BRANCHING LAWS FOR TENSOR MODULES OVER CLASSICAL LOCALLY FINITE LIE ALGEBRAS

by

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Abstract

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Given an embedding $\mathfrak{g}' \subset \mathfrak{g}$ of two Lie algebras and an irreducible $\mathfrak{g}$-module M, the branching problem is to determine the structure of M as a module over $\mathfrak{g}'$. In this thesis, we consider the case when $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of classical locally finite Lie algebras ([DP1]) and M is a simple tensor $\mathfrak{g}$-module ([PSt], [PSe]). The goal of the thesis is to solve the branching problem for these data. Since M is in general a not completely reducible $\mathfrak{g}'$-module, we determine the socle filtration of M over $\mathfrak{g}'$. Due to the description of embeddings of classical locally finite Lie algebras given in [DP1], when $\mathfrak{g}' \ncong \mathrm{gl}(\infty)$ our result holds for all possible embeddings $\mathfrak{g}' \subset \mathfrak{g}$.

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Chapter 1

Introduction

The structure and representation theory of locally finite Lie algebras has been a very active research area in the last 15 years and is nowadays an important part of the theory of infinite-dimensional Lie algebras and their applications. Locally finite Lie algebras of countable dimension are direct limits of finite-dimensional Lie algebras and can be considered as natural generalizations of finite-dimensional Lie algebras. Many notions from the finite-dimensional case have their "locally finite" counterparts. In particular, the notions of a (semi)simple Lie algebra (resp., solvable, nilpotent) are replaced by the notions of a locally (semi)simple (resp., locally solvable, locally nilpotent) Lie algebra. Furthermore, the notions of Cartan, Borel and parabolic subalgebras are introduced in a most natural way.

An important subclass of the general class of locally finite Lie algebras is the class of simple finitary Lie algebras, introduced by A. Baranov. In a series of papers Baranov classified, up to isomorphism, the simple finitary Lie algebras over $\mathbb{R}$ and $\mathbb{C}$ ([B1], [B2], [B3]). Baranov's result for simple finitary Lie algebras of countable dimension over $\mathbb{C}$ is very easy to state. Up to isomorphism, there are only three such Lie algebras: $\mathrm{sl}(\infty), \mathrm{sp}(\infty)$, and $\mathrm{so}(\infty)$ ([B1]). The analogous problem in group theory had been solved previously by Hall ([Ha]), and some notions from there have been adapted by Baranov to the case of Lie algebras.

The structure and representation theory of countable-dimensional finitary Lie algebras is nowadays quite developed. In particular, their Cartan, Borel and parabolic subalgebras have been classified ([DaPSn], [DP2], [Da], [DaP]). The current thesis solves the branching problem for simple tensor modules over the classical finitary Lie algebras $\mathrm{gl}(\infty)$, $\mathrm{sl}(\infty)$, $\mathrm{sp}(\infty)$, and $\mathrm{so}(\infty)$.

The branching problem is a classical problem in the theory of irreducible representations of finite-dimensional Lie algebras. Given a pair $\mathfrak{g}' \subset \mathfrak{g}$ of finite-dimensional Lie algebras and an irreducible $\mathfrak{g}$-module M, the branching problem is to determine the structure of M as a module over $\mathfrak{g}'$. When $\mathfrak{g}'$ is semisimple, in view of Weyl's semisimplicity theorem the branching problem reduces to finding the multiplicity of any simple $\mathfrak{g}'$-module M' as a direct summand of M. This is however not a simple task, due to the abundance of possible isomorphism classes of embeddings $\mathfrak{g}' \subset \mathfrak{g}$ ([Dy]). Therefore, even for the classical series of Lie algebras an explicit solution of the branching problem is known only for specific cases. Examples of such branching rules are the Gelfand-Tsetlin rule ([Z], [GW]) and certain branching rules for diagonal embeddings ([HTW]).

When we consider the classical locally finite Lie algebras $\mathrm{gl}(\infty), \mathrm{sl}(\infty), \mathrm{sp}(\infty)$, and $\mathrm{so}(\infty)$ the situation is quite different. On the one hand, one can give a description of the Lie algebra embeddings which is simpler than the classical description of Dynkin in the finite-dimensional case. On the other hand, the modules of interest, called simple tensor modules, are in general not completely reducible over the subalgebra. Therefore, the branching problem involves more than just determining the multiplicities of all simple constituents. One has to determine a semisimple filtration of the given module over the subalgebra. It is a natural choice to work with the socle filtration. In this way, the goal of the present work is to solve the following branching problem. Given an embedding $\mathfrak{g}' \subset \mathfrak{g}$ of two classical finitary Lie algebras and a simple tensor $\mathfrak{g}$-module M, find the socle filtration of M as a $\mathfrak{g}'$-module.

Four main articles lie in the background of this thesis, namely [DP1], [PSt], [PSe], and [DaPSe]. In [DP1] a description of all possible embeddings $\mathfrak{g}' \subset \mathfrak{g}$ is given, where $\mathfrak{g} \cong \mathrm{gl}(\infty), \mathrm{sl}(\infty), \mathrm{sp}(\infty)$, or $\mathrm{so}(\infty)$ and $\mathfrak{g}'$ is a simple locally finite subalgebra. Then, in [PSt] the structure of certain basic tensor $\mathfrak{g}$-modules is described. In [DaPSe] the general definition of tensor module is given and the category $\mathbb{T}_{\mathfrak{g}}$ of tensor modules for $\mathfrak{g} \cong \mathrm{sl}(\infty), \mathrm{sp}(\infty)$, or $\mathrm{so}(\infty)$ is introduced. Furthermore, [PSe] defines a larger category of $\mathfrak{g}$-modules for $\mathfrak{g} \cong \mathrm{sl}(\infty), \mathrm{sp}(\infty)$, or $\mathrm{so}(\infty)$, denoted by $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. Both categories $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ and $\mathbb{T}_{\mathfrak{g}}$ have the same simple objects, and those are precisely the modules for which we study the branching problem. A very recent work by Frenkel, Penkov, and Serganova shows that the category $\mathbb{T}_{\mathfrak{g}}$ is interesting not only for representation theoretic reasons, but moreover plays a major role in a categorification of the boson-fermion correspondence.

An important consequence from [PSe] is that whenever $\mathfrak{g}'$ is a locally simple subalgebra of $\mathfrak{g}$ and M is a simple tensor $\mathfrak{g}$-module, M considered as a $\mathfrak{g}'$-module is an object in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}'}$. Thus, the different embeddings $\mathfrak{g}' \subset \mathfrak{g}$ provide a tool for constructing a large variety of objects in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}'}$, some of which are not necessarily tensor modules. And the solution of the branching problem for $\mathfrak{g}' \subset \mathfrak{g}$ describes the structure of these objects.

We now describe the body of the thesis in detail. In Chapter 2 we define the basic notions related to locally finite Lie algebras of countable dimension and discuss the results from the papers [PSt], [DP1], and [PSe] mentioned above. Following the description of the possible embeddings given in [DP1], we introduce the notion of an embedding $\mathfrak{g}' \subset \mathfrak{g}$ of general tensor type. In the end of the chapter we present some branching rules for embeddings of finite-dimensional Lie algebras. These branching rules will be used throughout the thesis.

In Chapters 3 and 4 we consider embeddings $\mathfrak{g}' \subset \mathfrak{g}$ where $\mathfrak{g}' \cong \mathfrak{g}$. We decompose the embeddings of general tensor type into several intermediate embeddings of specific types and solve the branching problem for each of these intermediate embeddings. In particular, we show that the case $\mathrm{sl}(\infty) \subset \mathrm{sl}(\infty)$ is equivalent to the case $\mathrm{gl}(\infty) \subset \mathrm{gl}(\infty)$.

Chapter 5 focuses mainly on the proof of Theorem 5.2. This theorem shows that the branching problem for embeddings $\mathfrak{g}' \subset \mathfrak{g}$ of general tensor type can be reduced to branching problems for embeddings of simpler types. In the cases $\mathfrak{g}' \cong \mathfrak{g}$ these are exactly the types considered in Chapters 3 and 4. Thus, Theorem 5.2 justifies the choice of intermediate embeddings made in Chapters 3 and 4, and as a consequence solves fully the branching problem for embeddings $\mathfrak{g}' \subset \mathfrak{g}$ with $\mathfrak{g}' \cong \mathfrak{g}$.

In Chapter 6 we solve the branching problem for embeddings $\mathfrak{g}' \subset \mathfrak{g}$ such that $\mathfrak{g}'$ and $\mathfrak{g}$ are non-isomorphic. We start by reducing the number of possible pairs $\mathfrak{g}' \subset \mathfrak{g}$ via identification of equivalent pairs. Then, using Theorem 5.2, we further reduce the branching problem for embeddings of general tensor type to much simpler branching problems which we then solve explicitly.

Finally, Chapter 7 contains some remarks and observations on the results obtained in the thesis. In particular, we point out what are the invariants of an embedding $\mathfrak{g}' \subset \mathfrak{g}$ which determine completely the solution of the branching problem. We also derive some immediate corollaries of the main results of the thesis.

Chapter 2

Preliminaries

All Lie algebras considered will be defined over the field of complex numbers $\mathbb{C}$ and will be at most countable dimensional.

2.1 The classical locally finite Lie algebras

An infinite-dimensional Lie algebra $\mathfrak{g}$ is called *locally finite* if every finite subset of $\mathfrak{g}$ is contained in a finite-dimensional subalgebra. When $\mathfrak{g}$ is at most countable dimensional, being locally finite is equivalent to admitting an exhaustion $\mathfrak{g} = \bigcup_{i\in\mathbb{Z}_{>0}} \mathfrak{g}_i$ where

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_i \subset \ldots \tag{2.1}$$

is a sequence of nested finite-dimensional Lie algebras. A locally finite Lie algebra is called *locally simple* (respectively *locally semisimple*) if it admits an exhaustion (2.1) so that all $\mathfrak{g}_i$ are simple (resp. semisimple). It is an easy fact that any locally simple Lie algebra is simple. The first examples of infinite-dimensional locally simple Lie algebras are the Lie algebras $\mathrm{sl}(\infty)$, $\mathrm{sp}(\infty)$, and $\mathrm{so}(\infty)$. They are defined respectively as $\mathrm{sl}(\infty) = \bigcup_{i\in\mathbb{Z}_{>0}} \mathrm{sl}(i)$, $\mathrm{sp}(\infty) = \bigcup_{i\in\mathbb{Z}_{>0}} \mathrm{sp}(2i)$, and $\mathrm{so}(\infty) = \bigcup_{i\in\mathbb{Z}_{>0}} \mathrm{so}(i)$ via the natural inclusions $\mathrm{sl}(i) \subset \mathrm{sl}(i+1)$, $\mathrm{sp}(2i) \subset \mathrm{sp}(2i+2)$, and $\mathrm{so}(i) \subset \mathrm{so}(i+1)$. In contrast to the finite-dimensional case, here we do not distinguish the types B and D. The reason is that both the union of odd orthogonal Lie algebras and the union of even orthogonal Lie algebras under the natural inclusions are isomorphic to $\mathrm{so}(\infty)$.

A first example of a locally finite Lie algebra which is not locally simple is the Lie algebra $\mathrm{gl}(\infty)$. It is defined as the union $\mathrm{gl}(\infty) = \bigcup_{i\in\mathbb{Z}_{>0}} \mathrm{gl}(i)$ via the inclusions $\mathrm{gl}(i) \subset \mathrm{gl}(i+1)$. A Lie algebra is called *finitary* if it isomorphic to a subalgebra of gl(∞). As we already mentioned in the introduction, sl(∞), sp(∞), and so(∞) are, up to isomorphism, the only finitary locally simple Lie algebras ([B1]). Together with gl(∞), these four Lie algebras are usually referred to as the classical locally finite Lie algebras.

Now we give a quick overview of another approach to defining the Lie algebras gl(∞), sl(∞), sp(∞), and so(∞), which will be extensively used in the thesis. Let V and V_* be countable-dimensional vector spaces over $\mathbb{C}$ together with a non-degenerate bilinear pairing $\langle\cdot,\cdot\rangle : V \times V_* \to \mathbb{C}$. The vector space $V \otimes V_*$ is endowed with the structure of an associative algebra such that

$$(v_1 \otimes w_1)(v_2 \otimes w_2) = \langle v_2, w_1\rangle\, v_1 \otimes w_2$$

where $v_1, v_2 \in V$ and $w_1, w_2 \in V_*$. We denote by $\mathrm{gl}(V, V_*)$ the Lie algebra arising from the associative algebra $V \otimes V_*$, and by $\mathrm{sl}(V, V_*)$ we denote its commutator subalgebra $[\mathrm{gl}(V, V_*), \mathrm{gl}(V, V_*)]$. If $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ is an antisymmetric non-degenerate bilinear form, we define the Lie algebra $\mathrm{gl}(V, V)$ as above by taking $V_* = V$. In this case $S^2(V)$, the second symmetric power of V, is a Lie subalgebra of $\mathrm{gl}(V, V)$ and we denote it by $\mathrm{sp}(V)$. Similarly, if $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ is a symmetric non-degenerate bilinear form, we again define $\mathrm{gl}(V, V)$ by taking $V_* = V$ and then $\bigwedge^2(V)$ is a Lie subalgebra of $\mathrm{gl}(V, V)$, which we denote by $\mathrm{so}(V)$.

The vector spaces V and V_* are naturally modules over the Lie algebras defined above, such that $(v_1 \otimes w_1) \cdot v_2 = \langle v_2, w_1\rangle\, v_1$ and $(v_2 \otimes w_2) \cdot w_1 = -\langle v_2, w_1\rangle\, w_2$ for any $v_1, v_2 \in V$ and $w_1, w_2 \in V_*$. We call them respectively *the natural* and *the conatural representations*. In the cases of $\mathrm{sp}(V)$ and $\mathrm{so}(V)$ we have $V = V_*$.

By a result of Mackey [M], there always exist dual bases $\{\xi_i\}_{i\in I}$ of V and $\{\xi_i^*\}_{i\in I}$ of V_* indexed by a countable set I, so that $\langle \xi_i, \xi_j^*\rangle = \delta_{ij}$. Using these bases, we can identify $\mathrm{gl}(V, V_*)$ with the Lie algebra gl(∞), which we defined earlier. Similarly, $\mathrm{sl}(V, V_*) \cong \mathrm{sl}(\infty)$, $\mathrm{sp}(V) \cong \mathrm{sp}(\infty)$, and $\mathrm{so}(V) \cong \mathrm{so}(\infty)$.

In the rest of this section let $\mathfrak{g} \cong \mathrm{gl}(\infty), \mathrm{sl}(\infty), \mathrm{sp}(\infty)$, or $\mathrm{so}(\infty)$. We now turn our attention to the representation theory of $\mathfrak{g}$. We set $V^{\otimes(p,q)} = V^{\otimes p} \otimes V_*^{\otimes q}$, where V and V_* are as above ($V^{\otimes(p,q)} = V^{\otimes(p+q)}$ when $\mathfrak{g} \cong \mathrm{sp}(\infty), \mathrm{so}(\infty)$). The modules $V^{\otimes(p,q)}$ were studied first in [PSt]. In particular, it was shown in [PSt] that $V^{\otimes(p,q)}$ is a semisimple $\mathfrak{g}$-module only if $pq = 0$ for $\mathfrak{g} \cong \mathrm{gl}(\infty), \mathrm{sl}(\infty)$, and if $p + q \leq 1$ for $\mathfrak{g} \cong \mathrm{sp}(\infty), \mathrm{so}(\infty)$. As a result, to describe the structure of these modules one needs the language of socle filtrations.

The *socle filtration* of any module M over a ring or an algebra is defined in the following way:

$$0 \subset \mathrm{soc}M \subset \mathrm{soc}^{(2)}M \subset \cdots \subset \mathrm{soc}^{(r)}M \subset \cdots$$

where $\mathrm{soc}M$, called the *socle* of M, is the maximal semisimple submodule of M, or equivalently the sum of all simple submodules of M. The other terms in the filtration are defined inductively as follows: $\mathrm{soc}^{(r+1)}M = \pi_r^{-1}(\mathrm{soc}(M/\mathrm{soc}^{(r)}M))$, where $\pi_r : M \to M/\mathrm{soc}^{(r)}M$ is the natural projection. The semisimple modules $\overline{\mathrm{soc}}^{(r+1)}M := \mathrm{soc}^{(r+1)}M/\mathrm{soc}^{(r)}M$ are called the *layers* of the socle filtration. We say that M has *finite Loewy length* l if the socle filtration of M is finite and $l = \min\{r|\mathrm{soc}^{(r)}M = M\}$.

The following three properties of socle filtrations will be very useful in the sequel.

- If $N \subseteq M$, then for all r

$$\mathrm{soc}^{(r)}N = (\mathrm{soc}^{(r)}M) \cap N. \tag{2.2}$$

- If M_1 and M_2 are modules over the same ring or algebra and $M = M_1 \cap M_2$, then

$$\mathrm{soc}^{(r)}M = (\mathrm{soc}^{(r)}M_1) \cap (\mathrm{soc}^{(r)}M_2). \tag{2.3}$$

- If M and N are any two modules over the same ring or algebra, then

$$\mathrm{soc}^{(r)}(M \oplus N) = \mathrm{soc}^{(r)}M \oplus \mathrm{soc}^{(r)}N. \tag{2.4}$$

In what follows, if M is a module over the Lie algebra $\mathfrak{g}$ we will use the notation $\mathrm{soc}_{\mathfrak{g}}^{(r)}M$ instead of $\mathrm{soc}^{(r)}M$.

When $\mathfrak{g} \cong \mathrm{gl}(\infty)$, we set $V^{\{p,q\}} = \mathrm{soc}_{\mathfrak{g}}V^{\otimes(p,q)}$. It is also the maximal semisimple submodule of $V^{\otimes(p,q)}$ for $\mathfrak{g} \cong \mathrm{sl}(\infty)$ ([PSt]). For $\mathfrak{g} \cong \mathrm{sp}(\infty), \mathrm{so}(\infty)$ the maximal semisimple submodule of $V^{\otimes(p+q)}$ is denoted respectively by $V^{\langle p+q\rangle}$ and $V^{[p+q]}$.

By definition, a $\mathfrak{g}$-module M is called a *tensor module* if it is a subquotient of a finite direct sum of copies of $\bigoplus_{p+q\leq r} V^{\otimes(p,q)}$ for some integer r. If M is simple, being a tensor module is equivalent to being a submodule of $V^{\otimes(p,q)}$ for some p, q ([PSt]). Moreover, it is shown in [PSt] that there exists a choice of Borel subalgebra $\mathfrak{b}$ in $\mathfrak{g}$ such that all simple tensor modules are $\mathfrak{b}$-highest weight modules. Their highest weights are described using integer partitions. A *non-negative integer partition* λ with k parts is an integer sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. As in [HTW], we write $l(\lambda)$ to denote the length (or depth) of the partition λ (i.e. $l(\lambda) = k$) and $|\lambda|$ for the size of the partition (i.e. $|\lambda| = \sum_i \lambda_i$).

For $\mathfrak{g} \cong \mathrm{gl}(\infty), \mathrm{sl}(\infty)$ the simple tensor modules coincide. Their highest weights are given by pairs of non-negative integer partitions (λ, μ) and we denote them by $V_{\lambda,\mu}$. Moreover, if $V_{\lambda,\mu} \subset V^{\otimes(p,q)}$, then $|\lambda| = p$ and $|\mu| = q$. The modules $V_{\lambda,\mu}$ are constructed explicitly in [PSt] using a generalization of Weyl's construction for irreducible $\mathrm{gl}(n)$-modules (see [FH]). For $\mathfrak{g} \cong \mathrm{sp}(\infty), \mathrm{so}(\infty)$ the simple tensor modules are denoted respectively by $V_{\langle\lambda\rangle}$ and $V_{[\lambda]}$, where λ is a non-negative integer partition. If $V_{\langle\lambda\rangle} \subset V^{\otimes(p+q)}$ or respectively $V_{[\lambda]} \subset V^{\otimes(p+q)}$, then $|\lambda| = p + q$. It is shown in [PSt] that

$$V_{\langle\lambda\rangle} = V_{\lambda,0} \cap V^{\langle p+q\rangle}, \qquad V_{[\lambda]} = V_{\lambda,0} \cap V^{[p+q]}.$$

The tensor modules are natural analogues of the finite-dimensional representations of the classical Lie algebras $\mathrm{sl}(n)$, $\mathrm{sp}(2n)$, and $\mathrm{so}(n)$. The main motivation for this analogy is the fact that Weyl's construction provides a way to construct all irreducible finite-dimensional modules of $\mathrm{sl}(n)$ and $\mathrm{sp}(2n)$ and almost all such modules for $\mathrm{so}(n)$ (see [FH]). And as we mentioned above, the simple tensor modules are defined using a generalization of this construction.

In [DaPSe], the category $\mathbb{T}_{\mathfrak{g}}$ for $\mathfrak{g} \cong \mathrm{sl}(\infty), \mathrm{sp}(\infty), \mathrm{so}(\infty)$ is introduced. Its objects are precisely the tensor modules. Furthermore, the simple tensor modules are directly related to several larger categories of $\mathfrak{g}$-modules (see e.g. [PSe], [DaPSe]). Here we define two of these categories, the latter one being directly connected to the following work.

Let $\mathfrak{g} \cong \mathrm{sl}(\infty), \mathrm{sp}(\infty), \mathrm{so}(\infty)$. A $\mathfrak{g}$-module M is called *integrable* if $\dim \mathrm{span}\{m, g \cdot m, g^2 \cdot m, \dots\} < \infty$ for any $m \in M$ and $g \in \mathfrak{g}$. Since $\mathfrak{g}$ is locally simple, this definition is equivalent to the condition that, when restricted to any semisimple finite-dimensional subalgebra $\mathfrak{f}$ of $\mathfrak{g}$, M is isomorphic to a (not necessarily countable) direct sum of finite-dimensional $\mathfrak{f}$-modules. Following [PSe], we denote by $\mathrm{Int}_{\mathfrak{g}}$ the category of integrable $\mathfrak{g}$-modules.

The other category we define is the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. Its objects M are integrable modules of finite Loewy length such that the algebraic dual M^* is also integrable and of finite Loewy length. In other words, $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is the largest subcategory of $\mathrm{Int}_{\mathfrak{g}}$ which is closed under algebraic dualization and such that every object in it has finite Loewy length. Then, it is proven in [PSe] that the simple objects of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are precisely the simple tensor modules. Another important result in [PSe] is that $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is functorial with respect to any homomorphism $\varphi : \mathfrak{g}' \to \mathfrak{g}$ where $\mathfrak{g}', \mathfrak{g} \cong \mathrm{sl}(\infty), \mathrm{sp}(\infty), \mathrm{so}(\infty)$.

This means that, given such a homomorphism $\varphi : \mathfrak{g}' \to \mathfrak{g}$, any $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ considered as a $\mathfrak{g}'$-module is an object of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}'}$. In particular, for our purposes it is important that if we have an embedding $\mathfrak{g}' \subset \mathfrak{g}$ and M is a simple tensor $\mathfrak{g}$-module, then M has finite Loewy length as a module over $\mathfrak{g}'$ and all simple constituents in the socle filtration of M as a $\mathfrak{g}'$-module are simple tensor $\mathfrak{g}'$-modules.

Let again $\mathfrak{g} \cong \mathrm{gl}(\infty), \mathrm{sl}(\infty), \mathrm{sp}(\infty), \mathrm{so}(\infty)$. It is shown in [DP1] that any locally semisimple subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ is isomorphic to a direct sum of simple Lie algebras each of which is either finite-dimensional or is itself isomorphic to $\mathrm{sl}(\infty)$, $\mathrm{sp}(\infty)$, or $\mathrm{so}(\infty)$. As we mentioned above, the latter was also proven earlier by Baranov in [B1]. Furthermore, for any fixed $\mathfrak{g}'$, Dimitrov and Penkov describe the structure of the $\mathfrak{g}$-modules V and V_* as $\mathfrak{g}'$-modules. Here we consider only the case when $\mathfrak{g}'$ is a simple subalgebra of $\mathfrak{g}$. Then the following result holds.

Theorem 2.1. *[DP1] Let $\mathfrak{g} \cong \mathrm{gl}(\infty), \mathrm{sl}(\infty), \mathrm{sp}(\infty)$, or $\mathrm{so}(\infty)$ and let $\mathfrak{g}'$ be a simple infinite-dimensional subalgebra of $\mathfrak{g}$. Let V and V_* be respectively the natural and conatural represenations of $\mathfrak{g}$. Similarly, let V' and V'_* be the natural and conatural representations of $\mathfrak{g}'$. Then*

$$\begin{aligned} \mathrm{soc}_{\mathfrak{g}'} V &\cong kV' \oplus lV'_* \oplus N_a, & V/\mathrm{soc}_{\mathfrak{g}'} V &\cong N_b, \\ \mathrm{soc}_{\mathfrak{g}'} V_* &\cong lV' \oplus kV'_* \oplus N_c, & V_*/\mathrm{soc}_{\mathfrak{g}'} V_* &\cong N_d, \end{aligned} \tag{2.5}$$

where $k, l \in \mathbb{Z}_{\geq 0}$ such that at least one of them is in $\mathbb{Z}_{>0}$, and N_a, N_b, N_c, and N_d are finite- or countable-dimensional trivial $\mathfrak{g}'$-modules.

We denote $a = \dim N_a$, $b = \dim N_b$, $c = \dim N_c$, and $d = \dim N_d$.

Motivated by Theorem 2.1, in this thesis we consider pairs $\mathfrak{g}', \mathfrak{g}$ of two classical locally finite Lie algebras such that the embedding $\mathfrak{g}' \subset \mathfrak{g}$ satisfies property (2.5). We will refer to such embeddings as embeddings of *general tensor type*. In view of Theorem 2.1, when $\mathfrak{g}' \ncong \mathrm{gl}(\infty)$, (2.5) describes all possible embeddings $\mathfrak{g}' \subset \mathfrak{g}$.

2.2 Finite-dimensional branching laws

In this section we present some branching rules for embeddings of finite-dimensional Lie algebras. In view of Theorem 2.1 and the discussion above we are interested in the following type of embeddings.

Definition 2.1. *An embedding $\mathfrak{f}_1 \subset \mathfrak{f}_2$ of finite-dimensional classical Lie algebras is* *called diagonal if*

$$V_{2\downarrow\mathfrak{f}_1} \cong \underbrace{V_1 \oplus \cdots \oplus V_1}_{l} \oplus \underbrace{V_1^* \oplus \cdots \oplus V_1^*}_{r} \oplus \underbrace{N_1 \oplus \cdots \oplus N_1}_{z}$$

where V_i is the natural $\mathfrak{f}_i$-module (i=1,2), V_1^* is dual to V_1, and N_1 is the trivial one-dimensional $\mathfrak{f}_1$-module. The triple (l,r,z) is called the signature of the embedding.

In particular, we consider two types of branching rules. The first one is for embeddings of signature $(1, 0, r)$, which are often referred to as standard embeddings and the second one is for embeddings of signature $(k, 0, 0)$ often called proper diagonal embeddings.

We start by presenting branching rules for the Lie algebra $\mathrm{gl}(n)$. The irreducible finite-dimensional $\mathrm{gl}(n)$-modules are in one-to-one correspondence with n-tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_i \in \mathbb{C}$ and $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, n-1$. Such an n-tuple is called the *highest weight* of the corresponding representation. Let V_λ^n denote the irreducible highest weight $\mathrm{gl}(n)$-module with highest weight λ. We say that a weight $\sigma = (\sigma_1, \ldots, \sigma_{n-1})$, with $\sigma_i \in \mathbb{C}$ and $\sigma_i - \sigma_{i+1} \in \mathbb{Z}_{\geq 0}$ for all $i = 1, \ldots, n-2$, is *aligned with* λ in the sense of Gelfand-Tsetlin if $\lambda_i - \sigma_i \in \mathbb{Z}_{\geq 0}$ and $\sigma_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$ for all $i = 1, \ldots, n-1$. If λ and σ are integral weights, the above implies that

$$\lambda_1 \geq \sigma_1 \geq \lambda_2 \geq \sigma_2 \geq \cdots \geq \lambda_{n-1} \geq \sigma_{n-1} \geq \lambda_n.$$

One often says in this case that σ *interlaces* λ.

Now the following multiplicity-free branching rule holds.

Proposition 2.2. *[Z] Consider an embedding $\mathrm{gl}(n-1) \to \mathrm{gl}(n)$ of signature $(1, 0, 1)$. Then*

$$V^n_{\lambda\,\downarrow \mathrm{gl}(n-1)} \cong \bigoplus_{\sigma} V_\sigma^{n-1},$$

where $\sigma = (\sigma_1, \ldots, \sigma_{n-1})$ runs over all weights aligned with λ in the sense of Gelfand-Tsetlin.

A similar branching rule holds for the group $\mathrm{GL}(n, \mathbb{C})$. A detailed exposition about the analogous embeddings at the group level and the corresponding branching rules can be found in [GW].

The Gelfand-Tsetlin rule can be iterated to obtain a branching law for embeddings of $\mathrm{gl}(n)$ into $\mathrm{gl}(n + k)$. More precisely, we have the following proposition.

Proposition 2.3. *Let* $\mathrm{gl}(n) \to \mathrm{gl}(n+k)$ *be an embedding of signature* $(1,0,k)$*. Then*

$$V_\lambda^{n+k}{}_{\downarrow \mathrm{gl}(n)} \cong \bigoplus_\sigma m^k_{\lambda,\sigma} V^n_\sigma,$$

where the multiplicity $m^k_{\lambda,\sigma}$ *is the number of possible sequences of weights* $\sigma^1 = (\sigma^1_1, \ldots, \sigma^1_{n+k-1}), \ldots, \sigma^{k-1} = (\sigma^{k-1}_1, \ldots, \sigma^{k-1}_{n+1})$ *such that each* σ^{j+1} *is aligned with* σ^j *in the sense of Gelfand-Tsetlin,* σ^1 *is aligned with* λ*, and* σ *is aligned with* σ^{k-1}*. In other words, we have the following trapezoid of aligned weights, often referred to as Gelfand-Tsetlin trapezoid:*

$$\begin{array}{c} \lambda_1, \lambda_2, \ldots, \lambda_{n+k-1}, \lambda_{n+k} \\ \sigma^1_1, \sigma^1_2, \ldots, \sigma^1_{n+k-1} \\ \sigma^2_1, \ldots, \sigma^2_{n+k-2} \\ \cdots\cdots \\ \sigma_1, \ldots, \sigma_n. \end{array}$$

In what follows we will refer to the numbers $m^k_{\lambda,\sigma}$ as *Gelfand-Tsetlin multiplicities.*

Next, we consider embeddings of signature $(k, 0, 0)$. In order to obtain branching rules for these embeddings, we use the results from [HTW]. All branching rules below involve only irreducible modules with integral highest weights, and that is why we introduce the following notations. Let λ and μ be two non-negative integer partitions with p and q parts, where $p + q \leq n$. Let $V^n_{\lambda,\mu}$ denote the irreducible $\mathrm{gl}(n)$-module with highest weight $(\lambda, \mu) = (\lambda_1, \ldots \lambda_p, 0, \ldots 0, -\mu_q, \ldots, -\mu_1)$. We derive the desired branching rule in several steps. Consider first the block-diagonal subalgebra $\mathrm{gl}(n) \oplus \mathrm{gl}(m) \subset \mathrm{gl}(n+m)$. By Theorem 2.2.1 in [HTW] we have the following decomposition:

$$V^{n+m}_{\lambda,\mu}{}_{\downarrow \mathrm{gl}(n)\oplus\mathrm{gl}(m)} \cong \bigoplus_{\alpha^+,\beta^+,\alpha^-,\beta^-} c^{(\lambda,\mu)}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)} V^n_{\alpha^+,\alpha^-} \otimes V^m_{\beta^+,\beta^-},$$

where

$$c^{(\lambda,\mu)}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)} = \sum_{\gamma^+,\gamma^-,\delta} c^\lambda_{\gamma^+\delta} c^\mu_{\gamma^-\delta} c^{\gamma^+}_{\alpha^+\beta^+} c^{\gamma^-}_{\alpha^-\beta^-},$$

and the numbers $c^\alpha_{\beta,\gamma}$ are Littlewood-Richardson coefficients. Next, we consider the direct sum of $k > 2$ copies of $\mathrm{gl}(n)$ as a subalgebra of $\mathrm{gl}(kn)$ by block-diagonal inclusion. Then one can iterate the above branching rule to obtain the following:

$$V^{kn}_{\lambda,\mu\downarrow \mathrm{gl}(n)\oplus\cdots\oplus\mathrm{gl}(n)} \cong \bigoplus_{\substack{\beta_1^+,\ldots,\beta_k^+\\ \beta_1^-,\ldots,\beta_k^-}} C^{(\lambda,\mu)}_{(\beta_1^+,\ldots,\beta_k^+)(\beta_1^-,\ldots,\beta_k^-)} V^n_{\beta_1^+,\beta_1^-}\otimes\cdots\otimes V^n_{\beta_k^+,\beta_k^-}, \tag{2.6}$$

where

$$C^{(\lambda,\mu)}_{(\beta_1^+,\ldots,\beta_k^+)(\beta_1^-,\ldots,\beta_k^-)} = \sum_{\substack{\alpha_1^+,\ldots\alpha_{k-2}^+\\ \alpha_1^-,\ldots\alpha_{k-2}^-}} c^{(\lambda,\mu)}_{(\alpha_1^+,\alpha_1^-)(\beta_1^+,\beta_1^-)} c^{(\alpha_1^+,\alpha_1^-)}_{(\alpha_2^+,\alpha_2^-)(\beta_2^+,\beta_2^-)} \cdots c^{(\alpha_{k-3}^+,\alpha_{k-3}^-)}_{(\alpha_{k-2}^+,\alpha_{k-2}^-)(\beta_{k-2}^+,\beta_{k-2}^-)} c^{(\alpha_{k-2}^+,\alpha_{k-2}^-)}_{(\beta_{k-1}^+,\beta_{k-1}^-)(\beta_k^+,\beta_k^-)}.$$

The next step is to consider the $\mathrm{gl}(n)\oplus\mathrm{gl}(n)$-module $V^n_{\alpha^+,\alpha^-}\otimes V^n_{\beta^+,\beta^-}$. By Theorem 2.1.1 in [HTW], as a module over the subalgebra $\mathrm{gl}(n)=\{x\oplus x|x\in\mathrm{gl}(n)\}$, $V^n_{\alpha^+,\alpha^-}\otimes V^n_{\beta^+,\beta^-}$ has the decomposition

$$V^n_{\alpha^+,\alpha^-}\otimes V^n_{\beta^+,\beta^-\downarrow\mathrm{gl}(n)} \cong \bigoplus_{\lambda',\mu'} d^{(\lambda',\mu')}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)} V^n_{\lambda',\mu'},$$

where

$$d^{(\lambda',\mu')}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)} = \sum_{\substack{\alpha_1,\alpha_2,\beta_1,\beta_2\\ \gamma_1,\gamma_2}} c^{\alpha^+}_{\alpha_1\gamma_1} c^{\beta^-}_{\gamma_1\beta_2} c^{\alpha^-}_{\beta_1\gamma_2} c^{\beta^+}_{\gamma_2\alpha_2} c^{\lambda'}_{\alpha_2\alpha_1} c^{\mu'}_{\beta_2\beta_1}.$$

Iterating this branching rule, we obtain for $k>2$

$$V^n_{\beta_1^+,\beta_1^-}\otimes\cdots\otimes V^n_{\beta_k^+,\beta_k^-\downarrow\mathrm{gl}(n)} \cong \bigoplus_{\lambda',\mu'} D^{(\lambda',\mu')}_{(\beta_1^+,\ldots,\beta_k^+)(\beta_1^-,\ldots,\beta_k^-)} V^n_{\lambda',\mu'}, \tag{2.7}$$

where

$$D^{(\lambda,\mu)}_{(\beta_1^+,\ldots,\beta_k^+)(\beta_1^-,\ldots,\beta_k^-)} = \sum_{\substack{\alpha_1^+,\ldots\alpha_{k-2}^+\\ \alpha_1^-,\ldots\alpha_{k-2}^-}} d^{(\alpha_1^+,\alpha_1^-)}_{(\beta_1^+,\beta_1^-)(\beta_2^+,\beta_2^-)} d^{(\alpha_2^+,\alpha_2^-)}_{(\alpha_1^+,\alpha_1^-)(\beta_3^+,\beta_3^-)} \cdots d^{(\alpha_{k-2}^+,\alpha_{k-2}^-)}_{(\alpha_{k-3}^+,\alpha_{k-3}^-)(\beta_{k-1}^+,\beta_{k-1}^-)} d^{(\lambda,\mu)}_{(\alpha_{k-2}^+,\alpha_{k-2}^-)(\beta_k^+,\beta_k^-)}.$$

Now we can combine (2.6) and (2.7) in the following proposition.

Proposition 2.4. *Let* $\mathrm{gl}(n) \subset \mathrm{gl}(kn)$ *be an embedding of signature* $(k, 0, 0)$. *Then*

$$V^{kn}_{\lambda,\mu\downarrow \mathrm{gl}(n)} \cong \bigoplus_{\substack{\beta_1^+,\dots,\beta_k^+ \\ \beta_1^-,\dots,\beta_k^- \\ \lambda',\mu'}} C^{(\lambda,\mu)}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)} D^{(\lambda',\mu')}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)} V^n_{\lambda',\mu'} \text{ for } k > 2,$$

and

$$V^{2n}_{\lambda,\mu\downarrow \mathrm{gl}(n)} \cong \bigoplus_{\substack{\alpha^+,\beta^+,\alpha^-,\beta^- \\ \lambda',\mu'}} c^{(\lambda,\mu)}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)} d^{(\lambda',\mu')}_{(\alpha^+,\alpha^-),(\beta^+,\beta^-)} V^n_{\lambda',\mu'} \text{ for } k = 2.$$

In particular, from the properties of Littlewood-Richardson coefficients it follows that if $V^n_{\lambda',\mu'}$ *enters the decomposition of* $V^{2n}_{\lambda,\mu}$ *then* $|\lambda| - |\mu| = |\lambda'| - |\mu'|$.

Note that in the case $\mu = 0$ the two formulas from Proposition 2.4 reduce to the following two well-known branching rules:

$$V^{2n}_{\lambda,0\downarrow \mathrm{gl}(n)} \cong \bigoplus_{\beta_1,\beta_2,\lambda'} c^{\lambda}_{\beta_1,\beta_2} c^{\lambda'}_{\beta_1,\beta_2} V^n_{\lambda',0}$$

and

$$V^{kn}_{\lambda,0\downarrow \mathrm{gl}(n)} \cong \bigoplus_{\beta_1,\dots,\beta_k,\lambda'} c^{\lambda}_{\beta_1,\dots,\beta_k} c^{\lambda'}_{\beta_1,\dots,\beta_k} V^n_{\lambda',0},$$

where the numbers $c^{\lambda}_{\beta_1,\dots,\beta_k}$ are the generalized Littlewood-Richardson coefficients.

Notice that the coefficients $C^{(\lambda,\mu)}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)}$ and $D^{(\lambda',\mu')}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)}$ are defined only for $k > 2$. For convenience we extend these definitions to the case $k = 2$ by setting

$$C^{(\lambda,\mu)}_{(\beta_1^+,\beta_2^+)(\beta_1^-,\beta_2^-)} = c^{(\lambda,\mu)}_{(\beta_1^+,\beta_2^+),(\beta_1^-,\beta_2^-)},$$
$$D^{(\lambda',\mu')}_{(\beta_1^+,\beta_2^+)(\beta_1^-,\beta_2^-)} = d^{(\lambda',\mu')}_{(\beta_1^+,\beta_2^+),(\beta_1^-,\beta_2^-)}.$$

It is interesting to mention that many of the branching rules described in [HTW] had been known previously. However, in [HTW] the authors developed a new approach based on the theory of dual reductive pairs. This approach allowed them to relate branching rules for one symmetric pair to another and as a result to generalize many of the known branching rules for some symmetric pairs to all classical symmetric pairs.

Analogous branching rules hold for the Lie algebras $\mathrm{sp}(2n)$ and $\mathrm{so}(n)$. We will introduce them explicitly in the course of the thesis.

Chapter 3

Embeddings of $\mathrm{gl}(\infty)$ into $\mathrm{gl}(\infty)$ and of $\mathrm{sl}(\infty)$ into $\mathrm{sl}(\infty)$

In this chapter we describe a procedure for deriving branching rules for embeddings $\mathrm{gl}(\infty) \subset \mathrm{gl}(\infty)$ (respectively $\mathrm{sl}(\infty) \subset \mathrm{sl}(\infty)$) of general tensor type, i.e. embeddings which satisfy (2.5). Our procedure consists of the following steps. First, we decompose the embedding under consideration into several intermediate embeddings. Then in Sections 3.1, 3.2, and 3.3 we derive branching laws for each of the intermediate embeddings. The branching law for an embedding of general tensor type is then a corollary of Theorem 5.2 in Chapter 5.

Let $\mathfrak{g}' \cong \mathfrak{g}$ and both be isomorphic to one of $\mathrm{gl}(\infty)$ or $\mathrm{sl}(\infty)$. Let $\varphi : \mathfrak{g}' \to \mathfrak{g}$ be an embedding of general tensor type. Let

$$\begin{aligned} \mathrm{soc}_{\mathfrak{g}'} V &= \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_k \oplus \tilde{W}_1 \oplus \cdots \oplus \tilde{W}_l \oplus N_a, \\ \mathrm{soc}_{\mathfrak{g}'} V_* &= \tilde{V}_1^* \oplus \cdots \oplus \tilde{V}_k^* \oplus \tilde{W}_1^* \oplus \cdots \oplus \tilde{W}_l^* \oplus N_c, \end{aligned} \tag{3.1}$$

where $\tilde{V}_i$ is isomorphic to V' for each $i = 1, \ldots, k$ and $\tilde{W}_j \cong V'_*$ for each $j = 1, \ldots, l$. Similalry, $\tilde{V}_i^* \cong V'_*$ and $\tilde{W}_j^* \cong V'$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, l$. Let $\{v'_i\}_{i \in I}$, $\{v'^*_i\}_{i \in I}$ be a pair of dual bases of V' and V'_*. We construct the following bases of V and V_* respectively:

$$\begin{aligned} &\{\{v_i^j\}_{i \in I, j=1,\ldots,k} \cup \{w_i^j\}_{i \in I, j=1,\ldots,l} \cup \{z_i\}_{i \in I_a} \cup \{x_i\}_{i \in I_b}\}, \\ &\{\{v_i^{j*}\}_{i \in I, j=1,\ldots,k} \cup \{w_i^{j*}\}_{i \in I, j=1,\ldots,l} \cup \{t_i\}_{i \in I_c} \cup \{y_i\}_{i \in I_d}\}, \end{aligned}$$

where v_i^j denotes the image of v'_i into $\tilde{V}_j$ and similarly for the others. Furthermore, $A = \{z_i\}_{i \in I_a}$ and $C = \{t_i\}_{i \in I_c}$ are bases respectively for N_a and N_c as submodules respectively of V and V_*, where I_a and I_c are index sets with cardinalities $a = \dim N_a$ and $c = \dim N_c$. Similarly, $B = \{x_i\}_{i \in I_b}$ and $D = \{y_i\}_{i \in I_d}$ are bases of N_b and N_d considered as vector subspaces of V and V_*, where I_b and I_d are index sets with cardinalities $b = \dim N_b$ and $d = \dim N_d$.

Then we have the following proposition.

Proposition 3.1. *Let $\mathfrak{g}'$, $\mathfrak{g}$, and φ be as above. There exist decompositions of $\mathrm{soc}_{\mathfrak{g}'}V$ and $\mathrm{soc}_{\mathfrak{g}'}V_*$ as in (3.1) such that*

$$\left\langle v_{i_1}^{j_1}, v_{i_2}^{j_2*}\right\rangle = \delta_{i_1 i_2}\delta_{j_1 j_2}, \qquad \left\langle v_{i_1}^{j_1}, w_{i_2}^{j_2*}\right\rangle = 0,$$
$$\left\langle w_{i_1}^{j_1}, w_{i_2}^{j_2*}\right\rangle = \delta_{i_1 i_2}\delta_{j_1 j_2}, \qquad \left\langle w_{i_1}^{j_1}, v_{i_2}^{j_2*}\right\rangle = 0.$$

Furthermore, for each i in the respective index set, z_i pairs trivially with all elements from $\tilde{V}_1^* \oplus \cdots \oplus \tilde{V}_k^* \oplus \tilde{W}_1^* \oplus \cdots \oplus \tilde{W}_l^*$ and x_i pairs non-degenerately with infinitely many elements from $\tilde{V}_1^* \oplus \cdots \oplus \tilde{V}_k^* \oplus \tilde{W}_1^* \oplus \cdots \oplus \tilde{W}_l^*$. Similarly, t_i pairs trivially with all elements from $\tilde{V}_1 \oplus \cdots \oplus \tilde{V}_k \oplus \tilde{W}_1 \oplus \cdots \oplus \tilde{W}_l$ and y_i pairs non-degenerately with infinitely many elements from $\tilde{V}_1 \oplus \cdots \oplus \tilde{V}_k \oplus \tilde{W}_1 \oplus \cdots \oplus \tilde{W}_l$. Finally, if $\mathfrak{g}' \cong \mathfrak{g} \cong \mathrm{gl}(\infty)$ we have for all i, s

$$\varphi(v_i' \otimes v_s'^*) = v_i^1 \otimes v_s^{1*} + \cdots + v_i^k \otimes v_s^{k*} - w_s^1 \otimes w_i^{1*} - \cdots - w_s^l \otimes w_i^{l*},$$

and if $\mathfrak{g}' \cong \mathfrak{g} \cong \mathrm{sl}(\infty)$ we have for $i \neq s$

$$\varphi(v_i' \otimes v_s'^*) = v_i^1 \otimes v_s^{1*} + \cdots + v_i^k \otimes v_s^{k*} - w_s^1 \otimes w_i^{1*} - \cdots - w_s^l \otimes w_i^{l*},$$

$$\begin{aligned}\varphi(v_i' \otimes v_i'^* - v_s' \otimes v_s'^*) = & v_i^1 \otimes v_i^{1*} - v_s^1 \otimes v_s^{1*} + \cdots + v_i^k \otimes v_i^{k*} - v_s^k \otimes v_s^{k*} + \\ & w_s^1 \otimes w_s^{1*} - w_i^1 \otimes w_i^{1*} + \cdots + w_s^l \otimes w_s^{l*} - w_i^l \otimes w_i^{l*}.\end{aligned}$$

Proof. We fix decompositions of $\mathrm{soc}_{\mathfrak{g}'}V$ and $\mathrm{soc}_{\mathfrak{g}'}V_*$ as in (3.1). Then for i, j, m, n in the respective ranges we have

$$\left\langle v_i^j, t_m\right\rangle = \left\langle \varphi(v_i' \otimes v_n'^*) \cdot v_n^j, t_m\right\rangle = -\left\langle v_n^j, \varphi(v_i' \otimes v_n'^*) \cdot t_m\right\rangle = 0.$$

Suppose now that v_i^j pairs trivially with all elements from $\mathrm{soc}_{\mathfrak{g}'}V_*$. Then there exists y_m such that $\left\langle v_i^j, y_m\right\rangle \neq 0$. But then

$$\left\langle v_i^j, y_m\right\rangle = \left\langle \varphi(v_i' \otimes v_i'^*) \cdot v_i^j, y_m\right\rangle = -\left\langle v_i^j, \varphi(v_i' \otimes v_i'^*) \cdot y_m\right\rangle = \left\langle v_i^j, v\right\rangle$$

for some $v \in \mathrm{soc}_{\mathfrak{g}'}V_*$. Hence, $\left\langle v_i^j, v\right\rangle \neq 0$, which contradicts our assumption. This implies that $\tilde{V}_1 \oplus \cdots \oplus \tilde{V}_k \oplus \tilde{W}_1 \oplus \cdots \oplus \tilde{W}_l$ and $\tilde{V}_1^* \oplus \cdots \oplus \tilde{V}_k^* \oplus \tilde{W}_1^* \oplus \cdots \oplus \tilde{W}_l^*$ pair non-degenerately.

Furthermore,

$$\left\langle v_i^j, v_m^{*n}\right\rangle = \left\langle \varphi(v_i' \otimes v_s'^*) \cdot v_s^j, v_m^{*n}\right\rangle = -\left\langle v_s^j, \varphi(v_i' \otimes v_s'^*) \cdot v_m^{*n}\right\rangle = \delta_{im}\left\langle v_s^j, v_s^{*n}\right\rangle$$

for all i, j, m, n, s. Hence, $\left\langle v_i^j, v_m^{*n}\right\rangle = 0$ for all j, n and for $i \neq m$. In addition,

$$\left\langle v_i^j, v_i^{*n}\right\rangle = \left\langle v_s^j, v_s^{*n}\right\rangle$$

for $i = m$ and all s. In the same way we prove that $\left\langle w_i^j, w_m^{*n}\right\rangle = 0$ for all j, n and for $i \neq m$ and $\left\langle w_i^j, w_i^{*n}\right\rangle = \langle v_s^j, v_s^{*n}\rangle$ for $i = m$ and all s.

Now, for all i, j, m, n

$$\left\langle v_i^j, w_m^{*n}\right\rangle = \left\langle \varphi(v_i' \otimes v_i'^*) \cdot v_i^j, w_m^{*n}\right\rangle = -\left\langle v_i^j, \varphi(v_i' \otimes v_i'^*) \cdot w_m^{*n}\right\rangle = -\delta_{im}\left\langle v_i^j, w_m^{*n}\right\rangle.$$

Hence, $\left\langle v_i^j, w_m^{*n}\right\rangle = 0$ and in the same way $\left\langle w_i^j, v_m^{*n}\right\rangle = 0$.

Since the pairing between $\tilde{V}_1 \oplus \cdots \oplus \tilde{V}_k \oplus \tilde{W}_1 \oplus \cdots \oplus \tilde{W}_l$ and $\tilde{V}_1^* \oplus \cdots \oplus \tilde{V}_k^* \oplus \tilde{W}_1^* \oplus \cdots \oplus \tilde{W}_l^*$ is non-degenerate, the above considerations imply that for each j there exists m such that $\left\langle v_i^j, v_i^{*m}\right\rangle \neq 0$ for all $i \in I$. Similarly, for each j there exists m such that $\left\langle w_i^j, w_i^{*m}\right\rangle \neq 0$ for all $i \in I$. We can then renumerate the spaces $\tilde{V}_j$, $\tilde{V}_m^*$, $\tilde{W}_j$, $\tilde{W}_m^*$ in such a way that $\left\langle v_i^j, v_i^{*j}\right\rangle = 1$ and $\left\langle w_i^j, w_i^{*j}\right\rangle = 1$ for all j and all i. Then all the above implies in the case of $\mathrm{gl}(\infty)$ that

$$\varphi(v_i' \otimes v_s'^*) = v_i^1 \otimes v_s^{1*} + \cdots + v_i^k \otimes v_s^{k*} - w_s^1 \otimes w_i^{1*} - \cdots - w_s^l \otimes w_i^{l*}$$

for each $i, s \in I$. Similarly, when $\mathfrak{g} \cong \mathrm{sl}(\infty)$ we obtain that φ also has the desired form. □

Before we continue, we show that the case of embeddings $\mathrm{sl}(\infty) \subset \mathrm{sl}(\infty)$ reduces to the case of embeddings $\mathrm{gl}(\infty) \subset \mathrm{gl}(\infty)$.

Proposition 3.2. *Let $\varphi : \mathrm{sl}(V', V_*') \to \mathrm{sl}(V, V_*)$ be an embedding satisfying (2.5) with certain values of a, b, c, d, k, l and let M be a semisimple tensor $\mathrm{sl}(V, V_*)$-module. Then φ can be extended to an embedding $\varphi : \mathrm{gl}(V', V_*') \to \mathrm{gl}(V, V_*)$ which satisfies (2.5) with the same values of of a, b, c, d, k, l. Furthermore, the socle filtration of M over $\varphi(\mathrm{sl}(V', V_*'))$ is the same as the socle filtration of M considered as a $\mathrm{gl}(V, V_*)$-module over $\varphi(\mathrm{gl}(V', V_*'))$.*

Proof. We fix decompositions of the $\mathrm{sl}(V, V_*)$-modules V and V_* as in Proposition

3.1. Then $\varphi : \mathrm{sl}(V', V'_*) \to \mathrm{sl}(V, V_*)$ has the form

$$\varphi(v'_i \otimes v'^*_s) = v^1_i \otimes v^{1*}_s + \cdots + v^k_i \otimes v^{k*}_s - w^1_s \otimes w^{1*}_i - \cdots - w^l_s \otimes w^{l*}_i,$$

$$\begin{aligned}\varphi(v'_i \otimes v'^*_i - v'_s \otimes v'^*_s) =& v^1_i \otimes v^{1*}_i - v^1_s \otimes v^{1*}_s + \cdots + v^k_i \otimes v^{k*}_i - v^k_s \otimes v^{k*}_s + \\ & w^1_s \otimes w^{1*}_s - w^1_i \otimes w^{1*}_i + \cdots + w^l_s \otimes w^{l*}_s - w^l_i \otimes w^{l*}_i\end{aligned}$$

for all $i \neq s$. We can naturally extend φ to an embedding $\varphi : \mathrm{gl}(V', V'_*) \to \mathrm{gl}(V, V_*)$ by setting

$$\varphi(v'_1 \otimes v'^*_1) = v^1_1 \otimes v^{1*}_1 + \cdots + v^k_1 \otimes v^{k*}_1 - w^1_1 \otimes w^{1*}_1 - \cdots - w^l_1 \otimes w^{l*}_1.$$

Next, let M be as above. From [PSt] we know that the set of semisimple tensor $\mathrm{gl}(V, V_*)$-modules coincides with the set of semisimple tensor $\mathrm{sl}(V, V_*)$-modules. Then M is both a semisimple tensor $\mathrm{gl}(V, V_*)$-module and semisimple tensor $\mathrm{sl}(V, V_*)$-module. We will use the notation M_{gl} (resp., M_{sl}) to mark that we consider M as a $\mathrm{gl}(V, V_*)$ (resp., $\mathrm{sl}(V, V_*)$) module. Then, on the one hand, we have the chain of embeddings

$$\varphi(\mathrm{sl}(V', V'_*)) \subset \mathrm{sl}(V, V_*) \subset \mathrm{gl}(V, V_*).$$

This chain yields the following equality for every r:

$$\mathrm{soc}^{(r)}_{\varphi(\mathrm{sl}(V', V'_*))} M_{\mathrm{sl}} = \mathrm{soc}^{(r)}_{\varphi(\mathrm{sl}(V', V'_*))} M_{\mathrm{gl}}. \tag{3.2}$$

On the other hand, we have the chain of embeddings

$$\varphi(\mathrm{sl}(V', V'_*)) \subset \varphi(\mathrm{gl}(V', V'_*)) \subset \mathrm{gl}(V, V_*),$$

which yields the equality

$$\mathrm{soc}^{(r)}_{\varphi(\mathrm{sl}(V', V'_*))} M_{\mathrm{gl}} = \mathrm{soc}^{(r)}_{\varphi(\mathrm{gl}(V', V'_*))} M_{\mathrm{gl}}. \tag{3.3}$$

From (3.2) and (3.3) the statement follows. □

In view of Proposition 3.2 it is enough to consider embeddings $\mathrm{gl}(\infty) \subset \mathrm{gl}(\infty)$. Therefore, in the rest of the chapter $\mathfrak{g}'$ and $\mathfrak{g}$ will be both isomorphic to $\mathrm{gl}(\infty)$.

Proposition 3.3. *Let $\mathfrak{g}' \cong \mathfrak{g} \cong \mathrm{gl}(\infty)$ and let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of general tensor type, i.e. which satisfies (2.5). Then there exist intermediate subalgebras $\mathfrak{g}_1$*

and $\mathfrak{g}_2$ *such that* $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$ *and the following hold.*

(1) The embedding $\mathfrak{g}_1 \subset \mathfrak{g}$ *has the properties:*

$$\mathrm{soc}_{\mathfrak{g}_1} V \cong V_1 \oplus N_{a_1}, \qquad V/\mathrm{soc}_{\mathfrak{g}_1} V \cong N_b,$$
$$\mathrm{soc}_{\mathfrak{g}_1} V_* \cong V_{1*} \oplus N_{c_1}, \qquad V_*/\mathrm{soc}_{\mathfrak{g}_1} V_* \cong N_d,$$

where

$$N_{a_1} = \{v \in N_a | \, \langle v, N_c\rangle = 0\} \text{ and } N_{c_1} = \{w \in N_c | \, \langle N_a, w\rangle = 0\}.$$

(2) The embedding $\mathfrak{g}_2 \subset \mathfrak{g}_1$ *has the properties:*

$$V_1 \cong V_2 \oplus N_{a_2}, \qquad V_{1*} \cong V_{2*} \oplus N_{c_2},$$

where N_{a_2} *and* N_{c_2} *are such that* $N_a = N_{a_1} \oplus N_{a_2}$ *and* $N_c = N_{c_1} \oplus N_{c_2}$*.*

(3) The embedding $\mathfrak{g}' \subset \mathfrak{g}_2$ *has the properties:*

$$V_2 \cong kV' \oplus lV'_*, \qquad V_{2*} \cong lV' \oplus kV'_*.$$

Proof. We take decompositions of $\mathrm{soc}_{\mathfrak{g}'} V$ and $\mathrm{soc}_{\mathfrak{g}'} V_*$ as in Proposition 3.1. Set

$$V_2 = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_k \oplus \tilde{W}_1 \oplus \cdots \oplus \tilde{W}_l,$$
$$V_{2*} = \tilde{V}_1^* \oplus \cdots \oplus \tilde{V}_k^* \oplus \tilde{W}_1^* \oplus \cdots \oplus \tilde{W}_l^*.$$

Then Proposition 3.1 yields that V_2 and V_{2*} pair non-degenerately and we put $\mathfrak{g}_2 = V_2 \otimes V_{2*}$.

Next, let A and C be as above. Let $A_1 = \{z'_i\}_{i \in I_{a_1}}$ consist of those elements in A which pair trivially with all elements in C, and analogously let $C_1 = \{t'_i\}_{i \in I_{c_1}}$ consist of the elements in C which pair trivially with all vectors in A. Let $A_2 = A \setminus A_1$ and $C_2 = C \setminus C_1$ and denote their elements respectively with z''_i and t''_i and their index sets with I_{a_2} and I_{c_2}. Now set

$$V_1 = V_2 \oplus \mathrm{span}\{z''_i\}_{i \in I_{a_2}}, \qquad V_{1*} = V_{2*} \oplus \mathrm{span}\{t''_i\}_{i \in I_{c_2}}.$$

Then V_1 and V_{1*} pair non-degenerately and we put $\mathfrak{g}_1 = V_1 \otimes V_{1*}$.

The Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ satisfy the required properties.

□

Motivated by Proposition 3.3 we give the following definition.

Definition 3.1. *(i) An embedding* $\mathfrak{g}' \subset \mathfrak{g}$ *is said to be of type I if*

$$\mathrm{soc}_{\mathfrak{g}'} V \cong V' \oplus N_a, \qquad V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b,$$
$$\mathrm{soc}_{\mathfrak{g}'} V_* \cong V'_* \oplus N_c, \qquad V_*/\mathrm{soc}_{\mathfrak{g}'} V_* \cong N_d,$$

where N_a *and* N_c *pair trivially.*

(ii) An embedding $\mathfrak{g}' \subset \mathfrak{g}$ *is said to be of type II if*

$$V \cong V' \oplus N_a, \qquad V_* \cong V'_* \oplus N_c.$$

(iii) An embedding $\mathfrak{g}' \subset \mathfrak{g}$ *is said to be of type III if*

$$V \cong kV' \oplus lV'_*, \qquad V_* \cong lV' \oplus kV'_*.$$

In the following sections we derive branching rules for embeddings of the different types. In each case we start with determining the socle filtration over $\mathfrak{g}'$ of one of the $\mathfrak{g}$-modules $V^{\otimes p}$, $V_*^{\otimes q}$, $V^{\{p,q\}}$, $V^{\otimes(p,q)}$. Then for the simple $\mathfrak{g}$-submodules of these modules we use properties (2.2) and (2.3) of socle filtrations.

3.1 Branching laws for embeddings of type I

3.1.1 The modules $V^{\otimes p}$ and $V_*^{\otimes q}$

Let us consider embeddings $\mathfrak{g}' \subset \mathfrak{g}$ satisfying the following conditions:

$$\begin{aligned} \mathrm{soc}_{\mathfrak{g}'} V \cong V' \oplus N_a, \quad V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b, \\ \mathrm{soc}_{\mathfrak{g}'} V_* \cong V'_* \oplus N_c, \quad V_*/\mathrm{soc}_{\mathfrak{g}'} V_* \cong N_d. \end{aligned} \tag{3.4}$$

So far we make no assumptions about the pairing between N_a and N_c. Recall that we denote $\dim N_a = \dim(V'_*)^\perp = a$, $\dim N_b = b$, $\dim N_c = \dim(V')^\perp = c$, $\dim N_d = d$. Moreover, $\mathrm{codim}_V V' = a + b$ and $\mathrm{codim}_{V_*} V'_* = c + d$.

Our first goal is to compute the socle filtrations of the $\mathfrak{g}$-modules $V^{\otimes p}$ and $V_*^{\otimes q}$. We have the following short exact sequence of $\mathfrak{g}'$-modules

$$0 \to V' \oplus N_a \xrightarrow{i} V \xrightarrow{f} N_b \to 0. \tag{3.5}$$

For each index $1 \leq i \leq p$ we define

$$L_i : V^{\otimes p} \to V^{\otimes i-1} \otimes N_b \otimes V^{\otimes p-i} \cong V^{\otimes p-1} \otimes N_b,$$
$$L_i = \mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes f \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id},$$

where f appears at the i-th position in the tensor product.

Similarly, for each collection of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq p$ we define the homomorphism

$$L_{i_1,\dots,i_k} : V^{\otimes p} \to V^{\otimes i_1-1} \otimes N_b \otimes V^{\otimes i_2-i_1-1} \otimes N_b \otimes \cdots \otimes V^{\otimes p-i_k} \cong V^{\otimes p-k} \otimes N_b^{\otimes k},$$
$$L_{i_1,\dots,i_k} = \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes f \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes f \otimes \mathrm{id} \cdots \otimes \mathrm{id},$$

where the map f appears at positions i_1 through i_k in the tensor product.

Proposition 3.4. *For embeddings $\mathfrak{g}' \subset \mathfrak{g}$ which satisfy (3.4) we have*

$$\mathrm{soc}_{\mathfrak{g}'}^{(r+1)} V^{\otimes p} = \bigcap_{i_1 < \cdots < i_{r+1}} \ker L_{i_1,\dots,i_{r+1}}.$$

Moreover,

$$\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(r+1)} V^{\otimes p} \cong \binom{p}{r} N_b^{\otimes r} \otimes (V' \oplus N_a)^{\otimes p-r}.$$

Proof. We denote $S^{(r+1)} = \bigcap_{i_1 < \cdots < i_{r+1}} \ker L_{i_1,\dots,i_{r+1}}$. Note that $S^{(r+1)}$ consists of linear combinations of monomials from $V^{\otimes p}$ such that at most r terms in each monomial are outside of $\mathrm{soc}_{\mathfrak{g}'} V$. We need to check two properties:

(1) for any $u \in S^{(r+2)} \setminus S^{(r+1)}$ there exists $g \in U(\mathfrak{g}')$ such that $g \cdot u \in S^{(r+1)} \setminus S^{(r)}$;

(2) the quotient $S^{(r+1)}/S^{(r)}$ is semisimple.

Proof of (1): Take $u \in S^{(r+2)} \setminus S^{(r+1)}$, $u = \sum a_{i_1 \dots i_p} u_{i_1} \otimes \cdots \otimes u_{i_p}$ where $u_{i_j} \in V$. More precisely, denote $u_{i_j} = v_{i_j}$ for $u_{i_j} \in \mathrm{soc} V$ and $u_{i_j} = x_{i_j}$ for $u_{i_j} \notin \mathrm{soc} V$.

Now suppose that u_1 is a monomial of highest degree in the expression of u. In other words, u_1 has $r+1$ entries not in $\mathrm{soc}_{\mathfrak{g}'} V$. Consider an element $g_1 = w_1 \otimes w_1^* \in \mathfrak{g}'$ such that

- $\langle x_{i_j}, w_1^* \rangle \neq 0$ for at least one x_{i_j} from u_1;
- $\langle v_{i_j}, w_1^* \rangle = 0$ for all v_{i_j} which appear in the expression of u;
- $w_1 \neq v_{i_j}$ for all v_{i_j} that enter u.

The existence of such an element g_1 follows from the fact that all elements $x_{ij} \notin \mathrm{soc}_{\mathfrak{g}'} V$ pair non-degenerately with infinitely many elements from V'_*. Then $g_1 \cdot u \in S^{(r+1)} \setminus S^{(r)}$ or $g_1 \cdot u \in S^{(r+2)} \setminus S^{(r+1)}$. In the latter case there is a monomial $u_2 \in g_1 \cdot u$ with $r+1$ entries not in $\mathrm{soc}_{\mathfrak{g}'} V$. We proceed as above to find an element $g_2 = w_2 \otimes w_2^*$ analogous to g_1. Thus, after finitely many steps we obtain $(g_s \circ \cdots \circ g_2 \circ g_1) \cdot u \in S^{(r+1)} \setminus S^{(r)}$.
Proof of (2): The map

$$\bigoplus_{i_1 < \cdots < i_r} L_{i_1,\ldots,i_r} : S^{(r+1)}/S^{(r)} \to \bigoplus_{i_1 < \cdots < i_r} (\mathrm{soc}_{\mathfrak{g}'} V)^{\otimes p-r} \otimes (N_b)^{\otimes r}$$

is a well-defined isomorphism of $\mathfrak{g}'$-modules. □

In order to compute the socle filtration of $V_*^{\otimes q}$ we take the short exact sequence

$$0 \to V'_* \oplus N_c \xrightarrow{i} V_* \xrightarrow{g} N_d \to 0$$

and define

$$M_{i_1,\ldots,i_k} : V_*^{\otimes q} \to V_*^{\otimes i_1 - 1} \otimes N_d \otimes V_*^{\otimes i_2 - i_1 - 1} \otimes N_d \otimes \cdots \otimes V_*^{\otimes q - i_k} \cong V_*^{\otimes q-k} \otimes N_d^{\otimes k},$$
$$M_{i_1,\ldots,i_k} = \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes g \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes g \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id},$$

where the map g appears at positions i_1 through i_k in the tensor product.

Proposition 3.5. *The socle filtration of $V_*^{\otimes q}$ over $\mathfrak{g}'$ is*

$$\mathrm{soc}_{\mathfrak{g}'}^{(r)} V_*^{\otimes q} = \bigcap_{i_1 < \cdots < i_r} \ker M_{i_1,\ldots,i_r}.$$

3.1.2 Submodules of $V^{\otimes p}$ and $V_*^{\otimes q}$

Recall that property (2.2) of socle filtrations states that if N is a submodule of M then $\mathrm{soc}^{(r)} N = (\mathrm{soc}^{(r)} M) \cap N$. We will now use this to determime the socle filtration of any simple tensor $\mathfrak{g}$-module $V_{\lambda,0} \subset V^{\otimes p}$. To compute the multiplicity of each simple subquotient, we need to extend the definition of the Gelfand-Tsetlin multiplicity $m^k_{\lambda,\sigma}$ to the case $k = \infty$. More precisely, we define

$$m^\infty_{\lambda,\sigma} = \lim_{k \to \infty} m^k_{\lambda,\sigma}.$$

In other words,

$$m^{\infty}_{\lambda,\sigma} = \begin{cases} 0 & \text{if } m^k_{\lambda,\sigma} = 0 \text{ for all } k, \\ 1 & \text{if } \sigma = \lambda, \text{ i.e. } m^k_{\lambda,\sigma} = 1 \text{ for all } k, \\ \infty & \text{if } m^k_{\lambda,\sigma} > 1 \text{ for at least one } k. \end{cases}$$

Furthermore, for convenience we set $m^0_{\lambda,\sigma} = 0$ for $\lambda \neq \sigma$ and $m^0_{\lambda,\lambda} = 1$. We will refer to the values $m^k_{\lambda,\sigma}$ for $k \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ as the *extended Gelfand-Tsetlin multiplicities.*

Before stating the general result we need two lemmas.

Lemma 3.6. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding which satisfies (3.4) with $a = 0$. Then for any $V_{\lambda,0} \subset V^{\otimes p}$ we have*

$$\begin{aligned} &\mathrm{soc}_{\mathfrak{g}'} V_{\lambda,0} = V'_{\lambda,0}, \\ &\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\lambda,0} \cong \bigoplus_{|\lambda'|=|\lambda|-r} m^b_{\lambda,\lambda'} V'_{\lambda',0}, \end{aligned}$$

where $m^b_{\lambda,\lambda'}$ are the extended Gelfand-Tsetlin multiplicities.

Proof. From Proposition 3.4 above and from Theorem 2.1 in [PSt] it follows that

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V^{\otimes p} \cong \binom{p}{r} N_b^{\otimes r} \otimes (V')^{\otimes p-r} \cong \bigoplus_{|\lambda'|=p-r} c_{\lambda'} V'_{\lambda',0}$$

for some multiplicities $c_{\lambda'}$. Moreover, $\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\lambda,0} \subset \overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V^{\otimes p}$, hence

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\lambda,0} \cong \bigoplus_{|\lambda'|=p-r} c'_{\lambda'} V'_{\lambda',0} \tag{3.6}$$

for some unknown $c'_{\lambda'}$. Thus, we only need to compute the multiplicity with which each $V'_{\lambda',0}$ enters the decomposition of $V_{\lambda,0}$. Note that on distinct layers of the socle filtration non-isomorphic simple constituents $V'_{\lambda',0}$ appear.

Let $\{v_i\}_{i\in\mathbb{Z}_{>0}}$ and $\{v^*_i\}_{i\in\mathbb{Z}_{>0}}$ be a pair of dual bases in V' and V'_*, and $\{\xi_i\}_{i\in\mathbb{Z}_{>0}}$, $\{\xi^*_i\}_{i\in\mathbb{Z}_{>0}}$ be respectively a pair of dual bases in V and V_*. For each n, put $V_n = \mathrm{span}\{\xi_1, \dots, \xi_n\}$ and $V^*_n = \mathrm{span}\{\xi^*_1, \dots, \xi^*_n\}$. The pairing between V and V_* restricts to a non-degenerate pairing between V_n and V^*_n. Therefore we can define the Lie algebra $\mathfrak{g}_n = V_n \otimes V^*_n$. Furthermore, we set $\mathfrak{h}_n = \mathfrak{h}_{\mathfrak{g}} \cap \mathfrak{g}_n$ and $\mathfrak{b}_n = \mathfrak{b}_{\mathfrak{g}} \cap \mathfrak{g}_n$. It is clear that $\mathfrak{g}_n \cong \mathrm{gl}(n)$ and that $\mathfrak{h}_n$ (respectively, $\mathfrak{b}_n$) is a Cartan (respectively, Borel) subalgebra of $\mathfrak{g}_n$. Moreover, if we set $V^n_{\lambda,0} = V_{\lambda,0} \cap V_n^{\otimes p}$, then for $n \geq p$, $V^n_{\lambda,0}$ is a highest weight $\mathfrak{g}_n$-module with highest weight $(\lambda, 0)$ with respect to $\mathfrak{b}_n$.

Now we apply the same procedure to $\mathfrak{g}'$. We define $V'_n = \operatorname{span}\{v_1, \ldots, v_n\}$ and $V'^*_n = \operatorname{span}\{v_1^*, \ldots, v_n^*\}$. We set $\mathfrak{g}'_n = V'_n \otimes V'^*_n$, $\mathfrak{h}'_n = \mathfrak{h}_{\mathfrak{g}'} \cap \mathfrak{g}'_n$, $\mathfrak{b}'_n = \mathfrak{b}_{\mathfrak{g}'} \cap \mathfrak{g}'_n$, and $V'^n_{\lambda,0} = V'_{\lambda,0} \cap V'^{\otimes p}_n$. In this way we obtain a commutative diagram of inclusions

$$\begin{array}{ccccccccccc}
\mathfrak{g}'_1 & \longrightarrow & \mathfrak{g}'_2 & \longrightarrow & \cdots & \longrightarrow & \mathfrak{g}'_k & \longrightarrow & \cdots & \longrightarrow & \mathfrak{g}' \\
\downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\
\mathfrak{g}_{m_1} & \longrightarrow & \mathfrak{g}_{m_2} & \longrightarrow & \cdots & \longrightarrow & \mathfrak{g}_{m_k} & \longrightarrow & \cdots & \longrightarrow & \mathfrak{g}
\end{array}$$

in which all horizontal arrows are standard inclusions. Then for large k each embedding $\mathfrak{g}'_k \to \mathfrak{g}_{m_k}$ is diagonal, i.e. in our case

$$V_{m_k} \cong V'_k \oplus N'_k,$$

where N'_k is some finite-dimensional trivial $\mathfrak{g}'_k$-module. Moreover, if we set $n_k = \dim N'_k = \operatorname{codim}_{V_{m_k}} V'_k$ then $b = \operatorname{codim}_V V' = \lim_{k\to\infty} n_k$. Thus, when b is finite we obtain that for large k all vertical embeddings in the above diagram are of signature $(1, 0, b)$. In the case $b = \infty$, for large k all vertical embeddings are of signature $(1, 0, n_k)$ with $\lim_{k\to\infty} n_k = \infty$.

Let us consider first the case when b is finite. Then for large k we can use the Gelfand-Tsetlin rule for the embedding $\mathfrak{g}'_k \to \mathfrak{g}_{m_k}$ and the module $V^{m_k}_{\lambda,0} = V_{\lambda,0} \cap V^{\otimes p}_{m_k}$ to obtain the decomposition

$$V^{m_k}_{\lambda,0} \cong \bigoplus_{\lambda'} m^b_{\lambda,\lambda'} V'^k_{\lambda',0}. \tag{3.7}$$

Then (3.6) and (3.7) imply

$$(\operatorname{soc}^{(r+1)} V_{\lambda,0}) \cap V^{\otimes p}_{m_k} / (\operatorname{soc}^{(r)} V_{\lambda,0}) \cap V^{\otimes p}_{m_k} \cong \bigoplus_{|\lambda'|=p-r} m^b_{\lambda,\lambda'} V'^k_{\lambda',0},$$

and passing to the direct limit we obtain the statement.

Now let us consider the case $b = \infty$. Then from the Gelfand-Tsetlin rule we obtain

$$(\operatorname{soc}^{(r+1)} V_{\lambda,0}) \cap V^{\otimes p}_{m_k} / (\operatorname{soc}^{(r)} V_{\lambda,0} \cap V^{\otimes p}_{m_k}) \cong \bigoplus_{|\lambda'|=p-r} m^{n_k}_{\lambda,\lambda'} V'^k_{\lambda',0},$$

and passing to the direct limit we obtain the statement.

□

Lemma 3.7. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding which satisfies (3.4) with $b = 0$. Then* *any $V_{\lambda,0} \subset V^{\otimes p}$ is a completely reducible $\mathfrak{g}'$-module and*

$$V_{\lambda,0} \cong \bigoplus_{\lambda'} m^a_{\lambda,\lambda'} V'_{\lambda',0},$$

where $m^a_{\lambda,\lambda'}$ are again the extended Gelfand-Tsetlin multiplicities.

Proof. When $b = 0$ the map f from (3.5) is just the zero homomorphism, hence Proposition 3.4 implies that $V_{\lambda,0}$ is completely reducible. Therefore, we only need to compute the multiplicity of each $V'_{\lambda',0}$ in the expression of $V_{\lambda,0}$, and this is done in the same way as in Lemma 3.6.

□

Now we can state the branching rule for arbitrary values of a and b.

Theorem 3.8. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding which satisfies (3.4) with a and b being arbitrary non-negative integers or infinity. Then for any $V_{\lambda,0} \subset V^{\otimes p}$ we have*

$$\overline{\operatorname{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\lambda,0} \cong \bigoplus_{\lambda''} \bigoplus_{|\lambda'|=|\lambda''|-r} m^a_{\lambda,\lambda''} m^b_{\lambda'',\lambda'} V'_{\lambda',0}.$$

Proof. Let $\{v_i\}$ and $\{v_i^*\}$ be a pair of dual bases in V' and V'_*. Then

$$V = \operatorname{span}\{x_1, \dots, x_n, \dots, z_1, \dots, z_l, \dots, v_{k_1}, \dots, v_{k_n}, \dots\},$$
$$V_* = \operatorname{span}\{f_1, \dots, f_m, \dots, v^*_{k_1}, \dots, v^*_{k_n}, \dots\},$$

where $< z_i, v_j^* >= 0$ for all i, j, and $< x_i, v_j^* >\neq 0$ for all i and infinitely many j. We divide the f_i's into two groups: those which pair non-degenerately with $z_1, \dots, z_l, \dots$ we denote by f'_i, and the remaining ones we denote by f''_i. Next we define the subspaces

$$V'' = \operatorname{span}\{x_1, \dots x_n, \dots, v_{k_1}, \dots, v_{k_n}, \dots\},$$
$$V''_* = \operatorname{span}\{f''_1 \dots f''_m, \dots, v^*_{k_1}, \dots, v^*_{k_n}, \dots\}.$$

Then V'' and V''_* pair non-degenerately, and the four values which characterize the embedding $V'' \otimes V''_* \subset V \otimes V_*$ are $b'' = 0$, $a'' = a$, $d'' = k$, $c'' = l$ for some integers k and l. Therefore, this embedding satisfies the conditions of Lemma 3.7, hence every simple $\mathfrak{g}$-module $V_{\lambda,0}$ is completely reducible over $\mathfrak{g}'' = V'' \otimes V''_*$ and

$$V_{\lambda,0} \cong \bigoplus_{\lambda''} m^a_{\lambda,\lambda''} V''_{\lambda'',0}, \tag{3.8}$$

where by $V''_{\lambda'',0}$ we denote the simple tensor $\mathfrak{g}''$-modules.

To see now how $V_{\lambda,0}$ decomposes over $\mathfrak{g}'$ it is enough to see how each $V''_{\lambda'',0}$ decomposes over $\mathfrak{g}'$. The four values which characterize the embedding $\mathfrak{g}' \subset \mathfrak{g}''$ are $b' = b$, $a' = 0$, $d' = k$, $c' = l$, for some k and l. Therefore, this embedding satisfies the conditions of Lemma 3.6, and for every simple $\mathfrak{g}''$-module $V''_{\lambda'',0}$ we have

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V''_{\lambda'',0} \cong \bigoplus_{|\lambda'|=|\lambda''|-r} m^b_{\lambda'',\lambda'} V'_{\lambda',0}. \tag{3.9}$$

Then (3.8) and (3.9) imply the statement of the theorem. □

An analogous statement holds for submodules of $V_*^{\otimes q}$. Here is the result.

Theorem 3.9. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding which satisfies (3.4) such that c and d are arbitrary non-negative integers or infinity. Then for any $V_{0,\mu} \subset V_*^{\otimes q}$ we have*

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{0,\mu} \cong \bigoplus_{\mu''} \bigoplus_{|\mu'|=|\mu''|-r} m^c_{\mu,\mu''} m^d_{\mu'',\mu'} V'_{0,\mu'}.$$

3.1.3 The module $V^{\{p,q\}}$ and its submodules

Recall that in the previous two subsections we considered embeddings $\mathfrak{g}' \subset \mathfrak{g}$ which satisfy (3.4) without any conditions on the pairing between N_a and N_c. However, in order to compute the socle filtration of $V^{\{p,q\}}$ over $\mathfrak{g}'$ we have to consider separately embeddings of type I and II.

Theorem 3.10. *Consider an embedding $\mathfrak{g}' \subset \mathfrak{g}$ of type I, i.e. such that*

$$\mathrm{soc}_{\mathfrak{g}'} V \cong V' \oplus N_a, \qquad V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b,$$
$$\mathrm{soc}_{\mathfrak{g}'} V_* \cong V'_* \oplus N_c, \qquad V_*/\mathrm{soc}_{\mathfrak{g}'} V_* \cong N_d,$$

where $< z, t >= 0$ for all $z \in N_a$ and $t \in N_c$. Then for the socle filtration of $V^{\{p,q\}}$ we have

$$\mathrm{soc}_{\mathfrak{g}'} V^{\{p,q\}} = ((\mathrm{soc}_{\mathfrak{g}'} V^{\otimes p}) \otimes (\mathrm{soc}_{\mathfrak{g}'} V_*^{\otimes q})) \cap V^{\{p,q\}},$$
$$\mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V^{\{p,q\}} = (\sum_{m+n=r} ((\mathrm{soc}^{(m+1)}_{\mathfrak{g}'} V^{\otimes p}) \otimes (\mathrm{soc}^{(n+1)}_{\mathfrak{g}'} V_*^{\otimes q}))) \cap V^{\{p,q\}}.$$

Proof. Denote $S^{(m,n)} = (\mathrm{soc}^{(m)}_{\mathfrak{g}'} V^{\otimes p}) \otimes (\mathrm{soc}^{(n)}_{\mathfrak{g}'} V_*^{\otimes q})$. As in Proposition 3.4, we will prove the following two steps.

(1) For any

$$u \in (\sum_{m+n=r+1} S^{(m+1,n+1)}) \cap V^{\{p,q\}} \setminus (\sum_{m+n=r} S^{(m+1,n+1)}) \cap V^{\{p,q\}}$$

there exists $g \in U(\mathfrak{g}')$ such that

$$g \cdot u \in (\sum_{m+n=r} S^{(m+1,n+1)}) \cap V^{\{p,q\}} \setminus (\sum_{m+n=r-1} S^{(m+1,n+1)}) \cap V^{\{p,q\}}.$$

(2) The quotient $(\sum_{m+n=r} S^{(m+1,n+1)}) \cap V^{\{p,q\}} / (\sum_{m+n=r-1} S^{(m+1,n+1)}) \cap V^{\{p,q\}}$ is semisimple.

Proof of (2): Note that

$$(\sum_{m+n=r} S^{(m+1,n+1)}) \cap V^{\{p,q\}} \cong \sum_{m+n=r} (S^{(m+1,n+1)} \cap V^{\{p,q\}}).$$

Hence,

$$\begin{aligned}
&(\sum_{m+n=r} S^{(m+1,n+1)}) \cap V^{\{p,q\}} / (\sum_{m+n=r-1} S^{(m+1,n+1)}) \cap V^{\{p,q\}} \cong \\
&\sum_{m+n=r} (S^{(m+1,n+1)} \cap V^{\{p,q\}}) / \sum_{m+n=r-1} (S^{(m+1,n+1)} \cap V^{\{p,q\}}) \cong \\
&\bigoplus_{m+n=r} (S^{(m+1,n+1)} \cap V^{\{p,q\}} / \sum_{m+n=r-1} (S^{(m+1,n+1)} \cap V^{\{p,q\}})).
\end{aligned}$$

Now for any fixed m and n and sequences of numbers $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_n$ we consider the $\mathfrak{g}'$-module homomorphism

$$L_{i_1,\dots,i_m} \otimes M_{j_1,\dots,j_n} : S^{(m+1,n+1)} \to N_b^{\otimes m} \otimes N_d^{\otimes n} \otimes (\mathrm{soc} V)^{\otimes p-m} \otimes (\mathrm{soc} V_*)^{\otimes q-n}$$

and its restriction to the submodule $V^{\{p,q\}}$

$$\begin{aligned}
&L_{i_1,\dots,i_m} \otimes M_{j_1,\dots,j_n} : \\
&S^{(m+1,n+1)} \cap V^{\{p,q\}} \to N_b^{\otimes m} \otimes N_d^{\otimes n} \otimes ((\mathrm{soc}_{\mathfrak{g}'} V)^{\otimes p-m} \otimes (\mathrm{soc}_{\mathfrak{g}'} V_*)^{\otimes q-n} \cap V^{\{p,q\}}).
\end{aligned}$$

Notice that this is a well-defined homomorphism also on the quotient

$$\begin{aligned}
L_{i_1,\dots,i_m} \otimes M_{j_1,\dots,j_n} : S^{(m+1,n+1)} \cap V^{\{p,q\}} / &\sum_{m+n=r-1} (S^{(m+1,n+1)} \cap V^{\{p,q\}}) \to \\
&N_b^{\otimes m} \otimes N_d^{\otimes n} \otimes ((\mathrm{soc}_{\mathfrak{g}'} V)^{\otimes p-m} \otimes (\mathrm{soc}_{\mathfrak{g}'} V_*)^{\otimes q-n} \cap V^{\{p,q\}}).
\end{aligned}$$

Then for any fixed m and n the map

$$\bigoplus_{\substack{i_1<\dots<i_m\\ j_1<\dots<j_n}} L_{i_1,\dots,i_m}\otimes M_{j_1,\dots,j_n} : S^{(m+1,n+1)}\cap V^{\{p,q\}}/\sum_{m+n=r-1}(S^{(m+1,n+1)}\cap V^{\{p,q\}})\to$$
$$\bigoplus_{\substack{i_1<\dots<i_m\\ j_1<\dots<j_n}}(N_b^{\otimes m}\otimes N_d^{\otimes n}\otimes(((\mathrm{soc}_{\mathfrak{g}'}V)^{\otimes p-m}\otimes(\mathrm{soc}_{\mathfrak{g}'}V_*)^{\otimes q-n})\cap V^{\{p,q\}}))$$

is a well-defined injective homomorphism of $\mathfrak{g}'$-modules. Since N_a and N_c pair trivially, we obtain

$$N_b^{\otimes m}\otimes N_d^{\otimes n}\otimes(((\mathrm{soc}_{\mathfrak{g}'}V)^{\otimes p-m}\otimes(\mathrm{soc}_{\mathfrak{g}'}V_*)^{\otimes q-n})\cap V^{\{p,q\}})\cong$$
$$\bigoplus_{k=0,l=0}^{p-m,q-n}\binom{p-m}{k}\binom{q-n}{l}N_b^{\otimes m}\otimes N_d^{\otimes n}\otimes N_a^{\otimes k}\otimes N_c^{\otimes l}\otimes V'^{\{p-m-k,q-n-l\}},$$

which is a semisimple $\mathfrak{g}'$-module.

Proof of (1): Let $u=\sum_{i=1}^N a_iu_i\otimes u_i^*$ where each $u_i\otimes u_i^*$ is a monomial in some $(\mathrm{soc}_{\mathfrak{g}'}^{(m+1)}V^{\otimes p})\otimes(\mathrm{soc}_{\mathfrak{g}'}^{(n+1)}V_*^{\otimes q})$ with $m+n=r+1$ and $u\in V^{\{p,q\}}$. Then each u_i is a monomial in $V^{\otimes p}$ and we denote it by $u_i=u_{i_1}\otimes\cdots\otimes u_{i_p}$. As before we make the notation more precise by setting $v_{i_k}=u_{i_k}$ if $u_{i_k}\in\mathrm{soc}_{\mathfrak{g}'}V$ and $x_{i_k}=u_{i_k}$ if $u_{i_k}\notin\mathrm{soc}_{\mathfrak{g}'}V$. We use similar notations for u_i^*. Then we take $g_1=w_1\otimes w_1^*$ such that w_1 satisfies: $\langle w_1,x_{1j}^*\rangle\neq 0$ for at least one x_{1j}^* that enters the monomial u_1^*, $\langle w_1,v_{1j}^*\rangle=0$ for all v_{1j}^*, and w_1 does not appear in any u_i. Similarly, w_1^* satisfies: $\langle x_{1j},w_1^*\rangle\neq 0$ for at least one x_{1j} that enters the monomial u_1, $\langle v_{1j},w_1^*\rangle=0$ for all v_{1j}, and w_1^* does not appear in any u_i^*. Then

$$g_1\cdot(u_1\otimes u_1^*)\in\sum_{m+n=r}(S^{(m+1,n+1)}\cap V^{\{p,q\}})\setminus\sum_{m+n=r-1}(S^{(m+1,n+1)}\cap V^{\{p,q\}}).$$

After defining inductively $g_1,\dots g_{i-1}$, if $(g_{i-1}\circ\cdots\circ g_1)\cdot u_i\otimes u_i^*$ is not in the desired space, we define g_i in the same way as we defined g_1. Finally we set $g=g_N\circ\cdots\circ g_1$ and then $g\cdot u$ has the desired properties. □

Corollary 3.11. *Let $\mathfrak{g}'\subset\mathfrak{g}$ be again an embedding of type I. Then for any simple $\mathfrak{g}$-module $V_{\lambda,\mu}\subset V^{\{p,q\}}$ we have*

$$\mathrm{soc}_{\mathfrak{g}'}V_{\lambda,\mu}=(\mathrm{soc}_{\mathfrak{g}'}V_{\lambda,0}\otimes\mathrm{soc}_{\mathfrak{g}'}V_{0,\mu})\cap V^{\{p,q\}},$$
$$\mathrm{soc}_{\mathfrak{g}'}^{(r+1)}V_{\lambda,\mu}=(\sum_{m+n=r}((\mathrm{soc}_{\mathfrak{g}'}^{(m+1)}V_{\lambda,0})\otimes(\mathrm{soc}_{\mathfrak{g}'}^{(n+1)}V_{0,\mu})))\cap V^{\{p,q\}}.$$

Moreover,

$$\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(r+1)} V_{\lambda,\mu} \cong \bigoplus_{m+n=r} \bigoplus_{\lambda'',\mu''} \bigoplus_{\substack{|\lambda'|=|\lambda''|-m \\ |\mu'|=|\mu''|-n}} m^a_{\lambda,\lambda''} m^b_{\lambda'',\lambda'} m^c_{\mu,\mu''} m^d_{\mu'',\mu'} V'_{\lambda',\mu'}.$$

Proof. Note that

$$\begin{aligned}&(\sum_{m+n=r} ((\mathrm{soc}_{\mathfrak{g}'}^{(m+1)} V^{\otimes p}) \otimes (\mathrm{soc}_{\mathfrak{g}'}^{(n+1)} V_*^{\otimes q}))) \cap (V_{\lambda,0} \otimes V_{0,\mu}) = \\ &\sum_{m+n=r} (((\mathrm{soc}_{\mathfrak{g}'}^{(m+1)} V^{\otimes p}) \otimes (\mathrm{soc}_{\mathfrak{g}'}^{(n+1)} V_*^{\otimes q})) \cap (V_{\lambda,0} \otimes V_{0,\mu})) = \\ &\sum_{m+n=r} ((\mathrm{soc}_{\mathfrak{g}'}^{(m+1)} V_{\lambda,0}) \otimes (\mathrm{soc}_{\mathfrak{g}'}^{(n+1)} V_{0,\mu})).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\mathrm{soc}_{\mathfrak{g}'}^{(r+1)} V_{\lambda,\mu} = \mathrm{soc}_{\mathfrak{g}'}^{(r+1)} V^{\{p,q\}} \cap V_{\lambda,\mu} = \\ &(\sum_{m+n=r} ((\mathrm{soc}_{\mathfrak{g}'}^{(m+1)} V^{\otimes p}) \otimes (\mathrm{soc}_{\mathfrak{g}'}^{(n+1)} V_*^{\otimes q}))) \cap V^{\{p,q\}} \cap V_{\lambda,\mu} = \\ &(\sum_{m+n=r} ((\mathrm{soc}_{\mathfrak{g}'}^{(m+1)} V^{\otimes p}) \otimes (\mathrm{soc}_{\mathfrak{g}'}^{(n+1)} V_*^{\otimes q}))) \cap V^{\{p,q\}} \cap V_{\lambda,0} \otimes V_{0,\mu} = \\ &\sum_{m+n=r} (((\mathrm{soc}_{\mathfrak{g}'}^{(m+1)} V_{\lambda,0}) \otimes (\mathrm{soc}_{\mathfrak{g}'}^{(n+1)} V_{0,\mu})) \cap V^{\{p,q\}}),\end{aligned}$$

which proves the first part of the statement. To compute the layers of the socle filtration we notice the following. If $\{A_m\}$ and $\{B_n\}$ are families of modules over a ring or an algebra such that $A_m \subset A_{m+1}$ for all m and $B_n \subset B_{n+1}$ for all n, then

$$\begin{aligned}&(\sum_{m+n=r} A_m \otimes B_n)/(\sum_{m+n=r-1} A_m \otimes B_n) \cong \\ &\bigoplus_{m+n=r} A_m/A_{m-1} \otimes B_n/B_{n-1}.\end{aligned}$$

In our case, if we set $A_m = \mathrm{soc}_{\mathfrak{g}'}^{(m+1)} V_{\lambda,0}$ and $B_n = \mathrm{soc}_{\mathfrak{g}'}^{(n+1)} V_{0,\mu}$ we obtain

$$\begin{aligned}&(\sum_{m+n=r} (\mathrm{soc}_{\mathfrak{g}'}^{(m+1)} V_{\lambda,0}) \otimes (\mathrm{soc}_{\mathfrak{g}'}^{(n+1)} V_{0,\mu}))/(\sum_{m+n=r-1} (\mathrm{soc}_{\mathfrak{g}'}^{(m+1)} V_{\lambda,0}) \otimes (\mathrm{soc}_{\mathfrak{g}'}^{(n+1)} V_{0,\mu})) \cong \\ &\bigoplus_{m+n=r} ((\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(m+1)} V_{\lambda,0}) \otimes (\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(n+1)} V_{0,\mu})) \cong \\ &\bigoplus_{m+n=r} \bigoplus_{\lambda'',\mu''} \bigoplus_{\substack{|\lambda'|=|\lambda''|-m \\ |\mu'|=|\mu''|-n}} m^a_{\lambda,\lambda''} m^b_{\lambda'',\lambda'} m^c_{\mu,\mu''} m^d_{\mu'',\mu'} V'_{\lambda',0} \otimes V'_{0,\mu'},\end{aligned}$$

where the second isomorphism is a corollary of Theorems 3.8 and 3.9. Hence,

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\lambda,\mu} \subseteq \mathrm{soc}_{\mathfrak{g}'}\Big(\bigoplus_{m+n=r} \bigoplus_{\lambda'',\mu''} \bigoplus_{\substack{|\lambda'|=|\lambda''|-m \\ |\mu'|=|\mu''|-n}} m^a_{\lambda,\lambda''} m^b_{\lambda'',\lambda'} m^c_{\mu,\mu''} m^d_{\mu'',\mu'} V'_{\lambda',0} \otimes V'_{0,\mu'}\Big) \cong$$
$$\bigoplus_{m+n=r} \bigoplus_{\lambda'',\mu''} \bigoplus_{\substack{|\lambda'|=|\lambda''|-m \\ |\mu'|=|\mu''|-n}} m^a_{\lambda,\lambda''} m^b_{\lambda'',\lambda'} m^c_{\mu,\mu''} m^d_{\mu'',\mu'} V'_{\lambda',\mu'}.$$

To prove the opposite inclusion we notice that every element from

$$V^{\otimes(p,q)}/\Big(\sum_{m+n=r-1} (\mathrm{soc}^{(m+1)}_{\mathfrak{g}'} V_{\lambda,0}) \otimes (\mathrm{soc}^{(n+1)}_{\mathfrak{g}'} V_{0,\mu})\Big)\Big)$$

can be sent to

$$V^{\{p,q\}}/\Big(\sum_{m+n=r-1} ((\mathrm{soc}^{(m+1)}_{\mathfrak{g}'} V_{\lambda,0}) \otimes (\mathrm{soc}^{(n+1)}_{\mathfrak{g}'} V_{0,\mu})\Big)\Big)$$

by an element in $U(\mathfrak{g}')$. In other words,

$$\mathrm{soc}_{\mathfrak{g}'}\Big(V^{\otimes(p,q)}/\Big(\sum_{m+n=r-1} ((\mathrm{soc}^{(m+1)}_{\mathfrak{g}'} V_{\lambda,0}) \otimes (\mathrm{soc}^{(n+1)}_{\mathfrak{g}'} V_{0,\mu})))\Big)\Big) \subseteq$$
$$V^{\{p,q\}}/\Big(\sum_{m+n=r-1} ((\mathrm{soc}^{(m+1)}_{\mathfrak{g}'} V_{\lambda,0}) \otimes (\mathrm{soc}^{(n+1)}_{\mathfrak{g}'} V_{0,\mu}))\Big).$$

Therefore,

$$\mathrm{soc}_{\mathfrak{g}'}\Big(\Big(\sum_{m+n=r} ((\mathrm{soc}^{(m+1)}_{\mathfrak{g}'} V_{\lambda,0}) \otimes (\mathrm{soc}^{(n+1)}_{\mathfrak{g}'} V_{0,\mu}))\Big)/\Big(\sum_{m+n=r-1} ((\mathrm{soc}^{(m+1)}_{\mathfrak{g}'} V_{\lambda,0}) \otimes (\mathrm{soc}^{(n+1)}_{\mathfrak{g}'} V_{0,\mu})))\Big)\Big) \subseteq$$
$$\Big(\sum_{m+n=r} ((\mathrm{soc}^{(m+1)}_{\mathfrak{g}'} V_{\lambda,0}) \otimes (\mathrm{soc}^{(n+1)}_{\mathfrak{g}'} V_{0,\mu}))\Big) \cap V^{\{p,q\}}/\Big(\sum_{m+n=r-1} ((\mathrm{soc}^{(m+1)}_{\mathfrak{g}'} V_{\lambda,0}) \otimes (\mathrm{soc}^{(n+1)}_{\mathfrak{g}'} V_{0,\mu}))\Big) \cong$$
$$\cong \overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\lambda,\mu}.$$

□

3.2 Branching laws for embeddings of type II

In this section $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of type II, i.e. such that

$$V \cong V' \oplus N_a, \quad V_* \cong V'_* \oplus N_c.$$

Then N_a and N_c have the same dimension and there is a non-degenerate bilinear pairing between them, which is the restriction of the bilinear pairing between V and V_*.

By definition (see [PSt]), for any partitions λ and μ with $|\lambda| = p$ and $|\mu| = q$ we have $V_{\lambda,\mu} = V^{\{p,q\}} \cap (V_{\lambda,0} \otimes V_{0,\mu})$. Hence, we can compute the socle filtrations of $V_{\lambda,0} \otimes V_{0,\mu}$ and of $V^{\{p,q\}}$ and use property (2.3) of socle filtrations to obtain the socle filtration for $V_{\lambda,\mu}$.

Proposition 3.12. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of type II. Then*

(i) Every $V_{\lambda,0} \subset V^{\otimes p}$ and every $V_{0,\mu} \subset V_*^{\otimes q}$ is completely reducible over $\mathfrak{g}'$ and

$$V_{\lambda,0} \cong \bigoplus_{\lambda'} m^a_{\lambda,\lambda'} V'_{\lambda',0},$$
$$V_{0,\mu} \cong \bigoplus_{\mu'} m^c_{\mu,\mu'} V'_{0,\mu'}.$$

(ii)

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\lambda,0} \otimes V_{0,\mu} \cong \bigoplus_{\lambda',\mu'} \bigoplus_{\substack{|\lambda''|=|\lambda'|-r \\ |\mu''|=|\mu'|-r}} \bigoplus_{\gamma} m^a_{\lambda,\lambda'} m^c_{\mu,\mu'} c^{\lambda'}_{\lambda'',\gamma} c^{\mu'}_{\mu'',\gamma} V'_{\lambda'',\mu''}.$$

Proof. Part (i). Since V is semisimple over $\mathfrak{g}'$, then so is $V^{\otimes p}$ and similarly for $V_*^{\otimes q}$. Therefore, every $V_{\lambda,0} \subset V^{\otimes p}$ and every $V_{0,\mu} \subset V_*^{\otimes q}$ is semisimple as well. To obtain the exact multiplicities, we proceed as in the proof of Lemma 3.6.

Part (ii). From part (i) we have

$$V_{\lambda,0} \otimes V_{0,\mu} \cong \bigoplus_{\lambda',\mu'} m^a_{\lambda,\lambda'} m^c_{\mu,\mu'} V'_{\lambda',0} \otimes V'_{0,\mu'}.$$

Hence, property (2.4) of socle filtrations implies

$$\mathrm{soc}^{(r)}_{\mathfrak{g}'}(V_{\lambda,0} \otimes V_{0,\mu}) \cong \bigoplus_{\lambda',\mu'} m^a_{\lambda,\lambda'} m^c_{\mu,\mu'} \mathrm{soc}^{(r)}_{\mathfrak{g}'}(V'_{\lambda',0} \otimes V'_{0,\mu'}).$$

Then, using Theorem 2.3 in [PSt], we obtain the desired formula. □

To derive the socle filtration of the module $V^{\{p,q\}}$ we first need to give several definitions. Let $\{z_i\}_{i\in I_a}$ and $\{t_i\}_{i\in I_a}$ be a pair of dual bases for the trivial $\mathfrak{g}'$-modules

N_a and N_c. We define a new bilinear form:

$$\langle \cdot, \cdot \rangle_t : V \times V_* \to \mathbb{C}$$

such that $\langle z_i, t_j \rangle_t = \delta_{ij}$ and for all other pairs of basis elements from $V \times V_*$ the bilinear form is trivial. Then, for any pair of indices $I = (i, j)$ with $i \in \{1, 2, \dots, p\}$ and $j \in \{1, 2, \dots, q\}$, we define the contraction

$$\Phi_I : V^{\otimes(p,q)} \to V^{\otimes(p-1,q-1)},$$

$$\begin{aligned} &v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^* \mapsto \\ &\langle v_i, v_j^* \rangle_t \, v_1 \otimes \cdots \otimes \hat{v_i} \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes \hat{v_j^*} \otimes \cdots \otimes v_q^*. \end{aligned}$$

Similarly, for any collection of pairwise disjoint index pairs $I_1, \dots, I_r$ respectively from the sets $\{1, 2, \dots, p\}$ and $\{1, 2, \dots, q\}$, we define the r-fold contraction

$$\Phi_{I_1,\dots,I_r} : V^{\otimes(p,q)} \to V^{\otimes(p-r,q-r)}$$

in the obvious way.

Now we set $N_a^{\otimes(p,q)} = N_a^{\otimes p} \otimes N_c^{\otimes q}$. Let Φ_I^a denote the restriction of Φ_I to the submodule $N_a^{\otimes(p,q)}$ and set $N_a^{\{p,q\}} = \bigcap_I \ker \Phi_I^a$.

Proposition 3.13. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of type II. Then*

$$\mathrm{soc}^{(r)}_{\mathfrak{g}'} V^{\{p,q\}} = \bigcap_{I_1,\dots,I_r} \ker \Phi_{I_1,\dots,I_r}.$$

Moreover,

$$\mathrm{soc}_{\mathfrak{g}'} V^{\{p,q\}} \cong \bigoplus_{k=0}^{p} \bigoplus_{l=0}^{q} \binom{p}{k} \binom{q}{l} N_a^{\{k,l\}} \otimes V'^{\{p-k,q-l\}}$$

and, for $a > p + q - 2$,

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V^{\{p,q\}} \cong \bigoplus_{\{I_1,\dots,I_r\}} \bigoplus_{k=0}^{p-r} \bigoplus_{l=0}^{q-r} \binom{p-r}{k} \binom{q-r}{l} N_a^{\{k,l\}} \otimes V'^{\{p-r-k,q-r-l\}}.$$

Proof. Denote similarly as before $S_{p,q}^{(r)} = \bigcap_{I_1,\dots,I_r} \ker \Phi_{I_1,\dots,I_r}$. Clearly,

$$S_{p,q}^{(1)} \cong \bigoplus_{k=0}^{p} \bigoplus_{l=0}^{q} \binom{p}{k} \binom{q}{l} N_a^{\{k,l\}} \otimes V'^{\{p-k,q-l\}}.$$

Then for each disjoint collection of index pairs $I_1, \ldots, I_r$, if we restrict $\Phi_{I_1,\ldots,I_r}$ to $S^{(r+1)}_{p,q}$ we obtain a map

$$\Phi_{I_1,\ldots,I_r} : S^{(r+1)}_{p,q} \to S^{(1)}_{p-r,q-r}.$$

Moreover,

$$\bigoplus_{\{I_1,\ldots,I_r\}} \Phi_{I_1,\ldots,I_r} : S^{(r+1)}_{p,q}/S^{(r)}_{p,q} \to \bigoplus_{\{I_1,\ldots,I_r\}} S^{(1)}_{p-r,q-r}$$

is a well-defined and injective homomoprhism of $\mathfrak{g}'$-modules. This shows the semisimplicity of the consecutive quotients.

Now we will show that if $a > p+q-2$, then for each r the above homomoprhism is also surjective. Without loss of generality fix the following collection of index pairs $I_1 = (1, q-r+1), \ldots, I_r = (r, q)$. Let v be an indecomposable element of the copy of $S^{(1)}_{p-r,q-r}$ which corresponds to the chosen collection. By indecomposable here we mean that v cannot be decomposed as a sum $v = v' + v''$ such that all monomials in v' and v'' belong to v and each of v' and v'' is also an element of $S^{(1)}_{p-r,q-r}$. Then v contains at most $p+q-2r$ entries with distinct indices from the pair of dual bases $\{z_i\}$, $\{t_i\}$. Let $i_1, \ldots, i_r$ be indices from I_a such that neither z_{i_k} nor t_{i_k} for $k = 1, \ldots, r$ enters the expression of v. These exist thanks to the condition $a > p+q-2$. In addition, let v_k and v_k^* be a pair of dual elements respectively from V' and V'_* which do not enter the expression of v. Then the vector

$$u = z_{i_1} \otimes \cdots \otimes z_{i_r} \otimes v \otimes t_{i_1} \otimes \cdots \otimes t_{i_r} - u_1$$

where

$$\begin{aligned} u_1 = & v_k \otimes z_{i_2} \otimes \cdots \otimes z_{i_r} \otimes v \otimes v_k^* \otimes t_{i_2} \otimes \cdots \otimes t_{i_r} + \\ & z_{i_1} \otimes v_k \otimes z_{i_3} \otimes \cdots \otimes z_{i_r} \otimes v \otimes t_{i_1} \otimes v_k^* \otimes t_{i_3} \otimes \cdots \otimes t_{i_r} + \cdots + \\ & z_{i_1} \otimes \cdots \otimes z_{i_{r-1}} \otimes v_k \otimes v \otimes t_{i_1} \otimes \cdots \otimes t_{i_{r-1}} \otimes v_k^* \end{aligned}$$

belongs to $S^{(r+1)}_{p,q}$ and

$$\bigoplus_{\{I_1,\ldots,I_r\}} \Phi_{I_1,\ldots,I_r}(u) = v.$$

Thus, for $a > p+q-2$ we obtain an exact expression for the layers of the above semisimple filtration. To show that this filtration is indeed the socle filtration of $V^{\{p,q\}}$, we take $u \in S^{(r+2)}_{p,q} \setminus S^{(r+1)}_{p,q}$. Then without loss of generality, $u = u_1 + \cdots + u_s$

for some s such that

$$u_1 = z_{i_1} \otimes \cdots \otimes z_{i_{r+1}} \otimes u'_1 \otimes t_{i_1} \otimes \cdots \otimes t_{i_{r+1}} -$$
$$v_k \otimes z_{i_2} \otimes \cdots \otimes z_{i_{r+1}} \otimes u'_1 \otimes v_k^* \otimes t_{i_2} \otimes \cdots \otimes t_{i_{r+1}} -$$
$$z_{i_1} \otimes v_k \otimes z_{i_3} \otimes \cdots \otimes z_{i_{r+1}} \otimes u'_1 \otimes t_{i_1} \otimes v_k^* \otimes t_{i_3} \otimes \cdots \otimes t_{i_{r+1}} - \cdots -$$
$$z_{i_1} \otimes \cdots \otimes z_{i_r} \otimes v_k \otimes u'_1 \otimes t_{i_1} \otimes \cdots \otimes t_{i_r} \otimes v_k^* + u''_1$$

where v_k, v_k^* is a pair of dual elements from $V' \otimes V'_*$, $u'_1 \in V'^{\{p-r-1,q-r-1\}}$, and $u''_1 \in S_{p,q}^{(r+1)}$. The elements $u_2, \ldots, u_s$ have a similar form. Notice that if v_k appears at most k times in any monomial in the expression of u'_1, then it appears at most $k+1$ times in any monomial in u_1. Hence, for $j = 1, \ldots, k+1$, if we take

$$g_j = v_{i_j} \otimes v_k^*$$

such that v_{i_j} does not appear in the expression of u and pairs trivially with all entries in u'_1 for all j, then $(g_1 \circ \cdots \circ g_{k+1})(u_1) \in S_{p,q}^{(r+1)} \setminus S_{p,q}^{(r)}$. We can do the same procedure for $u_2, \ldots, u_s$ and this will complete the proof of the statement. □

In order to obtain an exact expression for the layers of the socle filtraiton of $V^{\{p,q\}}$ also in the cases when $a \leq p+q-2$ we will use another approach. This approach covers all cases in which a is finite. Therefore, in what follows, we fix $a \in \mathbb{Z}_{\geq 0}$. As is done in [PSt], for any index pair $I = (i,j)$ as above we define the inclusion

$$\Psi_I^a : N_a^{\{p-1,q-1\}} \to N_a^{\otimes(p,q)}$$

given by

$$x_1 \otimes \cdots \otimes x_{p-1} \otimes x_1^* \otimes \cdots \otimes x_{q-1}^* \mapsto$$
$$\sum_{k=1}^{a} \ldots x_{i_1} \otimes z_k \otimes x_{i+1} \otimes \cdots \otimes x_{j-1}^* \otimes t_k \otimes x_{j+1}^* \otimes \ldots,$$

where x_i is an arbitrary element in N_a and x_j^* is an arbitrary element in N_c. Similarly, for any disjoint collection of index pairs $I_1, \ldots, I_r$, where $r = 1, \ldots, \min(p,q)$, we define the inclusion

$$\Psi_{I_1,\ldots,I_r}^a : N_a^{\{p-r,q-r\}} \to N_a^{\otimes(p,q)}$$

as the sum of the r-fold insertions of all possible ordered collections of r terms of the form $z_i \otimes t_i$, including collections with repeating terms. Then, following [PSt], we denote

$$(N_a)_r^{\{p,q\}} = \sum_{\{I_1,\dots,I_r\}} \mathrm{im}\Psi^a_{I_1,\dots,I_r}.$$

It is stated in [PSt] that for all a we have the direct sum decomposition

$$N_a^{\otimes(p,q)} = N_a^{\{p,q\}} \oplus (N_a)_1^{\{p,q\}} \oplus \cdots \oplus (N_a)_l^{\{p,q\}},$$

where $l = \min(p,q)$.

Proposition 3.14. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of type II such that $a = \dim N_a \in \mathbb{Z}_{\geq 0}$. Then*

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V^{\{p,q\}} \cong \bigoplus_{k=0}^{p} \bigoplus_{l=0}^{q} \binom{p}{k}\binom{q}{l} (N_a)_r^{\{k,l\}} \otimes V'^{\{p-k,q-l\}}.$$

Proof. Note that every element $u \in \bigcap_{I_1,\dots,I_{r+1}} \ker \Phi_{I_1,\dots,I_{r+1}}$ can be written as $u = u_1 + u_2$, where

$$u_1 \in \bigoplus_{k=0}^{p} \bigoplus_{l=0}^{q} \binom{p}{k}\binom{q}{l} (N_a)_r^{\{k,l\}} \otimes V'^{\{p-k,q-l\}},$$

$$u_2 \in \bigcap_{I_1,\dots,I_r} \ker \Phi_{I_1,\dots,I_r}.$$

This proves the statement. □

Note that for k or l smaller than r we have $(N_a)_r^{\{k,l\}} = 0$ and so the formula in Proposition 3.14 can be rewritten as

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V^{\{p,q\}} \cong \bigoplus_{k=0}^{p-r} \bigoplus_{l=0}^{q-r} \binom{p}{k+r}\binom{q}{l+r} (N_a)_r^{\{k+r,l+r\}} \otimes V'^{\{p-k-r,q-l-r\}}.$$

Comparing this formula with the expression from Proposition 3.13 we obtain

$$\binom{p}{k+r}\binom{q}{l+r} (N_a)_r^{\{k+r,l+r\}} = \bigoplus_{\{I_1,\dots,I_r\}} \binom{p-r}{k}\binom{q-r}{l} N_a^{\{k,l\}}, \tag{3.10}$$

where $i \in \{i,\dots,p\}$ and $j \in \{1,\dots,q\}$. Having in mind that the set of all such collections of r disjoint index pairs $I_1,\dots,I_r$ has $\binom{p}{r}\binom{q}{r}r!$ elements, we can further rewrite (3.10) as

$$(N_a)_r^{\{k+r,l+r\}} = \bigoplus_{\{J_1,\dots,J_r\}} N_a^{\{k,l\}},$$

where the sum runs over all collections of disjoint index pairs $J_1, \dots, J_r$ with $i \in \{1, \dots, k+r\}$ and $j \in \{1, \dots, l+r\}$. This formula holds exactly in the cases when the images of $\Psi^a_{J_1,\dots,J_r}$ are all disjoint and this is precisely when $a > k+l$. And since $k+l \le p+q-2$ for all layers of the socle filtration, we have just proved that Proposition 3.13 and Proposition 3.14 give the same formulas for finite dimension a with $a > p+q-2$.

If we denote now $K_{k,l}^{(r+1)} = \dim(N_a)_r^{\{k,l\}}$, where $(N_a)_0^{\{k,l\}} = N_a^{\{k,l\}}$, we can rewrite the formulas from Proposition 3.13 and Proposition 3.14 in the following way:

$$\mathrm{soc}_{\mathfrak{g}'} V^{\{p,q\}} \cong \bigoplus_{k=0}^{p} \bigoplus_{l=0}^{q} \binom{p}{k} \binom{q}{l} K_{k,l}^{(1)} V'^{\{p-k,q-l\}};$$

for $a \in \mathbb{Z}_{\ge 0} \sqcup \{\infty\}$ and $a > p+q-2$,

$$\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(r+1)} V^{\{p,q\}} \cong \bigoplus_{\{I_1,\dots,I_r\}} \bigoplus_{k=0}^{p-r} \bigoplus_{l=0}^{q-r} \binom{p-r}{k} \binom{q-r}{l} K_{k,l}^{(1)} V'^{\{p-k-r,q-l-r\}};$$

for all $a \in \mathbb{Z}_{\ge 0}$,

$$\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(r+1)} V^{\{p,q\}} \cong \bigoplus_{k=0}^{p-r} \bigoplus_{l=0}^{q-r} \binom{p}{k+r} \binom{q}{l+r} K_{k,l}^{(r+1)} V'^{\{p-k-r,q-l-r\}}.$$

Therefore, the last step in the discussion about the socle filtration of $V^{\{p,q\}}$ is to determine the dimensions of the trivial $\mathfrak{g}'$-modules $N_a^{\{p,q\}}$ for all a, including $a = \infty$, and the dimensions of the modules $(N_a)_r^{\{p,q\}}$ for finite values of a. Notice here that $N_a^{\{p,q\}}$ has the same dimension as the gl(a)-module $V_a^{\{p,q\}}$. In particular, if $a = \infty$, then $V_a^{\{p,q\}}$ is just the gl(∞)-module $V^{\{p,q\}}$, which is obviously infinite dimensional. Similarly, $(N_a)_r^{\{p,q\}}$ has the same dimension as the gl(a)-module $(V_a)_r^{\{p,q\}}$ (see [PSt] for the notation). Thus, it is enough to determine the dimensions of the modules $V_a^{\{p,q\}}$ and $(V_a)_r^{\{p,q\}}$ for any finite a.

Schur-Weyl duality (see, e.g. [FH]) yields

$$V_a^{\otimes(p,q)} \cong \bigoplus_{\substack{|\lambda|=p \\ |\mu|=q}} V_{\lambda,0}^a \otimes V_{0,\mu}^a \otimes (H_\lambda \otimes H_\mu).$$

Here, H_λ (resp., H_μ) denotes the irreducible representation of the symmetric group $\mathfrak{S}_p$ (resp., $\mathfrak{S}_q$) corresponding to the partition λ (resp., μ). $V^a_{\lambda,0}$ denotes as before the irreducible gl(a)-module with highest weight $(\lambda, 0)$.

Furthermore, for $a \geq p+q$

$$V^a_{\lambda,0} \otimes V^a_{0,\mu} \cong \bigoplus_{\lambda',\mu',\gamma} c^{\lambda}_{\lambda',\gamma} c^{\mu}_{\mu',\gamma} V^a_{\lambda',\mu'}. \tag{3.11}$$

Formula (3.11) can be found e.g. in [K], [HTW] and the condition $a \geq p+q$ is important there. When $a < p+q$, it is shown in [K], that modification rules have to be applied to (3.11) in order to derive the correct branching rule. These modification rules are described in detail in [K]. The main problem comes from the fact that when $a < p+q$, terms $V^a_{\lambda,\mu}$ with $l(\lambda)+l(\mu) > a$ can appear in the expression (3.11) and they do not define actual representations. Representations of the form $V^a_{\lambda,\mu}$ with $l(\lambda)+l(\mu) > a$ are called inadmissible or nonstandard, and R. King shows that they cannot always be disregarded from the branching formula. Instead, the modification rules tell us how to find equivalent admissible representations, which then replace the inadmissible ones in the branching formula. One remark here is that when $a = p+q-1$ then all inadmissible representations vanish by the modification rules, so for this case formula (3.11) applies as well. Hence, the condition for the modified branching rule becomes $a \leq p+q-2$, which unsuprisingly appeared also as a condition in our considerations of the socle filtration of $V^{\{p,q\}}$.

Following the above discussion, we rewrite (3.11) as

$$V^a_{\lambda,0} \otimes V^a_{0,\mu} \cong \bigoplus_{\lambda',\mu'} \tilde{c}^{\lambda,\mu}_{\lambda',\mu'} V^a_{\lambda',\mu'},$$

where

$$\tilde{c}^{\lambda,\mu}_{\lambda',\mu'} = \bigoplus_{\gamma} c^{\lambda}_{\lambda',\gamma} c^{\mu}_{\mu',\gamma} \tag{3.12}$$

for $a > p+q-2$. For $a \leq p+q-2$, $\tilde{c}^{\lambda,\mu}_{\lambda',\mu'}$ can be obtained from (3.12) by the modification rules in [K]. Then

$$V_a^{\otimes(p,q)} \cong \bigoplus_{\substack{|\lambda|=p \\ |\mu|=q}} \bigoplus_{\lambda',\mu'} \tilde{c}^{\lambda,\mu}_{\lambda',\mu'} V^a_{\lambda',\mu'} \otimes (H_\lambda \otimes H_\mu).$$

Recall that

$$V_a^{\otimes(p,q)} = (V_a)_0^{\{p,q\}} \oplus (V_a)_1^{\{p,q\}} \oplus \cdots \oplus (V_a)_l^{\{p,q\}}$$

and

$$\bigcap_{I_1,\ldots,I_{r+1}} \ker \Phi^a_{I_1,\ldots,I_{r+1}} = (V_a)_0^{\{p,q\}} \oplus (V_a)_1^{\{p,q\}} \oplus \cdots \oplus (V_a)_r^{\{p,q\}}$$

for $r = 0, \ldots, l$.

Moreover, Schur-Weyl duality implies that a simple module $V^a_{\lambda,\mu}$ with $|\lambda| = p - r$ and $|\mu| = q - r$ cannot be realized as a submodule of $V_a^{\otimes(p-r-1,q-r-1)}$. Hence $V^a_{\lambda,\mu} \subset \bigcap_{I_1,\ldots,I_{r+1}} \ker \Phi^a_{I_1,\ldots,I_{r+1}}$. On the other hand, for each copy of the module $V^a_{\lambda,\mu}$ inside $V_a^{\otimes(p,q)}$ there exists a contraction $\Phi^a_{I_1,\ldots,I_r}$ such that $V^a_{\lambda,\mu}$ is a submodule of its image, hence $V^a_{\lambda,\mu} \notin \bigcap_{I_1,\ldots,I_r} \ker \Phi^a_{I_1,\ldots,I_r}$. Thus, we proved that each $(V_a)_r^{\{p,q\}}$ contains only simple modules $V^a_{\lambda,\mu}$ with $|\lambda| = p - r$ and $|\mu| = q - r$. Therefore,

$$(V_a)_r^{\{p,q\}} \cong \bigoplus_{\substack{|\lambda|=p\\|\mu|=q}} \bigoplus_{\substack{|\lambda'|=p-r\\|\mu'|=q-r}} \tilde{c}^{\lambda,\mu}_{\lambda',\mu'} V^a_{\lambda',\mu'} \otimes (H_\lambda \otimes H_\mu).$$

Finally,

$$K^{(r+1)}_{p,q} = \dim (V_a)_r^{\{p,q\}} = \sum_{\substack{|\lambda|=p\\|\mu|=q}} \sum_{\substack{|\lambda'|=p-r\\|\mu'|=q-r}} \tilde{c}^{\lambda,\mu}_{\lambda',\mu'} \dim V^a_{\lambda',\mu'} \dim H_\lambda \dim H_\mu.$$

In particular, for all $a \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$

$$K^{(1)}_{p,q} = \sum_{\substack{|\lambda|=p\\|\mu|=q}} \dim V^a_{\lambda,\mu} \dim H_\lambda \dim H_\mu.$$

We are now ready to formulate the branching law for any simple tensor module $V_{\lambda,\mu}$.

Theorem 3.15. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of type II, and let $V_{\lambda,\mu} \subset V^{\{p,q\}}$. Then*

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\lambda,\mu} \cong \bigoplus_{k=0}^{p-r} \bigoplus_{l=0}^{q-r} \bigoplus_{\substack{|\lambda'|=p-k\\|\mu'|=q-l}} \bigoplus_{\substack{|\lambda''|=|\lambda'|-r\\|\mu''|=|\mu'|-r}} T^{\lambda,\mu}_{\lambda',\mu',\lambda'',\mu''} V'_{\lambda'',\mu''},$$

where, for $a \in \mathbb{Z}_{\geq 0}$

$$T^{\lambda,\mu}_{\lambda',\mu',\lambda'',\mu''} = \min(\sum_{\gamma} m^a_{\lambda,\lambda'} m^c_{\mu,\mu'} c^{\lambda'}_{\lambda'',\gamma} c^{\mu'}_{\mu'',\gamma}, \binom{p}{k+r}\binom{q}{l+r} K^{(r+1)}_{k+r,l+r} \dim H_{\lambda''} \dim H_{\mu''}),$$

and for $a \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ *with* $a > p+q-2$

$$T^{\lambda,\mu}_{\lambda',\mu',\lambda'',\mu''} = \min(\sum_{\gamma} m^a_{\lambda,\lambda'} m^c_{\mu,\mu'} c^{\lambda'}_{\lambda'',\gamma} c^{\mu'}_{\mu'',\gamma}, r!\binom{p}{r}\binom{q}{r}\binom{p-r}{k}\binom{q-r}{l} K^{(1)}_{k,l} \dim H_{\lambda''} \dim H_{\mu''}).$$

The above discussion, in particular the two approaches to computing the layers of the socle filtration of $V^{\{p,q\}}$, leads to the following combinatorial identity, connecting the dimensions of certain simple representations of the symmetric group with Littlewood-Richardson coefficients.

Proposition 3.16. *For any two partitions* λ' *and* μ' *with* $|\lambda'| = p - r$, $|\mu'| = q - r$ *for some integers* p, q, r *the following holds:*

$$\binom{p}{r}\binom{q}{r} r! \dim H_{\lambda'} \dim H_{\mu'} = \sum_{\substack{|\lambda|=p \\ |\mu|=q}} \sum_{\gamma} c^{\lambda}_{\lambda',\gamma} c^{\mu}_{\mu',\gamma} \dim H_\lambda \dim H_\mu.$$

3.3 Branching laws for embeddings of type III

In this section we consider embeddings $\mathfrak{g}' \subset \mathfrak{g}$ of type III, i.e. for which

$$V \cong kV' \oplus lV'_*, \quad V_* \cong lV' \oplus kV'_*.$$

Proposition 3.17. *If we have an embedding* $\mathfrak{g}' \subset \mathfrak{g}$ *of type III, then*

$$\mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V^{\otimes(p,q)} \cong \bigoplus_{m=0}^{p} \bigoplus_{n=0}^{q} \binom{p}{m}\binom{q}{n} k^{(m+q-n)} l^{(n+p-m)} \mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V'^{\otimes(m+n,p+q-m-n)}.$$

Proof. This follows directly from property (2.4) of socle filtrations. □

Before considering the simple submodules of $V^{\otimes(p,q)}$, we need to derive the branching rule for diagonal embeddings $\mathrm{gl}(n) \subset \mathrm{gl}(kn + ln)$ of signature $(k, l, 0)$, i.e. for embeddings given by

$$V^{(k+l)n}_{\downarrow \mathrm{gl}(n)} = \underbrace{V^n \oplus \cdots \oplus V^n}_{k} \oplus \underbrace{V^{*n} \oplus \cdots \oplus V^{*n}}_{l}$$

where V^n is the natural representation of $\mathrm{gl}(n)$ and V^{*n} is its dual. We do this in several steps, using the formulas in [HTW]. First we decompose the embedding $\mathrm{gl}(n) \subset \mathrm{gl}(kn + ln)$ in the following standard way, suggested to us by R. King:

$$\mathrm{gl}(n) \subset \mathrm{gl}(n) \oplus \mathrm{gl}(n) \cong \mathrm{gl}(n) \oplus \mathrm{gl}(n) \subset \mathrm{gl}(kn) \oplus \mathrm{gl}(ln) \subset \mathrm{gl}(kn + ln), \tag{3.13}$$

where the isomorphism $\mathrm{gl}(n) \oplus \mathrm{gl}(n) \cong \mathrm{gl}(n) \oplus \mathrm{gl}(n)$ is given by

$$(A, B) \mapsto (A, -B^T)$$

for any $A, B \in \mathrm{gl}(n)$. The other maps in (3.13) are the obvious ones. Recall from Proposition 2.4 from Chapter 2 that for diagonal embeddings $\mathrm{gl}(n) \subset \mathrm{gl}(kn)$ the following branching rule holds:

$$V^{kn}_{\lambda,\mu \downarrow \mathrm{gl}(n)} \cong \bigoplus_{\substack{\beta_1^+,\dots,\beta_k^+ \\ \beta_1^-,\dots,\beta_k^- \\ \lambda',\mu'}} C^{(\lambda,\mu)}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)} D^{(\lambda',\mu')}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)} V^n_{\lambda',\mu'}, \tag{3.14}$$

where the coefficients $C^{(\lambda,\mu)}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)}$ and $D^{(\lambda',\mu')}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)}$ are as in Chapter 2. Thus, the decomposition (3.13), formulas 2.1.1 and 2.2.1 from [HTW], and equation (3.14) yield

$$\begin{aligned} V^{(k+l)n}_{\lambda,\mu \quad \downarrow \mathrm{gl}(n)} \cong \quad & \bigoplus c^{(\lambda,\mu)}_{(\gamma^+,\gamma^-),(\delta^+,\delta^-)} C^{(\gamma^+,\gamma^-)}_{(\alpha_1^+,\dots,\alpha_k^+)(\alpha_1^-,\dots,\alpha_k^-)} D^{(\sigma^+,\sigma^-)}_{(\alpha_1^+,\dots,\alpha_k^+)(\alpha_1^-,\dots,\alpha_k^-)} \\ & C^{(\delta^+,\delta^-)}_{(\beta_1^+,\dots,\beta_l^+)(\beta_1^-,\dots,\beta_l^-)} D^{(\tau^+,\tau^-)}_{(\beta_1^+,\dots,\beta_l^+)(\beta_1^-,\dots,\beta_l^-)} d^{(\lambda',\mu')}_{(\sigma^+,\sigma^-),(\tau^-,\tau^+)} V^n_{\lambda',\mu'}, \end{aligned} \tag{3.15}$$

where the sum is over all partitions $\gamma^+, \gamma^-, \delta^+, \delta^-$, $\alpha_1^+, \dots, \alpha_k^+$, $\alpha_1^-, \dots, \alpha_k^-$, σ^+, σ^-, $\beta_1^+, \dots, \beta_l^+$, $\beta_1^-, \dots, \beta_l^-$, τ^+, τ^-, λ', μ'.

Corollary 3.18. *Consider an embedding $\mathfrak{g}' \subset \mathfrak{g}$ of type III. Then for each $V_{\lambda,\mu} \subset V^{\otimes(p,q)}$ we have*

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\lambda,\mu} \cong \bigoplus_{m=0}^{p} \bigoplus_{n=0}^{q} \bigoplus_{\substack{|\lambda'|=m+n-r \\ |\mu'|=p+q-m-n-r}} A^{\lambda,\mu}_{\lambda',\mu'} V'_{\lambda',\mu'},$$

where

$$A^{\lambda,\mu}_{\lambda',\mu'} = \sum c^{(\lambda,\mu)}_{(\gamma^+,\gamma^-),(\delta^+,\delta^-)} C^{(\gamma^+,\gamma^-)}_{(\alpha_1^+,\dots,\alpha_k^+)(\alpha_1^-,\dots,\alpha_k^-)} D^{(\sigma^+,\sigma^-)}_{(\alpha_1^+,\dots,\alpha_k^+)(\alpha_1^-,\dots,\alpha_k^-)} C^{(\delta^+,\delta^-)}_{(\beta_1^+,\dots,\beta_l^+)(\beta_1^-,\dots,\beta_l^-)} D^{(\tau^+,\tau^-)}_{(\beta_1^+,\dots,\beta_l^+)(\beta_1^-,\dots,\beta_l^-)} d^{(\lambda',\mu')}_{(\sigma^+,\sigma^-),(\tau^-,\tau^+)}.$$

Proof. From Proposition 3.17 above and from Theorem 2.2 in [PSt] we obtain

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\lambda,\mu} \cong \bigoplus_{m=0}^{p} \bigoplus_{n=0}^{q} \bigoplus_{\substack{|\lambda'|=m+n-r \\ |\mu'|=p+q-m-n-r}} a^{\lambda,\mu}_{\lambda',\mu'} V'_{\lambda',\mu'}$$

for some multiplicities $a^{\lambda,\mu}_{\lambda',\mu'}$. To compute those multiplicities explicitly we proceed as before. Following Proposition 3.1, we take bases of V and V_* such that

$$V = \mathrm{span}\{v_1^1, \dots, v_n^1, \dots, \dots, v_1^k, \dots, v_n^k, \dots, w_1^1, \dots w_n^1, \dots, \dots, w_1^l, \dots, w_n^l, \dots\}$$
$$V_* = \mathrm{span}\{v_1^{1*}, \dots, v_n^{1*}, \dots, \dots, v_1^{k*}, \dots, v_n^{k*}, \dots, w_1^{1*}, \dots w_n^{1*}, \dots, \dots, w_1^{l*}, \dots, w_n^{l*}, \dots\}$$

and such that

$$\mathfrak{g}' \cong \mathrm{span}\{v_i^1 \otimes v_j^{1*} + \dots + v_i^k \otimes v_j^{k*} - w_j^1 \otimes w_i^{1*} - \dots - w_j^l \otimes w_i^{l*}\}.$$

Then we easily construct exhaustions of $\mathfrak{g}'$ and $\mathfrak{g}$ to obtain a commutative diagram of embeddings in which all vertical arrows have signature $(k, l, 0)$. Thus from (3.15) we obtain the values of the multiplicities. □

Chapter 4

Embeddings of $\mathrm{sp}(\infty)$ into $\mathrm{sp}(\infty)$ and of $\mathrm{so}(\infty)$ into $\mathrm{so}(\infty)$

The goal of this chapter is to obtain branching laws for all types of embeddings $\mathfrak{g}' \subset \mathfrak{g}$, where now $\mathfrak{g}'$ and $\mathfrak{g}$ are both isomorphic to $\mathrm{sp}(\infty)$ or $\mathrm{so}(\infty)$. The approach we use is analogous to the one in Chapter 3 and therefore the exposition is concise and focused mainly on the end results. We start with decomposing the embeddings of general tensor type into several intermediate embeddings.

Proposition 4.1. *Let both $\mathfrak{g}'$ and $\mathfrak{g}$ be isomorphic to $\mathrm{sp}(\infty)$ or to $\mathrm{so}(\infty)$. Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of general tensor type, i.e. such that*

$$\mathrm{soc}_{\mathfrak{g}'}V \cong kV' \oplus N_a, \quad V/\mathrm{soc}_{\mathfrak{g}'}V \cong N_b.$$

Then there are intermediate subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of the same type as $\mathfrak{g}$ and $\mathfrak{g}'$, such that $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$ and the following conditions hold.

(1) The embedding $\mathfrak{g}_1 \subset \mathfrak{g}$ satisfies

$$\mathrm{soc}_{\mathfrak{g}_1}V \cong V_1 \oplus N_{a_1}, \quad V/\mathrm{soc}_{\mathfrak{g}_1}V \cong N_b,$$

where $N_{a_1} = \{v \in N_a | \langle v, N_a \rangle = 0\}$.

(2) The embedding $\mathfrak{g}_2 \subset \mathfrak{g}_1$ has the property $V_1 \cong V_2 \oplus N_{a_2}$, where N_{a_2} is such that $N_a = N_{a_1} \oplus N_{a_2}$.

(3) The embedding $\mathfrak{g}' \subset \mathfrak{g}_2$ has the property $V_2 \cong kV'$.

Proof. We will prove the statement for the case when $\mathfrak{g}'$ and $\mathfrak{g}$ are isomorphic to $\mathrm{sp}(\infty)$. Let Ω be the non-degenerate antisymmetric bilinear form on V. Let V_2

be a submodule of $\mathrm{soc}_{\mathfrak{g}'}V$ isomorphic to kV'. Then, as in the proof of Proposition 3.1, we can show that the restriction of Ω to V_2 is a non-degenerate antisymmetric bilinear form on V_2. Thus, if we set $\mathfrak{g}_2 = S^2(V_2)$ then $\mathfrak{g}_2$ is a Lie algebra isomorphic to $\mathrm{sp}(\infty)$.

Let $\{v_i\}_{i\in\mathbb{Z}\setminus\{0\}}$ be a symplectic basis of V_2, i.e. such that $\Omega(v_i, v_j) = \mathrm{sign}(i)\delta_{i+j,0}$. Let also $A = \{z_j\}_{j\in I_a}$ be a basis of N_a, where I_a is an index set with cardinality $a = \dim N_a$. Let $A_1 = \{z'_j\}_{j\in I_{a_1}}$ be those elements in A which pair trivially with all elements in A. Let $A_2 = A \setminus A_1$ and denote its elements with z''_j and its index set with I_{a_2}. Now set

$$V_1 = \mathrm{span}\{v_i\}_{i\in\mathbb{Z}\setminus\{0\}} \oplus \mathrm{span}\{z''_j\}_{j\in I_{a_2}}$$

Then the restriction of Ω to V_1 is non-degenerate and we set $\mathfrak{g}_1 = S^2(V_1)$.

The Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ constructed in this way satisfy the required properties.

In the case of $\mathrm{so}(\infty)$ we follow the same construction. □

As in Chapter 3 we give the following definition.

Definition 4.1. *Let again both* $\mathfrak{g}'$ *and* $\mathfrak{g}$ *be isomorphic to* $\mathrm{sp}(\infty)$ *or to* $\mathrm{so}(\infty)$.

(i) An embedding $\mathfrak{g}' \subset \mathfrak{g}$ *is said to be of type I if*

$$\mathrm{soc}_{\mathfrak{g}'}V \cong V' \oplus N_a, \qquad V/\mathrm{soc}_{\mathfrak{g}'}V \cong N_b,$$

where any two vectors from N_a *pair trivially.*

(ii) An embedding $\mathfrak{g}' \subset \mathfrak{g}$ *is said to be of type II if* $V \cong V' \oplus N_a$.

(iii) An embedding $\mathfrak{g}' \subset \mathfrak{g}$ *is said to be of type III if* $V \cong kV'$.

4.1 Embeddings of sp(∞) into sp(∞) of types I, II, and III

In this section, unless otherwise stated, $\mathfrak{g} = \mathrm{sp}(V)$ and $\mathfrak{g}' = \mathrm{sp}(V')$ where V is a countable-dimensional complex vector space endowed with a non-degenerate antisymmetric bilinear form Ω and V' is a subspace of V on which Ω restricts non-degenerately. First we consider embeddings $\mathfrak{g}' \subset \mathfrak{g}$ of type I, i.e. such that

$$\mathrm{soc}_{\mathfrak{g}'}V \cong V' \oplus N_a, \quad V/\mathrm{soc}_{\mathfrak{g}'}V \cong N_b,$$

where N_a and N_b are trivial $\mathfrak{g}'$-modules of finite or countable dimension and any two vectors from N_a pair trivially. As in Proposition 3.2 we can prove that the embedding $\mathfrak{g}' \subset \mathfrak{g}$ extends to an embedding $\mathrm{gl}(V', V') \subset \mathrm{gl}(V, V)$ of type I with the same values for a and b. Moreover, from Chapter 3 we know the socle filtrations over $\mathrm{gl}(V', V')$ of $V^{\otimes d}$ and of any $V_{\lambda,0} \subset V^{\otimes d}$.

Theorem 4.2. *(i) The socle filtration of* $V^{\langle d \rangle}$ *over* $\mathfrak{g}'$ *is*

$$\mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V^{\langle d \rangle} = \mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V^{\otimes d} \cap V^{\langle d \rangle}.$$

(ii) For any simple $\mathfrak{g}$*-module* $V_{\langle \lambda \rangle} \subset V^{\langle d \rangle}$ *we have*

$$\mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V_{\langle \lambda \rangle} = \mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V_{\lambda,0} \cap V^{\langle d \rangle}.$$

Moreover, for the layers of the socle filtration we have

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\langle \lambda \rangle} \cong \bigoplus_{\lambda''} \bigoplus_{|\lambda'| = |\lambda''| - r} m^a_{\lambda,\lambda''} m^b_{\lambda'',\lambda'} V'_{\langle \lambda' \rangle},$$

where as before $m^a_{\lambda,\lambda''}$ *are the extended Gelfand-Tsetlin multiplicities.*

Proof. Proof of (i): From the previous chapter we know that

$$\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V^{\otimes d} = \bigcap_{i_1 < \cdots < i_{r+1}} \ker L_{i_1,\ldots,i_{r+1}},$$

where

$$L_{i_1,\ldots,i_k} : V^{\otimes d} \to V^{\otimes (d-k)} \otimes N_b^{\otimes k}$$

is defined as before for any set of indices $1 \le i_1 < \cdots < i_k \le d$. We take its restriction to $V^{\langle d \rangle}$, namely

$$L_{i_1,\ldots,i_k} : V^{\langle d \rangle} \to V^{\langle d-k \rangle} \otimes N_b^{\otimes k}.$$

Thus we obtain a well-defined map on the quotient

$$\bigoplus_{i_1 < \cdots < i_r} L_{i_1,\ldots,i_r} : \mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V^{\otimes d} \cap V^{\langle d \rangle} / \mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V^{\otimes d} \cap V^{\langle d \rangle} \to \bigoplus_{i_1 < \cdots < i_r} ((V' \oplus N_a)^{\otimes (d-r)} \cap V^{\langle d-r \rangle}) \otimes N_b^{\otimes k},$$

which is an injective homomorphism of $\mathfrak{g}'$-modules, whence the semisimplicity of the quotient.

To show that this is indeed the socle filtration we proceed in the very same way as in the proof of part (1) of Proposition 3.4.

Proof of (ii): From [PSt] we know that $V_{\langle\lambda\rangle} = V_{\lambda,0} \cap V^{\langle d\rangle}$. Therefore,

$$\begin{aligned}&\mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V_{\langle\lambda\rangle} = (\mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V^{\langle d\rangle}) \cap V_{\langle\lambda\rangle} = (\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V^{\otimes d}) \cap V^{\langle d\rangle} \cap V_{\langle\lambda\rangle} = \\ &(\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V^{\otimes d}) \cap V_{\lambda,0} \cap V^{\langle d\rangle} = (\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) \cap V^{\langle d\rangle}\end{aligned}$$

which proves the first part of the statement.

The above implies

$$\begin{aligned}&\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\langle\lambda\rangle} \cong (\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) \cap V^{\langle d\rangle} / (\mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) \cap V^{\langle d\rangle} \cong \\ &((\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) \cap V^{\langle d\rangle} + \mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) / (\mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) \subset \overline{\mathrm{soc}}^{(r+1)}_{\mathrm{gl}(V',V')} V_{\lambda,0}.\end{aligned}$$

Thus we obtain

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\langle\lambda\rangle} \subseteq \mathrm{soc}_{\mathfrak{g}'}(\overline{\mathrm{soc}}^{(r+1)}_{\mathrm{gl}(V',V')} V_{\lambda,0}). \tag{4.1}$$

To prove the opposite inclusion we will first prove that

$$\begin{aligned}&\mathrm{soc}_{\mathfrak{g}'}(\overline{\mathrm{soc}}^{(r+1)}_{\mathrm{gl}(V',V')} V^{\otimes d}) \subseteq \\ &((\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V^{\otimes d}) \cap V^{\langle d\rangle} + \mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V^{\otimes d}) / \mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V^{\otimes d}.\end{aligned} \tag{4.2}$$

Let $u \in \mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V^{\otimes d}$ satisfy

$$u + \mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V^{\otimes d} \notin ((\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V^{\otimes d}) \cap V^{\langle d\rangle} + \mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V^{\otimes d}) / \mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V^{\otimes d}.$$

Then, $u = u_1 + \cdots + u_s$, where without loss of generality $u_1 = x_{i_1} \otimes \cdots \otimes x_{i_r} \otimes u_1'$ and $u_1' \in V'^{\otimes(d-r)} \setminus V'^{\langle d-r\rangle}$. Consequently, there exists $g \in U(\mathfrak{g}')$ such that $g \cdot u_1' \in V'^{\langle d-r\rangle} \setminus \{0\}$. Hence, $g \cdot u_1 \in (\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V^{\otimes d}) \cap V^{\langle d\rangle}$ and $g \cdot u_1 \notin \mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V^{\otimes d}$.

This proves (4.2). From (4.2) it follows that

$$\begin{aligned}&\mathrm{soc}_{\mathfrak{g}'}(\overline{\mathrm{soc}}^{(r+1)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) \subseteq \\ &((\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) \cap V^{\langle d\rangle} + \mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) / (\mathrm{soc}^{(r)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) \cong \overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\langle\lambda\rangle}.\end{aligned} \tag{4.3}$$

Then (4.1) and (4.3) imply

$$\mathrm{soc}_{\mathfrak{g}'}(\overline{\mathrm{soc}}^{(r+1)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) \cong \overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\langle\lambda\rangle}.$$

Now we can apply Theorem 3.8 above and Theorem 3.3 from [PSt] to obtain

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\langle\lambda\rangle} \cong \mathrm{soc}_{\mathfrak{g}'}\Big(\bigoplus_{\lambda''} \bigoplus_{|\lambda'|=|\lambda''|-r} m^a_{\lambda,\lambda''} m^b_{\lambda'',\lambda'} V'_{\lambda',0}\Big) \cong \bigoplus_{\lambda''} \bigoplus_{|\lambda'|=|\lambda''|-r} m^a_{\lambda,\lambda''} m^b_{\lambda'',\lambda'} V'_{\langle\lambda'\rangle},$$

which proves the second part of the statement. □

The next step is to consider embeddings $\mathfrak{g}' \subset \mathfrak{g}$ of type II, i.e. such that $V \cong V' \oplus N_a$. Note that in this case N_a is an even-dimensional or countable-dimensional trivial $\mathfrak{g}'$-module on which the bilinear form Ω restricts non-degenerately. As in the case of embeddings of type I, V and V' are also respectively the natural modules of $\mathrm{gl}(V,V)$ and $\mathrm{gl}(V',V')$ with the same socle filtrations. To compute the socle filtration of any simple $\mathfrak{g}$-module $V_{\langle\lambda\rangle}$ we proceed as in Section 3.2. More precisely, we use property (2.3) of socle filtrations and the fact that $V_{\langle\lambda\rangle} = V^{\langle d\rangle} \cap V_{\lambda,0}$. We also need the following notations. If γ is the integer partition $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k$, then by γ^T we denote the transpose (or conjugate) partition of γ, i.e. such that $(\gamma)^T_i = |\{\gamma_j : \gamma_j \geq i\}|$, and by 2γ we denote the even partition $2\gamma_1 \geq 2\gamma_2 \geq \cdots \geq 2\gamma_k$.

We start with the following proposition.

Proposition 4.3. *For any partition λ with $|\lambda| = d$ the $\mathrm{gl}(V,V)$-module $V_{\lambda,0}$ has the following socle filtration over $\mathfrak{g}'$:*

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\lambda,0} \cong \bigoplus_{\lambda'} \bigoplus_{|\lambda''|=|\lambda'|-2r} \bigoplus_{\gamma} m^a_{\lambda,\lambda'} c^{\lambda'}_{\lambda''(2\gamma)^T} V'_{\langle\lambda''\rangle},$$

where as before $m^a_{\lambda,\lambda'}$ are the extended Gelfand-Tsetlin multiplicities and $c^{\lambda'}_{\lambda''(2\gamma)^T}$ are the Littlewood-Richardson coefficients.

Proof. We consider the chain of embeddings $\mathfrak{g}' \subset \mathrm{gl}(V',V') \subset \mathrm{gl}(V,V)$. From Section 3.1 it follows that $V_{\lambda,0}$ is completely reducible over $\mathrm{gl}(V',V')$ and

$$V_{\lambda,0} \cong \bigoplus_{\lambda'} m^a_{\lambda,\lambda'} V'_{\lambda',0}. \tag{4.4}$$

Furthermore, Theorem 3.3 from [PSt] implies that

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V'_{\lambda',0} \cong \bigoplus_{|\lambda''|=|\lambda'|-2r} \bigoplus_{\gamma} c^{\lambda'}_{\lambda''(2\gamma)^T} V'_{\langle\lambda''\rangle}. \tag{4.5}$$

Combining (4.4) and (4.5) we obtain the result. □

Next, let $\{z_i\}_{i\in\mathbb{Z}\setminus\{0\}}$ be a symplectic basis for N_a. As in section 3.2 we define a new bilinear form

$$\langle \cdot, \cdot \rangle_t : V \times V \to \mathbb{C}$$

such that $\langle z_i, z_j \rangle = \mathrm{sign}(i)\delta_{i+j,0}$ and all other basis elements from $V \times V$ pair trivially. Then for any $1 \leq r \leq d$ and any collection of pairwise disjoint indices $I_1, \ldots, I_r$ from the set $\{1, \ldots, d\}$ we define the r-fold contraction

$$\Phi_{I_1,\ldots,I_r} : V^{\otimes d} \to V^{\otimes (d-2r)}$$

with respect to the bilinear form $\langle \cdot, \cdot \rangle_t$. It is easy to check that $\Phi_{I_1,\ldots,I_r}$ is a $\mathfrak{g}'$-module homomorphism. Then if $\Phi^a_{I_1,\ldots,I_r}$ denotes the restriction of $\Phi_{I_1,\ldots,I_r}$ to the submodule $N_a^{\otimes d}$, we set

$$N_a^{\langle d \rangle} = \bigcap_I \ker \Phi^a_I.$$

Furthermore, for $a \in \mathbb{Z}_{\geq 0}$ we define the inclusion

$$\Psi^a_{I_1,\ldots,I_r} : N_a^{\langle d-2r \rangle} \to N_a^{\otimes d}$$

as the sum of the r-fold insertions of all possible ordered collections of r terms of the form $z_i \otimes z_{-i}$, including collections with repeating terms. Then we set

$$(N_a)_r^{\langle d \rangle} = \sum_{\{I_1,\ldots,I_r\}} \mathrm{im}\Psi^a_{I_1,\ldots,I_r}.$$

Now we have the following theorem.

Theorem 4.4. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of type II. Then*

$$\mathrm{soc}^{(r)}_{\mathfrak{g}'} V^{\langle d \rangle} = \bigcap_{I_1,\ldots,I_r} \ker \Phi_{I_1,\ldots,I_r}.$$

Moreover, if $a \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ and $a > 2d-2$, then

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V^{\langle d \rangle} \cong \bigoplus_{\{I_1,\ldots,I_r\}} \bigoplus_{s=0}^{d-2r} \binom{d-2r}{s} N_a^{\langle s \rangle} \otimes V'^{\langle d-2r-s \rangle}.$$

If a is finite (and necessarily even), then

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V^{\langle d\rangle} \cong \bigoplus_{s=0}^{d} \binom{d}{s} (N_a)_r^{\langle s\rangle} \otimes V'^{\langle d-s\rangle} = \bigoplus_{s=0}^{d-2r} \binom{d}{s+2r} (N_a)_r^{\langle s+2r\rangle} \otimes V'^{\langle d-s-2r\rangle}.$$

The proof is quite similar to the proofs of Propositions 3.13 and 3.14 and we skip it.

To finish the discussion about the socle filtration of $V^{\langle d\rangle}$ over $\mathfrak{g}'$ we need to determine the dimensions of the trivial modules $(N_a)_r^{\langle d\rangle}$ for all $a \in 2\mathbb{Z}_{\geq 0}$ and the dimension of $N_a^{\langle d\rangle}$ for $a \in 2\mathbb{Z}_{\geq 0} \sqcup \{\infty\}$. As in Section 3.2 we notice that this is the same as determining the dimensions of the sp(a)-modules $V_a^{\langle d\rangle}$ and $(V_a)_r^{\langle d\rangle}$ (see [PSt] for the notations).

Now let a be an even integer. Schur-Weyl duality for the gl(a)-module $V_a^{\otimes d}$ yields

$$V_a^{\otimes d} \cong \bigoplus_{|\lambda|=d} V^a_{\lambda,0} \otimes H_\lambda.$$

Moreover, the gl(a)-module $V^a_{\lambda,0}$ considered as an sp(a)-module has the decomposition

$$V^a_{\lambda,0} \cong \bigoplus_{\lambda'} \tilde{c}^{\lambda}_{\lambda'} V^a_{\langle\lambda'\rangle},$$

where $\tilde{c}^{\lambda}_{\lambda'} = \sum_\gamma c^{\lambda}_{\lambda'(2\gamma)^T}$ for $a \geq 2d$ (see e.g. [K], [HTW]). For $a < 2d$, $\tilde{c}^{\lambda}_{\lambda'}$ is obtained from $\sum_\gamma c^{\lambda}_{\lambda'(2\gamma)^T}$ by the modification rules in [K]. Thus,

$$V_a^{\otimes d} \cong \bigoplus_{|\lambda|=d} \bigoplus_{\lambda'} \tilde{c}^{\lambda}_{\lambda'} V^a_{\langle\lambda'\rangle} \otimes H_\lambda.$$

Moreover, as in Secton 3.2 we can prove that

$$(V_a)_r^{\langle d\rangle} \cong \bigoplus_{|\lambda|=d} \bigoplus_{|\lambda'|=d-2r} \tilde{c}^{\lambda}_{\lambda'} V^a_{\langle\lambda'\rangle} \otimes H_\lambda.$$

Hence, if we denote $K_d^{(r+1)} = \dim (N_a)_r^{\langle d\rangle}$, where $(N_a)_0^{\langle d\rangle} = N_a^{\langle d\rangle}$, we obtain

$$K_d^{(r+1)} = \sum_{|\lambda|=d} \sum_{|\lambda'|=d-2r} \tilde{c}^{\lambda}_{\lambda'} \dim V^a_{\langle\lambda'\rangle} \dim H_\lambda.$$

In particular, when $r=0$ the above formula holds also when $a=\infty$, i.e.

$$K_d^{(1)} = \sum_{|\lambda|=d} \dim V^a_{\langle\lambda\rangle} \dim H_\lambda,$$

where $a \in 2\mathbb{Z}_{\geq 0} \sqcup \{\infty\}$.

We are now ready to formulate the branching law for an arbitrary simple $\mathfrak{g}$-module $V_{\langle\lambda\rangle}$.

Theorem 4.5. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of type II and let $V_{\langle\lambda\rangle} \subset V^{\langle d\rangle}$. Then*

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\langle\lambda\rangle} \cong \bigoplus_{s=0}^{d-2r} \bigoplus_{|\lambda'|=d-s} \bigoplus_{|\lambda''|=|\lambda'|-2r} T^{\lambda}_{\lambda',\lambda''} V'_{\langle\lambda''\rangle},$$

where for finite even integers a

$$T^{\lambda}_{\lambda',\lambda''} = \min\{\sum_{\gamma} m^a_{\lambda,\lambda'} c^{\lambda'}_{\lambda''(2\gamma)^T}, \binom{d}{s+2r} K^{(r+1)}_{s+2r} \dim H_{\lambda''}\}.$$

And for $a \in 2\mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ with $a > 2d-2$

$$T^{\lambda}_{\lambda',\lambda''} = \min\{\sum_{\gamma} m^a_{\lambda,\lambda'} c^{\lambda'}_{\lambda''(2\gamma)^T}, r!\binom{d}{r}\binom{d}{r}\binom{d-2r}{s} K^{(1)}_{s} \dim H_{\lambda''}\}.$$

In the rest of this section we will consider embeddings $\mathfrak{g}' \subset \mathfrak{g}$ of type III, i.e. such that $V \cong kV'$. We will proceed as in Section 3.3. First, we will determine the branching rule for diagonal embeddings $\mathrm{sp}(2n) \subset \mathrm{sp}(2kn)$ of signature $(k,0,0)$. Using formulas 2.1.3. and 2.2.3. from [HTW] we define the following coefficients:

$$a^{\lambda}_{\mu,\nu} = \sum_{\delta,\gamma} c^{\gamma}_{\mu\nu} c^{\lambda}_{\gamma(2\delta)^T}, \qquad b^{\lambda}_{\mu,\nu} = \sum_{\alpha,\beta,\gamma} c^{\lambda}_{\alpha\beta} c^{\mu}_{\alpha\gamma} c^{\nu}_{\beta\gamma},$$

where $c^{\lambda}_{\mu\nu}$ are again the Littlewood-Richardson coefficients. Next, as in Section 2.2, for $k>2$ we define the generalized versions of the coefficients $a^{\lambda}_{\mu,\nu}, b^{\lambda}_{\mu,\nu}$:

$$A^{\lambda}_{\mu_1,\ldots,\mu_k} = \sum_{\alpha_1,\ldots,\alpha_{k-2}} a^{\lambda}_{\alpha_1,\mu_1} a^{\alpha_1}_{\alpha_2,\mu_2} \ldots a^{\alpha_{k-3}}_{\alpha_{k-2},\mu_{k-2}} a^{\alpha_{k-2}}_{\mu_{k-1},\mu_k},$$

$$B^{\lambda}_{\mu_1,\ldots,\mu_k} = \sum_{\alpha_1,\ldots,\alpha_{k-2}} b^{\alpha_1}_{\mu_1,\mu_2} b^{\alpha_2}_{\alpha_1,\mu_3} \ldots b^{\alpha_{k-2}}_{\alpha_{k-3},\mu_{k-1}} b^{\lambda}_{\alpha_{k-2},\mu_k}.$$

Then, if $V^{2kn}_{\langle\lambda\rangle}$ denotes the simple $\mathrm{sp}(2kn)$-module with highest weight λ, iterating the appropriate branching rules from [HTW], we obtain that

$$V^{2kn}_{\langle\lambda\rangle}{}_{\downarrow \mathrm{sp}(2n)} \cong \bigoplus_{\mu_1,\dots,\mu_k,\lambda'} A^{\lambda}_{\mu_1,\dots,\mu_k} B^{\lambda'}_{\mu_1,\dots,\mu_k} V^{2n}_{\langle\lambda'\rangle}. \tag{4.6}$$

Now we are ready to prove the following theorem.

Theorem 4.6. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of type III. Then*

(i) $\mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V^{\otimes d} \cong k^d \mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V'^{\otimes d}$*;*

(ii) for any $V_{\langle\lambda\rangle} \subset V^{\otimes d}$ we have

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\langle\lambda\rangle} \cong \bigoplus_{|\lambda'|=d-2r} C^{\lambda}_{\lambda'} V'_{\langle\lambda'\rangle},$$

where

$$C^{\lambda}_{\lambda'} = \sum_{\mu_1,\dots,\mu_k,\lambda'} A^{\lambda}_{\mu_1,\dots,\mu_k} B^{\lambda'}_{\mu_1,\dots,\mu_k}.$$

Proof. Part (i) follows directly from property (2.4) of socle filtrations.

To prove part (ii) we notice that part (i) and Theorem 3.2 from [PSt] imply

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\langle\lambda\rangle} \cong \bigoplus_{|\lambda'|=d-2r} c^{\lambda}_{\lambda'} V'_{\langle\lambda'\rangle}$$

for some unknown multiplicities $c^{\lambda}_{\lambda'}$. To determine the exact values of these multiplicities we use (4.6).

□

4.2 Embeddings of so(∞) into so(∞) of types I, II, and III

In this section $\mathfrak{g} \cong \mathrm{so}(V)$ and $\mathfrak{g}' \cong \mathrm{so}(V')$ where V is a countable-dimensional complex vector space together with a non-degenerate symmetric bilinear form Q and V' is a subspace of V on which Q restricts non-degenerately. As in the previous section each embedding $\mathfrak{g}' \subset \mathfrak{g}$ extends to an embedding $\mathrm{gl}(V', V') \subset \mathrm{gl}(V, V)$ of the same type. All statements and proofs in this section are analogous to those in Section 4.1 and therefore we state only the end results.

First, let us consider embeddings $\mathfrak{g}' \subset \mathfrak{g}$ of type I, i.e. such that

$$\mathrm{soc}_{\mathfrak{g}'} V \cong V' \oplus N_a, \qquad V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b,$$

where the restriction of Q to N_a is trivial. Then we have the following analogue of Theorem 4.2.

Theorem 4.7. *(i) The socle filtration of $V^{[d]}$ over $\mathfrak{g}'$ is*

$$\mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V^{[d]} = (\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V^{\otimes d}) \cap V^{[d]}.$$

(ii) For any simple $\mathfrak{g}$-module $V_{[\lambda]} \subset V^{[d]}$ we have

$$\mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V_{[\lambda]} = (\mathrm{soc}^{(r+1)}_{\mathrm{gl}(V',V')} V_{\lambda,0}) \cap V^{[d]}.$$

For the layers of the socle filtration we obtain

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{[\lambda]} \cong \bigoplus_{\lambda''} \bigoplus_{|\lambda'|=|\lambda''|-r} m^a_{\lambda,\lambda''} m^b_{\lambda'',\lambda'} V'_{[\lambda']},$$

where as before $m^a_{\lambda,\lambda''}$ are the extended Gelfand-Tsetlin multiplicities.

Next, we move to embeddings of type II, i.e, such that $V \cong V' \oplus N_a$. Then the restriction of Q to N_a is non-degenerate. Let $\{z_i\}_{i\in\mathbb{Z}\setminus\{0\}}$ be a basis for N_a with the property that $Q(z_i, z_j) = \delta_{i+j,0}$. Following Section 4.1, we define a new bilinear form

$$\langle\cdot,\cdot\rangle_t : V \times V \to \mathbb{C}$$

such that $\langle z_i, z_j\rangle_t = \delta_{i+j,0}$ and all other basis elements from $V \times V$ pair trivially. Then, for any $1 \leq r \leq d$ and any collection of pairwise disjoint indices $I_1, \dots, I_r$ from the set $\{1, \dots, d\}$, we define the r-fold contraction

$$\Phi_{I_1,\dots,I_r} : V^{\otimes d} \to V^{\otimes(d-2r)}$$

with respect to the bilinear form $\langle\cdot,\cdot\rangle_t$. It is easy to check that $\Phi_{I_1,\dots,I_r}$ is a $\mathfrak{g}'$-module homomorphism for any r and any $I_1, \dots, I_r$. Then if $\Phi^a_{I_1,\dots,I_r}$ denotes the restriction of $\Phi_{I_1,\dots,I_r}$ to the submodule $N_a^{\otimes d}$, we set

$$N_a^{[d]} = \bigcap_I \ker \Phi^a_I.$$

Furthermore, if a is a finite number we define the inclusion

$$\Psi^a_{I_1,\dots,I_r} : N_a^{[d-2r]} \to N_a^{\otimes d}$$

as the sum of the r-fold insertions of all possible ordered collections of r terms of the form $z_i \otimes z_{-i}$, including collections with repeating terms. Then we set

$$(N_a)_r^{[d]} = \sum_{\{I_1,\dots,I_r\}} \mathrm{im}\Psi^a_{I_1,\dots,I_r}.$$

Now we have the following analogue of Theorem 4.4.

Theorem 4.8. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of type II. Then*

$$\mathrm{soc}^{(r)}_{\mathfrak{g}'} V^{[d]} = \bigcap_{I_1,\dots,I_r} \ker \Phi_{I_1,\dots,I_r}.$$

Moreover, if $a \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ with $a > 2d-2$, then

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V^{[d]} \cong \bigoplus_{\{I_1,\dots,I_r\}} \bigoplus_{s=0}^{d-2r} \binom{d-2r}{s} N_a^{[s]} \otimes V'^{[d-2r-s]}.$$

If $a \in \mathbb{Z}_{\geq 0}$, then

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V^{[d]} \cong \bigoplus_{s=0}^{d} \binom{d}{s} (N_a)_r^{[s]} \otimes V'^{[d-s]} = \bigoplus_{s=0}^{d-2r} \binom{d}{s+2r} (N_a)_r^{[s+2r]} \otimes V'^{[d-s-2r]}.$$

Let $K_d^{(r+1)} = \dim(N_a)_r^{[d]}$ for $r > 0$ and for any finite integer a. In addition, let $K_d^{(1)} = \dim N_a^{[d]}$ for arbitrary a including $a = \infty$. Then the following proposition holds.

Proposition 4.9. *For $r > 0$ we have*

$$K_d^{(r+1)} = \sum_{|\lambda|=d} \sum_{|\lambda'|=d-2r} \tilde{c}^{\lambda}_{\lambda'} \dim V^a_{[\lambda']} \dim H_\lambda,$$

where for $a \geq 2d$ we have $\tilde{c}^{\lambda}_{\lambda'} = \sum_\gamma c^{\lambda}_{\lambda' 2\gamma}$, and for $a < 2d$, $\tilde{c}^{\lambda}_{\lambda'}$ is obtained from $\sum_\gamma c^{\lambda}_{\lambda' 2\gamma}$ by the modification rules described in [K].

Furhtermore,

$$K_d^{(1)} = \sum_{|\lambda|=d} \dim V_{[\lambda]}^a \dim H_\lambda.$$

For an idea of proof see Section 4.1.

Using the above notations we are ready to state the analogue of Theorem 4.5.

Theorem 4.10. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of type II and let $V_{[\lambda]} \subset V^{[d]}$. Then*

$$\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(r+1)} V_{[\lambda]} \cong \bigoplus_{s=0}^{d-2r} \bigoplus_{|\lambda'|=d-s} \bigoplus_{|\lambda''|=|\lambda'|-2r} T_{\lambda',\lambda''}^{\lambda} V'_{[\lambda'']},$$

where for $a \in \mathbb{Z}_{\geq 0}$

$$T_{\lambda',\lambda''}^{\lambda} = \min\{\sum_\gamma m_{\lambda,\lambda'}^a c_{\lambda'' 2\gamma}^{\lambda'}, \binom{d}{s+2r} K_{s+2r}^{(r+1)} \dim H_{\lambda''}\}.$$

And for $a \in \mathbb{Z}_{\geq 0} \sqcup \{\infty\}$ and $a > 2d-2$

$$T_{\lambda',\lambda''}^{\lambda} = \min\{\sum_\gamma m_{\lambda,\lambda'}^a c_{\lambda'' 2\gamma}^{\lambda'}, r!\binom{d}{r}\binom{d}{r}\binom{d-2r}{s} K_s^{(1)} \dim H_{\lambda''}\}.$$

Finally, in the rest of this section we consider embeddings $\mathfrak{g}' \subset \mathfrak{g}$ of type III, i.e. such that $V \cong kV'$. As in Section 4.1 we start with determining the branching rule for diagonal embeddings $\mathrm{so}(n) \subset \mathrm{so}(kn)$ of signature $(k, 0, 0)$. Using formulas 2.1.2 and 2.2.2 from [HTW] we define the coefficients

$$a_{\mu,\nu}^{\lambda} = \sum_{\delta,\gamma} c_{\mu\nu}^{\gamma} c_{\gamma 2\delta}^{\lambda}, \qquad b_{\mu,\nu}^{\lambda} = \sum_{\alpha,\beta,\gamma} c_{\alpha\beta}^{\lambda} c_{\alpha\gamma}^{\mu} c_{\beta\gamma}^{\nu},$$

where $c_{\mu\nu}^{\lambda}$ are again the Littlewood-Richardson coefficients. Next, for $k > 2$ we define the generalized versions of the coefficients $a_{\mu,\nu}^{\lambda}, b_{\mu,\nu}^{\lambda}$:

$$A_{\mu_1,\ldots,\mu_k}^{\lambda} = \sum_{\alpha_1,\ldots,\alpha_{k-2}} a_{\alpha_1,\mu_1}^{\lambda} a_{\alpha_2,\mu_2}^{\alpha_1} \ldots a_{\alpha_{k-2},\mu_{k-2}}^{\alpha_{k-3}} a_{\mu_{k-1},\mu_k}^{\alpha_{k-2}},$$

$$B_{\mu_1,\ldots,\mu_k}^{\lambda} = \sum_{\alpha_1,\ldots,\alpha_{k-2}} b_{\mu_1,\mu_2}^{\alpha_1} b_{\alpha_1,\mu_3}^{\alpha_2} \ldots b_{\alpha_{k-3},\mu_{k-1}}^{\alpha_{k-2}} b_{\alpha_{k-2},\mu_k}^{\lambda}.$$

Then, if $V^{kn}_{[\lambda]}$ denotes the simple so(kn)-module with highest weight λ, we have

$$V^{kn}_{[\lambda]}{}_{\downarrow \mathrm{so}(n)} \cong \bigoplus_{\mu_1,\ldots,\mu_k,\lambda'} A^{\lambda}_{\mu_1,\ldots,\mu_k} B^{\lambda'}_{\mu_1,\ldots,\mu_k} V^{n}_{[\lambda']}.$$

The following theorem is an analogue of Theorem 4.6.

Theorem 4.11. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of type III. Then*

(i) $\mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V^{\otimes d} \cong k^d \mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V'^{\otimes d}$*;*

(ii) for any $V_{[\lambda]} \subset V^{\otimes d}$ we have

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{[\lambda]} \cong \bigoplus_{|\lambda'|=d-2r} C^{\lambda}_{\lambda'} V'_{[\lambda']},$$

where

$$C^{\lambda}_{\lambda'} = \sum_{\mu_1,\ldots,\mu_k,\lambda'} A^{\lambda}_{\mu_1,\ldots,\mu_k} B^{\lambda'}_{\mu_1,\ldots,\mu_k}.$$

Chapter 5

Embeddings of general tensor type. Main statements

In this chapter, using the results from Chapters 3 and 4, we prove a theorem for embeddings $\mathfrak{g}' \subset \mathfrak{g}$ of general tensor type. Here $\mathfrak{g}$ and $\mathfrak{g}'$ denote any two classical locally finite Lie algebras, not necessarily of the same type.

Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of general tensor type, i.e. which satisfies the conditions

$$\begin{aligned} \mathrm{soc}_{\mathfrak{g}'} V &\cong kV' \oplus lV'_* \oplus N_a, \quad V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b, \\ \mathrm{soc}_{\mathfrak{g}'} V_* &\cong kV'_* \oplus lV' \oplus N_c, \quad V_*/\mathrm{soc}_{\mathfrak{g}'} V_* \cong N_d. \end{aligned}$$

If $\mathfrak{g} \cong \mathrm{sp}(\infty)$ or $\mathrm{so}(\infty)$, then $N_a = N_c$ and $N_b = N_d$. If $\mathfrak{g}' \cong \mathrm{sp}(\infty)$ or $\mathrm{so}(\infty)$, then the above reduces to

$$\begin{aligned} \mathrm{soc}_{\mathfrak{g}'} V &\cong (k+l)V' \oplus N_a, \quad V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b, \\ \mathrm{soc}_{\mathfrak{g}'} V_* &\cong (k+l)V' \oplus N_c, \quad V_*/\mathrm{soc}_{\mathfrak{g}'} V_* \cong N_d. \end{aligned}$$

Throughout this chapter we will use the following notations. Let

$$\{v_i^j\}_{i \in I, j=i,\dots,k} \cup \{w_i^j\}_{i \in I, j=1,\dots,l}$$

be a basis for the submodule $kV' \oplus lV'_*$ of V indexed by a countable index set I and similarly

$$\{v_i^{j*}\}_{i \in I, j=i,\dots,k} \cup \{w_i^{j*}\}_{i \in I, j=1,\dots,l}$$

a basis for $kV'_* \oplus lV'$ as a submodule of V_*. Let the two bases satisfy the conditions of Proposition 3.1. Let also $A = \{z_i\}_{i \in I_a}$ and $C = \{t_i\}_{i \in I_c}$ be bases respectively for N_a and N_c, where I_a and I_c are index sets with cardinalities $a = \dim N_a$ and $c = \dim N_c$. Similarly let $B = \{x_i\}_{i\in I_b}$ and $D = \{y_i\}_{i\in I_d}$ be bases of N_b and N_d considered as vector subspaces of V and V_*, where I_b and I_d are index sets with cardinalities $b = \dim N_b$ and $d = \dim N_d$. We have proven the existence of such bases for the pair $\mathfrak{g}', \mathfrak{g} \cong \mathrm{gl}(\infty)$ in Proposition 3.1. For the other cases such bases exist by similar arguments.

Before turning to the general theorem we will consider a special case. Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding such that

$$V \cong kV' \oplus lV'_* \text{ and } V_* \cong lV' \oplus kV'_*.$$

We define a bilinear form $\langle\cdot,\cdot\rangle_d : (V \oplus V_*) \otimes (V \oplus V_*) \to \mathbb{C}$ by setting

$$\begin{aligned} \left\langle v_i^{s_1}, v_j^{*s_2}\right\rangle_d &= \delta_{ij}, & \left\langle w_i^{*s_3}, w_j^{s_4}\right\rangle_d &= \delta_{ij},\\ \left\langle v_i^{s_1}, w_j^{s_3}\right\rangle_d &= \delta_{ij}, & \left\langle w_i^{*s_3}, v_j^{*s_1}\right\rangle_d &= \delta_{ij} \end{aligned}$$

for all $s_1, s_2 = 1,\dots,k$ and all $s_3, s_4 = 1,\dots,l$. In addition, all other basis elements pair trivially. Let $J_1,\dots,J_s$ be a collection of disjoint index pairs (i,j), where $i,j = 1,\dots p+q$. Let

$$\Phi'_{J_1,\dots,J_s} : V^{\otimes(p,q)} \to V^{\otimes(p-s',q-s'')}$$

be the contraction with respect to the bilinear form $\langle\cdot,\cdot\rangle_d$, where $s'+s'' = 2s$. In the case $\mathfrak{g} \cong \mathrm{sp}(\infty)$ or $\mathfrak{g} \cong \mathrm{so}(\infty)$ the above can be rewritten as

$$\Phi'_{J_1,\dots,J_s} : V^{\otimes d} \to V^{\otimes(d-2s)}.$$

It is not difficult to prove that $\Phi'_{J_1,\dots,J_s}$ is a $\mathfrak{g}'$-module homomorphism.

Let us denote

$$\begin{aligned} V &= \tilde{V}_1 \oplus \dots \oplus \tilde{V}_k \oplus \tilde{V}_{k+1} \oplus \dots \oplus \tilde{V}_{k+l},\\ V_* &= \tilde{V}_1^* \oplus \dots \oplus \tilde{V}_k^* \oplus \tilde{V}_{k+1}^* \oplus \dots \oplus \tilde{V}_{k+l}^* \end{aligned}$$

where $\tilde{V}_1,\dots,\tilde{V}_k$ are copies of V' inside V such that $\tilde{V}_j = \mathrm{span}\{v_i^j\}_{i\in I}$ for $j = 1,\dots,k$. Similarly, $\tilde{V}_{k+1},\dots,\tilde{V}_{k+l}$ are copies of V'_* inside V such that $\tilde{V}_{k+j} = \mathrm{span}\{w_i^j\}_{i\in I}$ for $j = 1,\dots,l$. In the same way, $\tilde{V}_1^*,\dots,\tilde{V}_k^*$ are copies of V'_* inside V_* with bases $\{v_i^{j*}\}_{i\in I}$ for $j = 1,\dots,k$ and $\tilde{V}_{k+1}^*,\dots,\tilde{V}_{k+l}^*$ are copies of V' inside V_* with bases

$\{w_i^{j*}\}_{i\in I}$ for $j = 1, \ldots, l$. Then

$$V^{\otimes(p,q)} = \bigoplus_{\substack{i_1,\ldots,i_p \\ j_1,\ldots,j_q}} \tilde{V}_{i_1} \otimes \cdots \otimes \tilde{V}_{i_p} \otimes \tilde{V}^*_{j_1} \otimes \cdots \otimes \tilde{V}^*_{j_q},$$

where $i_1, \ldots, i_p, j_1, \ldots, j_q = 1, \ldots, k + l$ are not necessarily distinct indices. Notice that each $\tilde{V}_{i_1} \otimes \cdots \otimes \tilde{V}_{i_p} \otimes \tilde{V}^*_{j_1} \otimes \cdots \otimes \tilde{V}^*_{j_q}$ is isomorphic to $V'^{\otimes(p',q')}$ for some p' and q' with $p' + q' = p + q$. For any $J_1, \ldots, J_s$ as above, we want to define a map $\tilde{\Phi}_{J_1,\ldots,J_s}$ on $V^{\otimes(p,q)}$ such that on each subspace $V'^{\otimes(p',q')}$ it is the contraction with respect to the standard bilinear form on $V' \times V'_*$.

We proceed in the following way. Let $\pi_{\substack{i_1,\ldots,i_p \\ j_1,\ldots,j_q}}$ denote the projection of $V^{\otimes(p,q)}$ onto the subspace $\tilde{V}_{i_1} \otimes \cdots \otimes \tilde{V}_{i_p} \otimes \tilde{V}^*_{j_1} \otimes \cdots \otimes \tilde{V}^*_{j_q}$. Then we set

$$\tilde{\Phi}_{J_1,\ldots,J_s} = \bigoplus_{\substack{i_1,\ldots,i_p \\ j_1,\ldots,j_q}} \Phi'_{J_1,\ldots,J_s} \circ \pi_{\substack{i_1,\ldots,i_p \\ j_1,\ldots,j_q}}.$$

Thus,

$$\tilde{\Phi}_{J_1,\ldots,J_s} : V^{\otimes(p,q)} \to \bigoplus_{\substack{i_1,\ldots,i_p \\ j_1,\ldots,j_q}} \Phi'_{J_1,\ldots,J_s}(\tilde{V}_{i_1} \otimes \cdots \otimes \tilde{V}_{i_p} \otimes \tilde{V}^*_{j_1} \otimes \cdots \otimes \tilde{V}^*_{j_q}) \cong \bigoplus_{\substack{i_1,\ldots,i_p \\ j_1,\ldots,j_q}} V'^{\otimes(p-s',q-s'')}$$

for some s', s'' such that $s' + s'' = 2s$. The last isomoprhism follows from the observation that $\Phi'_{J_1,\ldots,J_s}$ acts on $\tilde{V}_{i_1} \otimes \cdots \otimes \tilde{V}_{i_p} \otimes \tilde{V}^*_{j_1} \otimes \cdots \otimes \tilde{V}^*_{j_q}$ as the usual contraction with respect to the standard bilinear form on $V' \times V'_*$.

Then the following proposition holds.

Proposition 5.1. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding such that*

$$V \cong kV' \oplus lV'_* \text{ and } V_* \cong lV' \oplus kV'_*.$$

Then

$$\mathrm{soc}^{(r)}_{\mathfrak{g}'} V^{\otimes(p,q)} = \bigcap_{J_1,\ldots,J_r} \ker \tilde{\Phi}_{J_1,\ldots,J_r}.$$

Proof. Property (2.4) of socle filtration implies

$$\mathrm{soc}^{(r)}_{\mathfrak{g}'} V^{\otimes(p,q)} = \bigoplus_{\substack{i_1,\dots,i_p \\ j_1,\dots,j_q}} \mathrm{soc}^{(r)}_{\mathfrak{g}'} \tilde{V}_{i_1} \otimes \cdots \otimes \tilde{V}_{i_p} \otimes \tilde{V}^*_{j_1} \otimes \cdots \otimes \tilde{V}^*_{j_q}.$$

Moreover, from [PSt] it follows that

$$\mathrm{soc}^{(r)}_{\mathfrak{g}'} \tilde{V}_{i_1} \otimes \cdots \otimes \tilde{V}_{i_p} \otimes \tilde{V}^*_{j_1} \otimes \cdots \otimes \tilde{V}^*_{j_q} = \bigcap_{J_1,\dots,J_r} \ker \Phi'_{J_1,\dots,J_r}|_{\tilde{V}_{i_1} \otimes \cdots \otimes \tilde{V}_{i_p} \otimes \tilde{V}^*_{j_1} \otimes \cdots \otimes \tilde{V}^*_{j_q}}.$$

Then,

$$\mathrm{soc}^{(r)}_{\mathfrak{g}'} V^{\otimes(p,q)} = \bigoplus_{\substack{i_1,\dots,i_p \\ j_1,\dots,j_q}} \bigcap_{J_1,\dots,J_r} \ker \Phi'_{J_1,\dots,J_r}|_{\tilde{V}_{i_1} \otimes \cdots \otimes \tilde{V}_{i_p} \otimes \tilde{V}^*_{j_1} \otimes \cdots \otimes \tilde{V}^*_{j_q}} =$$

$$\bigcap_{\substack{i_1,\dots,i_p \\ j_1,\dots,j_q}} \bigcap_{J_1,\dots,J_r} \ker(\Phi'_{J_1,\dots,J_r} \circ \pi_{\substack{i_1,\dots,i_p \\ j_1,\dots,j_q}}) = \bigcap_{J_1,\dots,J_r} \ker \tilde{\Phi}_{J_1,\dots,J_r}.$$

□

Now we are ready to state the main theorem for embeddings of general tensor type.

Theorem 5.2. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of general tensor type. Let $M = V^{\{p,q\}}$ for $\mathfrak{g} \cong \mathrm{gl}(\infty)$, $M = V^{\langle d \rangle}$ for $\mathfrak{g} \cong \mathrm{sp}(\infty)$, and $M = V^{[d]}$ for $\mathfrak{g} \cong \mathrm{so}(\infty)$. Suppose that there exist intermediate subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, of the same type as $\mathfrak{g}$, such that $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$ and the following properties are satisfied.*

(1) For the embedding $\mathfrak{g}_1 \subset \mathfrak{g}$ one has

$$\mathrm{soc}V \cong V_1 \oplus N_{a_1}, \qquad V/\mathrm{soc}V \cong N_b,$$
$$\mathrm{soc}V_* \cong V_{1*} \oplus N_{c_1}, \qquad V_*/\mathrm{soc}V_* \cong N_d,$$

where

$$N_{a_1} = \{v \in N_a |\, \langle v, N_c \rangle = 0\} \text{ and } N_{c_1} = \{w \in N_c |\, \langle N_a, w \rangle = 0\}.$$

(2) For the embedding $\mathfrak{g}_2 \subset \mathfrak{g}_1$ one has

$$V_1 \cong V_2 \oplus N_{a_2}, \qquad V_{1*} \cong V_{2*} \oplus N_{c_2},$$

where N_{a_2} and N_{c_2} are such that $N_a = N_{a_1} \oplus N_{a_2}$ and $N_c = N_{c_1} \oplus N_{c_2}$.

(3) For the embedding $\mathfrak{g}' \subset \mathfrak{g}_2$ one has

$$V_2 \cong kV' \oplus lV'_*, \qquad\qquad V_{2*} \cong lV' \oplus kV'_*.$$

Then, for any $N \subseteq M$,

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} N \cong \bigoplus_{l+m+n=r} \overline{\mathrm{soc}}^{(l+1)}_{\mathfrak{g}'}(\overline{\mathrm{soc}}^{(m+1)}_{\mathfrak{g}_2}(\overline{\mathrm{soc}}^{(n+1)}_{\mathfrak{g}_1} N)).$$

Proof. We have the following short exact sequences:

$$0 \to kV' \oplus lV'_* \oplus N_{a_1} \oplus N_{a_2} \xrightarrow{i} V \xrightarrow{f} N_b \to 0,$$
$$0 \to lV' \oplus kV'_* \oplus N_{c_1} \oplus N_{c_2} \xrightarrow{i} V_* \xrightarrow{g} N_d \to 0$$

and homomorphisms $L_{i_1,\dots,i_{n_1}}$ and $M_{j_1,\dots,j_{n_2}}$. For $\mathfrak{g}, \mathfrak{g}_1 \cong \mathrm{sp}(\infty)$ or $\mathrm{so}(\infty)$ the exact sequences coincide and $M_{j_1,\dots,j_{n_2}} = L_{j_1,\dots,j_{n_2}}$.

Recall that if M_1 is a $\mathfrak{g}_1$-module, such that $M_1 = V_1^{\{p,q\}}$, $V_1^{\langle d\rangle}$ or $V_1^{[d]}$, then we defined contractions $\Phi_{I_1,\dots,I_k}$ with respect to the pairing $\langle\cdot,\cdot\rangle_t$ on $V_1 \times V_{1*}$. We can extend this pairing to a pairing $\langle\cdot,\cdot\rangle_t : V \times V_* \to \mathbb{C}$ by setting

$$\langle z_i, t_j\rangle_t = \delta_{ij}$$

and such that all other basis elements from $V \times V_*$ pair trivially. Then we obtain contraction maps

$$\Phi_{I_1,\dots,I_k} : M \to M',$$

where $M' = V^{\{p-k,q-k\}}$ for $M = V^{\{p,q\}}$, and similarly for the other cases. We claim that for any choice of k and of disjoint index pairs $I_1,\dots,I_k$, the maps $\Phi_{I_1,\dots,I_k}$ are homomorphisms of $\mathfrak{g}_2$-modules and hence also of $\mathfrak{g}'$-modules. We now prove this for $k = 1$, and by induction the statement will follow for any $k > 1$. Let $g \in \mathfrak{g}_2$, $I = (i,j)$, and $u = u_1 \otimes \cdots \otimes u_p \otimes u_1^* \otimes \cdots \otimes u_q^*$ be an arbitrary pure tensor in M. Then

$$\Phi_I(g \cdot u) = g \cdot \Phi_I(u) + \langle g \cdot u_i, u_j^*\rangle_t \tilde{u} + \langle u_i, g \cdot u_j^*\rangle_t \tilde{u},$$

where $\tilde{u} = u_1 \otimes \cdots \otimes \hat{u}_i \otimes \cdots \otimes u_p \otimes u_1^* \otimes \cdots \otimes \hat{u_j^*} \otimes \cdots \otimes u_q^*$. Moreover, if $g = a \otimes b$ then

$$\langle g \cdot u_i, u_j^*\rangle_t + \langle u_i, g \cdot u_j^*\rangle_t = \langle u_i, b\rangle \langle a, u_j^*\rangle_t - \langle a, u_j^*\rangle \langle u_i, b\rangle_t .$$

But $a \in V_2$ and $b \in V_{2*}$, hence $\left\langle a, u_j^* \right\rangle_t = 0$ for all u_j^*, and similarly $\langle u_i, b \rangle_t = 0$ for all u_i.

Similarly, we define $\Phi'_{J_1,\ldots,J_k} : M \to M''$ to be the contraction with respect to the bilinear form $\langle \cdot, \cdot \rangle_d$ defined by

$$\left\langle v_i^{s_1}, v_j^{*s_2} \right\rangle_d = \delta_{ij} \text{ and } \left\langle w_i^{s_1}, w_j^{*s_2} \right\rangle_d = \delta_{ij},$$

and such that all other basis elements from $V \times V_*$ pair trivially. This is an extension of the bilinear form on $V_2 \times V_{2*}$ to a bilinear form on $V \times V_*$. As in the settings for Proposition 5.1 we set

$$\pi_{\substack{i_1,\ldots,i_p \\ j_1,\ldots,j_q}} : M \to \tilde{V}_{i_1} \otimes \cdots \otimes \tilde{V}_{i_p} \otimes \tilde{V}^*_{j_1} \otimes \cdots \otimes \tilde{V}^*_{j_q}.$$

Note that the map $\pi_{\substack{i_1,\ldots,i_p \\ j_1,\ldots,j_q}}$ is not a $\mathfrak{g}'$-module homomorphism, it is just a linear map. Then as before we define

$$\tilde{\Phi}_{J_1,\ldots,J_s} = \bigoplus_{\substack{i_1,\ldots,i_p \\ j_1,\ldots,j_q}} \Phi'_{J_1,\ldots,J_l} \circ \pi_{\substack{i_1,\ldots,i_p \\ j_1,\ldots,j_q}},$$

which again is just a linear map.

Now, for any $r \geq 0$ we define

$$S^{(r)}(M) = \bigcap_{\substack{l+m+n=r \\ n_1+n_2=n \\ i_1,\ldots,i_{n_1},j_1,\ldots,j_{n_2} \\ I_1,\ldots,I_m \\ J_1,\ldots,J_l}} \ker \tilde{\Phi}_{J_1,\ldots,J_l} \circ \Phi_{I_1,\ldots,I_m} \circ (L_{i_1,\ldots,i_{n_1}} \otimes M_{j_1,\ldots,j_{n_2}}).$$

If $n_1 = 0$ (resp. $n_2 = 0$) we set $L_0 = \mathrm{id}$ (resp. $M_0 = \mathrm{id}$) and similarly if $m = 0$ we set $\Phi_0 = \mathrm{id}$ and if $l = 0$ we set $\tilde{\Phi}_0 = \mathrm{id}$. For shortness, we write $S^{(r)}$ instead of $S^{(r)}(M)$ when M is clear from the context.

Note that we defined the maps $\tilde{\Phi}_{J_1,\ldots,J_l}$ and $\Phi_{I_1,\ldots,I_m}$ for M being equal to $V^{\{p,q\}}$, $V^{\langle d \rangle}$, and $V^{[d]}$, but the definitions can be easily extended to modules of the form $\bigoplus_i V^{\{p_i,q_i\}}$, $\bigoplus_i V^{\langle d_i \rangle}$, and $\bigoplus_i V^{[d_i]}$.

Now the following three properties hold:

(1) $S^{(r)}$ is a $\mathfrak{g}'$-submodule of M for every r (see Lemma 5.3);

(2) for any $u \in S^{(r+2)} \setminus S^{(r+1)}$ there exists $g \in U(\mathfrak{g}')$ such that $g(u) \in S^{(r+1)} \setminus S^{(r)}$ (see Lemma 5.4);

(3) $S^{(r+1)}/S^{(r)}$ is a semisimple $\mathfrak{g}'$-module and

$$S^{(r+1)}/S^{(r)} \cong \bigoplus_{l+m+n=r} \overline{\mathrm{soc}}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)}M))$$

(see Lemma 5.7).

Furthermore, if N is a submodule of M, then Lemma 5.7 yields

$$(S^{(r+1)} \cap N)/(S^{(r)} \cap N) \cong \bigoplus_{l+m+n=r} \overline{\mathrm{soc}}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)}N)).$$

Thus the statement of the theorem follows.

□

Theorem 5.2 provides a major tool for determining branching rules for embeddings of general tensor type. When $\mathfrak{g}' \cong \mathfrak{g}$, Propositions 3.3 and 4.1 give decompositions of the embedding $\mathfrak{g}' \subset \mathfrak{g}$ which satisfy the properties of Theorem 5.2. Therefore, for $\mathfrak{g}' \cong \mathfrak{g}$ we can apply Theorem 5.2 and thus reduce the branching problem for embeddings of general tensor type to branching problems for embeddings of special types, for which we already know the answer. In the next chapter we will use a similar approach in the cases when $\mathfrak{g}'$ and $\mathfrak{g}$ are not isomorphic.

Below we give proofs of the key lemmas used in the proof of Theorem 5.2.

Lemma 5.3. *Let $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$, M, and $S^{(r)}$ be as in Theorem 5.2. Then $S^{(r)}$ is a $\mathfrak{g}'$-submodule of M for every r.*

Proof. Let $u \in S^{(r)}$. Suppose that there exists $g \in \mathfrak{g}'$ such that $g \cdot u' \notin S^{(r)}$. In other words, there exist integers l, m, n, n_1, n_2 with $l + m + n = r$ and $n_1 + n_2 = n$ and index sets $i_1, \dots, i_{n_1}$, $j_1, \dots, j_{n_2}$, $I_1, \dots, I_m$, and $J_1, \dots, J_l$ such that

$$\tilde{\Phi}_{J_1,\dots,J_l} \circ \Phi_{I_1,\dots,I_m} \circ (L_{i_1,\dots,i_{n_1}} \otimes M_{j_1,\dots,j_{n_2}})(g \cdot u) \neq 0. \tag{5.1}$$

We set $u' = \Phi_{I_1,\dots,I_m} \circ (L_{i_1,\dots,i_{n_1}} \otimes M_{j_1,\dots,j_{n_2}})(u)$. Since $L_{i_1,\dots,i_{n_1}} \otimes M_{j_1,\dots,j_{n_2}}$ and $\Phi_{I_1,\dots,I_m}$ are $\mathfrak{g}'$-module homomorphisms, (5.1) implies that $u' \neq 0$. Furthermore, $\tilde{\Phi}_{J_1,\dots,J_l}(u') = 0$, whereas $\tilde{\Phi}_{J_1,\dots,J_l}(g \cdot u') \neq 0$. This can only happen if an element x_{i_j} or y_{i_j} appears in a monomial in u' in one of the positions specified by $J_1, \dots, J_l$. Let without loss of generality $J_1 = (1, 1)$ and let u' have the form

$$u' = x_{i_j} \otimes u'_2 \otimes \cdots \otimes u'_{s_1} \otimes u'^*_1 \otimes \cdots \otimes u'^*_{s_2} + u''. \tag{5.2}$$

Then, $\tilde{\Phi}_{J_1,\dots,J_l}(u'') = 0$ and

$$g \cdot u' = (g \cdot x_{i_j}) \otimes u'_2 \otimes \cdots \otimes u'_{s_1} \otimes u'^*_1 \otimes \cdots \otimes u'^*_{s_2} + x_{i_j} \otimes u''' + g \cdot u''.$$

Moreover, $\tilde{\Phi}_{J_1,\dots,J_l}(x_{i_j} \otimes u''') = 0$. Assume at first that $\tilde{\Phi}_{J_1,\dots,J_l}(g \cdot u'') = 0$ as well. Then

$$\tilde{\Phi}_{J_1,\dots,J_l}(g \cdot u') = \tilde{\Phi}_{J_1}((g \cdot x_{i_j}) \otimes u'^*_1) \circ \tilde{\Phi}_{J'_2,\dots,J'_l}(u'_2 \otimes \cdots \otimes u'_{s_1} \otimes u'^*_2 \otimes \cdots \otimes u'^*_{s_2}) \neq 0.$$

Here, if $J_2 = (i, j)$ then $J'_2 = (i-1, j-1)$ and similarly for $J'_3, \dots, J'_l$.

Therefore, $\tilde{\Phi}_{J_2,\dots,J_l}(u') \neq 0$. The last inequality and (5.2) imply that there exists an index i_{n_1+1} such that

$$\tilde{\Phi}_{J_2,\dots,J_l} \circ \Phi_{I_1,\dots,I_m} \circ (L_{i_1,\dots,i_{n_1},i_{n_1+1}} \otimes M_{j_1,\dots,j_{n_2}})(u) \neq 0.$$

This is a contradiction with $u \in S^{(r)}$.

If $\tilde{\Phi}_{J_1,\dots,J_l}(g \cdot u'') \neq 0$ we can replace u' in the above discussion with u'', which has a strictly smaller number of monomials that u'. Thus in finitely many steps we will reach a contradiction with the choice of u, and this proves the statement.

□

Lemma 5.4. *Let $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$, M, and $S^{(r)}$ be as in Theorem 5.2. Then for any $u \in S^{(r+2)} \setminus S^{(r+1)}$ there exists $g \in U(\mathfrak{g}')$ such that $g(u) \in S^{(r+1)} \setminus S^{(r)}$.*

Proof. We order the triples of numbers (n, m, l) lexicographically. Let $u \in S^{(r+2)} \setminus S^{(r+1)}$ and let (n, m, l) with $l + m + n = r + 1$ be the largest triple for which there exist index sets $i_1, \dots, i_{n_1}$, $j_1, \dots, j_{n_2}$, $I_1, \dots, I_m$, and $J_1, \dots, J_l$ such that

$$\tilde{\Phi}_{J_1,\dots,J_l} \circ \Phi_{I_1,\dots,I_m} \circ (L_{i_1,\dots,i_{n_1}} \otimes M_{j_1,\dots,j_{n_2}})(u) \neq 0.$$

Suppose first that $n > 0$. Then at least one monomial in u has an entry x_i or y_j. Without loss of generality we may assume that x_i appears in u. Then we take $g \in \mathfrak{g}'$ of the form

$$g = v^1_i \otimes v^{*1}_j + \cdots + v^k_i \otimes v^{*k}_j - w^1_j \otimes w^{*1}_i - \cdots - w^k_j \otimes w^{*k}_i$$

for $\mathfrak{g}' \cong \mathrm{gl}(\infty)$. If $\mathfrak{g}' \cong \mathrm{sp}(\infty)$, we symmetrize the above expression for g, and if $\mathfrak{g}' \cong \mathrm{so}(\infty)$, we antisymmetrize the above expression for g. We choose v^{*s}_j and w^{*s}_i such that they do not appear in u, at least one of them pairs non-degenerately with x_i, and they pair trivially with all basis elements from V_2 which appear in u. Similarly we choose v_i^s and w_j^s such that they do not appear in u and such that they pair trivially with all basis elements from V_{2*} which appear in u. Then

$$\tilde{\Phi}_{J_1,\dots,J_l} \circ \Phi_{I_1,\dots,I_m} \circ (L_{i_1,\dots,i_{n_1}} \otimes M_{j_1,\dots,j_{n_2}})(g \cdot u) = 0$$

and there exist indices $i''_1, \dots, i''_{n_1-1}$ such that

$$\tilde{\Phi}_{J_1,\dots,J_l} \circ \Phi_{I_1,\dots,I_m} \circ (L_{i''_1,\dots,i''_{n_1-1}} \otimes M_{j_1,\dots,j_{n_2}})(g \cdot u) \neq 0.$$

Now suppose that there exists another triple (n', m', l') with $l' + m' + n' = r + 1$ for which there are index sets $i'_1, \dots, i'_{n'_1}$, $j'_1, \dots, j'_{n'_2}$, $I'_1, \dots, I'_{m'}$, and $J'_1, \dots, J'_{l'}$ such that

$$\tilde{\Phi}_{J'_1,\dots,J'_{l'}} \circ \Phi_{I'_1,\dots,I'_{m'}} \circ (L_{i'_1,\dots,i'_{n'_1}} \otimes M_{j'_1,\dots,j'_{n'_2}})(g \cdot u) \neq 0.$$

But then there exists an index $i'_{n'+1}$ such that

$$\tilde{\Phi}_{J'_1,\dots,J'_{l'}} \circ \Phi_{I'_1,\dots,I'_{m'}} \circ (L_{i'_1,\dots,i'_{n'_1},i'_{n'_1+1}} \otimes M_{j'_1,\dots,j'_{n'_2}})(u) \neq 0,$$

which contradicts with the choice of u. Hence, $g \cdot u \in S^{(r+1)} \setminus S^{(r)}$.

Now, suppose that $n = 0$. This means that $u \in \mathrm{soc}_{\mathfrak{g}_1} M$ and u consists only of elements from V_1 and V_{1*}. Notice that, when restricted to $\mathrm{soc}_{\mathfrak{g}_1} M$, both maps $\tilde{\Phi}_{J_1,\dots,J_l}$ and $\Phi_{I_1,\dots,I_m}$ are $\mathfrak{g}'$-module homomorphisms. Therefore, in this case the statement of the lemma reduces to the following claim. Let $l + m = r + 1$, and $I_1, \dots, I_m$, and $J_1, \dots, J_l$ be such that

$$\tilde{\Phi}_{J_1,\dots,J_l} \circ \Phi_{I_1,\dots,I_m}(u) \neq 0.$$

Then there exists $g \in U(\mathfrak{g}')$ such that

$$\tilde{\Phi}_{J_1,\dots,J_l} \circ \Phi_{I_1,\dots,I_m}(g \cdot u) = 0.$$

Moreover,

$$\tilde{\Phi}_{J'_1,\dots,J'_{l'}} \circ \Phi_{I'_1,\dots,I'_{m'}}(g \cdot u) \neq 0$$

for some $I'_1, \dots, I'_{m'}$, and $J'_1, \dots, J'_{l'}$ with $l' + m' = r$.

We consider two cases.

(1) Let $\tilde{\Phi}_{J_1,\dots,J_l} \circ \Phi_{I_1,\dots,I_m}(u) \neq 0$ with $l \geq 1$. Then we can apply Lemma 5.5 to the element $\Phi_{I_1,\dots,I_m}(u)$. This yields an element $g \in U(\mathfrak{g}')$ and disjoint index pairs $J'_1, \dots, J'_{l-1}$ such that

$$\tilde{\Phi}_{J'_1,\dots,J'_{l-1}} \circ \Phi_{I_1,\dots,I_m}(g \cdot u) \neq 0.$$

(2) Let $\tilde{\Phi}_0 \circ \Phi_{I_1,\dots,I_m}(u) \neq 0$. Then, we can apply Lemma 5.6 to u. Thus there exists $g \in U(\mathfrak{g}')$ with $\tilde{\Phi}_0 \circ \Phi_{I_1,\dots,I_m}(g \cdot u) = 0$, and there exist $I'_1, \dots, I'_{m-1}$ such that

$$\tilde{\Phi}_0 \circ \Phi_{I'_1,\dots,I'_{m-1}}(g \cdot u) \neq 0.$$

□

Lemma 5.5. *Let $u \in \mathrm{soc}_{\mathfrak{g}_1} M$ be such that $\tilde{\Phi}_{J_1,\dots,J_l}(u) \neq 0$ for some l and some $J_1, \dots, J_l$. Then there exists $g \in U(\mathfrak{g}')$ such that $\tilde{\Phi}_{J_1,\dots,J_l}(g \cdot u) = 0$ and there exists a collection $J'_1, \dots, J'_{l-1}$ such that $\tilde{\Phi}_{J'_1,\dots,J'_{l-1}}(g \cdot u) \neq 0$.*

Proof. (1) If u consists only of elements from V_2 and V_{2*} then the claim follows from Proposition 5.1.

(2) Let u contain elements from V_1 and V_{1*}. Then $u = u_1 + \cdots + u_t$ and without loss of generality

$$u_1 = u'_1 \otimes u''_1 = u'_1 \otimes (z_{i_1} \otimes \dots z_{i_s} \otimes t_{i_1} \otimes \dots t_{i_s} + u''),$$

where u'_1 contains only elements from V_2 and V_{2*}, and u'' contains less than s pairs $z_i \otimes t_i$. Then, either $\tilde{\Phi}_{J_1,\dots,J_l}(u_1) = \tilde{\Phi}_{J_1,\dots,J_l}(u'_1) \otimes u''_1$, or $\tilde{\Phi}_{J_1,\dots,J_l}(u_1) = \tilde{\Phi}_{J_1,\dots,J_l}(u'_1 \otimes u'')$. In both cases we reduce this situation to case (1).

□

Lemma 5.6. *Let $u \in \mathrm{soc}_{\mathfrak{g}_1} M$ be such that $\Phi_{I_1,\dots,I_m}(u) \neq 0$ for some $m > 0$ and some $I_1, \dots, I_m$. Then there exists $g \in U(\mathfrak{g}')$ such that $\Phi_{I_1,\dots,I_m}(g \cdot u) = 0$ and $\Phi_{I'_1,\dots,I'_{m-1}}(g \cdot u) \neq 0$ for some collection $I'_1, \dots, I'_{m-1}$.*

Proof. Let u be as in the statement of the lemma. Then $u = u_1 + \cdots + u_t$ and without loss of generality we can fix $I_1, \dots, I_m$ such that u_1 has the form

$$\begin{aligned} u_1 =& z_{i_1} \otimes z_{i_2} \otimes \cdots \otimes z_{i_m} \otimes u_1' \otimes t_{i_1} \otimes t_{i_2} \otimes \cdots \otimes t_{i_m} - \\ & v_1^1 \otimes z_{i_2} \otimes \cdots \otimes z_{i_m} \otimes u_1' \otimes v_1^{1*} \otimes t_{i_2} \otimes \cdots \otimes t_{i_m} - \cdots - \\ & z_{i_1} \otimes z_{i_2} \otimes \cdots \otimes z_{i_m} \otimes v_1^1 \otimes u_1' \otimes t_{i_1} \otimes t_{i_2} \otimes \cdots \otimes t_{i_m} \otimes v_1^{1*} + u_1'', \end{aligned}$$

where $t_{i_j} \in N_{c_2}$ is the dual to $z_{i_j} \in N_{a_2}$, and v_1^1, v_1^{1*} is a pair of dual basis elements from $V' \otimes V'_*$. Moreover, $\Phi_{I_1,\dots,I_m}(u_1'') = 0$. The elements $u_2, \dots, u_t$ have similar form. Notice that if the basis vectors v_1^j and w_1^{j*}, $j = 1, \dots, k$, appear in total at most s times in any monomial in u_1' then they appear at most $s+1$ times in any monomial in $v_1^1 \otimes z_{i_2} \otimes \cdots \otimes z_{i_{m+1}} \otimes u_1' \otimes v_1^{1*} \otimes t_{i_2} \otimes \cdots \otimes t_{i_{m+1}}$. Let us take $g_i \in \mathfrak{g}'$ of the form

$$g_i = v_i^1 \otimes v_1^{1*} + \dots v_i^k \otimes v_1^{k*} - w_1^1 \otimes w_i^{1*} - \cdots - w_1^l \otimes w_i^{l*}$$

if $\mathfrak{g}' \cong \mathrm{gl}(\infty)$, or respectively its symmetrization of antisymmetrization if $\mathfrak{g}' \cong \mathrm{sp}(\infty)$ or $\mathrm{so}(\infty)$. Then, if $v_{i_1}^j, \dots, v_{i_{s+1}}^j$ and $w_{i_1}^{j*}, \dots, w_{i_{s+1}}^{j*}$, $j = 1, \dots, k$, are vectors that do not appear at all in the expression of u, then $(g_{i_1} \circ \cdots \circ g_{i_{s+1}})(u_1)$ has the desired properties. We proceed in the same way with $u_2, \dots, u_t$ to obtain the desired result.

□

Lemma 5.7. *Let $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$, M, and $S^{(r)}$ be as in Theorem 5.2. Then $S^{(r+1)}/S^{(r)}$ is a semisimple $\mathfrak{g}'$-module and*

$$S^{(r+1)}/S^{(r)} \cong \bigoplus_{l+m+n=r} \overline{\mathrm{soc}}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M)).$$

Furthermore, if N is a submodule of M then

$$(S^{(r+1)} \cap N)/(S^{(r)} \cap N) \cong \bigoplus_{l+m+n=r} \overline{\mathrm{soc}}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} N)).$$

Proof. Let $S^{(n+1,m+1,l+1)}$ denote the $\mathfrak{g}'$-submodule of M of elements v with the following properties:

(1) $v \in \mathrm{soc}_{\mathfrak{g}_1}^{(n+1)} M$;

(2) $\pi_n(v) \in \mathrm{soc}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M)$, where $\pi_n : \mathrm{soc}_{\mathfrak{g}_1}^{(n+1)} M \to \overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M$;

(3) $\pi_{mn} \circ \pi_n(v) \in \mathrm{soc}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M))$, where

$$\pi_{mn} : \mathrm{soc}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M) \to \overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M).$$

Notice that $S^{(n'+1,m'+1,n'+1)} \subset S^{(n+1,m+1,l+1)}$ if and only if $(n',m',l') < (n,m,l)$ in the lexicographic order. Thus, we obtain a filtration on M

$$0 \subset S^{(1,1,1)} \subset S^{(1,1,2)} \subset \cdots \subset S^{(1,1,L_{31})} \subset S^{(1,2,1)} \subset \cdots \subset S^{(1,L_{21},L_{32})} \subset S^{(2,1,1)} \subset \cdots \subset S^{(L_1,L_2,L_3)},$$

where L_1 is the Loewy length of M as a $\mathfrak{g}_1$-module and the other L's denote the Loewy lengths of the respective modules. Next, we intersect this filtration with $S^{(r+1)}$. Moreover, we take a coarser filtration in which only elements $S^{(n+1,m+1,l+1)}$ with $l+m+n=r$ appear:

$$0 \subset S^{(1,1,r+1)} \cap S^{(r+1)} \subset S^{(1,2,r)} \cap S^{(r+1)} \subset \cdots \subset S^{(1,r+1,1)} \cap S^{(r+1)} \subset S^{(2,1,r)} \cap S^{(r+1)} \subset \cdots \subset S^{(2,r,1)} \cap S^{(r+1)} \subset \cdots \subset S^{(r+1,1,1)} \cap S^{(r+1)}. \tag{5.3}$$

Notice that the consecutive quotients in the filtration (5.3) have the following form:

- if $m \neq 0$, then $(S^{(n+1,m+1,l+1)} \cap S^{(r+1)})/(S^{(n+1,m,l+2)} \cap S^{(r+1)})$;
- if $m = 0$ and $n \neq 0$, then $(S^{(n+1,1,l+1)} \cap S^{(r+1)})/(S^{(n,l+2,1)} \cap S^{(r+1)})$;
- if $m = 0$ and $n = 0$, then $S^{(1,1,l+1)} \cap S^{(r+1)}$.

We build now the corresponding filtration on the quotient $S^{(r+1)}/S^{(r)}$.

$$0 \subset (S^{(1,1,r+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)} \subset \cdots \subset (S^{(r+1,1,1)} \cap S^{(r+1)} + S^{(r)}/S^{(r)}. \tag{5.4}$$

For any l, m, and n, we define the following maps:

$$K_n = \bigoplus_{n_1+n_2=n} \bigoplus_{\substack{i_1<\cdots<i_{n_1} \\ j_1<\cdots<j_{n_2}}} L_{i_1,\ldots,i_{n_1}} \otimes M_{j_1,\ldots,j_{n_2}};$$

$$K'_m = \bigoplus_{\{I_1,\ldots,I_m\}} \Phi_{I_1,\ldots,I_m};$$

$$K''_l = \bigoplus_{\{J_1,\ldots,J_l\}} \Phi'_{J_1,\ldots,J_l}.$$

Furthermore, we set $K_0 = K'_0 = K''_0 = \mathrm{id}$. Recall that K_n and K'_m are $\mathfrak{g}'$-module homomorphisms, whereas K''_l for $l > 0$ is just a linear map.

To prove that $S^{(r+1)}/S^{(r)}$ is a semisimple $\mathfrak{g}'$-module we proceed by induction. Notice first that

$$K_r'' \circ K_0' \circ K_0 : (S^{(1,1,r+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)} \to \overline{\mathrm{soc}}_{\mathfrak{g}'}^{(r+1)}(\mathrm{soc}_{\mathfrak{g}_2}(\mathrm{soc}_{\mathfrak{g}_1} M))$$

is an isomorphism of $\mathfrak{g}'$-modules. This follows from Proposition 5.1 and from the observation that K_r'' restricted to $S^{(1,1,r+1)}$ is a $\mathfrak{g}'$-module homomorphism. Thus we proved that $(S^{(1,1,r+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)}$ is semisimple.

Now suppose that $S^{(n'+1,m'+1,l'+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)}$ is semisimple for some $n' + m' + l' = r$ and that

$$\begin{aligned}(S^{(n'+1,m'+1,l'+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)} \cong \\ \bigoplus_{\substack{l''+m''+n''=r \\ (n'',m'',l'')\le(n',m',l')}} \overline{\mathrm{soc}}_{\mathfrak{g}'}^{(l''+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m''+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n''+1)} M)).\end{aligned} \tag{5.5}$$

Let (n, m, l) be the immediate succesor of (n', m', l') in the lexicographic order of triples of integers with sum equal to r. We prove next that $S^{(n+1,m+1,l+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)}$ is semisimple.

Take an element $u \in S^{(n+1,m+1,l+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)}$ of the form $u = u_1 + \cdots + u_s + S^{(r)}$. Let without loss of generality

$$u_1 = x_{i_1} \otimes \cdots \otimes x_{i_{n_1}} \otimes u_1' \otimes y_{j_1} \otimes \cdots \otimes y_{j_{n_2}} + u_1'',$$

where $n_1 + n_2 = n$ and u_1'' has less than n elements x_j and y_j. Furthermore,

$$u_1' = z_{s_1} \otimes \cdots \otimes z_{s_m} \otimes u_1''' \otimes t_{s_1} \otimes \cdots \otimes t_{s_m} + u_1^{(iv)},$$

where $u_1^{(iv)}$ has less that m terms of the form $z_i \otimes t_i$. Let the elements $u_2, \ldots, u_s$ have a similar form. Then for any $g \in \mathfrak{g}'$ either $g \cdot u = 0$ or $g \cdot u \notin S^{(n'+1,m'+1,l'+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)}$. Let U denote the submodule of $S^{(n+1,m+1,l+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)}$ generated by all such elements u. Then

$$(S^{(n+1,m+1,l+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)} = S^{(n'+1,m'+1,l'+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)} \oplus U. \tag{5.6}$$

Moreover, we claim that

$$K_l'' \circ K_m' \circ K_n : (S^{(n+1,m+1,l+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)} \to \overline{\mathrm{soc}}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M)) \tag{5.7}$$

is a well-defined surjective homomorphism of $\mathfrak{g}'$-modules with kernel $(S^{(n'+1,m'+1,l'+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)}$.

It is clear that the above map is well-defined. Moreover, it is a $\mathfrak{g}'$-module homomorphism since K_l'' is a $\mathfrak{g}'$-module homomorphism when restricted to $K_m' \circ K_n((S^{(n+1,m+1,l+1)})$. So we have to prove that the map is surjective and to compute its kernel.

Theorems 3.10, 4.2, 4.4, 4.7, and 4.8 and Propositions 3.13 and 5.1 imply that $K_l'' \circ K_m' \circ K_n = \pi_{lmn} \circ \pi_{mn} \circ \pi_n$ when restricted to $S^{(n+1,m+1,l+1)}$. Here, π_n and π_{mn} are defined as before and

$$\pi_{lmn} : \mathrm{soc}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M)) \to \overline{\mathrm{soc}}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M)).$$

This shows that the image lies in $\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M))$. Furthermore, every $[v]$ from the right-hand side has a representative $v \in M$ such that

- $v \in \mathrm{soc}_{\mathfrak{g}_1}^{(n+1)} M - \mathrm{soc}_{\mathfrak{g}_1}^{(n)} M$,
- $\pi_n(v) \in \mathrm{soc}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M) - \mathrm{soc}_{\mathfrak{g}_2}^{(m)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M)$,
- $\pi_{mn}(v) \in \mathrm{soc}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M)) - \mathrm{soc}_{\mathfrak{g}'}^{(l)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)} M))$.

But then $v \in S^{(n+1,m+1,l+1)} \cap S^{(r+1)}$. This proves surjectivity.

The last step is to compute the kernel of $K_l'' \circ K_m' \circ K_n$. Suppose first that $m \neq 0$. Then the immediate predecessor of $S^{(n+1,m+1,l+1)} \cap S^{(r+1)}$ in the filtration (5.3) is $S^{(n+1,m,l+2)} \cap S^{(r+1)}$ and every element from it belongs to $\ker K_l'' \circ K_m' \circ K_n$. Hence, $\ker K_l'' \circ K_m' \circ K_n \supseteq (S^{(n+1,m,l+2)} \cap S^{(r+1)} + S^{(r)})/S^{(r)}$. We need to prove the opposite inclusion.

Let $v \in \ker K_l'' \circ K_m' \circ K_n$. Suppose first that $K_n(v) = 0$. Then $v \in S^{(n,m_1+1,l_1+1)} \cap S^{(r+1)} \subset S^{(n+1,m,l+2)} \cap S^{(r+1)}$. Now, let v be such that $K_n(v) \neq 0$ and $K_m'(K_n(v)) = 0$. But then $v \in S^{(n+1,m,l+2)} \cap S^{(r+1)}$. Finally, if $K_m'(K_n(v)) \neq 0$, it follows that $K_l'' \circ K_m' \circ K_n(v) = 0$ and hence $v \in S^{(r)}$. Therefore, we proved that

$$\ker K_l'' \circ K_m' \circ K_n = (S^{(n+1,m,l+2)} \cap S^{(r+1)} + S^{(r)})/S^{(r)}.$$

Now, suppose that $m = 0$ and $n \neq 0$. Then the immediate predecessor of $S^{(n+1,1,l+1)} \cap S^{(r+1)}$ in the filtration (5.3) is $S^{(n,l+1,1)} \cap S^{(r+1)}$ and it clearly belongs to $\ker K_l'' \circ K_m' \circ K_n$. Hence, $(S^{(n,l+1,1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)} \subseteq \ker K_l'' \circ K_0' \circ K_n$. Vice versa, if $v \in \ker K_l'' \circ K_0' \circ K_n$ is such that $K_n(v) = 0$ then $v \in S^{(n,m_1+1,l_1+1)} \cap S^{(r+1)} \subset$

$S^{(n,l+1,1)} \cap S^{(r+1)}$. Hence,

$$\ker K_l'' \circ K_0' \circ K_n = (S^{(n,l+1,1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)}.$$

Thus, we proved that $(S^{(n+1,m+1,l+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)}$ is semisimple. Moreover, (5.5), (5.6), and (5.7) imply

$$(S^{(n+1,m+1,l+1)} \cap S^{(r+1)} + S^{(r)})/S^{(r)} \cong \bigoplus_{\substack{n'+m'+l'=r \\ (n',m',l')\leq(n,m,l)}} \overline{\mathrm{soc}}_{\mathfrak{g}'}^{(l'+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m'+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n'+1)}M)).$$

This holds for every element in the filtration (5.4), so in particular $S^{(r+1)}/S^{(r)}$ is semisimple and

$$S^{(r+1)}/S^{(r)} \cong \bigoplus_{n+m+l=r} \overline{\mathrm{soc}}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)}M)).$$

Note that if $N \subset M$ is any submodule of M, then we can interesect the above filtrations with N and obtain

$$(S^{(r+1)} \cap N)/(S^{(r)} \cap N) \cong \bigoplus_{l+m+n=r} \overline{\mathrm{soc}}_{\mathfrak{g}'}^{(l+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_2}^{(m+1)}(\overline{\mathrm{soc}}_{\mathfrak{g}_1}^{(n+1)}N)).$$

□

Chapter 6

Embeddings of non-isomorphic Lie algebras

The goal of this chapter is to apply Theorem 5.2 to embeddings $\mathfrak{g}' \subset \mathfrak{g}$, where $\mathfrak{g}'$ and $\mathfrak{g}$ are non-isomorphic. More precisely, in each specific case we decompose an embedding of general tensor type into intermediate embeddings in accordance with Theorem 5.2. In this way, we again reduce the branching problem for embeddings of general tensor type to a branching problem for embeddings of simpler types, which will be then easy to derive.

We say that two embeddings $\mathfrak{g}_1 \subset \mathfrak{g}_2$ and $\mathfrak{g}_3 \subset \mathfrak{g}_4$ of classical locally finite Lie algebras are *of the same type* if they satisfy the conditions of Theorem 5.2 with the same values $k, l, a_1, a_2, c_1, b, d$.

We start with an observation that considerably reduces the number of different embeddings we have to consider.

Proposition 6.1. *When the following groups of embeddings are of the same type, the branching laws are the same in each group:*

(1) $\mathrm{sl}(\infty) \subset \mathrm{gl}(\infty)$, $\mathrm{gl}(\infty) \subset \mathrm{sl}(\infty)$, $\mathrm{sl}(\infty) \subset \mathrm{sl}(\infty)$, *and* $\mathrm{gl}(\infty) \subset \mathrm{gl}(\infty)$,

(2) $\mathrm{sp}(\infty) \subset \mathrm{gl}(\infty)$ *and* $\mathrm{sp}(\infty) \subset \mathrm{sl}(\infty)$,

(3) $\mathrm{so}(\infty) \subset \mathrm{gl}(\infty)$ *and* $\mathrm{so}(\infty) \subset \mathrm{sl}(\infty)$,

(4) $\mathrm{gl}(\infty) \subset \mathrm{sp}(\infty)$ *and* $\mathrm{sl}(\infty) \subset \mathrm{sp}(\infty)$,

(5) $\mathrm{gl}(\infty) \subset \mathrm{so}(\infty)$ *and* $\mathrm{sl}(\infty) \subset \mathrm{so}(\infty)$.

Proof. Part (1) follows from Proposition 3.2. Parts (2) and (3) are trivial. The idea of proof of part (4) is to show as in Proposition 3.2 that every embedding $\mathrm{sl}(V', V'_*) \subset \mathrm{sp}(V)$ can be extended to an embedding $\mathrm{gl}(V', V'_*) \subset \mathrm{sp}(V)$ of the same type. Thus, we obtain a chain of embeddings

$$\mathrm{sl}(V', V'_*) \subset \mathrm{gl}(V', V'_*) \subset \mathrm{sp}(V)$$

which proves part (4). The proof of part (5) is analogous. □

As a result of Proposition 6.1, we can exclude from our considerations below the cases of embeddings which involve $\mathrm{sl}(\infty)$.

6.1 The cases $\mathrm{sp}(\infty) \subset \mathrm{gl}(\infty)$ and $\mathrm{so}(\infty) \subset \mathrm{gl}(\infty)$

Let $\mathfrak{g}'$ be isomorphic to one of $\mathrm{sp}(V')$ and $\mathrm{so}(V')$ and let $\mathfrak{g} \cong \mathrm{gl}(V, V_*)$. Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of general tensor type, i.e.

$$\begin{aligned} &\mathrm{soc}_{\mathfrak{g}'} V \cong kV' \oplus N_a, && V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b, \\ &\mathrm{soc}_{\mathfrak{g}'} V_* \cong kV' \oplus N_c, && V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_d. \end{aligned}$$

Proposition 6.2. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be as above. Then there exist intermediate subalgebras $\mathfrak{g}_1 = \mathrm{gl}(V_1, V_{1*})$ and $\mathfrak{g}_2 = \mathrm{gl}(V_2, V_{2*})$ such that the conditions of Theorem 5.2 are satisfied for the sequence $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$.*

Proof. Notice that $\mathfrak{g}' \subset \mathrm{gl}(V', V') \subset \mathfrak{g}$. Therefore, we can apply Proposition 3.3 to the embedding $\mathrm{gl}(V', V') \subset \mathfrak{g}$ and construct $\mathfrak{g}_1$ and $\mathfrak{g}_2$ with the desired properties. □

In view of Proposition 6.2 and Theorem 5.2 it is enough to consider embeddings $\mathfrak{g}' \subset \mathfrak{g}$ with the property

$$V \cong V_* \cong kV'. \tag{6.1}$$

The following is an easy corollary of property (2.4) of the socle filtration of a direct sum.

Proposition 6.3. *Let $\mathfrak{g}' \subset \mathfrak{g}$ satisfy (6.1). Then for the socle filtration of the $\mathfrak{g}$-module $V^{\otimes(p,q)}$ we obtain*

$$\mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V^{\otimes(p,q)} \cong k^{(p+q)} \mathrm{soc}^{(r+1)}_{\mathfrak{g}'} V'^{\otimes(p+q)}.$$

Corollary 6.4. *Let $\mathfrak{g}' \subset \mathfrak{g}$ satisfy (6.1) and $V_{\lambda,\mu} \subset V^{\otimes(p,q)}$. Then*

(i) if $\mathfrak{g}' \cong \mathrm{sp}(V')$, we have

$$\overline{\mathrm{soc}}_{\mathfrak{g}'} V_{\lambda,\mu} \cong \bigoplus_{|\sigma|=p+q-2r} A_\sigma^{\lambda,\mu} V'_{\langle\sigma\rangle},$$

where for $k=1$

$$A_\sigma^{\lambda,\mu} = \sum_{\alpha,\beta,\gamma} c_{\alpha\beta}^{\sigma} c_{\alpha(2\gamma)^T}^{\lambda} c_{\beta(2\delta)^T}^{\mu},$$

and for $k \geq 2$

$$A_\sigma^{\lambda,\mu} = \sum_{\substack{\alpha_1^+,\dots,\alpha_k^+\\ \alpha_1^-,\dots,\alpha_k^-}} \sum_{\substack{\lambda',\mu'\\ \alpha,\beta,\gamma}} C^{(\lambda,\mu)}_{(\alpha_1^+,\dots,\alpha_k^+)(\alpha_1^-,\dots,\alpha_k^-)} D^{(\lambda',\mu')}_{(\alpha_1^+,\dots,\alpha_k^+)(\alpha_1^-,\dots,\alpha_k^-)} c_{\alpha\beta}^{\sigma} c_{\alpha(2\gamma)^T}^{\lambda'} c_{\beta(2\delta)^T}^{\mu'};$$

here the coefficients $C^{(\lambda,\mu)}_{(\alpha_1^+,\dots,\alpha_k^+)(\alpha_1^-,\dots,\alpha_k^-)}$ and $D^{(\lambda',\mu')}_{(\alpha_1^+,\dots,\alpha_k^+)(\alpha_1^-,\dots,\alpha_k^-)}$ are defined as in Section 2.2;

(ii) if $\mathfrak{g}' \cong \mathrm{so}(V')$, we have

$$\overline{\mathrm{soc}}_{\mathfrak{g}'} V_{\lambda,\mu} \cong \bigoplus_{|\sigma|=p+q-2r} B_\sigma^{\lambda,\mu} V'_{[\sigma]},$$

where for $k=1$

$$B_\sigma^{\lambda,\mu} = \sum_{\alpha,\beta,\gamma} c_{\alpha\beta}^{\sigma} c_{\alpha(2\gamma)}^{\lambda} c_{\beta(2\delta)}^{\mu},$$

and for $k \geq 2$

$$B_\sigma^{\lambda,\mu} = \sum_{\substack{\alpha_1^+,\dots,\alpha_k^+\\ \alpha_1^-,\dots,\alpha_k^-}} \sum_{\substack{\lambda',\mu'\\ \alpha,\beta,\gamma}} C^{(\lambda,\mu)}_{(\alpha_1^+,\dots,\alpha_k^+)(\alpha_1^-,\dots,\alpha_k^-)} D^{(\lambda',\mu')}_{(\alpha_1^+,\dots,\alpha_k^+)(\alpha_1^-,\dots,\alpha_k^-)} c_{\alpha\beta}^{\sigma} c_{\alpha(2\gamma)}^{\lambda'} c_{\beta(2\delta)}^{\mu'}.$$

Proof. Part (i): Proposition 6.3 above and Theorem 3.2 from [PSt] imply

$$\overline{\mathrm{soc}}_{\mathfrak{g}'} V_{\lambda,\mu} \cong \bigoplus_{|\sigma|=p+q-2r} a_\sigma^{\lambda,\mu} V'_{\langle\sigma\rangle}$$

for some multiplicities $a_\sigma^{\lambda,\mu}$. To determine the value of $a_\sigma^{\lambda,\mu}$ we have to derive the branching law for embeddings $\mathrm{sp}(2n) \subset \mathrm{gl}(2kn)$ of signature $(k,0,0)$. There is the following chain of embeddings

$$\mathrm{sp}(2n) \subset \mathrm{gl}(2n) \subset \mathrm{gl}(2kn)$$

which, using Proposition 2.4.2 from [HTW] for the first embedding and Proposition 2.4 from Chapter 2 for the second embedding, yields the desired expression for $a_\sigma^{\lambda,\mu}$. □

6.2 The cases $\mathrm{gl}(\infty) \subset \mathrm{sp}(\infty)$ and $\mathrm{gl}(\infty) \subset \mathrm{so}(\infty)$

Let in this section $\mathfrak{g}' \cong \mathrm{gl}(V', V'_*)$, and $\mathfrak{g} \cong \mathrm{sp}(V)$ or $\mathfrak{g} \cong \mathrm{so}(V)$. Furthermore, let $\mathfrak{g}' \subset \mathfrak{g}$ satisfy the conditions

$$\mathrm{soc}_{\mathfrak{g}'} V = kV' \oplus lV'_* \oplus N_a, \quad V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b.$$

The first observation we make is that $k = l$, as

$$\mathrm{soc}_{\mathfrak{g}'} V_* = lV' \oplus kV'_* \oplus N_c, \quad V_*/\mathrm{soc}_{\mathfrak{g}'} V_* \cong N_d$$

by Theorem 2.1. However, in this case $V \cong V_*$, hence the socle filtrations of V and V_* must be equal, and in particular $k = l$.

Proposition 6.5. *Let $\mathfrak{g} \cong \mathrm{sp}(V)$ and $\mathfrak{g}' \subset \mathfrak{g}$ satifsy the conditions*

$$\mathrm{soc}_{\mathfrak{g}'} V = kV' \oplus kV'_* \oplus N_a, \quad V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b.$$

Then there exist subalgebras $\mathfrak{g}_1 \cong \mathrm{sp}(V_1)$ and $\mathfrak{g}_2 \cong \mathrm{sp}(V_2)$ such that the chain $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$ satisfies the conditions of Theorem 5.2.

Proof. We now construct a subalgebra $\mathfrak{g}_2 \cong \mathrm{sp}(V_2)$ such that $V_2 \cong kV' \oplus kV'_*$. Then the existence of $\mathfrak{g}_1$ will follow from Proposition 4.1.

Consider the submodule V_2 of V such that $V_2 \cong kV' \oplus kV'_*$. As in Proposition 3.1 we can show that there exists a decomposition

$$V_2 = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_k \oplus \tilde{W}_1 \oplus \cdots \oplus \tilde{W}_k$$

with basis $\{v_i^j\}_{i \in I, j=1,\ldots,k} \cup \{w_i^j\}_{i \in I, j=1,\ldots,k}$ such that $\Omega(v_i^j, w_k^l) = \delta_{ik}\delta_{jl}$ and such that every element $g \in \mathfrak{g}'$ lies in $S^2 V_2$. Thus, on the one hand, the restriction of the bilinear form Ω to V_2 is non-degenerate and we can define the Lie subalgebra $\mathfrak{g}_2 =$

$\mathrm{sp}(V_2) = S^2V_2$. Then trivially $\mathfrak{g}_2 \subset \mathfrak{g} = S^2V$. On the other hand, $\mathfrak{g}' \subset S^2V_2 = \mathfrak{g}_2$. In this way we obtain a subalgebra $\mathfrak{g}_2$ as desired. □

An analogous statement holds for $\mathfrak{g} \cong \mathrm{so}(V)$.

Proposition 6.6. *Let* $\mathfrak{g} \cong \mathrm{so}(V)$ *and* $\mathfrak{g}' \subset \mathfrak{g}$ *satifsy the conditions*

$$\mathrm{soc}_{\mathfrak{g}'} V = kV' \oplus kV'_* \oplus N_a, \quad V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b.$$

Then there exist subalgebras $\mathfrak{g}_1 \cong \mathrm{so}(V_1)$ *and* $\mathfrak{g}_2 \cong \mathrm{so}(V_2)$ *such that the chain* $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$ *satisfies the conditions of Theorem 5.2.*

In view of Propositions 6.5 and 6.6 and Theorem 5.2 it is enough to consider embeddings $\mathfrak{g}' \subset \mathfrak{g}$ which satisfy the property

$$V \cong kV' \oplus kV'_*. \tag{6.2}$$

Proposition 6.7. *Let* $\mathfrak{g}' \subset \mathfrak{g}$ *satisfy (6.2). Then*

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V^{\otimes d} = \bigoplus_{m=0}^{d} \binom{d}{m} k^d \overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V'^{\otimes(m,d-m)}.$$

Corollary 6.8. *Let* $\mathfrak{g}' \subset \mathfrak{g}$ *satisfy (6.2).*

(i) If $\mathfrak{g} \cong \mathrm{sp}(V)$ *then, for any simple* $\mathfrak{g}$*-module* $V_{\langle\lambda\rangle} \subset V^{\otimes d}$,

$$\overline{\mathrm{soc}}^{(r+1)}_{\mathfrak{g}'} V_{\langle\lambda\rangle} = \bigoplus_{m=0}^{d} \bigoplus_{\substack{|\lambda'|=m-r \\ |\mu'|=d-m-r}} A^{\lambda}_{\lambda',\mu'} V'_{\lambda',\mu'},$$

where for $k = 1$

$$A^{\lambda}_{\lambda',\mu'} = \sum_{\gamma,\delta} c^{\gamma}_{\lambda',\mu'} c^{\lambda}_{\gamma,2\delta},$$

and for $k \geq 2$

$$A^{\lambda}_{\lambda',\mu'} = \sum_{\gamma,\delta,\lambda_1,\mu_1} \sum_{\substack{\beta_1^+,\ldots,\beta_k^+ \\ \beta_1^-,\ldots,\beta_k^-}} c^{\gamma}_{\lambda_1,\mu_1} c^{\lambda}_{\gamma,2\delta} C^{(\lambda_1,\mu_1)}_{(\beta_1^+,\ldots,\beta_k^+)(\beta_1^-,\ldots,\beta_k^-)} D^{(\lambda',\mu')}_{(\beta_1^+,\ldots,\beta_k^+)(\beta_1^-,\ldots,\beta_k^-)}.$$

The coefficients $C^{(\lambda_1,\mu_1)}_{(\beta_1^+,\ldots,\beta_k^+)(\beta_1^-,\ldots,\beta_k^-)}$ *and* $D^{(\lambda',\mu')}_{(\beta_1^+,\ldots,\beta_k^+)(\beta_1^-,\ldots,\beta_k^-)}$ *are defined as before.*

(ii) If $\mathfrak{g} \cong \mathrm{so}(V)$ *then, for any simple* $\mathfrak{g}$*-module* $V_{[\lambda]} \subset V^{\otimes d}$,

$$\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(r+1)} V_{[\lambda]} = \bigoplus_{m=0}^{d} \bigoplus_{\substack{|\lambda'|=m-r \\ |\mu'|=d-m-r}} B^{\lambda}_{\lambda',\mu'} V'_{\lambda',\mu'},$$

where for $k = 1$

$$B^{\lambda}_{\lambda',\mu'} = \sum_{\gamma,\delta} c^{\gamma}_{\lambda',\mu'} c^{\lambda}_{\gamma,(2\delta)^T},$$

and for $k \geq 2$

$$B^{\lambda}_{\lambda',\mu'} = \sum_{\gamma,\delta,\lambda_1,\mu_1} \sum_{\substack{\beta_1^+,\ldots,\beta_k^+ \\ \beta_1^-,\ldots,\beta_k^-}} c^{\gamma}_{\lambda_1,\mu_1} c^{\lambda}_{\gamma,(2\delta)^T} C^{(\lambda_1,\mu_1)}_{(\beta_1^+,\ldots,\beta_k^+)(\beta_1^-,\ldots,\beta_k^-)} D^{(\lambda',\mu')}_{(\beta_1^+,\ldots,\beta_k^+)(\beta_1^-,\ldots,\beta_k^-)}.$$

Proof. Part (i). Proposition 6.7 above and Theorem 2.2 in [PSt] imply

$$\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(r+1)} V_{\langle\lambda\rangle} = \bigoplus_{m=0}^{d} \bigoplus_{\substack{|\lambda'|=m-r \\ |\mu'|=d-m-r}} a^{\lambda}_{\lambda',\mu'} V'_{\lambda',\mu'}$$

for some unknown multiplicities $a^{\lambda}_{\lambda',\mu'}$. To determine those multiplicities we need to derive the branching rule for an embedding $\mathrm{gl}(n) \subset \mathrm{sp}(2kn)$ of signature $(k, k, 0)$. We decompose such an embedding as $\mathrm{gl}(n) \subset \mathrm{gl}(kn) \subset \mathrm{sp}(2kn)$. Then we use Proposition 2.4 from Chapter 2 for the first embedding, and Proposition 2.3.2 from [HTW] for the second embedding, to obtain the desired value for $a^{\lambda}_{\lambda',\mu'}$.

□

6.3 The cases $\mathrm{sp}(\infty) \subset \mathrm{so}(\infty)$ and $\mathrm{so}(\infty) \subset \mathrm{sp}(\infty)$

First we consider the case $\mathfrak{g}' \cong \mathrm{so}(V')$ and $\mathfrak{g} \cong \mathrm{sp}(V)$, where the embedding $\mathfrak{g}' \subset \mathfrak{g}$ is of general tensor type, i.e.

$$\mathrm{soc}_{\mathfrak{g}'} V \cong kV' \oplus N_a, \qquad V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b.$$

As in the previous sections we have the following proposition.

Proposition 6.9. *Let* $\mathfrak{g}' \subset \mathfrak{g}$ *be as above. Then necessarily* $k \in 2\mathbb{Z}_{\geq 0}$. *Moreover, there exist subalgebras* $\mathfrak{g}_1 \cong \mathrm{sp}(V_1)$ *and* $\mathfrak{g}_2 \cong \mathrm{sp}(V_2)$*, such that the chain* $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$ *satisfies the conditions of Theorem 5.2.*

Proof. To show that k is even we will employ the same ideas as in the proof of Proposition 3.1. Let φ denote the embedding $\mathfrak{g}' \to \mathfrak{g}$. Let $\{v_i'\}_{i\in\mathbb{Z}\setminus\{0\}}$ be a basis of V' with the property that $Q(v_i', v_j') = \delta_{i+j,0}$, where Q denotes the non-degenerate symmetric bilinear form on $\mathfrak{g}'$. Consider the submodule $V_2 \cong kV'$ of V. Let

$$V_2 = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_k$$

be a decomposition of V_2 such that $\tilde{V}_i \cong V'$ for all $i = 1, \ldots, k$ and let $v_i^j \in \tilde{V}_i$ be the image of v_i' under the isomorphism $\tilde{V}_i \cong V'$. Then, for all i, j, m, n and for any positive s such that $s \neq m$ and $s \neq -i$, we have

$$\Omega(v_i^j, v_m^n) = \Omega(\varphi(v_i' \otimes v_{-s}' - v_{-s}' \otimes v_i') \cdot v_s^j, v_m^n) = -\Omega(v_s^j, \varphi(v_i' \otimes v_{-s}' - v_{-s}' \otimes v_i') \cdot v_m^n) = \operatorname{sign}(i)\delta_{i+m,0}\Omega(v_s^j, v_{-s}^n).$$

Hence, for $m \neq -i$ we have $\Omega(v_i^j, v_m^n) = 0$, and for $m = -i$ we have $\Omega(v_i^j, v_{-i}^n) = \operatorname{sign}(i)\Omega(v_s^j, v_{-s}^n)$ for all positive integers $s \neq -i$ and for all j, n. Since the restriction of Ω to V_2 is non-degenerate, for each j there exists n such that $\Omega(v_i^j, v_m^n) = \operatorname{sign}(i)\delta_{i+m,0}$. We can adjust the $\tilde{V}_i$'s such that n is unique and different for each j.

However, now we will show that n cannot be equal to j. Suppose that for some j we have $n = j$. Then $\{v_i^j\}_{i\in\mathbb{Z}\setminus\{0\}}$ is on the one hand a basis for $\tilde{V}_j$ which satisfies $\Omega(v_i^j, v_m^j) = \operatorname{sign}(i)\delta_{i+m,0}$ and on the other hand a basis which satisfies $Q(v_i^j, v_m^j) = \delta_{i+m,0}$. But this is not possible. Hence each $\tilde{V}_j$ is an isotropic subspace for Ω and pairs non-degenerately with a unique $\tilde{V}_n$. This implies that k is even. We can further renumerate the subspaces $\tilde{V}_j$ so that $\tilde{V}_j$ pairs non-degenerately with $\tilde{V}_{j+\frac{k}{2}}$. Then the embedding φ has the form

$$\varphi(v_i' \otimes v_j' - v_j' \otimes v_i') = \sum_{s=1}^{\frac{k}{2}} v_i^s \otimes v_j^{s+\frac{k}{2}} - v_j^s \otimes v_i^{s+\frac{k}{2}} + v_j^{s+\frac{k}{2}} \otimes v_i^s - v_i^{s+\frac{k}{2}} \otimes v_j^s. \quad (6.3)$$

Now we can define the Lie subalgebra $\mathfrak{g}_2 = \operatorname{sp}(V_2)$ as S^2V_2. Clearly, $\mathfrak{g}_2 \subset \mathfrak{g} = S^2V$ and moreover (6.3) implies that $\mathfrak{g}' \subset S^2V_2 = \mathfrak{g}_2$. The existence of $\mathfrak{g}_1$ then follows from Proposition 4.1. □

In what follows we replace k by $2k$. Thus, in view of Proposition 6.9 and Theorem 5.2, it is enough to consider embeddings $\mathfrak{g}' \subset \mathfrak{g}$ with the property

$$V \cong 2kV' \quad (6.4)$$

Theorem 6.10. *Let $\mathfrak{g}' \subset \mathfrak{g}$ satisfy (6.4). Then the following hold.*

(i) The socle filtration of the $\mathfrak{g}$*-module* $V^{\otimes d}$ *is*

$$\mathrm{soc}_{\mathfrak{g}'}^{(r+1)} V^{\otimes d} = (2k)^d \mathrm{soc}_{\mathfrak{g}'}^{(r+1)} V'^{\otimes d}.$$

(ii) For any simple $V_{\langle\lambda\rangle} \subset V^{\otimes d}$ *we have*

$$\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(r+1)} V_{\langle\lambda\rangle} \cong \bigoplus_{|\lambda'|=d-2r} A_{\lambda'}^{\lambda} V'_{[\lambda']}$$

where for $k = 1$

$$A_{\lambda'}^{\lambda} = \sum_{\substack{\gamma,\delta,\lambda_1,\mu_1 \\ \alpha,\beta,\sigma,\tau}} c_{\lambda_1,\mu_1}^{\gamma} c_{\gamma,2\delta}^{\lambda} c_{\alpha,\beta}^{\lambda'} c_{\alpha,2\sigma}^{\lambda_1} c_{\beta,2\tau}^{\mu_1},$$

and for $k \geq 2$

$$A_{\lambda'}^{\lambda} = \sum_{\substack{\beta_1^+,\dots,\beta_k^+ \\ \beta_1^-,\dots,\beta_k^-}} \sum_{\substack{\gamma,\delta, \\ \lambda_1,\mu_1,\lambda_2,\mu_2 \\ \alpha,\beta,\sigma,\tau}} c_{\lambda_1,\mu_1}^{\gamma} c_{\gamma,2\delta}^{\lambda} C^{(\lambda_1,\mu_1)}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)} D^{(\lambda_2,\mu_2)}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)} c_{\alpha,\beta}^{\lambda'} c_{\alpha,2\sigma}^{\lambda_2} c_{\beta,2\tau}^{\mu_2}.$$

The coefficients $C^{(\lambda_1,\mu_1)}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)}$ *and* $D^{(\lambda',\mu')}_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)}$ *are defined as before.*

Proof. Part (ii): From part (i) and Theorem 4.2 in [PSt] it follows that

$$\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(r+1)} V_{\langle\lambda\rangle} \cong \bigoplus_{|\lambda'|=d-2r} a_{\lambda'}^{\lambda} V'_{[\lambda']}$$

for some multiplicities $a_{\lambda'}^{\lambda}$. To obtain the values of $a_{\lambda'}^{\lambda}$ we construct the following commutative diagram. Let $\{v_i'\}_{i\in\mathbb{Z}\setminus\{0\}}$ and $\{v_i^j\}_{\substack{i\in\mathbb{Z}\setminus\{0\} \\ j=1,\dots,2k}}$ be respectively the bases of V' and of V from Proposition 6.9. Let

$$V'_{2n} = \mathrm{span}\{v_i'\}_{i=\pm1,\dots,\pm n}$$
$$V_{4kn} = \mathrm{span}\{v_i^j\}_{\substack{j=1,\dots,2k \\ i=\pm1,\dots,\pm n}}.$$

We set $\mathfrak{g}'_{2n} = \bigwedge V'_{2n} \cong \mathrm{so}(2n)$, $\mathfrak{g}''_{2n} = V'_{2n} \otimes V'_{2n} \cong \mathrm{gl}(2n)$, and $\mathfrak{g}_{4kn} = S^2 V_{4kn} \cong \mathrm{sp}(4kn)$. Then we have an embedding

$$\varphi : \bigwedge V'_{2n} \to S^2 V_{4kn}$$

given by (6.3) with k replaced by $2k$. Notice that φ extends to an embedding

$\varphi : V'_{2n} \otimes V'_{2n} \to S^2 V_{4kn}$ given by

$$\varphi(v'_i \otimes v'_j) = \sum_{s=1}^{k} v_i^s \otimes v_j^{s+k} + v_j^{s+k} \otimes v_i^s.$$

Therefore we obtain the following commutative diagram of embeddings

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \mathfrak{g}'_{2n} & \longrightarrow & \mathfrak{g}'_{4n} & \longrightarrow \dots \longrightarrow & \mathfrak{g}'_{2mn} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \mathfrak{g}''_{2n} & \longrightarrow & \mathfrak{g}''_{4n} & \longrightarrow \dots \longrightarrow & \mathfrak{g}''_{2mn} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \mathfrak{g}_{4kn} & \longrightarrow & \mathfrak{g}_{8kn} & \longrightarrow \dots \longrightarrow & \mathfrak{g}_{4kmn} & \longrightarrow & \dots
\end{array}$$

We have to compute the branching laws for the vertical embeddings in the above diagram. Each vertical embedding is of the form so($2n$) $\subset$ gl($2n$) $\subset$ sp($4kn$). Then, using Formula 2.4.1 from [HTW] for the first embedding and Section 6.2 above for the second embedding, we obtain the desired multiplicities. □

Remark. In Theorem 6.10 we actually prove that whenever we have an embedding so(V') $\subset$ sp(V) satisfying (6.4) there is an intermediate subalgebra isomorphic to gl(V'). In other words, we have the following chain of embeddings

$$\mathrm{so}(V') \subset \mathrm{gl}(V') \subset \mathrm{sp}(V).$$

Moreover, in the notations of Proposition 6.9, each $\tilde{V}_i$ is an isotropic subspace of V stabilized by so(V') and by gl(V'). Thus

$$\mathrm{so}(V') \subset \mathrm{gl}(V') \subset \mathfrak{m} \subset \mathrm{sp}(V),$$

where $\mathfrak{m} = \mathrm{Stab}_{\mathrm{so}(V')} \tilde{V}_i$ and by Theorem 5.1 in [DP1] $\mathfrak{m}$ is a maximal subalgebra of sp(V).

The case of $\mathfrak{g}' \cong$ so(∞) and $\mathfrak{g} \cong$ sp(∞) is treated in the same way and here we just state the end results.

Let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of general tensor type, i.e.

$$\mathrm{soc}_{\mathfrak{g}'} V \cong kV' \oplus N_a, \qquad V/\mathrm{soc}_{\mathfrak{g}'} V \cong N_b.$$

Then we have the following analogue of Proposition 6.9.

Proposition 6.11. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be as above. Then necessarily $k \in 2\mathbb{Z}_{\geq 0}$. Moreover, there exist subalgebras $\mathfrak{g}_1 \cong \mathrm{so}(V_1)$ and $\mathfrak{g}_2 \cong \mathrm{so}(V_2)$, such that the chain $\mathfrak{g}' \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}$ satisfies the conditions of Theorem 5.2.*

Thus, in what follows we replace k with $2k$.

Theorem 6.12. *Let $\mathfrak{g}' \subset \mathfrak{g}$ satisfy $V \cong 2kV'$. Then the following hold.*

(i) The socle filtration of the $\mathfrak{g}$-module $V^{\otimes d}$ is

$$\mathrm{soc}_{\mathfrak{g}'}^{(r+1)} V^{\otimes d} = (2k)^d \mathrm{soc}_{\mathfrak{g}'}^{(r+1)} V'^{\otimes d}.$$

(ii) For any simple $V_{[\lambda]} \subset V^{\otimes d}$ we have

$$\overline{\mathrm{soc}}_{\mathfrak{g}'}^{(r+1)} V_{[\lambda]} \cong \bigoplus_{|\lambda'|=d-2r} A_{\lambda'}^{\lambda} V'_{\langle\lambda'\rangle}$$

where for $k = 1$

$$A_{\lambda'}^{\lambda} = \sum_{\substack{\gamma,\delta,\lambda_1,\mu_1\\ \alpha,\beta,\sigma,\tau}} c_{\lambda_1,\mu_1}^{\gamma} c_{\gamma,(2\delta)^T}^{\lambda} c_{\alpha,\beta}^{\lambda'} c_{\alpha,2\sigma}^{\lambda_1} c_{\beta,(2\tau)^T}^{\mu_1},$$

and for $k \geq 2$

$$A_{\lambda'}^{\lambda} = \sum_{\substack{\beta_1^+,\dots,\beta_k^+\\ \beta_1^-,\dots,\beta_k^-}} \sum_{\substack{\gamma,\delta,\\ \lambda_1,\mu_1,\lambda_2,\mu_2\\ \alpha,\beta,\sigma,\tau}} c_{\lambda_1,\mu_1}^{\gamma} c_{\gamma,(2\delta)^T}^{\lambda} C_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)}^{(\lambda_1,\mu_1)} D_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)}^{(\lambda_2,\mu_2)} c_{\alpha,\beta}^{\lambda'} c_{\alpha,2\sigma}^{\lambda_2} c_{\beta,(2\tau)^T}^{\mu_2}.$$

The coefficients $C_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)}^{(\lambda_1,\mu_1)}$ and $D_{(\beta_1^+,\dots,\beta_k^+)(\beta_1^-,\dots,\beta_k^-)}^{(\lambda',\mu')}$ are defined as before.

Chapter 7

Corollaries and further results

In this chapter we show what invariants determine the solution of the branching problem completely. We also address two further questions of interest and show how the results in this thesis give preliminary partial answers to these questions.

Corollary 7.1. *Let $\mathfrak{g}', \mathfrak{g}$ be classical locally finite Lie algebras and let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding satisfying (2.5). Then Theorem 5.2 implies that the branching law for a fixed simple tensor $\mathfrak{g}$-module M depends only on the values $k, l, a_1, b, c_1, d,$ and $a_2 = c_2$.*

Let $\mathfrak{g}' \subset \mathfrak{g}$ be as above and let M' and M be simple tensor modules for $\mathfrak{g}'$ and $\mathfrak{g}$ respectively. Denote by $[M, M']$ the total multiplicity of M' in the socle filtration of M as a $\mathfrak{g}'$-module. Since M has finite Loewy length over $\mathfrak{g}'$ (see [PSe] or the chapters above), $[M, M']$ is the multiplicity of M' as a $\mathfrak{g}'$-module subquotient of M. Our first goal is to determine the value of $[M, M']$.

Let us consider the case $\mathfrak{g}' \cong \mathfrak{g} \cong \mathrm{gl}(\infty)$. We fix $M = V_{\lambda,\mu}$ and $M' = V'_{\lambda',\mu'}$. First we observe that if $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of type III, then isomorphic $\mathfrak{g}'$-modules appear in a single layer of the socle filtration of M as a $\mathfrak{g}'$-module. Therefore

$$[V_{\lambda,\mu}, V'_{\lambda',\mu'}] = A^{\lambda,\mu}_{\lambda',\mu'}, \tag{7.1}$$

where $A^{\lambda,\mu}_{\lambda',\mu'}$ is as in Corollary 3.18. In particular, $[V_{\lambda,\mu}, V'_{\lambda',\mu'}]$ is finite.

If now $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of type I, we can sum up the multiplicities of all occurencies of the module $V'_{\lambda',\mu'}$ on the distinct layers of the socle filtration of $V_{\lambda,\mu}$

as determined in Corollary 3.11. In this case we obtain

$$[V_{\lambda,\mu}, V'_{\lambda',\mu'}] = \sum_{r=0}^{p+q} \sum_{|\lambda''|+|\mu''|=|\lambda'|+|\mu'|+r} m^a_{\lambda,\lambda''} m^b_{\lambda'',\lambda'} m^c_{\mu,\mu''} m^d_{\mu'',\mu'} = m^{a+b}_{\lambda,\lambda'} m^{c+d}_{\mu,\mu'}. \tag{7.2}$$

When $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of type II, it is much easier to compute the multiplicity $[V_{\lambda,\mu}, V'_{\lambda',\mu'}]$ directly, rather than use the socle filtration of $V_{\lambda,\mu}$ over $\mathfrak{g}'$. Therefore, in this case we proceed as follows.

Let $\{v_i\}_{i\in I}$, $\{v_i^*\}_{i\in I}$ be a pair of dual bases respectively for V' and V'_* as submodules of V and V_* and let $\{z_i\}_{i\in I_a}$, $\{t_i\}_{i\in I_c}$ be a pair of dual bases respectively for N_a and N_c. Let as before $V'_k = \operatorname{span}\{v_1, \ldots, v_k\}$ and $V'^*_k = \operatorname{span}\{v_1^*, \ldots, v_k^*\}$. Similarly let $V_{m_k} = \operatorname{span}\{v_1, \ldots, v_k, z_1, \ldots, z_{m_k-k}\}$ and $V^*_{m_k} = \operatorname{span}\{v_1^*, \ldots, v_k^*, t_1, \ldots, t_{m_k-k}\}$. Then as before we can construct a commutative diagram of embeddings

$$\begin{array}{ccccccccc}
\mathfrak{g}'_1 & \longrightarrow & \mathfrak{g}'_2 & \longrightarrow \ldots \longrightarrow & \mathfrak{g}'_k & \longrightarrow \ldots \longrightarrow & \mathfrak{g}' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}_{m_1} & \longrightarrow & \mathfrak{g}_{m_2} & \longrightarrow \ldots \longrightarrow & \mathfrak{g}_{m_k} & \longrightarrow \ldots \longrightarrow & \mathfrak{g}
\end{array}$$

such that all vertical arrows are of signature $(1, 0, n_k)$ with $n_k = m_k - k$. If a is finite we can construct the diagram so that $n_k = a$ for all k. If $a = \infty$ then $\lim_{k\to\infty} n_k = a$. We set $V^{m_k}_{\lambda,\mu} = V^{m_k}_{\lambda,\mu} \cap (V^{\otimes p}_{m_k} \otimes (V^*)^{\otimes q}_{m_k})$ and similarly for $V'^k_{\lambda',\mu'}$. Then the Gelfand-Tsetlin rule implies

$$[V^{m_k}_{\lambda,\mu}, V'^k_{\lambda',\mu'}] = m^{n_k}_{\lambda,\lambda'} m^{n_k}_{\mu,\mu'}$$

and thus

$$[V_{\lambda,\mu}, V'_{\lambda',\mu'}] = m^a_{\lambda,\lambda'} m^c_{\mu,\mu'}. \tag{7.3}$$

(7.1), (7.2), and (7.3) imply the following statement.

Corollary 7.2. *Let $\mathfrak{g}' \cong \mathfrak{g} \cong \mathrm{gl}(\infty)$ and let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of general tensor type. Then for any simple tensor $\mathfrak{g}$-module $V_{\lambda,\mu}$ and any simple tensor $\mathfrak{g}'$-module $V'_{\lambda',\mu'}$ we have*

$$[V_{\lambda,\mu}, V'_{\lambda',\mu'}] = \sum_{\lambda_2,\mu_2} m^{a+b}_{\lambda,\lambda_2} m^{c+d}_{\mu,\mu_2} A^{\lambda_2,\mu_2}_{\lambda',\mu'}.$$

Proof. We have the following equalities:

$$[V_{\lambda,\mu}, V'_{\lambda',\mu'}] = \sum_{\substack{\lambda_1,\mu_1 \\ \lambda_2,\mu_2}} m^{a_1+b}_{\lambda,\lambda_1} m^{c_1+d}_{\mu,\mu_1} m^{a_2}_{\lambda_1,\lambda_2} m^{c_2}_{\mu_1,\mu_2} A^{\lambda_2,\mu_2}_{\lambda',\mu'} =$$

$$\sum_{\lambda_2,\mu_2} m^{a_1+a_2+b}_{\lambda,\lambda_2} m^{c_1+c_2+d}_{\mu,\mu_2} A^{\lambda_2,\mu_2}_{\lambda',\mu'} = \sum_{\lambda_2,\mu_2} m^{a+b}_{\lambda,\lambda_2} m^{c+d}_{\mu,\mu_2} A^{\lambda_2,\mu_2}_{\lambda',\mu'}.$$

□

Thus, the total multiplicity of $V'_{\lambda',\mu'}$ in the decomposition of $V_{\lambda,\mu}$ depends only on the values $a+b$, $c+d$, k and l. Notice that when $\mu = 0$ Corollary 7.2 yields

$$[V_{\lambda,0}, V'_{\lambda',\mu'}] = \sum_{\lambda_2} m^{a+b}_{\lambda,\lambda_2} A^{\lambda_2,\mu}_{\lambda',\mu'}.$$

The above statement leads to the following observation. If $M = V_{\lambda,\mu}$ with $\lambda, \mu \neq 0$, then there are simple constituents with infinite multiplicity in the decomposition of M as a $\mathfrak{g}'$-module if and only if any of a, b, c, or d is infinite. In the special case when $M = V_{\lambda,0}$ (resp., $M = V_{0,\mu}$), the total multiplicity $[M, M']$ does not depend on $c+d$ (resp., $a+b$), and $V_{\lambda,0}$ (resp., $V_{0,\mu}$) has simple constituents with infinite multiplicity as a $\mathfrak{g}'$-module if and only if a or b (resp., c or d) is infinite.

More generally, the following statement holds.

Corollary 7.3. *Let $\mathfrak{g}'$ and $\mathfrak{g}$ be any two classical locally finite Lie algebras and let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of general tensor type. Fix a simple tensor $\mathfrak{g}$-module M. Then,*

(1) if $M = V_{\lambda,\mu}$ with $\lambda, \mu \neq 0$, there are simple constituents with infinite multiplicity in the decomposition of M as a $\mathfrak{g}'$-module if and only if any of the values a, b, c, or d is infinite;

(2) if $M = V_{\lambda,0}$ (resp., $M = V_{0,\mu}$), M has simple constituents with infinite multiplicity as a $\mathfrak{g}'$-module if and only if any of a, b (resp., c, d) is infinite;

(3) in any other case $a = c$, $b = d$, hence M has simple constituents with infinite multiplicity as a $\mathfrak{g}'$-module if and only if a or b is infinite.

Notice that to prove Corollary 7.3 it is enough to determine the multiplicity $[V_{\langle\lambda\rangle}, V'_{\langle\lambda'\rangle}]$ for embeddings $\mathrm{sp}(\infty) \subset \mathrm{sp}(\infty)$ of types I and II and the multiplicity $[V_{[\lambda]}, V'_{[\lambda']}]$ for embeddings $\mathrm{so}(\infty) \subset \mathrm{so}(\infty)$ of types I and II. By Theorem 5.2 all other cases reduce to these and to Corollary 7.2.

We now go back to the discussion of the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. Let $\mathfrak{g}' \subset \mathfrak{g}$ be a pair of simple classical locally finite Lie algebras and let M be a simple tensor $\mathfrak{g}$-module. Then, as we mentioned in Chapter 2, M as a module over $\mathfrak{g}'$ is an object of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}'}$ ([PSe]). A natural question to ask is when M is a tensor $\mathfrak{g}'$-module. The description of tensor modules from [PSt] implies that each simple constituent comes with a finite multiplicity. Thus, Corollary 7.3 gives a necessary condition for M to be a tensor $\mathfrak{g}'$-module.

Let again $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of general tensor type of any two classical locally finite Lie algebras and let M be a simple tensor $\mathfrak{g}$-module. Now we address the question when M is indecomposable as a $\mathfrak{g}'$-module. Indecomposable modules play a major role in [PSe] and [DaPSe]. One sufficient condition for indecomposability is that the socle of M be a simple $\mathfrak{g}'$-module. Using the results from the previous chapters we can check under what conditions on the embedding $\mathfrak{g}' \subset \mathfrak{g}$ this holds.

Let us again consider the case $\mathfrak{g}' \cong \mathfrak{g} \cong \mathrm{gl}(\infty)$ and $M = V_{\lambda,\mu}$. If $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of type I, then Corollary 3.11 implies that $\mathrm{soc}_{\mathfrak{g}'} M$ is simple if and only if $a = c = 0$.

If $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of type II, then Theorem 3.15 implies that $\mathrm{soc}_{\mathfrak{g}'} M$ is never simple.

If $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of type III, then Corollary 3.18 implies that $\mathrm{soc}_{\mathfrak{g}'} M$ is simple if and only if either $k = 1$ and $l = 0$ or $k = 0$ and $l = 1$.

Using Theorem 5.2, we can combine the above observations in the following proposition.

Proposition 7.4. *Let $\mathfrak{g}' \cong \mathfrak{g} \cong \mathrm{gl}(\infty)$ and let $\mathfrak{g}' \subset \mathfrak{g}$ be an embedding of general tensor type. Then, for any simple $\mathfrak{g}$-module $V_{\lambda,\mu}$ with $\lambda, \mu \neq 0$, $\mathrm{soc}_{\mathfrak{g}'} V_{\lambda,\mu}$ is a simple $\mathfrak{g}'$-module if and only if $a = c = 0$ and either $k = 1$ and $l = 0$ or $k = 1$ and $l = 0$. In addition, $\mathrm{soc}_{\mathfrak{g}'} V_{\lambda,0}$ is a simple $\mathfrak{g}'$-module if and only if $a = 0$ and either $k = 1$ and $l = 0$ or $k = 1$ and $l = 0$.*

In particular, Proposition 7.4 shows that $\mathrm{soc}_{\mathfrak{g}'} V_{\lambda,\mu}$ for $\lambda, \mu \neq 0$ is simple if and only if $\mathrm{soc}_{\mathfrak{g}'} V$ and $\mathrm{soc}_{\mathfrak{g}'} V_*$ are simple. When $\mu = 0$, $\mathrm{soc}_{\mathfrak{g}'} V_{\lambda,0}$ is simple if and only if $\mathrm{soc}_{\mathfrak{g}'} V$ is simple. Similar results can be obtained in the other cases too. Thus the following corollary holds.

Corollary 7.5. *If $\mathfrak{g}'$ and $\mathfrak{g}$ are any two classical locally finite Lie algebras and $\mathfrak{g}' \subset \mathfrak{g}$ is an embedding of general tensor type such that $\mathrm{soc}_{\mathfrak{g}'} V$ and $\mathrm{soc}_{\mathfrak{g}'} V_*$ are simple $\mathfrak{g}'$-modules, then any simple tensor $\mathfrak{g}$-module M is indecomposable over $\mathfrak{g}'$.*

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