otimes ((L_1^{x_{v+1}} L_2^{x_{v+2}} \dots L_r^{x_{v+r}}) E^m (R_1^{z_{w+1}} R_2^{z_{w+2}} \dots R_t^{z_{w+t}})).$$

Proposition 2.9. [PP1] *Let $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$. An integrable ideal of $U(\mathfrak{g}(\infty))$ is prime if and only if it is primitive.*

For any integrable ideal $I \subset U(\mathfrak{g}(\infty))$, let

$$Q(I)_n := \{z \in Irr_n \mid I \cap U(\mathfrak{g}(n)) \subset \operatorname{Ann}_{U(\mathfrak{g}(n))} z\}.$$

The collection $\{Q(I)_n\}$ is a well-defined c.l.s., which we denote by $Q(I)$.

For $\mathfrak{g}(\infty) = \mathfrak{sl}(\infty)$ we denote by $Q_l(I)$ the union of all irreducible components of $Q(I)$ which are left irreducible c.l.s.

Theorem 2.10. [PP1]

1) If $\mathfrak{g}(\infty) = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$, then the maps

$$I \longrightarrow Q(I),$$

$$Q \longrightarrow I(Q)$$

are mutually inverse bijections (which reverse the inclusion relation) between the set of integrable ideals in $U(\mathfrak{g}(\infty))$ and the set of c.l.s. for $\mathfrak{g}(\infty)$.

2) In case $\mathfrak{g}(\infty) = \mathfrak{sl}(\infty)$, then the maps

$$I \longrightarrow Q_l(I)$$

$$Q \longrightarrow I(Q)$$

are mutually inverse bijection (which reverse the relation of inclusion) between the set of prime ideals in $U(\mathfrak{g}(\infty))$ and the set of left irreducible c.l.s. for $\mathfrak{g}(\infty)$.

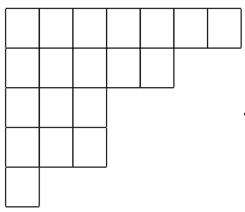
3) Each integrable ideal of $U(\mathfrak{sl}(\infty))$ has the form $I(Q)$ for some left c.l.s. Q.

2.12. Robinson–Schensted algorithm

A *partition* λ of an integer $n \in \mathbb{Z}_{\geq 0}$ is a nonincreasing finite sequence $\lambda_1 \geq \lambda_2 \geq \dots$ of positive integers, whose sum $|\lambda| = \Sigma\lambda_i$ equals n. The terms λ_i of this sequence are called *the parts* of the partition λ. Let P_n be the (obviously finite) set of all partitions of n, and P be the union of all P_n for $n \in \mathbb{Z}_{\geq 0}$.

To each $\lambda \in P_n$ one can associate a subset of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, called *Young diagram* $Y(\lambda)$; it is defined by $(i, j) \in Y(\lambda) \Longleftrightarrow j < \lambda_i$ (so that $\#Y(\lambda) = |\lambda|$). The elements of a Young diagram will be called boxes, and we may correspondingly depict the Young diagram as rows of boxes of respective lengths λ_i, aligned by their left ends, the row of length λ_i lying higher then the row of length λ_j for all $i < j$.

As an example we consider the partition $\lambda = (7, 5, 3, 3, 1)$ in P_{19}, and draw its Young diagram

$Y(\lambda) =$  .

Clearly, a partition $\lambda \in P$ is determined by $Y(\lambda)$. The principal reason for referring to the elements of a Young diagram $Y(\lambda)$ as boxes (rather than as points), is that it allows one to represent maps $f : Y(\lambda) \to \mathbb{Z}$ by filling each box $s \in Y(\lambda)$ with the number $f(s)$. We shall call such a filled Young diagram a *standard Young tableau* (or simply a standard tableau) *of shape* λ if it satisfies the following condition: all numbers $f(s)$ strictly decrease along each row and weakly decrease along each column.

Let's describe the Robinson–Schensted (or Robinson–Schensted–Knuth) algorithm. It starts from an ordered set $d = \{d_i\}$ positive integer, where $1 \le i \le n$, and produces as output two Young tableaux: the insertion tableau Y and the recording tableau Y'. This algorithm is based on a procedure of inserting a new positive integer into a Young tableau, displacing certain entries, and creating a tableau with one more box than the original one.

The starting Young tableaux $Y_0 = Y'_0 := \{\varnothing\}$ are empty. We set the counter of steps s to be equal to 1. From this moment, we will perform subsequent steps until we reach $s = n + 1$. The algorithm is as follows.

1) If the current step is $s = n + 1$, then we finish the algorithm. If $s < n$, we name $e := d_s$ *the current number*. Furthermore, we name the first row the *current row* and assign $r := 1$ (r is the number of the current row).

2) Find the leftmost number l which is less or equal than the current number in the current row. If such l exists, then go to step (3). If there is no such an element, then add a box filled by e to the end of the current row of Y_{s-1}, denote this new Young tableau by Y_s and add $n - s + 1$ to the end of the current row of Y'_{s-1}. Set $s := s + 1$. Return to step (1).

3) Change l in Y_{s-1} by the current number e, assign $e := l$, and change the current row to the next row (even if it the latter empty) by putting $r := r + 1$. Return to step (2).

The Young tableaux Y_n and Y'_n obtained at the last step constitute the output of the Robinson–Schensted algorithm.

Here is an example. Set $\{d_1 = 5, d_2 = 1, d_3 = 3, d_4 = 2, d_5 = 3, d_6 = 6, d_7 = 4\}$. We start with current step $s = 0$, current row $r = 1$, current number $e = d_1 = 5$, and $Y_0 = Y'_0 = \varnothing$.

Below we list all Y_i and Y'_i, which we obtain in the course of the Robinson–Schensted algorithm.

$$Y_0 = \varnothing,\ Y'_0 = \varnothing,$$

$$Y_1 = \boxed{5},\ Y'_1 = \boxed{7},$$

$$Y_2 = \boxed{5}\boxed{1},\ Y'_2 = \boxed{7}\boxed{6},$$

$$Y_3 = \begin{ytableau} 5 & 3 \\ 1 \end{ytableau}, \quad Y_3' = \begin{ytableau} 7 & 6 \\ 5 \end{ytableau},$$

$$Y_4 = \begin{ytableau} 5 & 3 & 2 \\ 1 \end{ytableau}, \quad Y_4' = \begin{ytableau} 7 & 6 & 4 \\ 5 \end{ytableau},$$

$$Y_5 = \begin{ytableau} 5 & 3 & 2 \\ 3 \\ 1 \end{ytableau}, \quad Y_5' = \begin{ytableau} 7 & 6 & 4 \\ 5 \\ 3 \end{ytableau},$$

$$Y_6 = \begin{ytableau} 6 & 3 & 2 \\ 5 \\ 3 \\ 1 \end{ytableau}, \quad Y_6' = \begin{ytableau} 7 & 6 & 4 \\ 5 \\ 3 \\ 2 \end{ytableau},$$

$$Y_7 = \begin{ytableau} 6 & 4 & 2 \\ 5 & 3 \\ 3 \\ 1 \end{ytableau}, \quad Y_7' = \begin{ytableau} 7 & 6 & 4 \\ 5 & 1 \\ 3 \\ 2 \end{ytableau}.$$

We also can apply the Robinson–Schensted algorithm to the elements of the permutation group S_n. Let

$$\delta = \begin{pmatrix} 1 & 2 & \dots & n \\ \delta(1) & \delta(2) & \dots & \delta(n) \end{pmatrix}$$

be a permutation. Then we apply the Robinson–Schensted algorithm to the sequence

$$\{\delta(1), \delta(2), \dots, \delta(n)\}.$$

We denote the output Young tableaux by $Y(\delta)$ (insertion tableau) and $Y'(\delta)$ (recording tableau).

3. Primitive ideals of $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$

Our main result in this section is that every primitive ideal of $U(\mathfrak{o}(\infty))$ or $U(\mathfrak{sp}(\infty))$ is weakly bounded. This implies that, every primitive ideal of $U(\mathfrak{o}(\infty))$ is a locally integrable.

Let U stand for $U(\mathfrak{o}(\infty))$ or $U(\mathfrak{sp}(\infty))$, $\mathfrak{g}(2n)$ stand for $\mathfrak{o}(2n)$ or $\mathfrak{sp}(2n)$, and $W(2n)$ stand for the Weyl group of $\mathfrak{g}(2n)$. From now on, we slightly change the notation: we will denote $\mathfrak{o}(2n)$ and $\mathfrak{sp}(2n)$ by $\mathfrak{g}(2n)$, while before we used the notation $\mathfrak{g}(n)$. This is needed in order to simplify some formulas.

3.1. Symbols

Define a *symbol of type* C_n to be a collection of nonnegative integers

$$\Lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_1, & \dots, & \alpha_{m+1} \\ \beta_1, & \dots, & \beta_m \end{pmatrix},$$

such that $\alpha_i < \alpha_{i+1}$, $\beta_i < \beta_{i+1}$ and $\sum\limits_{i=1}^{m+1} \alpha_i + \sum\limits_{i=1}^{m} \beta_i = n + m^2$. We consider the following equivalence relation on the set of symbols of type C_n

$$\begin{pmatrix} \alpha_1, & \dots, & \alpha_{m+1} \\ \beta_1, & \dots, & \beta_m \end{pmatrix} \sim \begin{pmatrix} 0, & \alpha_1+1, & \dots, & \alpha_{m+1}+1 \\ 0, & \beta_1+1, & \dots, & \beta_m+1 \end{pmatrix}.$$

If $\alpha_i \leq \beta_i \leq \alpha_{i+1}$ for $1 \leq i \leq m$, then the symbol Λ is *special.* The set of special symbols in a natural one-to-one correspondence with the set of nilpotent orbits of $\mathfrak{sp}(2n)$ (see [BV]). Take the set $\{2\alpha_i, 2\beta_j + 1\}$ for $1 \leq i \leq m+1$, $1 \leq j \leq m$, order its elements in increasing order, and denote it by $\{\nu_j\}_{j=1}^{2m+1}$. Then $\nu_C(\Lambda) := \{\nu_j - j + 1\}$ is a partition of $2n$.

Define a *symbol of type* D_n as a collection of nonnegative integers

$$\Lambda = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_1, & \dots, & \alpha_m \\ \beta_1, & \dots, & \beta_m \end{pmatrix},$$

such that $\alpha_i < \alpha_{i+1}$, $\beta_i < \beta_{i+1}$ and $\sum\limits_{i=1}^{m} \alpha_i + \sum\limits_{i=1}^{m} \beta_i = n + m(m-1)$. Introduce the equivalence relation on the set of symbols of type D_n :

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \sim \begin{pmatrix} \beta \\ \alpha \end{pmatrix},$$

$$\begin{pmatrix} \alpha_1, & \dots, & \alpha_m \\ \beta_1, & \dots, & \beta_m \end{pmatrix} \sim \begin{pmatrix} 0, & \alpha_1+1, & \dots, & \alpha_m+1 \\ 0, & \beta_1+1, & \dots, & \beta_m+1 \end{pmatrix}.$$

If $\beta_i \le \alpha_i \le \beta_{i+1}$ or $\alpha_i \le \beta_i \le \alpha_{i+1}$ for $1 \le i \le m-1$, and respectively also $\beta_i \le \alpha_i$ or $\alpha_i \le \beta_i$, then we call the symbol Λ *special*. The set of special symbols in one-to-one correspondence with the set of nilpotent orbits of $\mathfrak{o}(2n)$ [BV]. Take the set $\{2\alpha_i+1, 2\beta_i\}$ for all i, order its elements in increasing order, and denote it by $\{\nu_j\}_{j=1}^{2m}$. Then $\nu_D(\Lambda) := \{\nu_j - j + 1\}$ is a partition of $2n$.

3.2. Primitive ideals of $U(\mathfrak{o}(2n))$ and $U(\mathfrak{sp}(2n))$

In this section we recall the classification of primitive ideals of $U(\mathfrak{o}(2n))$ and $U(\mathfrak{sp}(2n))$.

Let $L(\lambda)$ be the unique irreducible $\mathfrak{g}(2n)$-module with highest weight λ. By $W(2n)$ and $\Delta(2n)$ we denote the Weyl group and the root system of $\mathfrak{g}(2n)$ respectively. We fix a set of positive roots $\Delta(2n)^+$ as in Subsection 2.2. Then we denote $I(\lambda) = \mathrm{Ann}(L(\lambda - \rho_{\mathfrak{g}(2n)}))$. Recall that our notation for the Killing form is $(\cdot,\cdot)$.

Let w be an element of $W(2n)$. Recall of the Young tableau $Y(w)$ from Subsection 2.12. Let $q_1, q_2, \dots, q_s$ be the lengths of the rows of $Y(w)$. Note that the Young tableaux $Y(w)$ and $Y(w^{-1})$ have the same shape. We consider the set $\{q_i\}$ as a partition $p(w)$ of $2n$.

Proposition 3.1. [BV, Proposition 17] *Given $w \in W(2n)$, there exists a unique symbol $\Lambda = \Lambda(w)$ such that $p(w) = \nu_D(\Lambda)$ (respectively, $\nu_C(\Lambda)$) for $\mathfrak{g}(2n) = \mathfrak{o}(2n)$ (respectively, $\mathfrak{sp}(2n)$).*

Let us describe the construction of $\Lambda(w)$. For $\mathfrak{g}(2n) = \mathfrak{sp}(2n)$, we consider the set $\{q_1, q_2, \dots, q_s\}$ for odd s, and the set $\{0, q_1, q_2, \dots, q_s\}$ for even s. We put $\mu_i := q_i + i - 1$ and split the set $\{\mu_i\}$ into two subsets: the set $\{\bar\alpha_j\}$ of even μ_i's and the set $\{\bar\beta_j\}$ of odd μ_i's. The symbol $\Lambda(w)$ is then defined as

$$\begin{pmatrix} \bar\alpha_1/2 & \dots & \bar\alpha_{[s/2]+1}/2 \\ (\bar\beta_1 - 1)/2 & \dots & (\bar\beta_{[s/2]} - 1)/2 \end{pmatrix}.$$

For $\mathfrak{g}(2n) = \mathfrak{o}(2n)$, we consider the set $\{q_1, q_2, \dots, q_s\}$ for even s, and the set $\{0, q_1, q_2, \dots, q_s\}$ for odd s. We let $\mu_i := q_i + i - 1$ and split the set $\{\mu_i\}$ into two subsets: the set $\{\bar\alpha_j\}$ of even μ_i's and the set $\{\bar\beta_j\}$ of odd μ_i's. The symbol $\Lambda(w)$ is then defined as

$$\Lambda(w) = \begin{pmatrix} (\bar\alpha_1 - 1)/2 & \dots & (\bar\alpha_{[s+1/2]} - 1)/2 \\ \bar\beta_1/2 & \dots & \bar\beta_{[s+1/2]}/2 \end{pmatrix}.$$

In what follows we put $\nu_{C,D}(w) := \nu_{C,D}(\Lambda(w))$.

From now on, we fix λ with the property that $-\lambda$ is dominant. Then we put

$$\Delta_\lambda := \{\alpha \in \Delta(2n) \mid \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \in \mathbb{Z}\},$$

$$\Delta_\lambda^+ := \Delta(2n)^+ \cap \Delta_\lambda,$$

$$W_\lambda := W(\Delta_\lambda) \subseteq W(2n).$$

One can show that $W_\lambda = W_{[\lambda]}$.

Let $\lambda = \lambda_{-n}\bar{\varepsilon}_{-n} + \cdots + \lambda_{-1}\bar{\varepsilon}_{-1} + \lambda_1\bar{\varepsilon}_1 + \cdots + \lambda_n\bar{\varepsilon}_n$ (where $\lambda_{-k} = -\lambda_k$) be a weight of $\mathfrak{g}(2n)$. We now introduce an equivalence relation on the set of indices

$$[\pm n] := \{-n, -n+1, \ldots, n-1, n\}.$$

By definition two indices i and j are equivalent if $\lambda_i - \lambda_j \in \mathbb{Z}$. Denote by $E_1$the equivalence class of indices of $\lambda_i \in \mathbb{Z}$, by E_2 the equivalence class of indices of $\lambda_i \in \mathbb{Z}+1/2$, and by $E_3, E_4, \ldots$ all other equivalence classes. Note that the classes E_i are invariant under the action of W_λ. We can represent W_λ as the direct product $W_1 \times W_2 \times \cdots \times W_s$, where

$$W_i = \{w \in W_\lambda \mid \; w|_{[\pm n]\setminus E_i} = \mathrm{id}\}.$$

Let Δ_i be the root subsystem of Δ_λ, which corresponds to the subgroup W_i and $\Delta_i^+ = \Delta^i \cap \Delta_\lambda^+$. Then each element $w \in W_\lambda$ can be uniquely expressed as $w = w_1 w_2 \ldots w_s$ where $w_i \in W_i$.

We define the *symbol* $\Lambda^\lambda(w)$ of an element $w = w_1 w_2 \ldots w_s \in W_\lambda$ to be the pair of symbols $\Lambda^\lambda(w) = (\Lambda(w_1), \Lambda(w_2))$ We call $\Lambda^\lambda(w)$ *special* if both symbols $\Lambda(w_1)$ and $\Lambda(w_2)$ are special. If

$$\Lambda(w) = \begin{pmatrix} \alpha_1, & \ldots, & \alpha_s \\ \beta_1, & \ldots, & \beta_m \end{pmatrix}, \; \Lambda(w') = \begin{pmatrix} \alpha'_1, & \ldots, & \alpha'_{s'} \\ \beta'_1, & \ldots, & \beta'_{m'} \end{pmatrix}$$

are symbols (where $s = m+1$ or $s = m$ and $s' = m'+1$ or $s' = m'$), then we say that $\Lambda(w')$ is a *permutation* of $\Lambda(w)$ if the sets $\{\alpha_1, \alpha_2, \ldots \alpha_s, \beta_1, \beta_2, \ldots, \beta_m\}$ and $\{\alpha'_1, \alpha'_2, \ldots \alpha'_{s'}, \beta'_1, \beta'_2, \ldots, \beta'_{m'}\}$ coincide.

Recall that the dot action of $W(2n)$ on $\mathfrak{h}^*_{\mathfrak{g}(2n)}$ is defined in Subsection 2.4 by setting $w \cdot \lambda = w(\lambda + \rho_{\mathfrak{g}(2n)}) - \rho_{\mathfrak{g}(2n)}$. For an element $w \in W(2n)$, we denote $I(w) := I(w \cdot \lambda)$. Two elements w_1 and w_2 of $W(2n)$ are called *equivalent*, written $w_1 \sim w_2$, if $I(w_1) = I(w_2)$.

Theorem 3.2. [BV, Theorem 18] *Let $\Delta(2n)$ be the root system of type D_n or C_n, $W(2n)$ be the Weyl group of $\Delta(2n)$, and w, w_1, w_2 be elements of $W(2n)$. Then the following holds.*

1) The elements w_1 and w_2 have the same tableaux $Y(w_1) = Y(w_2)$ if and only if $w_1 \sim w_2$.

2) There exists $w' \in W(2n)$ such that $w' \sim w$ and the symbol $\Lambda(w')$ of w' is special and is a permutation of $\Lambda(w)$.

Let Σ_λ be the set of simple roots in Δ_λ^+, and let $w \in W_\lambda$. Put

$$S_\lambda(w) := \{\alpha \in \Delta_\lambda^+ \mid w \cdot \alpha \notin \Delta_\lambda^+\},$$
$$\tau_\lambda(w) := S_\lambda(w) \cap \Sigma_\lambda.$$

Proposition 3.3. [J2] *Let $\alpha \in \Sigma_\lambda$ and $w \in W_\lambda$. Suppose $\alpha \in \tau_\lambda(w^{-1})$ is such that $\tau_\lambda(w^{-1}s_\alpha) \nsubseteq \tau_\lambda(w^{-1})$, where s_α is the reflection corresponding to the root α. Then*

$$I(s_\alpha w) = I(w).$$

We should also note that, for $\mathfrak{g}(2n) = \mathfrak{sp}(2n)$, W_1 is a Weyl group of type C, W_2 is a Weyl group of type D, and W_i for $i \neq 1, 2$ is of type A. For the case $\mathfrak{g}(2n) = \mathfrak{o}(2n)$, W_1 and W_2 are Weyl groups of type D, and W_i is of type A for $i \neq 1, 2$.

Corollary 3.4. *Let W_j be of type D_n or C_n and let $s_\alpha = s_{-i,-i+1} \in W_j$ be the simple reflection corresponding to a root $\alpha = \varepsilon_{-i} - \varepsilon_{-i+1}$, for $-n+1 \leq i \leq -1$. Denote $v = w^{-1}$ for $w \in W_j$. Then*

$$I(s_{-i,-i+1}w) = I(w)$$

whenever one of the following inequalities holds

1) $v(i-1) > v(i+1) > v(i) > 0$ where $-n < i < -1$,

2) $v(i) > v(i+1) > v(i-1) > 0$ where $-n < i < -1$,

3) $v(i-1) > v(i-2) > v(i) > 0$ where $-n+1 < i < 0$,

4) $v(i) > v(i-2) > v(i-1) > 0$ where $-n+1 < i < 0$,

5) $v(i) > 0, v(i-1) < 0$ and $v(i-1) > v(i+1)$ where $-n < i < -1$,

6) $v(i) < 0, v(i-1) > 0$ and $v(i+1) > v(i)$ where $-n < i < -1$,

7) $v(i) > 0, v(i-1) < 0$ and $v(i-2) > v(i)$ where $-n+1 < i < -1$,

8) $v(i) < 0, v(i-1) > 0$ and $v(i-2) > v(i-1)$ where $-n+1 < i < -1$.

Proof. Recall the choice of the sets of positive and simple roots from Subsection 2.2. We will argue simultaneously in both cases D_n and C_n. This is possible because in the proof we only use short roots.

Assume $\varepsilon_{-i} - \varepsilon_{-i+1} \in \tau_\lambda(v)$, $v(\varepsilon_{-i} - \varepsilon_{-i+1}) \notin \Delta_\lambda^+$. This implies exactly one of the three inequalities:

a) $v(i-1) > v(i) > 0$,

b) $v(i) < v(i-1) < 0$,

c) $v(i) > 0 > v(i-1)$.

Clearly, we have $I(s_{-i,-i+1}w) = I(w)$ if $\tau_\lambda(vs_\alpha) \not\subseteq \tau_\lambda(v)$, i.e., if there exists a simple root $\beta = \varepsilon_{-j} - \varepsilon_{-j+1}$ such that $vs_\alpha \cdot \beta \notin \Delta_\lambda^+$ and $v \cdot \beta \in \Delta_\lambda^+$.

First, assume that v satisfies inequality 1), and hence also inequality a). We have,

$$v \cdot (\varepsilon_{-i-1} - \varepsilon_{-i}) = -\varepsilon_{v(i+1)} + \varepsilon_{v(i)} \in \Delta_\lambda^+,$$

because $v(i-1) > v(i-2) > 0$, and

$$vs_{-i,-i+1} \cdot (\varepsilon_{-i-1} - \varepsilon_{-i}) = v \cdot (\varepsilon_{-i-1} - \varepsilon_{-i+1}) = -\varepsilon_{v(i+1)} + \varepsilon_{v(i-1)} \notin \Delta_\lambda^+,$$

because $v(i-1) > v(i+1) > 0$.

Next, assume that v satisfies inequality 3), and hence also inequality a). Then,

$$v \cdot (\varepsilon_{-i+1} - \varepsilon_{-i+2}) = -\varepsilon_{-v(i-1)} + \varepsilon_{v(i-2)} \in \Delta_\lambda^+,$$

because $v(i-2) > v(i) > 0$, and

$$vs_{-i,-i+1} \cdot (\varepsilon_{-i+1} - \varepsilon_{-i+2}) = v \cdot (\varepsilon_{-i} - \varepsilon_{-i+2}) = -\varepsilon_{v(i)} + \varepsilon_{v(i-2)} \notin \Delta_\lambda^+,$$

because $v(i-2) > v(i) > 0$.

Now, assume that v satisfies inequality 5), hence also inequality c). In this case,

$$v \cdot (\varepsilon_{-i-1} - \varepsilon_{-i}) = \varepsilon_{-v(i+1)} + \varepsilon_{v(i)} \in \Delta_\lambda^+,$$

because $v(i) > 0 > v(i+1)$, and

$$vs_{-i,-i+1} \cdot (\varepsilon_{-i-1} - \varepsilon_{-i}) = v \cdot (\varepsilon_{-i-1} - \varepsilon_{-i+1}) = \varepsilon_{-v(i+1)} - \varepsilon_{-v(i-1)} \notin \Delta_\lambda^+,$$

because $v(i+1) < v(i-1) < 0$.

Finally, assume that v satisfies inequality 7), hence also inequality c). Then,

$$v \cdot (\varepsilon_{-i+1} - \varepsilon_{-i+2}) = \varepsilon_{-v(i-1)} + \varepsilon_{v(i-2)} \in \Delta_\lambda^+,$$

because $v(i-2) > 0 > v(i-1)$, and

$$vs_{i,i+1} \cdot (\varepsilon_{-i+1} - \varepsilon_{-i+2}) = v \cdot (\varepsilon_{-i} - \varepsilon_{-i+2}) = -\varepsilon_{v(i)} + \varepsilon_{v(i+2)} \notin \Delta^+_\lambda,$$

because $v(i+2) > 0 > v(i)$.

Thus we proved the corollary for the inequalities $1), 3), 5), 7)$. Note that if $I(s_{i,i+1}w) = I(w)$ then $I(s^2_{i,i+1}w) = I(s_{i,i+1}w)$. Hence, the element w satisfies inequality $2), 4), 6$ or $8)$ if and only of the element $ws_{i,i+1}$ satisfies inequality $1), 3), 5), 7)$ respectively. The proof is complete.

□

3.3. Primitive ideals of $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$

In this section we show that every primitive ideal of $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$ is weakly bounded.

Let $\mathfrak{g}(\infty)$ be equal to $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$, and $\mathfrak{g}(2n)$ be the Lie algebra $\mathfrak{o}(2n)$ or $\mathfrak{sp}(2n)$ respectively. As usual, denote by $\mathrm{SL}(2n, \mathbb{C})$ the group of $2n \times 2n$ complex matrices with determinant equal to 1. Recall that $\mathfrak{g}(2n) = \{X \in \mathfrak{gl}(2n) \mid XF + FX^t = 0\}$, where F is defined in Subsection 2.3. Put

$$G(2n) = \{g \in SL(2n, \mathbb{C}) \mid g^t F g = F\}.$$

Then $\mathfrak{g}(2n)$ is the Lie algebra of the Lie group $G(2n)$.

Set $U := U(\mathfrak{g})$ and $U_{2n} := U(\mathfrak{g}(2n))$.

Our goal in this subsection is to prove the following proposition.

Proposition 3.5. *Let I be an ideal of U, and let $I_{2n} = I \cap U_{2n}$. Then there exists $r \in \mathbb{Z}_{>0}$ such that, for $n \gg 0$ (i.e, for each sufficiently large n) the intersection $J(2n) \cap U_{2f(2n)}$ for an arbitrary primitive ideal $J(2n)$ containing I_{2n}, is a bounded ideal of $U_{2f(2n)}$ where $f(2n) = \left[\frac{n-3r/2}{r+1}\right] - r/2$.*

For the proof of Proposition 3.5 we need to discuss some facts related to the associated variety of a primitive ideal defined in Subsection 2.18.

It is clear that

$$\text{if } J_1 \subset J_2 \text{ then } \mathrm{Var}(J_2) \subset \mathrm{Var}(J_1).$$

If I is an ideal of U, then the intersections $I_{2n} = I \cap U_{2n}$ determine a sequence of $G(2n)$-stable varieties $\mathrm{Var}(I_{2n}) \subset \mathfrak{g}(2n)^*$, and we have

$$\phi_{2m,2n}(\mathrm{Var}(I_{2m})) \subset \mathrm{Var}(I_{2n}) \tag{6}$$

for $m \geq n$, where the map $\phi_{2m,2n} \colon \mathfrak{g}(2m)^* \to \mathfrak{g}(2n)^*$ is induced by the natural inclusion $\mathfrak{g}(2n) \hookrightarrow \mathfrak{g}(2m)$.

For any $n \geq 2$ and any $r' \in \mathbb{Z}_{\geq 0}$ we put

$$\mathfrak{g}(2n)^{\leq r'} := \{x \in \mathfrak{g}(2n) \mid \mathrm{rk}(x) \leq r'\},$$

where rk refers to the rank of a matrix. We identify $\mathfrak{g}(2n)$ and $\mathfrak{g}(2n)^*$ via the Killing form, and so we consider $\mathfrak{g}(2n)^{\leq r'}$ as a subset of $\mathfrak{g}(2n)^*$.

Lemma 3.6. *Let I be a nonzero ideal of U. Then there exists $r \in \mathbb{Z}_{\geq 0}$ such that*

$$\mathrm{Var}(I_{2n}) \subset (\mathfrak{g}(2n))^{\leq r}$$

for all $n \gg 0$.

Proof. If I is nonzero then $\mathrm{Var}(I_{2m}) \neq \mathfrak{g}(2m)^*$ for some $m \geq 2$. For every $n \geq m$ and every $X \in \mathrm{Var}(I_{2m})$, formula (6) shows that

$$\phi_{2n,2m}(G(2n) \cdot X) \subset \mathrm{Var}(I_{2m}) \neq \mathfrak{g}(2m)^*,$$

where $G(2n) \cdot X$ is the coadjoint orbit of X in $\mathfrak{g}(2n)^*$. Hence $\phi_{2n,2m}(G(2n) \cdot X)$ is not dense in $\mathfrak{g}(2n)^*$. This, together with [PP2, Lemma 4.12], implies the required result for $r = m$ under the assumption that $n > 3m$. $\square$

Further, without loss of generality, we assume that the number r is even (we reassign $r := r + 1$ in the case of odd r).

A well-known theorem of A. Joseph [J1] implies that the associated variety of a primitive ideal $J(2n) \subset U_{2n}$ equals the closure of a nilpotent coadjoint orbit. The natural inclusion $\mathfrak{g}(2n) \hookrightarrow \mathfrak{gl}(2n)$ induces the surjection $\mathfrak{gl}^*(2n) \twoheadrightarrow \mathfrak{g}^*(2n)$. The conjugacy classes of nilpotent $(2n \times 2n)$-matrices surject naturally to the nilpotent coadjoint orbits of $\mathfrak{g}(2n)$. Moreover, these conjugacy classes are related to partitions of $2n$: the partition attached to a conjugacy class comes from the Jordan normal form of a representative of this class. In this way we attach the partition $s(J(2n))$ of $2n$ to $J(2n)$. By $p(J(2n))$ we denote the partition conjugate to that partition. Let $r(J(2n))$ be the difference between $2n$ and the maximal element of $p(J(2n))$. It is easy to check that $r(2n) \colon = r(J(2n))$ equals the rank of an arbitrary element in the orbit defined by the partition $p(2n) := p(J(2n))$. Note that, given $w \in W$ we have $\mu_{C,D}(\Lambda(w)) = p(I(w))$ if $\Lambda(w)$ is special, [BV, Theorem 7].

Lemma 3.7. *Let $X \in \mathfrak{g}(2n)^{\leq r}$ be a nilpotent matrix and $p(2n)$ be the partition attached to the conjugacy class of X. Then $r(2n) \leq r$.*

Proof. We have $\mathrm{rk}(X) \leq r$. Then $r(2n) = \mathrm{rk} X \leq r$ $\square$

Let $J(2n) = \mathrm{Ann}(L(\lambda' - \rho_{\mathfrak{g}(2n)}))$ be a primitive ideal as in Proposition 3.5 for a weight $\lambda' \in \mathfrak{h}^*_{\mathfrak{g}(2n)}$. Then there exists $w \in W$ such that $\lambda = w^{-1}\lambda'$ and $-\lambda$ is a dominant weight. Decompose the subgroup $W_\lambda \simeq W_1 \times W_2 \times \cdots \times W_s$ as in Subsection 3.2. In what follows, given an element $w' \in W_\lambda$, we write $w' = w'_1 w'_2 \dots w'_s$ for $w'_i \in W_i$. In particular, $w = w_1 w_2 \dots w_s$ for $w_i \in W_i$. Let $Y(w_i)$ be the Young tableaux defined in Subsection 2.8. Lemmas 3.6, 3.5 and Theorem 3.2 allow us to conclude that $2n - r$ is the number of Jordan blocks of the conjugacy class corresponding to the nilpotent orbit whose closure is the associated variety of $I(\lambda)$. Next, let l_i be the length of the longest row of $Y(w_i)$. Then $\max\{l_1, \dots, l_s\} = 2n - r$, and one can easily see that in fact $2n - r$ equals l_1 or l_2.

Choose an element $w' = w'_1 w'_2 \dots w'_s$ such that $w \sim w'$ and the symbol $\Lambda^\lambda(w')$ is special (i.e., $\Lambda(w'_1)$ and $\Lambda(w'_2)$ are special). Since $w \sim w'$, we have

$$l := \max\{l_1, \dots, l_s\} = \max\{l'_1, \dots, l'_s\} = 2n - r,$$

where $l'_1, \dots, l'_s$ are defined with respect to the decomposition $w' = w'_1 w'_2 \dots w'_s$. Moreover $l = l'_1$ or $l = l'_2$. Therefore $l'_1 \neq l'_2$ for $2n > 2r$. In what follows we assume that $Y(w'_c)$, for $c = 1$ or 2, has a row of length l . Note that tableaux of types C and D have even number of elements, and let $2h$ be the number of elements of $Y(w'_c)$.

In order to state the next result we need the following definition.

Definition 3.1. *A weight $\gamma = \sum \gamma_i \varepsilon_i \in \mathfrak{h}^*_{\mathfrak{g}(2n)}$ is* half-integral *if $\gamma_i - \gamma_j \in \mathbb{Z}$ and $\gamma_i + \gamma_j \in \mathbb{Z}$ for all i, j.*

Lemma 3.8. *There exist an element $\tilde{w} \in W_\lambda$, satisfying $\tilde{w} \sim w'$, and an integer $k(2n) \in 2\mathbb{Z}_{>0}$ such that, after erasing the first and last $k(2n) + r$ coordinates of $\tilde{w}\lambda$, as well as the $2n - 2k(2n) - 2f(2n) - 2r$ central coordinates, we obtain a half-integral dominant regular weight of $\mathfrak{g}(2f(2n))$ (where $f(2n) = [\frac{n-3r/2}{r+1}] - r/2$ as in Proposition 3.3).*

Proof. According to Theorem 3.2 we can suppose without loss of generality that the symbols $\Lambda(w'_1)$ and $\Lambda(w'_2)$ are special. Recall that the set of indices of the weight λ is $\{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}$. Note that interchanging of coordinates of λ without changing the order within the classes E_i preserves the primitive ideal $I(\lambda)$. Therefore we can assume that the equivalence class E_c has the form

$$\left\{ \frac{-\#E_c}{2}, \frac{-\#E_c}{2} + 1, \dots, -1, 1, \dots, \frac{\#E_c}{2} - 1, \frac{\#E_c}{2} \right\}.$$

Lemma 3.7 implies $\#E_c \geq 2n - r$, hence after erasing the first and last $r/2$ coordinates of λ we obtain a half-integral weight λ'.

Let $r' = 2h - l$. Note that $r' \leq r$. The length of the longest row of $Y(w_c)$ equals the length of the longest decreasing subsequence of the sequence

$$a = (w'_c(-h), w'_c(-h+1), \dots, w'_c(-1), w'_c(1), \dots, w'_c(h-1), w'_c(h)).$$

Set $k = \#\{i \mid w'_c(i) > 0, i > 0\}$. Obviously, the length of the longest decreasing subsequence of a is less or equal to $2h - k$, because a cannot contain both $w'(i)$ and $w'(-i)$ if $i > 0$ and $w'(i) < 0$.

Note that the simple reflections satisfying one of conditions $1) - 4)$ of Corollary 3.4 preserve the shape of the Young tableau $Y(w'_c)$ (see [K]). For each $t \in \mathbb{Z}_{>0}$ and each $w \in W(2t)$, we define the sequence of integers

$$\sigma(w) = \{w(-t), w(-t+1) \dots w(-1), w(1), \dots, w(t-1), w(t)\}.$$

Let s_α be a simple reflection satisfying one of conditions $1) - 4)$ of Corollary 3.4. Then, by the very definition of the transformation s_α, we can change $\sigma(w'_c)$ to $\sigma(s_\alpha w'_c)$ without changing the ideal $I(w')$. Furthermore, $s_\alpha(\sigma(w'_c)) = \sigma(s_\alpha w'_c)$. Denote $h(\delta) := \{\delta(-t), \delta(-t+1), \dots, \delta(-1)\}$ for $\delta \in W(2t)$ or $\delta \in W(2t)$. Observe that $h(w'_c)$ determines $\sigma(\delta)$ since $\delta(j) = -\delta(-j)$.

We call a subsequence A of a finite sequence of numbers $\{a_1, a_2 \dots, a_v\}$ *positive interval* if A has the form $\{a_u, a_{u+1}, \dots a_{w-1}, a_w\}$ for $1 \leq u < w \leq v$ and any $a_j \in A$ is positive. The notation $|A|$ stands for the number of elements of a positive interval A. Note that $\max_{A \subset h(w'_c)} |A| \geq [\frac{n-3r/2}{r+1}]$. Indeed, the sequence $h(w'_c)$ consists of $n - r/2$ elements with no more than r negative elements. Consequently, $h(w'_c)$ contains at least $n - 3/2r$ positive elements and at most $r+1$ positive intervals. Hence, there exists at least one positive interval with at least $[\frac{n-3r/2}{r+1}]$ elements.

Next, we find the leftmost maximal (by inclusion) positive interval A_0 of the sequence $h(w'_c)$ such that $|A_0| \geq [\frac{n-3r/2}{r+1}]$. Then we apply the Robinson–Schensted algorithm to the interval A_0 of $h(w'_c)$, and denote by Y_0 the output insertion Young tableau. Starting from the bottom left corner of Y_0, we place all rows one after another in increasing length order. This transforms the sequence $h(w'_c)$ to a new sequence h', and the interval A_0 to an interval A'_0. As a next step, we express the sequence h' as $h(\tilde{w}_c)$ for some $\tilde{w}_c \in W(2h)$.

Now, we extend $\tilde{w}_c$ to $\tilde{w} \in W(2n)$ by putting $\tilde{w}_i = w'_i$ for all $i \neq c$. Then $I(\tilde{w}) = I(w')$, because $\tilde{w}(w')^{-1}$ equals a product of simple reflections satisfying some condition among $1) - 4)$ (see [K]).

After that, we erase in $h(\tilde{w}_c)$ all elements to the left of A'_0 and denote the number of these elements by $k(2n)$. Next, we erase all elements to the right of A'_0 and denote the number of these elements by $m(2n)$. Note that

$$m(2n) < n - r/2 - k(2n) - [\frac{n - 3r/2}{r+1}] = n - k(2n) - f(2n).$$

In this way we get a sequence which equals A_0'. Since A_0' consists of positive integers or half-integers ordered as described above (A_0' is comprised of the rows of the Young tableau Y_0 in the opposite order), Lemmas 3.6, 3.7 show that after erasing the first $r/2$ elements of A_0' we obtain a strictly decreasing sequence A_0''. This means precisely that after erasing the first and last $k(2n)+r$ coordinates, as well as the $2n-2k(2n)-2f(2n)$ central coordinates of $\tilde{w}\lambda$, we obtain a half-integral dominant regular weight of $\mathfrak{g}(2f(2n))$. $\square$

Let $\tilde{w}$ be as in Lemma 3.8. Denote by $\bar{\lambda}$ the weight obtained from $\tilde{w}\lambda$ via replacing by zeros the first and last $k(2n)+r$ coordinates, as well as the $2n-2k(2n)-2f(2n)$ central coordinates.

Denote by $\mathfrak{g}(\bar{\lambda})$ the root subalgebra of $\mathfrak{g}(2n)$ whose dual Cartan subalgebra $\mathfrak{h}(\bar{\lambda})$ is spanned by all elements ε_i such that $(\bar{\lambda},\varepsilon_i)\neq 0$.

Let I be a primitive ideal of $U(\mathfrak{g}(\infty))$, and $J(2n)$ be as in Proposition 3.5.

Corollary 3.9. *In the notation of Proposition 3.5, $J(2n)\cap U(\mathfrak{g}(\bar{\lambda}))$ is a bounded ideal of $U(\mathfrak{g}(\bar{\lambda}))$. Moreover, this ideal is integrable if $\mathfrak{g}(2n)=\mathfrak{o}(2n)$ or if $\bar{\lambda}_i^{2f(2n)}\in\mathbb{Z}$.*

Proof. By Lemma 3.8 we can construct an element $\tilde{w}\in W_{\mathfrak{g}(2n)}$ satisfying

$$J(2n)=\operatorname{Ann}_{U_{2n}} L(\tilde{w}\lambda-\rho_{\mathfrak{g}(2n)}),$$

and such that after erasing the first and last $k(2n)+r$ coordinates, as well as the $2n-2k(2n)-2f(2n)$ central coordinates of $\tilde{w}\lambda$, we obtain a half-integral dominant weight $\bar{\lambda}^{2f(2n)}$ of $\mathfrak{g}(2f(2n))$ (so that $\bar{\lambda}|_{\mathfrak{h}_{\mathfrak{g}(2f(2n))}}=\lambda^{2f(2n)}$).

The module $L(\bar{\lambda}^{2f(2n)}-\rho_{\mathfrak{g}(2f(2n))})$ is a simple $\mathfrak{g}(2f(2n))$-module with highest weight $\bar{\lambda}^{2f(2n)}-\rho_{\mathfrak{g}(2f(2n))}$. The ideal $J(2n)\cap U(\mathfrak{g}(\bar{\lambda}))$ is a bounded ideal of $U(\mathfrak{g}(\bar{\lambda}))$. Indeed, in the case of $\mathfrak{g}(2n)=\mathfrak{o}(2n)$ a half-integral dominant weight is integral dominant, and hence the ideal $J(2n)\cap U(\mathfrak{g}(\bar{\lambda}))$ is integrable. If $\mathfrak{g}(2n)=\mathfrak{sp}(2n)$ and $\bar{\lambda}_i^{2f(2n)}\in\mathbb{Z}$, then the weight $\bar{\lambda}^{2f(2n)}$ is integral dominant hence $J(2n)\cap U(\mathfrak{g}(\bar{\lambda}))$ is an integrable ideal. For the last case, where $\mathfrak{g}(2n)=\mathfrak{sp}(2n)$ and $\bar{\lambda}_i^{2f(2n)}\in\mathbb{Z}-\frac{1}{2}$, the fact that $J(2n)\cap U(\mathfrak{g}(\bar{\lambda}))$ is bounded follows from Lemma 2.4. $\square$

Let I be an ideal of an associative algebra A. We denote by $\sqrt{I}$ the intersection of all primitive ideals of A containing I. Note that $\sqrt{I}$ is the pullback in A of the Jacobson radical of the ring A/I. If I is a primitive ideal then $I=\sqrt{I}$.

Lemma 3.10. [PP4] *Assume that the dimension of A is finite or countable. Then the following conditions on an element $z \in A$ are equivalent:*

1) $z \in \sqrt{I}$,

2) for every $a \in A$ there is $k \in \mathbb{Z}_{>0}$, such that $(az)^k \in I$.

Proof. The fact that 1) implies 2) follows from [MR, Corollary 1.8]. We will show that 2) implies 1).

Let $z \in A$ satisfy 2), and let $\bar{x}$ be the image of $x \in A$ in A/I. Assume to the contrary that there exists a simple A/I-module M such that $\bar{z} \cdot M \neq 0$. Pick $m \in M$ with $\bar{z} \cdot m \neq 0$. There is $a \in A$ such that $\bar{a} \cdot (\bar{z} \cdot m) = m$. Let $k \in \mathbb{Z}_{>0}$ satisfy $(\bar{a}\bar{z})^k = 0$. Then $0 = (\bar{z}(\bar{a}\bar{z})^k) \cdot m = \bar{z} \cdot m \neq 0$. This contradiction concludes the proof. $\square$

Now we have all tools to prove Proposition 3.5.

Proof. Consider the primitive ideal $J(2n) = \mathrm{Ann}_{U_{2n}} L(\tilde{w}\lambda - \rho_{\mathfrak{g}(2n)})$ such that $I_{2n} \subset J(2n)$. The associated variety $\mathrm{Var}(J(2n))$ is the closure of a nilpotent orbit, and there exists an integer r such that the rank of any $X \in \mathrm{Var}(J(2n))$ is less or equal r for all n. More precisely, according to Lemma 3.6, there exists $r \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{Var}(I_{2n}) \subset \mathfrak{g}(2n)^{\leq r}$ for $n \gg 0$. Since $J(2n) \supset I_{2n}$, we have

$$\mathrm{Var}(J(2n)) \subset \mathrm{Var}(I_{2n}) \subset \mathfrak{g}(2n)^{\leq r}. \tag{7}$$

Corollary 3.9 states that $J(2n) \cap U(\mathfrak{g}(\bar{\lambda}))$ is a bounded ideal of $U(\mathfrak{g}(\bar{\lambda}))$, where $\mathfrak{g}(\bar{\lambda}) \simeq \mathfrak{g}(2f(2n))$ for $f(2n) = [\frac{n-3r/2}{r+1}] - r/2$. In order to conclude that the ideal $J(2n) \cap U(2f(2n))$ is bounded, it suffices to observe that the root subalgebra $\mathfrak{g}(\bar{\lambda})$ of $\mathfrak{g}(2n)$ is conjugate to $\mathfrak{g}(2f(2n))$ naturally embedded in $\mathfrak{g}(2n)$. $\square$

Theorem 3.11. *If $\mathfrak{g}(\infty) = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ then any primitive ideal $I \subset U(\mathfrak{g}(\infty))$ is weakly bounded. Moreover, each primitive ideal of $U(\mathfrak{o}(\infty))$ is locally integrable.*

Proof. Note that $2f((r+1)(2n+r)+3r) = 2n$. Hence, Proposition 3.5 implies the existence of $r \geq 0$ such that $\sqrt{I_{(r+1)(2n+r)+3r}} \cap U_{2n}$ is a bounded (integrable for $\mathfrak{g}(\infty) = \mathfrak{o}(\infty)$) ideal of U_{2n} for $n \gg 0$. Next, Lemma 3.10 shows that $(\sqrt{I})_{2n} = \cap_{2n' \geq 2n} \sqrt{I_{2n'}}$ for all $n \geq 2$. However,

$$\cap_{2n' \geq 2n} \sqrt{I_{2n'}} = (\cap_{2n' \geq (r+1)(2n+r)+3r} \sqrt{I_{2n'}}) \cap U_{2n}.$$

Being bounded (integrable for $\mathfrak{g}(\infty) = \mathfrak{o}(\infty)$) in $U_{2f(n')}$, the ideal $\sqrt{I_{2n'}} \cap U_{2f(n')}$ is an intersection of bounded ideals (ideals of finite codimension for $\mathfrak{g}(\infty) = \mathfrak{o}(\infty)$) in $U_{2f(2n')}$, hence

$$(\sqrt{I})_{2n} = (\cap_{2n' \geq (r+1)(2n+r)+3r} \sqrt{I_{2n'}}) \cap U_{2n}$$

is an intersection of bounded ideals (ideals of finite codimension for $\mathfrak{g}(\infty) = \mathfrak{o}(\infty)$) in U_{2n}. This means that the ideal $(\sqrt{I})_{2n}$ is bounded (integrable for $\mathfrak{g}(\infty) = \mathfrak{o}(\infty)$) for $n \gg 0$. A very similar argument shows that $(\sqrt{I})_{2n}$ is bounded (integrable for $\mathfrak{g}(\infty) = \mathfrak{o}(\infty)$) for all $n \geq 2$. $\square$

4. Integrable ideals and c.l.s. for $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$

In this section we recall the notion of precoherent local system. It allows us to show that each primitive locally integrable ideal of $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$ is integrable.

4.1. Precoherent local systems and integrability of locally integrable ideals

Let $\mathfrak{g}(\infty) = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$. Our next goal is to establish that the conditions on an ideal I in $U(\mathfrak{g}(\infty))$ to be integrable and locally integrable are equivalent, see Theorem 4.1 below.

Let I be a locally integrable ideal of $U(\mathfrak{g}(\infty))$. For every $n \in \mathbb{Z}_{\geq 0}$, $I \cap U(\mathfrak{g}(2n))$ is an intersection of ideals of finite codimension in $U(\mathfrak{g}(2n))$. Thus, $I \cap U(\mathfrak{g}(2n))$ is an intersection of annihilators of finite-dimensional $\mathfrak{g}(2n)$-modules. Since any finite-dimensional module of a semisimple finite-dimensional Lie algebra is semisimple, it follows that $I \cap U(\mathfrak{g}(2n))$ is an intersection of annihilators of simple finite-dimensional $U(\mathfrak{g}(2n))$-modules. Recall that by Irr_{2n} we denote the set of classes of isomorphism of simple $\mathfrak{g}(2n)$-modules.

Definition 4.1. *A* precoherent local system of modules *(further p.l.s.) for $\mathfrak{g}(\infty)$ is a collection of sets*

$$\{Q_n\}_{n\in\mathbb{Z}_{\geq 1}} \subset \Pi_{n\in\mathbb{Z}_{\geq 1}} Irr_{2n}$$

such that $Q_m \supset \langle Q_n \rangle_m$ for any $n \geq m$, where $\langle Q_n \rangle_m$ denotes the set of all simple $\mathfrak{g}(2m)$-constituents of the $\mathfrak{g}(2n)$-modules from Q_n.

The fact that, for any locally integrable ideal I, the intersection $I \cap U(\mathfrak{g}(2n))$ is an intersection of annihilators of simple finite-dimensional $U(\mathfrak{g}(2n))$-modules implies that we can extend the definition of $Q(I)$ (see Section 2.11) to locally integrable ideals. Namely, if $I \subset U(\mathfrak{g}(\infty))$ is locally integrable ideal we define the p.l.s. $Q(I)$ by putting

$$Q(I)_n := \{z \in Irr_{2n} \mid I \cap U(\mathfrak{g}(2n)) \subset \mathrm{Ann}_{U(\mathfrak{g}(2n))}\, z\}.$$

It is clear that $Q(I)$ is a p.l.s., because if $I \cap U(\mathfrak{g}(2n))$ annihilates a module M from a class $z \in Irr_{2n}$ then $I \cap U(\mathfrak{g}(2m))$ annihilates all simple constituents of M as $\mathfrak{g}(2m)$-module, for $n \geq m$.

Moreover, we claim that $I(Q(I)) = I$ (see Subsection 2.9 for the definition $I(Q)$). Indeed, from the definitions of $Q(I)$ and $I(Q)$ we have that $I(Q(I)) \cap U(\mathfrak{g}(2n))$ equals the intersection of the annihilators of simple modules annihilated by $I \cap U(\mathfrak{g}(2n))$ and, possibly, some annihilators of isomorphism classes $\bar{z}$ such that $\mathrm{Ann}_{U(\mathfrak{g}(2n))}\,\bar{z} \supset I \cap U(\mathfrak{g}(2n))$. Since

$$I \cap U(\mathfrak{g}(2n)) \cap \mathrm{Ann}_{U(\mathfrak{g}(2n))}\,\bar{z} = I \cap U(\mathfrak{g}(2n))$$

for any $\bar{z}$, we conclude $I(Q(I)) \cap U(\mathfrak{g}(2n)) = I \cap U(\mathfrak{g}(2n))$.

Now we are ready to formulate the main result of this section. This is an analogue of I. Penkov's and A. Petukhov's result for $\mathfrak{sl}(\infty)$ [PP4, Theorem 4.2].

Theorem 4.1. *If $I \subset U(\mathfrak{g}(\infty))$ is a locally integrable ideal then I is integrable.*

Since $I(Q(I)) = I$ for any locally integrable ideal I, Theorem 4.1 follows from the following proposition.

Proposition 4.2. *If Q is a p.l.s. then $I(Q)$ is an integrable ideal.*

Definition 4.2. *Two p.l.s. Q and Q' are* equivalent *if there exists an integer n such that $Q_{n'} = Q'_{n'}$ for any $n' > n$.*

It follows directly from the definition of equivalence of p.l.s. that $I(Q) = I(Q')$ whenever Q and Q' are equivalent p.l.s. Thus, to prove Proposition 4.2 it is enough to prove the following proposition.

Proposition 4.3. *For any p.l.s. Q, there exists a c.l.s. Q' such that Q and Q' are equivalent.*

The rest of this section is devoted to the proof of Proposition 4.2.

4.2. Equivalence of p.l.s. and c.l.s.

In this subsection we provide a somewhat explicit construction of a c.l.s. Q' which is equivalent to a given p.l.s. Q, and in this way give a proof of Proposition 4.3.

Finite-dimensional $\mathfrak{o}(2n)$-modules are in one-to-one correspondence with n-tuples $\lambda = (\lambda_1, ..., \lambda_n)$ of numbers λ_i which are simultaneously either integers or half-integers and satisfy

$$\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_{n-1} \geqslant |\lambda_n|. \tag{8}$$

Finite-dimensional $\mathfrak{sp}(2n)$-modules are in one-to-one correspondence with n-tuples $\lambda = (\lambda_1, ..., \lambda_n)$ of integers λ_i satisfying

$$\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_{n-1} \geqslant \lambda_n \geqslant 0. \tag{9}$$

This correspondence is nothing but the assigment of the highest weight λ to a simple finite-dimensional module L so that $L = L(\lambda)$. In what follows we call n-tuples satisfying (8) in the case of $\mathfrak{o}(2n)$ and (9) in the case of $\mathfrak{sp}(2n)$ *admissible n-tuples.* We refer to n as the *width* of λ, and write $\#\lambda = n$.

The Gelfand–Tsetlin rule [Mo] claims that, for two admissible n-tuples λ and μ with $\#\lambda = n$, $\#\mu = n - 1$, the following conditions are equivalent.

- $\mathrm{Hom}_{\mathfrak{g}(2\#\mu)}(L(\mu), L(\lambda)) \neq 0$.
- There exists an n-tuple of integers $\nu = (\nu_1, \dots, \nu_n)$ which satisfy the inequalities

$$\begin{aligned}&\lambda_1 \geqslant \nu_1 \geqslant \lambda_2 \geqslant \nu_2 \geqslant \dots \geqslant \lambda_{n-1} \geqslant \nu_{n-1} \geqslant \lambda_n \geqslant \nu_n \geqslant 0,\\ &\nu_1 \geqslant \mu_1 \geqslant \nu_2 \geqslant \mu_2 \geqslant \dots \geqslant \nu_{n-1} \geqslant \mu_{n-1} \geqslant \nu_n \geqslant 0\end{aligned}$$

  in the case of $\mathfrak{sp}(2n)$.
- There exists a $(n-1)$-tuple of integers or half-integers $\nu = (\nu_1, \dots, \nu_n)$ which satisfy the inequalities

$$\begin{aligned}&\lambda_1 \geqslant \nu_1 \geqslant \lambda_2 \geqslant \nu_2 \geqslant \dots \geqslant \nu_{n-2} \geqslant \lambda_{n-1} \geqslant \nu_{n-1} \geqslant |\lambda_n|,\\ &\nu_1 \geqslant \mu_1 \geqslant \nu_2 \geqslant \mu_2 \geqslant \dots \geqslant \mu_{n-2} \geqslant \nu_{n-1} \geqslant |\mu_{n-1}|\end{aligned}$$

  in the case of $\mathfrak{o}(2n)$.

We note that the set of admissible n-tuples for $\mathfrak{sp}(2n)$ is a subset of the set of admissible n-tuples for $\mathfrak{o}(2n)$. Furthermore, the Gelfand–Tsetlin rule for the set of admissible n-tuples for $\mathfrak{sp}(2n)$ can be obtained by restriction of the Gelfand–Tsetlin rule for $\mathfrak{o}(2n)$. We will write $\lambda > \mu$ whenever a pair (λ, μ) as above satisfies the Gelfand–Tsetlin rule. For tuples λ and μ with $\#\lambda \geqslant \#\mu$, the Gelfand–Tsetlin rule implies that the following conditions are equivalent.

- $\mathrm{Hom}_{\mathfrak{g}(2\#\mu)}(L(\mu), L(\lambda)) \neq 0$
- There exists a sequence of admissible tuples $\lambda = \lambda^0, \lambda^1, \dots, \lambda^{\#\lambda - \#\mu} = \mu$ such that $\#\lambda^i = \#\lambda - i$ and

$$\lambda = \lambda^0 > \lambda^1 > \dots > \lambda^{\#\lambda - \#\mu} = \mu$$

  We write $\lambda \succ \mu$ whenever these conditions hold.

By writing $\{a_1, a_2\} \geqslant \{b_1, b_2\}$ we indicate the validity of all inequalities $a_i \geqslant b_j$ for all i and j. Using this notation we can rewrite the Gelfand–Tsetlin rule in a more convenient form. It is easy to check that the following two conditions are equivalent:

- $\lambda \succ \mu$ and $\#\lambda - \#\mu = 1$,
- $\{\lambda_1, \lambda_1\} \geqslant \{\lambda_2, \mu_1\} \geqslant \{\lambda_3, \mu_2\} \geqslant \ldots \geqslant \{\lambda_{n-1}, \mu_{n-2}\} \geqslant \{|\lambda_n|, |\mu_{n-1}|\}$.

We can now rephrase the definitions of p.l.s. and c.l.s.

a) The following conditions are equivalent:

- Q is a p.l.s.
- for all λ, μ such that $\lambda \succ \mu$ and $\lambda \in Q_{\#\lambda}$, we have $\mu \in Q_{\#\mu}$.

b) The following conditions are equivalent:

- Q a is c.l.s.
- Q a is p.l.s. and for every $\mu \in Q_{\#\mu}$ there is $\lambda \in Q_{\#\mu}$ such that $\lambda \succ \mu$.

We denote by $Q^\vee(\lambda)$ the largest p.l.s. Q which does not contain a given tuple λ.

Proposition 4.4. *For any admissible n-tuple λ, the p.l.s. $Q^\vee(\lambda)$ is equivalent to the c.l.s.*

$$Q(\lambda) := \bigcup_{1 \leqslant k \leqslant \#\lambda} Q(k, \lambda_k),$$

where the collection of sets $Q(k,a)$ for $k \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}/2$ is defined by putting

$$Q(k,a)_m := \{\mu \in Irr_{2m} \mid \mu_k < a, \ if \ k \leq \#\mu\}.$$

Note that Proposition 4.4 implies Proposition 4.3. Indeed, let Q be a p.l.s. Then

$$Q = \cap_{\lambda \notin Q} Q^\vee(\lambda).$$

According to Proposition 4.4, each p.l.s. $Q^\vee(\lambda)$ is equivalent to a c.l.s. $Q(\lambda)$. The lattice of c.l.s. is artinian [Zh3], and therefore we conclude that the p.l.s. Q is equivalent to the c.l.s.

$$Q(\lambda_1) \cap Q(\lambda_2) \cap \ldots \cap Q(\lambda_s)$$

for some finite set of elements $\lambda_1, \lambda_2, \ldots, \lambda_s \notin Q$.

It remains to prove Proposition 4.4. It is clear that Proposition 4.4 follows from the following lemma.

Lemma 4.5. *Let λ and μ be admissible tuples such that $\#\mu \geqslant 2\#\lambda$. Then the following conditions are equivalent:*

1) $\mu \succ \lambda$,

2) $\mu_k \geqslant \lambda_k$ *for each* $1 \leqslant k \leqslant \#\lambda$.

Without loss of generality, we can consider our admissible tuples as admissible tuples of integers. Indeed, if $\lambda \succ \mu$ and both admissible tuples consist of half-integers, we can add 1/2 to all entires of the corresponding admissible tuples.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an admissible tuple and $k \in \mathbb{Z}$. Set

$$R(\lambda, k) := \begin{cases} (\lambda_1, \dots, \lambda_i, k, \lambda_{i+2}, \dots \lambda_n) & \text{if } k \geq |\lambda_n| \\ \lambda & \text{if } k < |\lambda_n|, \end{cases}$$

so that $R(\lambda, k)$ is an admissible tuple and $i+1$ is maximal possible such that $k \geq \lambda_{i+1}$. Set

$$L(\lambda, k) := \begin{cases} (\lambda_1, \dots, \lambda_i, k, \lambda_{i+2}, \dots \lambda_n) & \text{if } k \leq \lambda_1 \\ \lambda & \text{if } k > \lambda_1, \end{cases}$$

so that $L(\lambda, k)$ is an admissible tuple and i is maximal possible such that $k \leq \lambda_{i+1}$.

Let us prove the following technical lemma.

Lemma 4.6. *Let λ and μ be admissible tuples such that $\#\mu - \#\lambda = 1$, $\mu > \lambda$ and let $k \in \mathbb{Z}$. Then*

1) $R(\mu, k) > R(\lambda, k)$,

2) $L(\mu, k) > L(\lambda, k)$,

3) $L(\mu, k) > R(\lambda, k)$ whenever one of the following conditions is satisfied

$$\mu_{i+1} \geqslant k > \mu_{i+2},$$

$$\mu_{i+2} \geqslant k \geqslant \mu_{i+3}.$$

for i such that $\lambda_i \geqslant k \geqslant \lambda_{i+1}$.

Proof. 1) Obviously, $\mu_i \geqslant k \geqslant \mu_{i+3}$. There are three possibilities:

$$\mu_i \geqslant k > \mu_{i+1,} \tag{$*$}$$

$$\mu_{i+1} \geqslant k > \mu_{i+2}, \tag{$**$}$$

$$\mu_{i+2} \geqslant k \geqslant \mu_{i+3}. \tag{$***$}$$

In the case $(*)$ we have $R(\lambda, k) = \{\lambda_1, \ldots, \lambda_i, k, \lambda_{i+2}, \ldots, \lambda_n\}$, $R(\mu, k) = \{\mu_1, \ldots, \mu_i, k, \mu_{i+2}, \ldots, \mu_n\}$. To check that $R(\mu, k) > R(\lambda, k)$, we need to check that the corresponding inequalities hold. But all inequalities not involving k follow from the inequality $\mu > \lambda$. Hence it remains to check that

$$\{\mu_i, \lambda_{i-1}\} \geqslant \{k, \lambda_i\} \geqslant \{\mu_{i+2}, k\} \geqslant \{\mu_{i+3}, \lambda_{i+2}\}.$$

These inequalities are implied by $(*)$. The cases $(**)$ and $(***)$ can be verified in a similar way.

2) The proof is similar to case 1).

3) We have

$$\lambda_i \geqslant k \geqslant \lambda_{i+1}$$

and one of the three cases $(*)$, $(**)$, $(***)$ holds.

For cases $(**)$ and $(***)$ one can easily check the following inequalities

$$\{\mu_i, \lambda_{i-1}\} \geqslant \{k, \lambda_i\} \geqslant \{\mu_{i+2}, k\} \geqslant \{\mu_{i+3}, \lambda_{i+2}\},$$

$$\{\mu_{i+1}, \lambda_i\} \geqslant \{k, k\} \geqslant \{\mu_{i+3}, \lambda_{i+2}\}.$$

So, $L(\mu, k) > R(\lambda, k)$ whenever one of the conditions $(**)$ and $(***)$ is satisfied. $\square$

Note that in the case $(*)$ the inequality $L(\mu, k) > R(\lambda, k)$ may be false. Indeed

$$\{\mu_{i-1}, \lambda_{i-2}\} \geqslant \{k, \lambda_{i-1}\} \geqslant \{\mu_{i+1}, \lambda_i\} \geqslant \{\mu_{i+2}, k\} \geqslant \{\mu_{i+3}, \lambda_{i+2}\}.$$

We see that $\lambda_i \geqslant k$ and $k \geqslant \lambda_i$. So $k = \lambda_i$, which is false in the general case.

We are now ready to prove Lemma 4.5. We may assume without loss of generality that $\#\mu = 2\#\lambda$. Indeed, the entries λ_i are independent of the entries μ_{2n+j} for $i, j \in \mathbb{Z}_{>0}$.

Proof of Lemma 4.5. The implication 1) $\Rightarrow$ 2) is obvious. We need to show that 2) $\Rightarrow$ 1).

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $k \leqslant \lambda_n$. Put $\lambda[k] = (\lambda_1, \lambda_2, \ldots, \lambda_n, k)$. Obviously, if $\lambda \prec \mu$ then $\lambda[k] < \mu[k]$. We will proceed by induction on $\#\lambda$. The base: if $\#\lambda = 1$, $\#\mu = 2$ and $\mu_1 > \lambda_1$, then, clearly, $\mu > \lambda$.

To perform the inductive step, we introduce the following notation: for admissible tuples $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{2n})$ such that $\mu \succ \lambda$, and for $\mu_{n+1} \geqslant \lambda_{n+1}$, $\mu_{2n} \geq \mu_{2n+1} \geq \mu_{2n+2}$, we set

$$\begin{aligned}
\lambda^* &:= (\lambda_1, \lambda_2, \ldots, \lambda_n, \lambda_{n+1}), \\
\mu^* &:= (\mu_1, \mu_2, \ldots, \mu_{2n}, \mu_{2n+1}, \mu_{2n+2}).
\end{aligned}$$

The inequality $\mu \succ \lambda$ means that we have a chain of admissible tuples

$$\mu = \lambda^0 > \lambda^1 > \ldots > \lambda^n = \lambda.$$

Without loss of generality, we can assume that the last entries of all λ^i for $0 \leqslant i \leqslant n-1$ all equal to μ_{2n}. Denote μ_{2n} by m. Put

$$\mu[m] = \lambda^0[m] > \lambda^1[m] > \ldots > \lambda^n[m] = \lambda[m].$$

By Lemma 4.6 we have

$$\begin{aligned}
R(\mu[m], \lambda_{n+1}) =& R(\lambda^0[m], \lambda_{n+1}) > R(\lambda^1[m], \lambda_{n+1}) > \ldots \\
& > R(\lambda^n[m], \lambda_{n+1}) = R(\lambda[m], \lambda_{n+1}), \\
L(\mu[m], \lambda_{n+1}) =& L(\lambda^0[m], \lambda_{n+1}) > L(\lambda^1[m], \lambda_{n+1}) > \ldots \\
& > L(\lambda^n[m], \lambda_{n+1}) = L(\lambda[m], \lambda_{n+1}).
\end{aligned} \tag{10}$$

Now we are going to show that there exists a pair $(i-1, i)$, where $1 \leqslant i \leqslant n$, for which λ^{i-1} and λ^i satisfy conditions $(**)$ or $(***)$, i.e., if $\lambda^{i-1}[m]_k \geq \lambda_{n+1} \geq \lambda^{i-1}[m]_{k+1}$ then

$$\lambda^i[m]_{k+1} \geq \lambda_{n+1} \geq \lambda^i[m]_{k+2}$$

or

$$\lambda^i[m]_{k+2} \geq \lambda_{n+1} \geq \lambda^i[m]_{k+3}.$$

Indeed, if for all pairs $(i-1, i)$ condition $(*)$ is satisfied, then

$$\lambda_{n+1} \geqslant \lambda[m]^{n-1}_{n+1}, \lambda_{n+1} \geqslant \lambda[m]^{n-2}_{n+1}, \ldots, \lambda_{n+1} \geqslant \lambda[m]^{0}_{n+1}.$$

We can rewrite the last inequality as $\lambda_{n+1} \geqslant \mu_{n+1}$. But we also have $\lambda_{n+1} \leqslant \mu_{n+1}$. This is a contradiction.

Therefore, a pair $(i-1, i)$ exists as required. This means that $L(\mu[m], \lambda_{n+1}) \succ R(\lambda[m], \lambda_{n+1})$. If $\lambda_{n+1} \geqslant m$ then $R(\lambda[m], \lambda_{n+1}) = \lambda^*$, because $\lambda_n \geqslant \lambda_{n+1} \geqslant m$. If $\lambda_{n+1} \leqslant m$, we replace m by λ_{n+1}, and then $L(\mu[m], \lambda_{n+1}) \succ \lambda^*$. This implies

$$L(\mu[m], \lambda_{n+1}) = \{\mu_1, \ldots, \mu_{j-1}, \lambda_{n+1}, \mu_{j+1}, \ldots, \mu_{2n}, m = \mu_{2n+1}\},$$

where

$$\mu_j \geqslant \lambda_{n+1} \geqslant \mu_{j+1} \tag{11}$$

for some $j \geqslant n+1$.

The last thing we need to prove is that $\mu^* \succ L(\mu[m], \lambda_{n+1})$. For this we need to check the inequalities

$$\begin{gathered}\mu_1 \geqslant \{\mu_2, \mu_1\} \geqslant \ldots \geqslant \{\mu_j, \mu_{j-1}\} \geqslant \{\mu_{j+1}, \lambda_{n+1}\} \geqslant \\ \{\mu_{j+2}, \mu_{j+1}\} \geqslant \ldots \geqslant \{\mu_{2n+1}, \mu_{2n}\} \geqslant \{\mu_{2n+2}, \mu_{2n+1}\}.\end{gathered}$$

All inequalities except

$$\mu_j \geq \lambda_{n+1},\ \mu_j - 1 \geq \lambda_{n+1},\ \mu_j + 1 \leq \lambda_{n+1},\ \mu_j + 2 \leq \lambda_{n+1}$$

are obvious, and these latter inequalities follow from inequality (3). This proves the inductive step and hence the lemma. $\square$

5. Coherent local systems of bounded ideals for $U(\mathfrak{sp}(\infty))$

In this section we generalize the construction of c.l.s. of simple finite-dimensional $\mathfrak{sp}(2n)$-modules to collections of bounded ideals in $U(\mathfrak{sp}(2n))$ when n runs over $\mathbb{Z}_{>0}$. To do this, we use some tools from Kazhdan–Lusztig theory.

5.1. Kazhdan–Lusztig theory

Here we introduce some basic definitions and facts related to Coxeter groups and Kazhdan–Lusztig theory.

Definition 5.1. *Let G be a group with identity 1_G. For a (not necessarily finite) subset S of G, we say that G is a* (generalized) Coxeter group with respect to S*, or that (G, S) is a* (generalized) Coxeter system*, if G is generated by S with a presentation of the form*

$$G = \langle S \mid (st)^{m_{s,t}} = 1_G \text{ for } s, t \in S \rangle ,$$

where, for each $s, t \in S$, $m_{s,t} = m_{t,s}$ is a positive integer or ∞, and, for all $s \in S$, $m_{s,s} = 1$. The condition $m_{s,t} = \infty$ means that no relation of the form $(st)^m = 1_G$ should be imposed.

The Coxeter matrix *of G is given by $[m_{s,t}]_{s,t\in S}$. We write $\bar{S}$ for the set*

$$\left\{gsg^{-1} \mid s \in S \text{ and } g \in G\right\} .$$

Definition 5.2. *The* Bruhat length ℓ^G *of a Coxeter system (G, S) is given by the function $\ell^G \colon G \to \mathbb{Z}_{\geq 0}$ defined in the following way: for all $g \in G$, $\ell^G(g)$ is the smallest integer $k \geq 0$ such that $g = s_1 s_2 \cdots s_k$ for some $s_1, s_2, \ldots, s_k \in S$. We say that $g = s_1 s_2 \cdots s_k$ is a* reduced expression *for $g \in G$ if $s_1, s_2, \ldots, s_k \in S$ and $k = \ell^G(g)$.*

Definition 5.3. *Let (G, S) be a Coxeter system and $g, h \in G$. Then, we write $g \preccurlyeq h$ if there is a reduced expression $h = s_1 s_2 \ldots s_k$, where $k \in \mathbb{Z}_{\geq 0}$ and $s_1, s_2, \ldots, s_k \in S$, such that g is a* subword *of $s_1 s_2 \ldots s_k$, i.e., there exist $j \in \mathbb{Z}_{\geq 0}$ and integers $i_1, i_2, \ldots, i_j$ with $1 \leq i_1 < i_2 < \ldots < i_j \leq k$ such that $g = s_{i_1} s_{i_2} \ldots s_{i_j}$. The relation $\preccurlyeq$ is called the* Bruhat order *on G. It is a well-known fact that this is a partial order.*

We will write $g \prec h$ if $g \preccurlyeq h$ and $g \neq h$.

Definition 5.4. *Let (G, S) be a Coxeter system and q be an indeterminate. The ring $\mathbb{Z}\left[q^{-\frac{1}{2}}, q^{+\frac{1}{2}}\right]$ of Laurent polynomials in $q^{\frac{1}{2}}$ is denoted by $\mathcal{A}$. The* Hecke algebra $\mathcal{H}$ *is an associative algebra which is a free module over $\mathcal{A}$ with generating set $\{T_g \mid g \in G\}$, such that the multiplicative identity of $\mathcal{H}$ is $1_{\mathcal{H}} = T_{1_G}$ and that the following multiplicative relations are satisfied:*

$$T_s^2 = (q-1)\,T_s + q\,T_{1_G}\,,$$

$$T_g\,T_s = T_{gs} \ \ \text{if } g \prec gs\,,$$

$$T_s\,T_g = T_{sg} \ \ \text{if } g \prec sg\,,$$

for each $s \in S$ and $g \in G$.

This algebra has an involution $v \mapsto \bar{v}$ for $v \in \mathcal{H}$ which sends $q^{1/2}$ to $q^{-1/2}$ and each T_s to T_s^{-1}.

Theorem 5.1. [KL] *Let (G, S) be a Coxeter system with associated Hecke algebra $\mathcal{H}$. Denote $l(g) = l^G(g)$ for $g \in G$. For each $g \in G$, there is a unique $C_g^G \in \mathcal{H}$ fixed by the involution on $\mathcal{H}$ and satisfying the condition*

$$C_g^G = q^{-\frac{\ell(g)}{2}} \sum_{x \preccurlyeq g} (-1)^{\ell(x)-\ell(g)} q^{\ell(x)-\ell(g)}\, \overline{P_{x,g}^G(q)}\, T_x\,, \tag{12}$$

where, for each $x, y \in G$,

(i) $P_{x,y}^G(q) \in \mathbb{Z}[q]$,

(ii) $P_{x,y}^G(q) = 0$ *if* $x \not\preccurlyeq y$,

(iii) $P_{x,x}^G(q) = 1$, *and*

(iv) $\deg\left(P_{x,y}^G(q)\right) \leq \frac{\ell(y)-\ell(x)-1}{2}$ *if* $x \prec y$.

The polynomials $P_{x,y}^G(q)$, where $x, y \in G$ for $x \preccurlyeq y$, are known as the Kazhdan–Lusztig (KL) polynomials *of G.*

5.2. Kazhdan–Lusztig multiplicities for $\mathfrak{sp}(2n)$ and $\mathfrak{o}(2n)$

In this subsection we recall the equalities which are commonly known as the Kazhdan-Lusztig conjecture. This will allow us to establish a connection between simple bounded infinite-dimensional highest weight modules of $\mathfrak{sp}(2n)$ and simple finite-dimensional modules with half-integral highest weights of $\mathfrak{o}(\infty)$.

Let $\mathfrak{g}(2n)$ equal $\mathfrak{o}(2n)$ or $\mathfrak{sp}(2n)$. Let λ be an integral dominant weight with half-integral entries for the Lie algebra $\mathfrak{o}(2n)$, and let λ be a dominant integral weight, or a weight satisfying the conditions of Lemma 2.4, for the Lie algebra $\mathfrak{sp}(2n)$. Denote by M_w the Verma module with highest weight $w \cdot \lambda$, and by L_w the unique simple quotient of M_w. If V is a weight $\mathfrak{g}(2n)$-module then the formal character $ch(V)$ equals the formal sum $\sum_\mu m_\mu e^\mu$, where m_μ is the dimension of the weight space V^μ for $m_\mu \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. We can define in a similar way the formal character of a weight module over any reductive finite-dimensional Lie algebra.

The following equality is often referred to as *Kazhdan–Lusztig conjecture*:

$$ch(L_w) = \sum_{y \succcurlyeq w} (-1)^{-\ell(w)-\ell(y)} P^W_{yw_0, ww_0}(1) ch(M_y),$$

$$ch(M_w) = \sum_{y \succcurlyeq w} P^W_{w,y}(1) ch(L_y),$$

where W is the Weyl group of $\mathfrak{g}(2n)$, the Coxeter system is defined by the set of simple reflections, and $w_0 \in W$ is the element of maximal length. The Kazhdan-Lusztig conjecture was proved independently in [BB] and [BK].

We put $\varepsilon(2n) =' \rho_{\mathfrak{sp}(2n)} - \rho'_{\mathfrak{o}(2n)} = (1, 1, \ldots, 1)$.

The subgroup of W which preserves λ is called *the isotropy group of* λ.

Theorem 5.2. [H, page 267] *Let $\mathfrak{g}'$ and $\mathfrak{g}''$ be finite-dimensional semisimple Lie algebras, with respective Weyl groups W' and W''. Fix weights λ' for $\mathfrak{g}'$ and λ'' for $\mathfrak{g}''$, with corresponding blocks $\mathcal{O}'_{\lambda'}$ and $\mathcal{O}''_{\lambda''}$ and reflection subgroups $W'_{[\lambda']}$ and $W''_{[\lambda'']}$. If there is an isomorphism between these Weyl groups as Coxeter groups, which sends the isotropy group of λ' to the isotropy group of λ'', then the category $\mathcal{O}'_{\lambda'}$ is equivalent to $\mathcal{O}''_{\lambda''}$, with $L(\lambda')$ sent to $L(\lambda'')$ and $M(\lambda')$ sent to $M(\lambda'')$.*

Let λ' be a $\mathfrak{sp}(2n)$-weight with $\lambda'_i \in \mathbb{Z} + \frac{1}{2}$ satisfying the conditions of Lemma 2.4 and let $\lambda'' = \lambda' + \varepsilon(2n)$. Put $W' = W_{\mathfrak{sp}(2n)}$ and $W'' = W_{\mathfrak{o}(2n)}$. We consider λ'' as an integral dominant $\mathfrak{o}(2n)$-weight. Moreover, the reflection subgroup $W''_{[\lambda'']}$ equals W'', and $W'_{[\lambda']}$ is a subgroup of index 2 of the group W'consisting of permutations $w \in S_{2n}$ such that $w(-i) = -w(i)$, $1 \leq i \leq n$, for which the number $\#\{i > 0 \mid w(i) < 0\}$ is even (here we consider W' as a subgroup of S_{2n}, see Subsection 2.2). The Coxeter groups $W''_{[\lambda'']}$ and $W'_{[\lambda']}$ are isomorphic. These facts and the definition of Kazhdan–Lusztig polynomials imply the following

Corollary 5.3. *After identifying $W'_{[\lambda']}$ with $W''_{[\lambda'']} = W''$, one has $P^{W'_{[\lambda']}}_{w,v}(q) = P^{W''_{[\lambda'']}}_{w,v}(q)$ for all elements $w, v \in W''$.*

Let $L(\lambda)$ be a simple finite-dimensional $\mathfrak{sp}(2n)$-module with highest weight λ. Then we set $\varpi_i := \sum_{j=1}^{i} \varepsilon_j$, for $1 \leq j \leq n$. Then $\lambda = \sum_{i=1}^{n} v_i \varpi_i$ for some $v_i \in \mathbb{Z}_{\geq 0}$, where $\varpi_i = \sum_{j=1}^{i} \varepsilon_i$. Also we denote by T_λ^j the set of all weights of the form $\lambda - \sum_{i=1}^{n} d_i \varepsilon_i$, where d_i are nonnegative integers, $\sum_{i=1}^{n} d_i$ is even, $0 \leq d_i \leq v_i$ for $1 \leq i \leq n-1$ and $0 \leq d_n + \delta_1^j \leq 2v_n + 1$, δ_1^j being the Kronecker delta.

Lemma 5.4. [BHL, Theorem 5.5] *Denote $\nu_0 = -\frac{1}{2}\varpi_n$ and $\nu_1 = \varpi_{n-1} - \frac{3}{2}\varpi_n$. The $\mathfrak{sp}(2n)$-module $L(\nu_j) \otimes L(\lambda)$ is completely reducible with the decomposition*

$$L(\nu_j) \otimes L(\lambda) \simeq \bigoplus_{\kappa \in T_\lambda^j} L(\nu_j + \kappa).$$

Lemmas 2.4 and 5.4 imply that each simple bounded highest weight infinite-dimensional module could be constructed as a simple constituent of the above tensor product for some λ. Then it follows that the tensor product of $L(\nu_j)$ with any simple bounded infinite-dimensional highest weight module is completely reducible.

Corollary 5.5. *Let $L(\lambda)$ be a simple bounded $\mathfrak{sp}(2n)$-module with highest weight λ. Then the module $L(\lambda)$ is completely reducible as a module over $\mathfrak{sp}(2n-2)$, where the embedding $\psi_{2n-2} \colon \mathfrak{sp}(2n-2) \to \mathfrak{sp}(2n)$ is described in Subsection 2.3.*

Proof. Recall the definition of Shale-Weil modules from Subsection 2.8. Set $\lambda_0 = \frac{1}{2}\varpi_n$. Then there exists a simple finite-dimensional module $L(\mu)$ with highest weight μ, such that $L(\lambda)$ is a simple constituent of $SW^+(2n) \otimes L(\mu)$, because of Lemma 5.4 and the fact that the highest weight of $SW^+(2n)$ equals $-\lambda_0$. Decompose $L(\mu) = \bigoplus_{j=1}^{k} L(\bar{\mu}_j)$ for some simple finite-dimensional $\mathfrak{sp}(2n-2)$-modules $L(\bar{\mu}_j)$. Next, one can check directly that $L(\lambda_0)$ as $\mathfrak{sp}(2n-2)$-module is isomorphic to a countable direct sum of copies of $SW^+(2n-2) \oplus SW^-(2n-2)$. Therefore the tensor product $SW^+(2n) \otimes L(\mu)$ is isomorphic to a direct sum of countably many copies of $SW^+(2n-2) \otimes L(\bar{\mu}_j)) \oplus (SW^-(2n-2) \otimes L(\bar{\mu}_j))$.

The tensor products $SW^+(2n-2) \otimes L(\bar{\mu}_j)$ and $SW^-(2n-2) \otimes L(\bar{\mu}_j)$ are completely reducible by Lemma 5.4. In this way, we see that $L(\mu) \otimes SW^+(2n)$ is completely reducible over $\mathfrak{sp}(2n-2)$, and the same holds for its submodule $L(\lambda)$. □

Next, we consider the Lie subalgebra $\mathrm{Im}(\psi_{2n-2}) + \mathfrak{h}_{\mathfrak{sp}(2n)}$ of $\mathfrak{sp}(2n)$. We set $\mathfrak{sp}^{\mathfrak{h}}(2n-2) = \mathrm{Im}(\psi_{2n-2}) + \mathfrak{h}(2n)$. Analogously, we define the Lie subalgebra $\mathfrak{o}^{\mathfrak{h}}(2n-2)$ of $\mathfrak{o}(2n)$.

Proposition 5.6. *Let λ be a weight of $\mathfrak{sp}(2n)$ satisfying the conditions of Lemma 2.4, and μ be a weight of $\mathfrak{sp}(2n-2)$ satisfying Lemma 2.4. Moreover, let $\lambda' = \lambda + \varepsilon(2n)$ be an integral dominant weight of $\mathfrak{o}(2n)$ with half-integral marks, and $\mu' = \mu + \varepsilon(2n-2)$ be an integral dominant weight of $\mathfrak{o}(2n-2)$ with half-integral marks. Then* $\mathrm{Hom}_{\mathfrak{sp}(2n-2)}(L(\mu), L(\lambda)) \neq 0$ *if and only if* $\mathrm{Hom}_{\mathfrak{o}(2n-2)}(L(\mu'), L(\lambda')) \neq 0$.

Proof. Consider the decomposition

$$L(\lambda') = \bigoplus_{j=1}^{k} L(\lambda'^j),$$

over the Lie algebra $\mathfrak{o}^{\mathfrak{h}}(2n-2)$, where each $L(\lambda'^j)$ is a simple finite-dimensional representation of $\mathfrak{o}^{\mathfrak{h}}(2n-2)$ with highest weight λ'^j. Note, that as an $\mathfrak{o}(2n-2)$-module $L(\lambda'^j)$ is isomorphic to the simple finite-dimensional $\mathfrak{o}(2n-2)$-module $L(\bar{\lambda}'^j)$ for $\bar{\lambda}'^j = \sum_{i=2}^{n} \lambda_i^j \varepsilon_{i-1}$, where $\lambda'^j = \sum_{i=1}^{n} \lambda_i^j \varepsilon_i$. This implies

$$ch(L(\lambda')) = \sum_{j=1}^{k} ch(L(\lambda'^j)). \tag{13}$$

We apply the Kazhdan–Lusztig conjecture for $w = \mathrm{id} \in W_{\mathfrak{o}_{2n-2}}$ to each $\mathfrak{o}(2n-2)$-module $L(\bar{\lambda}'^j)$. Since each $\bar{\lambda}'^j$ is an integral dominant weight, we get

$$ch(L(\bar{\lambda}'^j)) = \sum_{y \in W_{\mathfrak{o}(2n-2)}} (-1)^{-\ell(y)} P^{W_{\mathfrak{o}(2n-2)}}_{yw_0,w_0}(1) ch(M(y \cdot \bar{\lambda}'^j)).$$

Now we denote by $M^{\mathfrak{h}}(\tau)$ the $\mathfrak{o}^{\mathfrak{h}}(2n-2)$-module with highest weight τ such that as $\mathfrak{o}(2n-2)$-module $M^{\mathfrak{h}}(\tau)$ is isomorphic to $M(\bar{\tau})$, where $\bar{\tau} = \sum_{i=2}^{n} \tau_i \varepsilon_{i-1}$. By $M^{\mathfrak{h}}(\alpha, \gamma)$ we denote the Verma module over $\mathfrak{o}^{\mathfrak{h}}(2n-2)$ with highest weight $\alpha\varepsilon_1 + \sum_{i=2}^{n} \gamma_{i-1}\varepsilon_i$, where $\gamma = \sum_{i=1}^{n-1} \gamma_i \varepsilon_i$ is a weight of $\mathfrak{o}(2n-2)$ for $\alpha \in \mathbb{C}$. Clearly, $M(\tau) = M^{\mathfrak{h}}(\tau_1, \bar{\tau})$. Therefore we can write the decomposition

$$ch(L(\lambda'^j)) = \sum_{y \in W_{\mathfrak{o}(2n-2)}} (-1)^{-\ell(y)} P^{W_{\mathfrak{o}(2n-2)}}_{yw_0,w_0}(1) ch(M^{\mathfrak{h}}(\lambda_1'^j, y \cdot \bar{\lambda}'^j)) \tag{14}$$

Combining formulas (13) and (14), we obtain

$$ch(L(\lambda')) = \sum_{j=1}^{k} \sum_{y \in W_{\mathfrak{o}(2n-2)}} (-1)^{-\ell(y)} P^{W_{\mathfrak{o}(2n-2)}}_{yw_0,w_0}(1) ch(M^{\mathfrak{h}}(\lambda_1'^j, y \cdot \bar{\lambda}'^j)) \tag{15}$$

On the other hand, we may apply the Kazhdan–Lusztig conjecture to the $\mathfrak{o}(2n)$-module $L(\lambda')$. This yields

$$ch(L(\lambda')) = \sum_{y \in W_{\mathfrak{o}(2n)}} (-1)^{-\ell(y)} P^{W_{\mathfrak{o}(2n)}}_{yw_0, w_0}(1) ch(M(y \cdot \lambda')). \tag{16}$$

Next, for an $\mathfrak{o}(2n)$-weight τ we set

$$\tau(a_1, a_2, \ldots, a_{2n-2}) := \tau - \sum_{i=1}^{n-1} a_i(\varepsilon_1 - \varepsilon_{i+1}) - \sum_{i=n}^{2n-2} a_i(\varepsilon_1 + \varepsilon_{i-n+2})$$

for $a_i \in \mathbb{Z}_{\geq 0}$. We decompose the character of each Verma module $M(y \cdot \lambda')$ over $\mathfrak{o}^{\mathfrak{h}}(2n-2)$:

$$ch(M(y \cdot \lambda')) = \sum_{a_1, a_2 \ldots, a_{n-1} = 0}^{\infty} ch(M^{\mathfrak{h}}((y \cdot \lambda')(a_1, a_2, \ldots, a_{n-1}))). \tag{17}$$

This is a direct consequence of the definition of Verma module. Note that, if we consider the restriction of (17) to each weight subspace of $M(y \cdot \lambda')$ as an equality of dimensions then the left-hand side is a positive integer, while the right-hand side is a sum of positive integers. This means that each such restriction has only finitely many terms.

Combining formulas (16) and (17), we get

$$\begin{aligned} &ch(L'(\lambda)) = \\ &= \sum_{y \in W_{\mathfrak{o}(2n)}} \sum_{a_1, a_2 \ldots, a_{n-1} = 0}^{\infty} (-1)^{-\ell(y)} P^{W_{\mathfrak{o}(2n)}}_{yw_0, w_0}(1) ch(M^{\mathfrak{h}}((y \cdot \lambda')(a_1, a_2, \ldots, a_{n-1}))). \end{aligned} \tag{18}$$

From equations (15) and (18) we obtain

$$\begin{aligned} &\sum_{j=1}^{k} \sum_{y \in W_{\mathfrak{o}(2n-2)}} (-1)^{-\ell(y)} P^{W_{\mathfrak{o}(2n-2)}}_{yw_0, w_0}(1) ch(M^{\mathfrak{h}}(\lambda'^j_1, y \cdot \bar{\lambda}'^j)) = \\ &= \sum_{y \in W_{\mathfrak{o}(2n)}} \sum_{a_1, a_2 \ldots, a_{n-1} = 0}^{\infty} (-1)^{-\ell(y)} P^{W_{\mathfrak{o}(2n)}}_{yw_0, w_0}(1) ch(M^{\mathfrak{h}}((y \cdot \lambda')(a_1, a_2, \ldots, a_{n-1}))). \end{aligned} \tag{19}$$

We rewrite this equation as

$$\sum_{j=1}^{k} \sum_{y\in W_{\mathfrak{o}(2n-2)}} (-1)^{-\ell(y)} P^{W_{\mathfrak{o}(2n-2)}}_{yw_0,w_0}(1) ch(M^{\mathfrak{h}}(\lambda'^j_1, y\cdot \bar{\lambda}'^j))-$$

$$-\sum_{y\in W_{\mathfrak{o}(2n)}} \sum_{a_1,a_2\dots,a_{n-1}=0}^{\infty} (-1)^{-\ell(y)} P^{W_{\mathfrak{o}(2n)}}_{yw_0,w_0}(1) ch(M^{\mathfrak{h}}((y\cdot\lambda')(a_1,a_2,\dots,a_{n-1}))) = 0. \tag{20}$$

Now we will show that the sum of coefficients in front of $ch(M^{\mathfrak{h}}(\gamma))$, for each γ appearing in formula (20), equals 0. If a weight γ_0 appears in (20) and is maximal (for the order defined by the fixed Borel subalgebra), then the above claim is obvious. Moreover, finitely many maximal weights γ_0 exist because of formulas (16) and (17). Therefore we can erase from formula (20) all terms of the form $ch(M(\gamma_0))$ for maximal γ_0. For any fixed γ we prove our claim after finitely many iterations.

In this way, we see that in equation (19) the coefficients in front of every character of Verma module at the left-hand and right-hand sides of the equation are equal.

Next, we apply the Kazhdan–Lusztig conjecture to the $\mathfrak{sp}(2n)$-module $L(\lambda)$. By Corollary 5.3 we have that the Kazhdan–Lusztig polynomials appearing in (21) below are the same as in formula (16) for $L(\lambda')$:

$$ch(L(\lambda)) = \sum_{y\in W_{[\lambda]}} (-1)^{-\ell(y)} P^{W_{[\lambda]}}_{yw_0,w_0}(1) ch(M(y\cdot\lambda)). \tag{21}$$

For a $\mathfrak{sp}(2n)$-weight τ we set

$$\tau(a_1,a_2,\dots,a_n) := \tau - 2a_1\varepsilon_1 - \sum_{i=2}^{n} a_i(\varepsilon_1-\varepsilon_i).$$

Also we denote by $M^{\mathfrak{h}}(\tau)$ the $\mathfrak{sp}^{\mathfrak{h}}(2n-2)$-module with highest weight τ. As $\mathfrak{sp}(2n-2)$-module, $M^{\mathfrak{h}}(\tau)$ is isomorphic to $M(\bar{\tau})$ for $\bar{\tau} = \sum_{i=2}^{n} \tau_i\varepsilon_{i-1}$. Now we decompose the character of each Verma module $M(y\cdot\lambda)$ over $\mathfrak{sp}^{\mathfrak{h}}(2n-2)$ similarly to formula (17):

$$ch(M(y\cdot\lambda)) = \sum_{a_1,a_2,\dots,a_n=0}^{\infty} ch(M^{\mathfrak{h}}((y\cdot\lambda)(a_1,a_2,\dots,a_n))). \tag{22}$$

Analogously to the case of $\mathfrak{o}(2n)$ we combine formulas (21) and (22):

$$ch(L(\lambda)) =$$

$$= \sum_{y\in W_{[\lambda]}} (-1)^{-\ell(y)} P^{W_{[\lambda]}}_{yw_0,w_0}(1) \sum_{a_1,a_2,\ldots,a_n=0}^{\infty} ch(M^{\mathfrak{h}}((y\cdot\lambda)(a_1,a_2,\ldots,a_n))) =$$

$$= \sum_{a_1=0}^{\infty} \sum_{y\in W_{[\lambda]}} \sum_{a_2,\ldots,a_n=0}^{\infty} (-1)^{-\ell(y)} P^{W_{[\lambda]}}_{yw_0,w_0}(1) ch(M^{\mathfrak{h}}((y\cdot\lambda - 2a_1\varepsilon_1)(a_2,\ldots,a_n))). \tag{23}$$

From the fact that the Kazhdan–Lusztig polynomials are the same as for $\mathfrak{o}(2n)$, and from our observation concerning the coefficients of each Verma module appearing in formula (19), we can rewrite formula (23) as

$$ch(L(\lambda)) = \sum_{a_1=0}^{\infty} \sum_{j=1}^{k} \sum_{y\in W_{[\bar\lambda^j]}} (-1)^{-\ell(y)} P^{W_{[\lambda]}}_{yw_0,w_0}(1) ch(M^{\mathfrak{h}}(\lambda^j_1 - 2a_1, y\cdot\bar\lambda^j)), \tag{24}$$

where $\lambda^j = \lambda'^j - \varepsilon(2n)$ and $\bar\lambda^j = \bar\lambda'^j - \varepsilon(2n-2)$. Now we apply the Kazhdan–Lusztig conjecture to each inner sum, and keeping in mind that the Kazhdan–Lusztig polynomials here and for $\mathfrak{o}(2n-2)$ are the same, we obtain

$$ch(L(\lambda)) = \sum_{a_1=0}^{\infty} \sum_{j=1}^{k} ch(L(\lambda_j - 2a_1\varepsilon_1)). \tag{25}$$

It is easy to check that if M is a weight module over $\mathfrak{sp}^{\mathfrak{h}(2n-2)}$, and is semisimple as an $\mathfrak{sp}(2n-2)$-module, then M is semisimple over $\mathfrak{sp}^{\mathfrak{h}}(2n-2)$. Therefore, $L(\lambda)$ is semisimple over $\mathfrak{sp}^{\mathfrak{h}}(2n-2)$.

We will show that $L(\lambda_j - 2a_1\varepsilon_1)$ is a simple constituent of $L(\lambda)$ for all j. Denote by A_1 the set of all weights $\lambda_j - 2a_1\varepsilon_1$ for $1 \le j \le k$, $j, a_1 \in \mathbb{Z}_{\ge 0}$. Note that this set is partially ordered ($\alpha \ge \beta \iff \alpha - \beta$ is a sum of positive roots from $\Delta_{\mathfrak{sp}(2n)}$). Also this set is bounded from above. Moreover, it is clear that if a weight $\zeta \in A_1$ is maximal then $\zeta = \lambda_p$ for some p, and therefore there are finitely many maximal weights in A_1. Denote this set of maximal weights by T_1. Consider now the set $A_1 \backslash T_1$ and repeat the procedure for this set. We obtain a set $T_2 \subset A_1 \backslash T_1$, and after $i-1$ steps — a set T_i. Let $A_i := A_1 \backslash \bigcup_{j<i} T_j$. By definition, T_i is the set of all maximal weights of A_i.

Next, we consider the subspace M^{T_1} of $L(\lambda)$ spanned by all weight spaces with weights $\chi_p \in T_1$ and choose a weight basis B of M^{T_1}. By each vector $b \in B$ we generate the submodule L^b of $L(\lambda)$. Each such module is a highest weight module with respective highest weight $\chi_p \in T_1$. The semisimplicity of $L(\lambda)$ implies that L^b is simple, and we set $L^b = L(\chi_p)$.

Denote by L^{T_1} the submodule generated by M^{T_1}. Each vector $u \in L_1^T$ can be obtained as $u = gv$ for some $v \in M_1^T$ and some $g \in U(\mathfrak{sp}^{\mathfrak{h}}(2n-2))$. Since v is contained in the span of B, the vector u lies in the sum of modules L^b. Therefore, L^{T_1} is isomorphic to the direct sum of the $\mathfrak{sp}^{\mathfrak{h}}(2n-2)$-modules $L(\chi_p)$. Consider a complement L_1 in $L(\lambda)$ to L^{T_1}. This is a submodule of $L(\lambda)$ with character

$$\sum_{a_1=0}^{\infty}\sum_{j=1}^{k} ch(L(\lambda_j - 2a_1\varepsilon_1)) - \sum_{p\in T_1} ch(L(\chi_p)) = \sum_{p\in A_2} ch(L(\chi_p)).$$

This character is well-defined because all weight spaces are finite-dimensional.

Denote by $L^{T_{e+1}}$ the submodule of L_e generated by $M^{T_{e+1}}$, where $M^{T_{e+1}}$ is the subspace spanned by all weight spaces with weights $\chi_j \in T_{e+1}$. Let L_{e+1} be a complement to $L^{T_{e+1}}$ in L_e. One can show that the following formula holds for any $e \in \mathbb{Z}_{\geq 1}$ (by repeating above decomposition of $L(\lambda)$ for all L_j, $j \leq e$):

$$L(\lambda) = \bigoplus_{i=1}^{e} L^{T_i} \oplus L_e.$$

Since the set A_1 is ordered as described above, each $\chi \in A_1$ is an element of T_i for some i. Therefore, $L(\lambda_j - 2a_1\varepsilon_1)$ is a simple constituent of L^{T_i} for some i as well as a simple constituent of $L(\lambda)$. Hence,

$$F = \bigoplus_{a_1=0}^{\infty}\bigoplus_{j=1}^{k} L(\lambda_j - 2a_1\varepsilon_1)$$

is a submodule of $L(\lambda)$. However, F has the same character as $L(\lambda)$ therefore $L(\lambda) = F$.

Now, we note that the modules $L(\lambda_j - 2a_1\varepsilon_1)$ and $L(\bar{\lambda}^j)$ are isomorphic as $\mathfrak{sp}(2n-2)$-modules for $\bar{\lambda}^j = \sum_{i=2}^{n} \lambda_i \varepsilon_{i-1}$. We obtain an isomorphism of $\mathfrak{sp}(2n-2)$-modules

$$L(\lambda) \simeq \bigoplus_{j=1}^{k} m_j L(\bar{\lambda}_j),$$

where m_j equals the cardinality $\aleph_0$ for all j.

In this way we proved that

$$\mathrm{Hom}_{\mathfrak{o}(2n-2)}(L(\mu'), L(\lambda')) \neq 0 \Longleftrightarrow \mu' = \bar{\lambda}'^j, \text{for some } j \Longleftrightarrow$$

$$\iff L(\mu) \text{ is a simple constituent of } L(\lambda) \text{ over } \mathfrak{sp}(2n-2) \iff$$

$$\iff \mathrm{Hom}_{\mathfrak{sp}(2n-2)}(L(\mu), L(\lambda)) \neq 0$$

Thus we proved the proposition. □

We have now shown that, for weights λ and μ respectively of $\mathfrak{sp}(2n)$ and $\mathfrak{sp}(2n-2)$, satisfying Lemma 2.4, the following conditions are equivalent:

- $\mathrm{Hom}_{\mathfrak{sp}(2n-2)}(L(\mu), L(\lambda)) \neq 0$.
- There exists an n-tuple of half-integers $\nu = (\nu_1, \dots, \nu_n)$ which satisfy the inequalities

$$\lambda_1 + 1 \geqslant \nu_1 \geqslant \lambda_2 + 1 \geqslant \nu_2 \geqslant \dots \geqslant \lambda_{n-1} + 1 \geqslant \nu_{n-1} \geqslant |\lambda_n + 1| \geqslant \nu_n \geqslant \frac{1}{2},$$
$$\nu_1 \geqslant \mu_1 + 1 \geqslant \nu_2 \geqslant \mu_2 + 1 \geqslant \dots \geqslant \nu_{n-1} \geqslant |\mu_{n-1} + 1| \geqslant \nu_n \geqslant \frac{1}{2}.$$

In conclusion we would like to recall the following proposition. The degree of a weight module is given in Definition 2.22.

Proposition 5.7. [M, Theorem 12.2(ii)] *Let $L(\lambda)$ be a simple module of $\mathfrak{sp}(2n)$ with highest weight λ satisfying Lemma 2.4 and $L(\lambda+\varepsilon)$ be the simple finite-dimensional module of $\mathfrak{o}(2n)$ with highest weight $\lambda + \varepsilon(2n)$. Then*

$$\deg(L(\lambda)) = \dim(L(\lambda + \varepsilon(2n)))/2^{n-1}.$$

5.3. Coherent local systems of bounded ideals: definition and classification

In this subsection we introduce the notion of coherent local systems of bounded ideals, which we abbreviate as c.l.s.b. This is a generalization of the notion of a c.l.s. Also we obtain a classification of irreducible c.l.s.b. based on Zhilinskii's classification of irreducible of c.l.s. As above, $\mathfrak{g}(2n)$ denotes the Lie algebra $\mathfrak{o}(2n)$ or $\mathfrak{sp}(2n)$, and $\mathfrak{g}(\infty)$ denotes the Lie algebra $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$. Also we assume that splitting Cartan subalgebras and Borel subalgebras of $\mathfrak{g}(2n)$ are fixed as in Subsection 2.2.

Recall that simple bounded highest weight $\mathfrak{o}(2n)$-modules are finite dimensional, and that simple bounded highest weight $\mathfrak{sp}(2n)$-modules are either finite dimensional, or are modules with highest weights satisfying Lemma 2.4.

Let J_n denote the set of annihilators of simple bounded highest weight $\mathfrak{g}(2n)$-modules (in fact, J_n coincides with the set of annihilators of all bounded $\mathfrak{g}(2n)$-modules). Next, let R_n denote the set of isomorphism classes of simple bounded highest weight modules. Note that the annihilator A of a simple bounded $\mathfrak{o}(2n)$-module determines the simple module annihilated by A up to isomorphism. Also, the annihilator A of a finite-dimensional $\mathfrak{sp}(2n)$-module determines this module up to isomorphism. However, one can show that, for a fixed splitting Borel subalgebra of $\mathfrak{sp}(2n)$, there are precisely two simple bounded infinite-dimensional highest weight $\mathfrak{sp}(2n)$-modules with a given annihilator $A \in J_n$.

Definition 5.5. *A* coherent local system of bounded ideals *(further c.l.s.b.) for $\mathfrak{g}(\infty)$ is a collection of sets*

$$\{\mathbb{I}_n\}_{n\in\mathbb{Z}_{\geq 2}} \subset \Pi_{n\in\mathbb{Z}_{\geq 2}} J_n$$

such that $\mathbb{I}_m = \langle \mathbb{I}_n \rangle_m$ for $n > m$, where $\langle \mathbb{I}_n \rangle_m$ denotes the set of all annihilators of simple $\mathfrak{g}(2m)$-constituents of the $\mathfrak{g}(2n)$-modules which are annihilated by at least one ideal from $\mathbb{I}_n$.

Definition 5.6. *A c.l.s.b. $\mathbb{I}$ is* irreducible *if $\mathbb{I} \neq \mathbb{I}' \cup \mathbb{I}''$ with $\mathbb{I}' \not\subset \mathbb{I}''$ and $\mathbb{I}' \not\supset \mathbb{I}''$, $\mathbb{I}'$, $\mathbb{I}''$ being c.l.s.b.*

Any annihilator $z \in J_n$ of a simple bounded $\mathfrak{g}(2n)$-module corresponds to one or two classes in R_n (i.e., either to an integral dominant weight or to two half-integral weights of $\mathfrak{g}(2n)$). Denote by $\{\lambda(z)\}$ the set of weights λ such that $L(\lambda)$ is annihilated by z. If the set $\{\lambda(z)\}$ contains only one weight, then we denote this weight by $\lambda(z)$ (otherwise $\#\{\lambda(z)\} = 2$).

Let $z_1, z_2 \in J_n$ and $\#\{\lambda(z_1)\} = 1$. We denote by $z_1 z_2$ the set if annihilators of the modules $L(\lambda(z_1) + \mu)$ for $\mu \in \{\lambda(z_2)\}$. For $S_1, S_2 \subset J_n$ we put

$$S_1 S_2 := \{z \in J_n \mid z \in z_1 z_2 \text{ for some } z_1 \in S_1 \text{ with} \#\{\lambda(z_1)\} = 1 \text{ and } z_2 \in S_2\}.$$

Let Q' and Q'' be c.l.s.b.. We denote by $Q'Q''$ the smallest c.l.s.b. such that $(Q')_n(Q'')_n \subset (Q'Q'')_n$. By definition, $Q'Q''$ is the *product* of Q' and Q''.

For any ideal $I \subset U(\mathfrak{g}(\infty))$, define the collection of sets $Q(I)$ by putting

$$Q(I)_n := \{z \in J_n \mid I \cap U(\mathfrak{g}(2n)) \subset z\}.$$

Recall that the natural $\mathfrak{g}(\infty)$-module V is the direct limit $\varinjlim V_n$, where V_n is the natural $\mathfrak{g}(2n)$-module. Furthermore, $\Lambda^\bullet(M)$ and $S^\bullet(M)$ denote respectively the symmetric and the exterior algebras of a module M, and $\Lambda^p(M)$ and $S^p(M)$ denote respectively the pth symmetric and the pth exterior powers M.

For simplicity we will use the following notations: given $p \in \mathbb{Z}_{\geq 0}$,

$E := Q(\mathrm{Ann}(\Lambda^\bullet V)),\ L_p := Q(\mathrm{Ann}(\Lambda^p V)),\ L_p^\infty := Q(\mathrm{Ann}(S^\bullet(V \otimes \mathbb{C}^p))),$
$E^\infty := \{\text{annihilators of all modules with integral highest weight}\},$

$$R := \begin{cases} \{\text{annihilators of spinor modules}\} \text{ for } \mathfrak{o}(\infty) \\ \{ \text{ or annihilators of Shale–Weil modules for } \mathfrak{sp}(\infty)\} \end{cases}$$

Then the following table describes the set of basic c.l.s.b. for the Lie algebras $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$.

Lie algebra	C.l.s.b.
$\mathfrak{o}(\infty)$	E, L_p, L_p^∞, E^∞, R
$\mathfrak{sp}(\infty)$	E, L_p, L_p^∞, E^∞, R

Proposition 5.8. *Any irreducible c.l.s.b. can be expressed uniquely as a product as follows:*

$$(L_v^\infty L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}})E^m$$

or

$$(L_v^\infty L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}})E^m R$$

where

$$r, v \in \mathbb{Z}_{\geq 0},\ x_i \in \mathbb{Z}_{\geq 0} \text{ for } v+1 \leq i \leq v+r.$$

Here, for $v = 0$, L_v^∞ is assumed to be the empty c.l.s.b.

Proof. A. Zhilinskii proved the analogous statement for c.l.s., see Theorem 2.8. Hence, for $\mathfrak{o}(\infty)$ the proposition is obvious because in this case c.l.s. and c.l.s.b. are the same objects.

In Proposition 5.6 we showed that the $\mathfrak{sp}(2n-2)$-branching of a bounded $\mathfrak{sp}(2n)$-module $L(\lambda)$ yields a set of highest weights which is obtained by translation via $\sum_{i=1}^n \varepsilon_i$ from the set of highest weights obtained from the $\mathfrak{o}(2n-2)$-branching of the $\mathfrak{o}(2n)$-module $L(\lambda + \sum_{i-1}^n \varepsilon_i)$.

Thus, there is a one-to-one correspondence between the set of c.l.s. for $\mathfrak{o}(\infty)$ and the set of c.l.s.b. for $\mathfrak{sp}(\infty)$, that respects the relation of inclusion, and the product operation. This completes the proof. $\square$

5.4. Classification of precoherent local systems of bounded ideals

In this subsection we show that the collection of sets $Q(I)$ corresponding to a primitive ideal $I \in U(\mathfrak{g}(\infty))$ is equivalent to a c.l.s.b. The respective notion of equivalence is defined below. Recall that every primitive ideal of $U(\mathfrak{g}(\infty))$ is a weakly bounded ideal, as proved in Theorem 3.11.

Definition 5.7. *A* precoherent local system of bounded ideals *(further p.l.s.b.) for $\mathfrak{g}(\infty)$ is a collection of sets*

$$\{\mathbb{I}_n\}_{n\in\mathbb{Z}_{\geq 2}} \subset \Pi_{n\in\mathbb{Z}_{\geq 2}} J_n$$

such that $\mathbb{I}_m \supset \langle \mathbb{I}_n \rangle_m$ for any $n > m$, where $\langle \mathbb{I}_n \rangle_m$ denotes the set of all annihilators of simple $\mathfrak{g}(2m)$-constituents of the $\mathfrak{g}(2n)$-module which are annihilated by at least one ideal from $\mathbb{I}_n$.

The definition of a weakly bounded ideal implies that $Q(I)$ is a p.l.s.b whenever I is a weakly bounded ideal.

Definition 5.8. *Two p.l.s.b. $\mathbb{I}$ and $\mathbb{I}'$ are* equivalent *if there exists an integer n such that $\mathbb{I}_{n'} = \mathbb{I}'_{n'}$ for any $n' > n$.*

As we pointed out above, there is a one-to-one correspondence between the set of c.l.s. for $\mathfrak{o}(\infty)$ and the set of c.l.s.b. for $\mathfrak{sp}(\infty)$ which respects the relation of inclusion, and product operation. If we consider a c.l.s.b. Q as a purely combinatorial object (i.e., as a set of highest $\mathfrak{sp}(2n)$-weights for $n \geq 2$) then we can describe the corresponding c.l.s. as follows. If our c.l.s.b. consists of integral $\mathfrak{sp}(2n)$-weights for $n \geq 2$ then the corresponding c.l.s. be the same set of weights considered as $\mathfrak{o}(2n)$-weights. If our c.l.s.b. consists of $\mathfrak{sp}(2n)$-weights with half-integral entries for $n \geq 2$ then the corresponding c.l.s. is obtained by adding $\varepsilon(2n)$ to each $\mathfrak{sp}(2n)$-weight and considering new weights as $\mathfrak{o}(2n)$-weights. Hence, the proofs of the Lemmas 4.4, 4.5, 4.6 are precisely the same as in Subsection 4.2 (we us all the notion from this subsection).

Lemma 5.9. *For any admissible n-tuple λ, the p.l.s. $Q^\vee(\lambda)$ is equivalent to the c.l.s.*

$$Q(\lambda) := \bigcup_{1\leqslant k\leqslant \#\lambda} Q(k, \lambda_k),$$

where the collection of sets $Q(k,a)$ for $k \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}/2$ is defined by putting

$$Q(k,a)_m := \{\mu \in A_m \mid \mu_k < a,\ if\ k \leq \#\mu\}.$$

Lemma 5.10. *Let λ and μ be admissible tuples such that $\#\mu \geqslant 2\#\lambda$. Then the following conditions are equivalent:*

1) $\mu \succ \lambda$*,*

2) $\mu_k \geqslant \lambda_k$ *for each* $1 \leqslant k \leqslant \#\lambda$*.*

Lemma 5.11. *Let λ and μ be admissible tuples such that $\#\mu - \#\lambda = 1$, $\mu > \lambda$ and let $k \in \mathbb{Z}$. Then*

1) $R(\mu, k) > R(\lambda, k)$*,*

2) $L(\mu, k) > L(\lambda, k)$*,*

3) $L(\mu, k) > R(\lambda, k)$ *whenever one of the following conditions is satisfied*

$$\mu_{i+1} \geqslant k > \mu_{i+2}, \tag{$*\,*$}$$

$$\mu_{i+2} \geqslant k \geqslant \mu_{i+3}. \tag{$*\,*\,*$}$$

for i such that $\lambda_i \geqslant k \geqslant \lambda_{i+1}$.

The following proposition is a corollary of the above three lemmas.

Proposition 5.12. *For any p.l.s.b. $\mathbb{I}$ there exists a c.l.s.b. $\mathbb{I}'$ such that $\mathbb{I}$ and $\mathbb{I}'$ are equivalent.*

Theorem 5.13. *Let I be a primitive ideal of $U(\mathfrak{g}(\infty))$. Then $Q(I)$ is equivalent to a c.l.s.b.*

Proof. Follows from Theorem 3.11 and Proposition 5.12. □

6. Classification of primitive ideals of $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$

Here we introduce a set of modules such that each primitive ideal of $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$ equals the annihilator of a unique module from this set. Also, for a module from this set we formulate a criterion for integrability of its annihilator.

In this subsection $\mathfrak{g}(2n)$ denotes the Lie algebra $\mathfrak{o}(2n)$ or $\mathfrak{sp}(2n)$, and $\mathfrak{g}(\infty)$ denotes the Lie algebra $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$.

Let Z be a Young diagram with row lengths

$$l_1 \geq l_2 \geq \cdots \geq l_s > 0.$$

Then we denote by $V_Z(n)$ the $\mathfrak{g}(2n)$-module with highest weight

$$\underbrace{(l_1,\ l_2,\ \ldots,\ l_s,\ 0,\ 0,\ \ldots,\ 0)}_{n \text{ numbers}}.$$

The $\mathfrak{g}(2n)$-module $V_Z(n)$ is isomorphic to a simple direct constituent of the tensor product

$$S^{l_1}(V(n)) \otimes S^{l_2}(V(n)) \otimes \cdots \otimes S^{l_s}(V(n)),$$

where $S^{l_i}(V(n))$ denotes the l_ith symmetric power of the natural module $V(n)$. In this way, the $\mathfrak{g}(\infty)$-module V_Z is defined as the direct limit $\varinjlim V_Z(n)$. We denote by R the $\mathfrak{o}(\infty)$-module which is equal to the direct limit $\varinjlim R(2n)$, where $R(2n)$ is the $\mathfrak{o}(2n)$-module with highest weight $(\frac{1}{2}\sum_1^n \varepsilon_i)$. Also we denote by R the $\mathfrak{sp}(\infty)$-module which is equal to the direct limit $\varinjlim SW^+(2n)$.

Proposition 6.1. [PP3] *Any nonzero prime integrable ideal $I \subsetneq \mathrm{U}(\mathfrak{g}(\infty))$ is the annihilator of a unique $\mathfrak{g}(\infty)$-module of the form*

$$\begin{array}{ll} (\mathrm{S}^\bullet(V))^{\otimes x} \otimes (\Lambda^\bullet(V))^{\otimes y} \otimes V_Z & \textit{for } \mathfrak{g}(\infty) = \mathfrak{sp}(\infty), \\ (\mathrm{S}^\bullet(V))^{\otimes x} \otimes (\Lambda^\bullet(V))^{\otimes y} \otimes V_Z \ \textit{or} & \\ (\mathrm{S}^\bullet(V))^{\otimes x} \otimes (\Lambda^\bullet(V))^{\otimes y} \otimes V_Z \otimes R & \textit{for } \mathfrak{g}(\infty) = \mathfrak{o}(\infty), \end{array}$$

where $x, y \in \mathbb{Z}_{\geq 0}$, and Z is an arbitrary Young diagram.

Definition 6.1. *Let $S_1, S_2 \subset J_n$ and $s_1, s_2 \subset R_n$ be respective sets of isomorphism classes of $\mathfrak{g}(2n)$-modules. Put*

$$S_1 \otimes S_2 := \{z \in J_n \mid z = \operatorname{Ann} r \text{ for some } r \in R_n \text{ such that}$$
$$\operatorname{Hom}_{\mathfrak{g}(2n)}(r, r_1 \otimes r_2) \neq 0 \ \text{ for some } r_1 \in s_1 \text{ and } r_2 \in s_2\}$$

Let $\mathbb{I}'$ be a c.l.s.b. of the form

$$(L_v^{\infty} L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}})E^m$$

and $\mathbb{I}''$ be a c.l.s.b. of the form

$$(L_v^{\infty} L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}})E^m R \text{ or } (L_v^{\infty} L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}})E^m$$

Then the tensor product $\mathbb{I}' \otimes \mathbb{I}''$ *of this two c.l.s.b. is the collection of sets defined by* $(\mathbb{I}' \otimes \mathbb{I}'')_i = \mathbb{I}'_i \otimes \mathbb{I}''_i$.

Lemma 6.2. *Let Q be a c.l.s.b of Lie algebra $\mathfrak{sp}(\infty)$ which can be expressed as*

$$Q = (L_1^{\infty})^{\otimes v} \otimes (L_1^{x_{v+1}} L_2^{x_{v+2}} \dots L_r^{x_{v+r}})E^m.$$

Then tensor product of c.l.s.b. $Q \otimes R$ is a c.l.s.b.

Proof. One can show that the statement follows from Lemma 5.4. □

One can easily deduce that

$$(L_v^{\infty} L_{v+1}^{x_{v+1}} L_{v+2}^{x_v+2} \dots L_{v+r}^{x_{v+r}})E^m = (L_1^{\infty})^{\otimes v} \otimes (L_1^{x_{v+1}} L_2^{x_{v+2}} \dots L_r^{x_{v+r}})E^m,$$

$$(L_v^{\infty} L_{v+1}^{x_{v+1}} L_{v+2}^{x_{v+2}} \dots L_{v+r}^{x_{v+r}})E^m R = (L_1^{\infty})^{\otimes v} \otimes (L_1^{x_{v+1}} L_2^{x_{v+2}} \dots L_r^{x_{v+r}})E^m \otimes R.$$

For every c.l.s.b. $Q = \{Q_n\}_{n \in \mathbb{Z}_{\geq 1}}$ we can define the ideal

$$I(Q) := \bigcup_m (\bigcap_{z \in Q_m} z) \subset U(\mathfrak{g}(\infty)).$$

We say that $I(Q)$ is the *globalization* of Q.

Let Z be a Young diagram with row lengths

$$l_1 \geq l_2 \geq \dots \geq l_s > 0.$$

For each positive number x we denote by $\{x\}$ the fractional part of x. Let $V(x, y, Z)(2n)$, for $x \in \mathbb{Z}_{\geq 0}$, $y \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, denote the simple $\mathfrak{g}(2n)$-module with highest weight

$$\sum_{i=1}^{x} (n + \{y\})\varepsilon_i + \sum_{i=x+1}^{x+s} (l_{i-x} + y)\varepsilon_i + \sum_{i=x+s+1}^{n} y\varepsilon_i$$

for $\mathfrak{g}(2n) = \mathfrak{o}(2n)$,

$$\sum_{i=1}^{x} (n - \{y\})\varepsilon_i + \sum_{i=x+1}^{x+s} (l_{i-x} + y - 2\{y\})\varepsilon_i + \sum_{i=x+s+1}^{n} (y - 2\{y\})\varepsilon_i$$

for $\mathfrak{g}(2n) = \mathfrak{sp}(2n)$, where $n \geq x+s$. It is clear that we can embed $V(x,y,Z)(2n) \hookrightarrow V(x,y,Z)(2n+2)$ as $\mathfrak{g}(2n)$ submodule. Now, we can define $\mathfrak{g}(\infty)$-module $V(x,y,Z)$ as the direct limit $\varinjlim V(x,y,Z)(2n)$. Let Q be a c.l.s.b of the form

- $(L_x^{\infty} L_{x+1}^{l_1} L_{x+2}^{l_2} \dots L_{x+s}^{l_s}) E^y$, then $I(Q) = \mathrm{Ann}(V(x,y,Z)) \subset U(\mathfrak{g}(\infty))$ for $y \in \mathbb{Z}_{\geq 0}$,
- $(L_x^{\infty} L_{x+1}^{l_1} L_{x+2}^{l_2} \dots L_{x+s}^{l_s}) E^{y-\frac{1}{2}} R$, then $I(Q) = \mathrm{Ann}(V(x,y,Z)) \subset U(\mathfrak{g}(\infty))$ for $y \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$.

The following is the main result of this section.

Theorem 6.3. a) *Any nonzero primitive ideal $I \subsetneq \mathrm{U}(\mathfrak{g}(\infty))$ is the annihilator $I(x,y,Z)$ of an $\mathfrak{sp}(\infty)$-module of the form*

$$(\mathrm{S}^{\bullet}(V))^{\otimes x} \otimes (\Lambda^{\bullet}(V))^{\otimes y} \otimes V_Z \text{ for } y \in \mathbb{Z}_{\geq 0},$$
$$(\mathrm{S}^{\bullet}(V))^{\otimes x} \otimes (\Lambda^{\bullet}(V))^{\otimes y} \otimes V_Z \otimes R \text{ for } y \in \mathbb{Z}_{\geq 0} + \frac{1}{2},$$

where $x \in \mathbb{Z}_{\geq 0}$, and Z is an arbitrary Young diagram. Moreover, $I(x_1,y_1,Z_1) = I(x_2,y_2,Z_2)$ if and only if $x_1 = x_2$, $y_1 = y_2$ and $Z_1 = Z_2$.

b) *The ideal $I(x,y,Z)$ is integrable if and only if $y \in \mathbb{Z}_{\geq 0}$.*

Theorem 6.3 follows form Proposition 6.4, Propositon 6.5 and Corollary 6.6 which we prove below.

Proposition 6.4. *Every primitive ideal $I \subset U(\mathfrak{sp}(\infty))$ is of the form $I(x,y,Z)$ for some $x \in \mathbb{Z}_{\geq 0}$, some $y \in \mathbb{Z}_{\geq 0}/2$, and some Young diagram Z.*

Proof. We claim that $I = I(Q(I))$. Indeed, the p.l.s.b. $Q(I)$ consists of all bounded ideals $z \supset I \cap U(\mathfrak{sp}(2n))$. Since I is weakly bounded, i.e., is such that every intersection $I \cap U(\mathfrak{sp}(2n))$ is an intersection of bounded ideals, we have $\bigcap_{z \in Q(I)_n} z = I \cap U(\mathfrak{sp}(2n))$. Thus,

$$I(Q(I)) = \bigcup_n I \cap U(\mathfrak{sp}(2n)) = I.$$

Recall, that for every p.l.s.b. $Q(I)$ we can find an equivalent c.l.s.b. $Q(I)'$. Therefore, there exists a bijection ϕ between the set of primitive ideals $I \subset U(\mathfrak{sp}(2n))$ and the set of c.l.s.b. Q, such that if $\phi(I) = Q$ then $I(Q) = I$. Lemma 6.2 and the above classification of c.l.s.b. in Proposition 5.8 complete the proof.

□

For our next proposition we need some preliminary considerations.

Let $Z(U(\mathfrak{sp}(2n)))$ be the center of $U(\mathfrak{sp}(2n))$, i.e., the set of all elements $g \in U(\mathfrak{sp}(2n))$ such that $gu = ug$ for each $u \in U(\mathfrak{sp}(2n))$.

Definition 6.2. *Let $L(\lambda)$ be a simple $\mathfrak{sp}(2n)$-module with highest weight λ, a wnd let v^+ be a highest weight vector of $L(\lambda)$. For $z \in Z(U(\mathfrak{sp}(2n)))$ we have $z \cdot v^+ = \chi(\lambda)(z) \cdot v^+$ for $\chi(\lambda)(z) \in \mathbb{C}$. Since z is a central element, it acts as $\chi(\lambda)(z)$ on $L(\lambda)$. The map*

$$\chi(\lambda)\colon Z(U(\mathfrak{sp}(2n))) \mapsto \mathbb{C}$$

is called the central character of weight λ.

For a Young diagram Z, we denote the length of the ith row of Z by Z_i. Let $I(x, y, Z)$ be a primitive ideal of $U(\mathfrak{sp}(\infty))$. One can check that the work of Zhiliskii [Zh3], together with Lemma 2.4 and Proposition 5.6 imply the following fact: $Q(I(x, y, Z))_n$ consists of all weights $\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i$ such that $\lambda_i - \lambda_j \in \mathbb{Z}_{\geq 0}$ for $i > j$, $\lambda_{n-1} + \lambda_n \geq -2$ for $\lambda_n \in \mathbb{Z} + \frac{1}{2}$, $\lambda_n \in \mathbb{Z}_{\geq 0}$ for $\lambda_n \in \mathbb{Z}$, and $y + Z_i - \lambda_{x+i} \in \mathbb{Z}_{\geq 0}$ for all $i \in \mathbb{Z}_{\geq 1}$. We call the c.l.s.b. $Q(I)$ *integral* whenever the entries of any $\lambda \in Q(I)$ are integers. If the entries of any $\lambda \in Q(I)$ are half-integers, then $Q(I)$ is *half-integral*.

Proposition 6.5. *Two primitive ideals $I_1 = I(x_1, y_1, Z_1), I_2 = I(x_2, y_2, Z_2)$ of $U(\mathfrak{sp}(\infty))$ are equal if and only if $x_1 = x_2$, $y_1 = y_2$ and $Z_1 = Z_2$.*

Proof. Assume that the triplet (x_1, y_1, Z_1) does not equal (x_2, y_2, Z_2). We will consider $Q(I)$ as a set of weights as in Subsections 5.3 and 5.4. Then there exists such n that $Q(I_1)_n \neq Q(I_2)_n$. This is equivalent to the following. We may assume without loss of generality that $x_1 \geq x_2$ with $n > x_2$, and that there exist a weight $\lambda' \in Q_n(I_1)$ and an integer $x_2 < k \leq n$, such that $\lambda'_i \geq \lambda_i$ for $x_2 < i < k$ and $\lambda'_k > \lambda_k$ for any $\lambda \in Q(I_2)_n$. Since the choice of λ' is not unique, we choose one with minimal possible k. Also, without loss of generality, we assume that $k = n$ because of the fact that a $\mathfrak{sp}(2m)$-module $L(\beta)$ always has an $\mathfrak{sp}(2l)$-submodule isomorphic to $L(\gamma)$, for $m > l$ and $\gamma = \sum_{i=1}^{l} \beta_i \varepsilon_i$.

Suppose that $I_1 = I_2$. Obviously, then $I_1 \cap U(\mathfrak{sp}(2n)) = I_2 \cap U(\mathfrak{sp}(2n))$. This implies

$$\bigcap_{\lambda \in Q(I_1)_n} \operatorname{Ann} L(\lambda) = \bigcap_{\lambda \in Q(I_2)_n} \operatorname{Ann} L(\lambda). \tag{26}$$

Next, we consider the intersections $I_1 \cap Z(U(\mathfrak{sp}(2n)))$ and $I_2 \cap Z(U(\mathfrak{sp}(2n)))$. These intersections are equal because $I_1 = I_2$. By Harish-Chandra's Theorem, $Z(U(\mathfrak{sp}(2n)))$ is isomorphic to the polynomial algebra $\mathbb{C}[a_1, a_2, \dots, a_n]$

in n variables. A maximal ideal F of $Z(U(\mathfrak{sp}(2n)))$ has the form

$$F = F(a) = \{f \in Z(U(\mathfrak{sp}(2n))) \mid f(a) = 0\},$$

for some $a \in \mathbb{C}^n$. Moreover, it is well known that the intersection of the annihilator of a simple highest weight $\mathfrak{sp}(2n)$-module $L(\lambda)$ with $Z(U(\mathfrak{sp}(2n)))$ equals to the maximal ideal $F(\chi(\lambda))$ of $Z(U(\mathfrak{sp}(2n)))$. This fact and formula (26) imply

$$\bigcap_{\lambda \in Q(I_1)_n} F(\chi(\lambda)) = \bigcap_{\lambda \in Q(I_2)_n} F(\chi(\lambda)).$$

The latter holds if and only if the Zariski closures of the sets $\{\chi(\lambda) \mid \lambda \in Q(I_1)_n\}$ and $\{\chi(\lambda) \mid \lambda \in Q(I_2)_n\}$ coincide.

We now choose the variables $a_1, a_2, \ldots, a_n$ to equal the independent Casimir elements G_s for $1 \leq s \leq n$, which act on the module $L(\lambda)$ by the constants

$$g_s(\lambda) = (-1)^s \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_s \leq n} \Pi_{j=1}^{s}((\lambda_{i_j} + n - i_j + 1)^2 - (i_j - j + 1)^2).$$

The expressions $g_s(\lambda)$ are symmetric polynomials in the variables $(\lambda_{i_j} + n - i_j + 1)^2$, see [IMR, Theorem 3.8]. Therefore,

$$\chi(\lambda) = (g_1(\lambda), g_2(\lambda), \ldots, g_n(\lambda)).$$

Next, we define the following equivalence relation on the set $Q(I_2)_n$:

$$\lambda \sim_{x_2} \mu \Longleftrightarrow \lambda_i = \mu_i \text{ for } i > x_2,$$

where $\lambda, \mu \in Q(I_2)_n$. In this way, we obtain finitely many equivalence classes. The class of λ is denoted by $K[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$.

Given $u, m, j \in \mathbb{Z}_{\geq 0}$, $m > x_2$, we set

$$\begin{aligned} d_{m,j} &= ((\lambda_m + n - m + 1)^2 - (m - j + 1)^2), \\ b_{u,j} &= (u - j + 1)^2. \end{aligned} \tag{27}$$

Furthermore, we consider the subset $S[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$ of $\mathbb{C}^n$ defined as the set of all points $(a_1, a_2, \ldots, a_n)$ such that

$$a_s = f_s = (-1)^s \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_s \leq n} \Big(\prod_{i_j \leq x_2} (y_{i_j} - b_{i_j,j}) \prod_{i_j > x_2} d_{i_j,j}\Big)$$

for all $1 \leq s \leq n$, and for all $(y_1, y_2, \ldots, y_{x_2}) \in \mathbb{C}^{x_2}$. It is clear that if $\lambda \in K[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$, then $\chi(\lambda)$ belongs to $S[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$.

Now, we will show that $S[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$ is an affine subspace of $\mathbb{C}^n$. The fact that the polynomials $g_s(\lambda)$ are symmetric implies that the polynomials f_s are symmetric in the variables $y_1, y_2, \ldots, y_{x_2}$. Note that the degree of f_s for $s \leq x_2$ is equal to s. In addition, the polynomial f_s is linear in each y_j, $1 \leq j \leq x_2$. Thus, f_s for $1 \leq s \leq x_2$ are linearly independent.

Hence, the polynomials f_s for $1 \leq s \leq n$ are linear combinations of the polynomials f_t for $1 \leq t \leq x_2$. Let s satisfy $x_2 < s \leq n$. Denote by $l_s(a_1, a_2 \ldots, a_{x_2})$ the affine-linear function such that $f_s = l_s(f_1, f_2 \ldots, f_{x_2})$. Then $S[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$ coincides with the affine subspace of $\mathbb{C}^n$ defined by the system of equations

$$\begin{cases} a_{x_2+1} = l_{x_2+1}(a_1, a_2, \ldots, a_{x_2}), \\ a_{x_2+2} = l_{x_2+2}(a_1, a_2, \ldots, a_{x_2}), \\ \ldots, \\ a_n = l_n(a_1, a_2, \ldots, a_{x_2}). \end{cases} \tag{28}$$

Next, we will show that $S[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$ is the Zariski closure of $K[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$. Indeed, assume that there is an equation $f(a_1, a_2, \ldots, a_n) = 0$ such that $f(\chi_\lambda) = 0$ for every $\lambda \in K[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$. Then we consider the function

$$\bar{f}(a_1, a_2 \ldots, a_{x_2}) = (a_1, \ldots, a_{x_2}, l_{x_2+1}(a_1, \ldots, a_{x_2}), \ldots, l_n(a_1, \ldots, a_{x_2})),$$

and note that $\bar{f}(\chi(\lambda)_1, \chi(\lambda)_2, \ldots, \chi(\lambda)_{x_2}) = 0$ for all $\lambda \in K[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$. However,the Combinatorial Nullstellensatz [A] claims that such polynomial is equal to 0. Thus $S[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$ is the Zariski closure of $K[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$.

Now, we consider the functions

$$u_t = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_s \leq x_2} (\prod_{j=1}^{s} (y_{i_j} - b_{i_j,j})$$

for $1 \leq t \leq x_2$. Each u_t can be expressed as linear combination of the functions $f_1, f_2, \ldots f_t$ with a nonzero coefficient of f_t. Hence, the functions u_t for $1 \leq t \leq x_2$ are linearly independent. Note that $d_{i_j,j} \neq 0$ for $\lambda_i \notin \mathbb{Z}$. This and formula (28) implies that the functions f_t for $n - x_2 < t \leq n$ are linearly independent whenever $Q(I_2)$ is half-integral.

Consider the point h^0 of $S[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$ defined by the parameters $y_i = b_{i,1}$. One can check that $h^0 = (\sum_{i>x_2} d_{i,1}, c_2, c_3, \ldots, c_{n-x_2}, 0, \ldots, 0)$ for some $c_1, c_2, \ldots, c_{n-x_2} \in \mathbb{C}$. The fact, that f_s for $n - x_2 < s \leq n$ are linearly independent whenever $Q(I_2)$ is half-integral, implies that a point $h^1 \in S[\lambda_{x_2+1}, \lambda_{x_2+2}, \ldots, \lambda_n]$ with $h^1_i = 0$ for $i > n - x_2$ must be equal to

h^0 whenever $Q(I_2)$ is half-integral. Denote $d_{i,1}$ by $d_i(\lambda)$ and h^0 by $h^0(\lambda)$. Clearly, $d_i(\lambda^1) > d_i(\lambda^2)$ if $\lambda_i^1 > \lambda_i^2$.

Recall that the proposition we are proving has already been proved by Zhilinskii in the work [Zh3] under the assumption that the c.l.s.b. $Q(I_1)$ and $Q(I_2)$ are integral, i.e. $Q(I_1)$ and $Q(I_2)$ are c.l.s. Hence it remains to prove our proposition for I_1 and I_2 such that at least one of the c.l.s.b. $Q(I_1)$ and $Q(I_2)$ is half-integral.

Denote by $Z(S)$ the Zariski closure of S in $\mathbb{C}^n$. Let V be a set of $\mathfrak{sp}(2n)$-weights. We denote the respective set of central characters χ_λ for $\lambda \in V$ by $H(V)$.

In what follows we denote by $\sim_{x_2}$ the equivalence relation on the set $Q(I_1)_n$, constructed exactly as the equivalence relation on $Q(I_2)_n$ denoted above by $\sim_{x_2}$. We abbreviate

$$K[\lambda] := K[\lambda_{n-x_2+1}, \lambda_{n-x_2+2}, \dots, \lambda_n]$$

and

$$S[\lambda] := S[\lambda_{n-x_2+1}, \lambda_{n-x_2+2}, \dots, \lambda_n].$$

Finally, we prove that $I_1 \neq I_2$ by considering the following cases:

- $Q(I_1)$ is integral, $Q(I_2)$ is half-integral, and $x_1 > x_2$. It is clear that $n = x_2 + 1$. This implies that $Z(H(Q(I_1)_n)) = 0$, while the Zariski closure $Z(H(Q(I_2)_n))$ is finite union of proper affine subspaces, and hence $I_1 \neq I_2$.

- $Q(I_1)$ is half-integral, $Q(I_2)$ is integral or half-integral, and $x_1 > x_2$. We obtain that $n = x_2 + 1$ and that $Z(H(Q(I_1)_n)) = 0$, $Z(H(Q(I_2)_n)) \neq 0$ similarly to the case when $Q(I_1)$ is integral, $Q(I_2)$ is half-integral, and $x_1 > x_2$.

- $Q(I_1)$ is half-integral, $Q(I_2)$ is integral or half-integral, and $x_1 = x_2$. Then $Z(H(Q(I_1)_n)) = \bigcap_{i=1}^{y} S(\lambda^i)$ for some $\lambda^i \in Q(I_1)_n$, and denote $S_i := S(\lambda^i)$. Also we obtain $Z(H(Q(I_2)_n)) = \bigcap_{j=1}^{u} S(\lambda^j)$ for some $\lambda^j \in Q(I_2)_n$, and put $S'_j = S(\lambda^j)$. Therefore, $Z(H(Q(I_1)_n)) = Z(H(Q(I_2)_n))$ if and only if the sets $\{S_1, S_2, \dots, S_y\}$ and $\{S'_1, S'_2, \dots, S'_u\}$ are equal. We show that $S(\lambda') \neq S'_j$ for any $1 \leq j \leq u$. There are two possibilities for S'_j: first, the coordinates with indices $n - x_2 + 1, n - x_2 + 2, \dots, n$ of points in S' are not linearly independent; second, the coordinates with indices $n - x_2 + 1, n - x_2 + 2, \dots, n$ of points in S' are linearly independent. In the first case, we have $S(\lambda') \neq S_j$ because the last x_2 coordinates of points in $S(\lambda')$ are linearly independent. In the second case, we note that $S(\lambda')$ contains the point $h^0(\lambda') =$

$(\sum_{i>x_2} d_i(\lambda'), c'_2, c'_3, \ldots, c'_{n-x_2}, 0, \ldots, 0)$. Since $\lambda'_i \geq \lambda_i$ for $x_2 \leq i \leq n$, we have $\sum_{i>x_2} d_i(\lambda') > \sum_{i>x_2} d_i(\lambda)$ where $\lambda \in Q(I_2)_n$. Therefore $h_0(\lambda^j) \neq h_0(\lambda')$ for $\lambda^j \in Q(I_2)_n$. As $h_0(\lambda^j)$ is the unique point with $\lambda^j_s = 0$ for $n - x_2 + 1 \leq s \leq n$, we conclude that $S(\lambda)$ does not coincide with S'_j for all j, and hence $I_1 \neq I_2$.

- $Q(I_1)$ is integral, $Q(I_2)$ is half-integral, and $x_1 = x_2$. Then it is clear that $k = n = x_2 + 1$. Note that, if $d_{n,n}(\lambda') = 0$ then $a_n = 0$ for each $a \in S(\lambda')$. On the other hand, formula (27) implies $d_{n,n}(\lambda')$ is equal to zero if and only if $\lambda'_n = 0$. This implies $Q(I_1) = S(\lambda')$. Thus the Zariski closures $Z(H(Q(I_2)_n))$ and $Z(H(Q(I_1)_n))$ do not coincide because there exists $a \in Z(H(Q(I_2)))$ such that $a_n \neq 0$. For $d_{n,n}(\lambda') \neq 0$ proof is the same as for the case when $Q(I_1)$ is half-integral, $Q(I_2)$ is integral or half-integral, and $x_1 = x_2$.

□

Corollary 6.6. *Every primitive ideal $I = I(x, y, Z) \subset U(\mathfrak{sp}(\infty))$ with $y \in \mathbb{Z} + \frac{1}{2}$, is nonintegrable.*

Proof. Follows from the Proposition 6.5 and the classification of integrable ideals given by Zhilinskii in [Zh1], [Zh2] and [Zh3]. □

We conclude this thesis by the remark that we have now established that the primitive ideals of $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$ are described by the same triplets (x, y, Z). This follows from a direct comparison of Proposition 4.8 in [PP3] and Theorem 6.3 above. The only difference between the two cases that the primitive ideals $I(x, y, Z)$ with $y \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ are integrable in the case of $U(\mathfrak{o}(\infty))$, and nonintegrable in the case of $U(\mathfrak{sp}(\infty))$. This remark is a strong hint for the conjecture that the isomorphism of the lattices of ideals in $U(\mathfrak{o}(\infty))$ and $U(\mathfrak{sp}(\infty))$ constructed in [PP3] preserves primitivity.

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