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Ph.D. Thesis

A geometric approach to the study
of $(\mathfrak{g}, \mathfrak{k})$ -modules of finite type

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1. NOTATION AND DEFINITIONS

Throughout this thesis \mathfrak{g} will be a reductive Lie algebra over the algebraically closed field \mathbb{C} of characteristic 0 and $\mathfrak{k} \subset \mathfrak{g}$ will be a reductive in \mathfrak{g} subalgebra. Let $U(\mathfrak{k})$ be the universal enveloping algebra of \mathfrak{k} , G be the adjoint group of $[\mathfrak{g}, \mathfrak{g}]$, K be a connected subgroup of G with Lie algebra $\mathfrak{k} \cap [\mathfrak{g}, \mathfrak{g}]$ and B be a Borel subgroup of K . We work in the category of algebraic varieties over \mathbb{C} . All Lie algebras considered are finite-dimensional.

Definition 1.1. A $(\mathfrak{g}, \mathfrak{k})$ -module is a \mathfrak{g} -module for which action of \mathfrak{k} is *locally finite*, i.e. for which $\dim(U(\mathfrak{k})m) < \infty$ for any $m \in M$, where $U(\mathfrak{k})m := \{m' \in M \mid m' = um \text{ for some } u \in U(\mathfrak{k}) \text{ and } m \in M\}$.

Definition 1.2. Let M be a *locally finite* \mathfrak{k} -module, i.e a $(\mathfrak{k}, \mathfrak{k})$ -module. We say that M *has finite type over \mathfrak{k}* if all isotypic components of \mathfrak{k} are finite-dimensional. We say that a $(\mathfrak{g}, \mathfrak{k})$ -module is of *finite type* if M has finite type over \mathfrak{k} .

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$.

Definition 1.3. We say that a \mathfrak{g} -module M *affords a central character* if for some homomorphism of algebras $\chi : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ we have $zm = \chi(z)m$ for all $z \in Z(\mathfrak{g})$ and $m \in M$.

Definition 1.4. We say that a \mathfrak{g} -module M *affords a generalized central character* if for some homomorphism

$$\chi : Z(\mathfrak{g}) \rightarrow \mathbb{C}$$

and some $n \in \mathbb{Z}_{>0}$ we have

$$(z - \chi(z))^n m = 0$$

for all $m \in M$ and $z \in Z(\mathfrak{g})$.

Let M be a $(\mathfrak{g}, \mathfrak{k})$ -module of finite type and V be a simple \mathfrak{k} -module. We denote by $[M : V]_{\mathfrak{k}}$ the supremum over all finite-dimensional \mathfrak{k} -submodules $M' \subset M$ of the Jordan-Hölder multiplicities $[M' : V]_{\mathfrak{k}}$. By $[M : \cdot]_{\mathfrak{k}}$ we denote the corresponding function from the set of simple \mathfrak{k} -modules to $\mathbb{Z}_{\geq 0}$. This is nothing but the \mathfrak{k} -character of M .

Definition 1.5. A *bounded* $(\mathfrak{g}, \mathfrak{k})$ -module M is a $(\mathfrak{g}, \mathfrak{k})$ -module which is *bounded* as a \mathfrak{k} -module, i.e. for which the function $[M : \cdot]_{\mathfrak{k}}$ is uniformly bounded by some constant C_M .

Definition 1.6. A *multiplicity-free* $(\mathfrak{g}, \mathfrak{k})$ -module M is a $(\mathfrak{g}, \mathfrak{k})$ -module which is multiplicity-free as a \mathfrak{k} -module, i.e. for which the function $[M : \cdot]_{\mathfrak{k}}$ is uniformly bounded by 1.

For a finitely generated \mathfrak{g} -module we denote by $V(M)$ the associated variety of M , by $\text{Ann}M$ the annihilator of M in $U(\mathfrak{g})$, by $\text{GV}(M)$ the zero set of $\text{gr}(\text{Ann}M)$ (see Subsection 3.5). We call a G -orbit Gu in \mathfrak{g}^* *nilpotent* if $0 \in \overline{Gu}$. If \mathfrak{g} is semi-simple we identify \mathfrak{g} and \mathfrak{g}^* .

For a variety X we denote by $\mathcal{O}(X)$ the structure sheaf of X , by $\mathbb{C}[X]$ the algebra of regular functions on X , by $\mathcal{D}(X)$ the sheaf of differential operators on X , by $D(X)$ the algebra of global sections of $\mathcal{D}(X)$. We denote by TX (respectively, T^*X) the total space of the tangent (respectively, cotangent) bundle to X . If \mathcal{M} is a coherent $\mathcal{D}(X)$ -module, $\mathcal{V}(\mathcal{M}) \subset T^*X$ stands for *the singular support* of \mathcal{M} [Bo] (see also Subsection 3.4). All $\mathcal{D}(X)$ -modules considered are quasicoherent.

For a finite-dimensional vector space W we set $n_W := \dim W$.

Definition 1.7. We call an n_W -tuple $\bar{\lambda} := (\lambda_1, \dots, \lambda_{n_W})$, $\lambda_i \in \mathbb{C}$, *decreasing* if $\lambda_i - \lambda_j \in \mathbb{Z}_{\geq 0}$ for $i \geq j$. We call $\bar{\lambda}$ *semi-decreasing* if it is not decreasing but becomes decreasing when we remove one coordinate (cf. with O. Mathieu's [M] definitions of ordered/semi-ordered tuples).

Assume $n_W \geq 3$. Let $\bar{\lambda} = (\lambda_1, \dots, \lambda_{n_W-1})$, $\lambda_j \in \mathbb{C}$, be a decreasing tuple and $t \in \mathbb{C}$. By adding an additional coordinate t to $\bar{\lambda}$ (at any position) we obtain a semi-decreasing n_W -tuple $\bar{\lambda}^+$.

Definition 1.8. *The monodromy* $m(\bar{\lambda}^+)$ of $\bar{\lambda}^+$ is the number $e^{2\pi i(t-\lambda_1)}$.

Definition 1.9. We call an n_W -tuple $\bar{\lambda} := (\lambda_1, \dots, \lambda_{n_W})$, $\lambda_i \in \mathbb{C}$, *integral* if $\lambda_i - \lambda_j \in \mathbb{Z}$ for any i, j . We call $\bar{\lambda}$ *semi-integral* if it is not integral but becomes integral when we remove one term. We call a tuple $\bar{\lambda}$ *regular* if $\lambda_i \neq \lambda_j$ for all pairs $i \neq j$.

Any semi-decreasing n_W -tuple $\bar{\lambda}$ is regular integral, singular integral, or semi-integral. If $\bar{\lambda}$ is integral then $m(\bar{\lambda}) = 1$; if $\bar{\lambda}$ is not integral then $m(\bar{\lambda}) \neq 1$.

Definition 1.10. We call an n_W -tuple a *Shale-Weil tuple* if $\mu_i > \mu_j$ for $i > j$, $\mu_{n_W-1} > |\mu_{n_W}|$ and $\mu_i \in \frac{1}{2} + \mathbb{Z}$ for all $i \in \{1, \dots, n_W\}$.

For a K -variety X and a point $x \in X$ we denote by K_x the stabilizer of x in K and by \mathfrak{k}_x the Lie algebra of K_x . If there exists a subgroup $H \subset K$ such that H is conjugate to K_x for all x from some open subset $\tilde{X} \subset X$, we call H a *generic stabilizer* of K on X ; we call the Lie algebra of H a *generic isotropy subalgebra*. By definition, X is a *K -spherical variety* if X is irreducible and has an open B -orbit.

By SL_n , SO_n , SP_n we denote respectively the special linear, orthogonal and symplectic groups of n -dimensional vector space (SP_n is defined only when $n = 2k$). By \mathfrak{sl}_n , \mathfrak{so}_n , \mathfrak{sp}_n we denote the corresponding Lie algebras.

2. INTRODUCTION AND BRIEF STATEMENTS OF RESULTS

There are two well-known categories of $(\mathfrak{g}, \mathfrak{k})$ -modules: the category of Harish-Chandra modules and the category \mathcal{O} . In the first case \mathfrak{k} is a symmetric subalgebra of \mathfrak{g} (i.e. \mathfrak{k} coincides with the fixed points of an involution of \mathfrak{g}), and in the second case \mathfrak{k} is a Cartan subalgebra $\mathfrak{h}_{\mathfrak{g}}$ of \mathfrak{g} . For both types of pairs $(\mathfrak{g}, \mathfrak{k})$ the $(\mathfrak{g}, \mathfrak{k})$ -modules in question are of finite type. I. Penkov, V. Serganova, and G. Zuckerman have proposed to study, and attempt to classify, simple $(\mathfrak{g}, \mathfrak{k})$ -modules of finite type for arbitrary reductive in \mathfrak{g} subalgebras \mathfrak{k} , [PSZ], [PZ] (see also [Mi]).

Let X be the variety of all Borel subalgebras of \mathfrak{g} . Let $\lambda \in H^1(X, \Omega^{1, \text{cl}})$ be a cohomology class ($\Omega^{1, \text{cl}}$ is the sheaf of closed holomorphic 1-forms on X) and $\mathcal{D}^\lambda(X)$ be the corresponding sheaf of twisted differential operators on X [Be]. The cohomology class λ defines functors of 'localization' (Loc: \mathfrak{g} -modules to $\mathcal{D}^\lambda(X)$ -modules) and 'global sections' (GSec: $\mathcal{D}^\lambda(X)$ -modules to \mathfrak{g} -modules). For any central character χ there exists a cohomology class λ such that GSec(Loc) is the identity functor after restriction to the category of \mathfrak{g} -modules which affords the central character χ [BeBe2]. Any simple \mathfrak{g} -module affords a central character [Dix].

We fix $\lambda \in H^1(X, \Omega^{1, \text{cl}})$. In the category of $\mathcal{D}^\lambda(X)$ -modules there is a distinguished full subcategory of holonomic sheaves of modules, or simply holonomic modules. Informally, holonomic modules are $\mathcal{D}^\lambda(X)$ -modules of minimal growth (see Definition 3.20). The simple holonomic modules are in one-to-one correspondence with the set of pairs (L, S) , where L is an irreducible closed subvariety of X and S is sheaf of $\mathcal{D}^\lambda(L')$ -modules which is $\mathcal{O}(L')$ -coherent after restriction to a suitable open subset $L' \subset L$. Moreover, a coherent holonomic module S is locally free on L' , and one could think about it as a vector bundle S_B over L' with a flat connection. Note that flat local sections of S_B are not necessarily algebraic.

Our first result is the following theorem, which we prove in Section 4 (Corollary 4.1).

Theorem 2.1. Let M be a simple $(\mathfrak{g}, \mathfrak{k})$ -module of finite type. Then $\text{Loc}(M)$ is a holonomic $\mathcal{D}^\lambda(X)$ -module.

The following theorem is a 'geometric twin' of the previous one and is proved also in Section 4. The definitions needed for the statement of the theorem see in Subsection 3.1 and Subsection 3.7, or in [VP].

Theorem 2.2. Let $\mathcal{Z} \subset \mathfrak{g}^*$ be a nilpotent G -orbit, \mathfrak{k}^\perp be the annihilator of \mathfrak{k} in \mathfrak{g}^* , $N_K(\mathfrak{g}^*)$ be the K -null-cone in \mathfrak{g}^* . Then the irreducible components of $\mathcal{Z} \cap \mathfrak{k}^\perp \cap N_K(\mathfrak{g}^*)$ are isotropic subvarieties of \mathcal{Z} .

Let $V_{\mathfrak{g},\mathfrak{k}}$ be the set of all irreducible components of all possible intersections of $N_K(\mathfrak{k}^\perp)$ with G -orbits in $N_G(\mathfrak{g}^*)$. This finite set of subvarieties of \mathfrak{g}^* determines a finite set $\mathcal{V}_{\mathfrak{g},\mathfrak{k}}$ of subvarieties of T^*X and a finite set $L_{\mathfrak{g},\mathfrak{k}}$ of subvarieties of X , see Section 4.

Theorem 2.3. Let M be a finitely generated $(\mathfrak{g}, \mathfrak{k})$ -module of finite type which affords a central character, and let (L, S) the pair corresponding to $\text{Loc}M$ as introduced above. Then L is an element of $L_{\mathfrak{g},\mathfrak{k}}$.

We prove this theorem in Section 4 (Corollary 4.1). Any simple Harish-Chandra module is holonomic [BeBe2] and of finite-type [HC]¹. For the literature on Harish-Chandra modules, see [KV] and references therein.

In the classification of simple $(\mathfrak{g}, \mathfrak{h}_{\mathfrak{g}})$ -modules of finite type the bounded simple modules play a crucial role. Based on this, and on the experience with Harish-Chandra modules, I. Penkov and V. Serganova have proposed to study bounded $(\mathfrak{g}, \mathfrak{k})$ -modules for general reductive subalgebras \mathfrak{k} . A question arising in this context is, given \mathfrak{g} , to describe all reductive in \mathfrak{g} bounded subalgebras, i.e. reductive in \mathfrak{g} subalgebras \mathfrak{k} for which at least one infinite-dimensional simple bounded $(\mathfrak{g}, \mathfrak{k})$ -module exists. In [PS] I. Penkov and V. Serganova give a partial answer to this problem, and in particular proved an important inequality which restricts severely the class of possible \mathfrak{k} . They also give the complete list of bounded reductive subalgebras of $\mathfrak{g} = \mathfrak{sl}_n$ which are maximal subalgebras.

In the work [Pe] we prove the following theorem.

Theorem 2.4. There exists an infinite-dimensional simple bounded $(\mathfrak{sl}(V), \mathfrak{k})$ -module if and only if V is a $K \times \mathbb{C}^*$ -spherical variety.

We reproduce the proof of this theorem in Section 5. There are well-known pairs of algebras which admit an infinite-dimensional simple bounded module:

- (1) \mathfrak{k} is a Cartan subalgebra of a simple Lie algebra \mathfrak{g} of type A or C [F];
- (2) \mathfrak{k} is a symmetric subalgebra of a reductive Lie algebra \mathfrak{g} .

Bounded $(\mathfrak{g}, \mathfrak{h}_{\mathfrak{g}})$ -modules are nothing but weight modules with bounded weight multiplicities, and all such simple modules have been classified by O. Mathieu [M].

As far as we know, the simple bounded Harish-Chandra modules have not been singled explicitly out within the category of all Harish-Chandra modules. For a given symmetric pair $(\mathfrak{g}, \mathfrak{k})$, Harish-Chandra

¹In the literature on Harish-Chandra modules the condition of being of finite type is often called admissibility.

modules admit only finitely many support varieties, and a Harish-Chandra module M is spherical if and only if any irreducible component of the support variety $V(M)$ is spherical. There is some progress in singling out the spherical varieties among the support varieties of Harish-Chandra modules [Pan2], [Ki].

Let I be a two-sided ideal in $U(\mathfrak{g})$. By definition, I is *primitive* if I is the annihilator of some simple \mathfrak{g} -module M . Let W be a finite-dimensional \mathbb{C} -vector space. To any n_W -tuple $\bar{\lambda} := (\lambda_1, \dots, \lambda_{n_W})$ one assigns a weight λ (see Subsection 3.9) and a primitive ideal $I(\lambda) \subset U(\mathfrak{sl}(W))$ (this is the annihilator of the simple $\mathfrak{sl}(W)$ -module with highest weight λ), and any primitive ideal of $U(\mathfrak{sl}(W))$ arises in this way from some tuple $\bar{\lambda}$.

Let M_1, M_2 be simple $(\mathfrak{g}, \mathfrak{k})$ -modules. Let $\text{Ann}M_1, \text{Ann}M_2$ be the annihilators in $U(\mathfrak{g})$ of M_1 and M_2 respectively. I. Penkov and V. Serganova [PS] proved that, if $\text{Ann}M_1 = \text{Ann}M_2$ and M_1 is a bounded $(\mathfrak{g}, \mathfrak{k})$ -module, then M_2 is \mathfrak{k} -bounded too (see also Theorem 3.13). Moreover, a $(\mathfrak{g}, \mathfrak{k})$ -module is bounded if and only if the algebra $U(\mathfrak{g})/\text{Ann}M$ satisfies certain relations. In Section 5 we prove the following weaker geometric version of this result.

Theorem 2.5. A simple $(\mathfrak{sl}(W), \mathfrak{k})$ -module M is bounded if and only if the associated variety $\text{GV}(M)$ is K -coisotropic.

This shows that 'boundedness' is not a property of a module M but of the ideal $\text{Ann}M$, and moreover of the nilpotent orbit $\text{GV}(M) \subset \mathfrak{sl}(W)^*$. Theorem 2.5 also motivates our interest in the classification of K -coisotropic nilpotent G -orbits in \mathfrak{g}^* . In the case $\mathfrak{g} = \mathfrak{sl}(W)$, the set of K -coisotropic nilpotent orbits in $\mathfrak{sl}(W)^*$ is naturally identified with the set of partition equivalence classes of K -spherical partial W -flag varieties (see Subsection 3.3). We work out the classification of K -spherical flag varieties in Section 6.

Furthermore, we make the following conjecture.

Conjecture 2.1. A simple $(\mathfrak{g}, \mathfrak{k})$ -module M is bounded if and only if the associated variety $\text{GV}(M)$ is K -coisotropic.

This statement is closely related with [Pan, Question, p. 191] and we believe that this question will be answered soon [ZT].

In the rest of the thesis we consider in greater detail four special pairs $(\mathfrak{g}, \mathfrak{k})$:

$$\begin{array}{cc}
 \text{a) } W = S^2V & \text{b) } W = \Lambda^2V \\
 \hline
 \text{1a) } (\mathfrak{sl}(W), \mathfrak{sl}(V)) & \text{1b) } (\mathfrak{sl}(W), \mathfrak{sl}(V)) \\
 \text{2a) } (\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V)) & \text{2b) } (\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))
 \end{array} \tag{1}$$

We hope that our results about these cases shed some light on how the general theory of bounded modules looks like.

In the rest of Section 2 V is a finite-dimensional vector space and $W=S^2V$ (in this case $n_V \geq 3$) or $W = \Lambda^2V$ (in this case $n_V = 2k$ and $n_V \geq 5$).

For a simple $\mathfrak{sl}(W)$ -module M we have $\dim V(M) \geq n_W - 1$ or M is finite-dimensional. A simple $\mathfrak{sl}(W)$ -module M is *of small growth* if $\dim V(M) \leq n_W - 1$ (the definition of a module of small growth which is not necessarily simple is given in Section 8). The following theorem is proved in Subsection 9.3.

Theorem 2.6. Any simple bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -module is of small growth.

We note that all bounded weight modules are also of small growth. The following result is inspired by the corresponding result of O. Mathieu for weight modules. The proof of this result is presented in Subsection 9.3.

Theorem 2.7. Let M be a simple bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -module and $\bar{\lambda}$ be an n_W -tuple such that $\text{Ann}M=I(\lambda)$. Then $\bar{\lambda}$ is semi-decreasing.

Theorem 2.8. Let M be a finitely generated $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -module and suppose $I(\lambda) \subset \text{Ann}M$ for some semi-decreasing tuple $\bar{\lambda}$. Then M is $\mathfrak{sl}(V)$ -bounded.

We prove Theorem 2.8 in Section 9.3. The primitive ideals which correspond to the semi-decreasing sequences are called Joseph ideals (see Section 7).

Fix $t \in \mathbb{C}$. Let $P_{e^{2\pi it}}(W)$ be the cardinality of the set of isomorphism classes of simple perverse sheaves on W (with respect to the stratification by $GL(V)$ -orbits) which have fixed monodromy $e^{2\pi it}$ and are neither supported at 0 nor are smooth on W . The above simple perverse sheaves on W are described, following [BR], in quiver terms in the Appendix. In particular, there are only finitely many isomorphism classes of simple perverse sheaves with a given monodromy. The following theorem 'counts' simple bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -modules. We prove this theorem in Subsection 9.3.

Theorem 2.9. Let $\bar{\lambda}$ be a semi-decreasing tuple. Then there exist precisely $P_{m(\lambda)}(W)$ non-isomorphic infinite-dimensional simple $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -modules annihilated by $I(\lambda)$.

Let $\bar{\lambda}$ be a decreasing tuple. Let \mathcal{J}_λ be the set of infinite-dimensional simple bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -modules annihilated by $\text{Ker}\chi_\lambda$. Let $\langle \mathcal{J}_\lambda \rangle$ be the free vector space generated by \mathcal{J}_λ . The set of n_W -tuples carries

a natural action of S_{n_W} . This action induces an action of S_{n_W} on $\langle \mathcal{J}_\lambda \rangle$. The S_{n_W} -module $\langle \mathcal{J}_\lambda \rangle$ is isomorphic to a direct sum of $P_1(W)$ -copies of an $(n_W - 1)$ -dimensional module of S_{n_W} .

We now turn our attention to row 2 of table (1). The space $W \oplus W^*$ has a natural symplectic form

$$\omega(\cdot, \cdot) : ((x_1, l_1), (x_2, l_2)) \mapsto x_1(l_2) - x_2(l_1) \in \mathbb{C}.$$

There is an inclusion $\mathfrak{gl}(W) \subset \mathfrak{sp}(W \oplus W^*)$. The following theorem relates bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -modules and bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules. We prove it in Subsection 9.3.

Theorem 2.10. a) Let \tilde{M} be a bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -module. Then any simple $\mathfrak{sl}(W)$ -subquotient of \tilde{M} is an $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -bounded module.

b) Let M be a simple bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -module. Then there exists a simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -module \tilde{M} such that M is an $\mathfrak{sl}(W)$ -subquotient of \tilde{M} .

Any tuple $\bar{\mu} = (\mu_1, \dots, \mu_{n_W})$ determines a two-sided ideal $I_{\mathfrak{sp}}(\bar{\mu})$ of $U(\mathfrak{sp}(W \oplus W^*))$ and any primitive ideal of $U(\mathfrak{sp}(W \oplus W^*))$ is determined by some tuple $\bar{\mu}$.

For a simple $\mathfrak{sp}(W \oplus W^*)$ -module M , we have either $\dim V(M) \geq n_W$ or $\dim M < \infty$. A simple $\mathfrak{sp}(W \oplus W^*)$ -module M is of *small growth* if $\dim V(M) \leq n_W$ (the definition of a module of small growth which is not necessarily simple is given in Section 8). The following theorems are 'sp-twins' of Theorem 2.6, Theorem 2.7, Theorem 2.8.

Theorem 2.11. Any simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -module is of small growth.

Theorem 2.12. Let M be an infinite-dimensional simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -module and $\bar{\mu}$ be an n_W -tuple such that $\text{Ann} M = I_{\mathfrak{sp}}(\bar{\mu})$. Then $\bar{\mu}$ is a Shale-Weil tuple.

Theorem 2.13. Let M be a finitely generated $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -module and $I_{\mathfrak{sp}}(\bar{\mu}) \subset \text{Ann} M$ for some Shale-Weil tuple $\bar{\mu}$. Then M is bounded.

We prove these three theorems in Subsection 9.4. The primitive ideals which correspond to the Shale-Weil sequences are called Joseph ideals (see Section 7).

We call a Shale-Weil n_W -tuple $\bar{\mu}$ *positive* if $\mu_{n_W} > 0$ and *negative* otherwise. Let \mathfrak{h}_W be the \mathbb{C} -affine space of n_W -tuples. The automorphism

$$\sigma : \mathfrak{h}_W^* \rightarrow \mathfrak{h}_W^*, (\mu_1, \mu_2, \dots, \mu_{n_W}) \mapsto (\mu_1 \cdot \mu_2, \dots, -\mu_{n_W})$$

interchanges the sets of positive and negative Shale-Weil tuples. Furthermore, $I_{\mathfrak{sp}}(\bar{\mu}) = I_{\mathfrak{sp}}(\sigma\bar{\mu})$. Set $\bar{\mu}_0 := (n_W - \frac{1}{2}, n_W - \frac{3}{2}, \dots, \frac{1}{2})$. The following theorem shows that the categories of bounded modules annihilated by different ideals are equivalent. We prove it in Subsection 9.4.

Theorem 2.14. a) For any two positive Shale-Weil tuples $\bar{\mu}_1, \bar{\mu}_2$ the categories of bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules annihilated by $I_{\mathfrak{sp}}(\bar{\mu}_1)$ and $I_{\mathfrak{sp}}(\bar{\mu}_2)$ are equivalent.

b) In particular, the set of simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules is naturally identified with the set of pairs $(\bar{\mu}, M)$, where $\bar{\mu}$ is a positive Shale-Weil n_W -tuple and M is a simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -module annihilated by $I_{\mathfrak{sp}}(\bar{\mu}_0)$.

Let $\bar{\mu}$ be a positive Shale-Weil n_W -tuple. The functor $\mathcal{H}_{\bar{\mu}}^{\sigma\bar{\mu}}$ is exact and involutive. The category of modules annihilated by $I_{\mathfrak{sp}}(\bar{\mu})$ is stable under $\mathcal{H}_{\bar{\mu}}^{\sigma\bar{\mu}}$. We denote $\mathcal{H}_{\bar{\mu}_0}^{\sigma\bar{\mu}_0}$ by Inv .

The Dynkin diagram of Spin_{2n_W} has a nontrivial involution (it is unique unless $n = 4$) and it induces an involution σ on Spin_{2n_W} . Furthermore there is a unique central element $z \in \text{Spin}_{2n_W}$ such that $z^2 = 1$ and $\text{Spin}_{2n_W}/\{1, z\} = \text{SO}_{2n_W}$ (for $n = 4$, z is uniquely determined by the additional requirement $\sigma(z) = z$). An *odd pair* of simple Spin_{2n_W} -modules is a pair $\{L^{\bar{\mu}}, L^{\sigma\bar{\mu}}\}$ of simple Spin_{2n_W} -modules which are conjugate by σ and such that z acts by -1 on $L^{\bar{\mu}}$ and $L^{\sigma\bar{\mu}}$. Obviously, $L^{\bar{\mu}}$ and $L^{\sigma\bar{\mu}}$ has the same dimension. Any positive Shale-Weil n_W -tuple $\bar{\mu}$ determines an odd pair $\{L^{\bar{\mu}}, L^{\sigma\bar{\mu}}\}$ of Spin_{2n_W} -modules (cf. [M]).

Let $\{L, L^{\sigma}\}$ be the unique odd pair of minimal dimension. The following theorem should be understood as a mnemonic rule corresponding to the results of Subsection 9.4. It is related to the $\mathfrak{gl}(V)$ -characters of bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules.

Theorem 2.15. Let $(\bar{\mu}, M)$ be a pair as in Theorem 2.14 and let $\bar{\mu}M$ be the corresponding simple $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -module. Then

$$\frac{[\bar{\mu}M : \cdot]_{\mathfrak{k}} + [\bar{\mu}\text{Inv}M : \cdot]_{\mathfrak{k}}}{[M : \cdot]_{\mathfrak{k}} + [\text{Inv}M : \cdot]_{\mathfrak{k}}} = \frac{[L^{\bar{\mu}} : \cdot]_{\mathfrak{h}} + [L^{\sigma\bar{\mu}} : \cdot]_{\mathfrak{h}}}{[L : \cdot]_{\mathfrak{h}} + [L^{\sigma} : \cdot]_{\mathfrak{h}}} \quad (2.1)$$

and

$$\frac{[\bar{\mu}M : \cdot]_{\mathfrak{k}} - [\bar{\mu}\text{Inv}M : \cdot]_{\mathfrak{k}}}{[M : \cdot]_{\mathfrak{k}} - [\text{Inv}M : \cdot]_{\mathfrak{k}}} = \frac{[L^{\bar{\mu}} : \cdot]_{\mathfrak{h}} - [L^{\sigma\bar{\mu}} : \cdot]_{\mathfrak{h}}}{[L : \cdot]_{\mathfrak{h}} - [L^{\sigma} : \cdot]_{\mathfrak{h}}} \quad (2.2)$$

The functions $[L : \cdot]_{\mathfrak{h}}, [L^{\sigma} : \cdot]_{\mathfrak{h}}, [L^{\bar{\mu}} : \cdot]_{\mathfrak{h}}, [L^{\sigma\bar{\mu}} : \cdot]_{\mathfrak{h}}$ are known since the modules

$$L, L^{\sigma}, L^{\bar{\mu}}, L^{\sigma\bar{\mu}}$$

are finite-dimensional. Therefore the formulas (2.1), (2.2) provide two linear equations for the four unknowns.

Let M_1, M_2 be non-isomorphic simple $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules annihilated by $I_{\mathfrak{sp}}(\mu_0)$. Then both M_1, M_2 are multiplicity-free $\mathfrak{gl}(V)$ -modules and the functions

$$[M_1 : \cdot]_{\mathfrak{gl}(V)} \text{ and } [M_2 : \cdot]_{\mathfrak{gl}(V)}$$

are pairwise disjoint, i.e. their product is the zero-function. Moreover, there are infinitely-many non-isomorphic simple $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules annihilated by $I_{\mathfrak{sp}}(\mu_0)$. We relate the category of bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules with a suitable category of perverse sheaves in Subsection 9.4.

Theorem 2.16. Let $\bar{\mu}$ be a positive Shale-Weil n_W -tuple. Then the category of $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules annihilated by $I_{\mathfrak{sp}}(\bar{\mu})$ is equivalent to the direct sum of two copies of the category of perverse sheaves on W with respect to the stratification by $GL(V)$ -orbits.

For both cases $W = \Lambda^2 V, S^2 V$, the category of perverse sheaves on W with respect to the stratification by $GL(V)$ -orbits is equivalent to a category of representations of an explicitly described quiver with relations [BG], see also the Appendix. The simple objects of this category are enumerated by pairs (S, Y) , where S is a $GL(V)$ -orbit in W and Y is a simple $GL(V)$ -equivariant local system on S , i.e. a simple representation of the fundamental group $\pi_1(S)$. As S has to be $GL(V)$ -spherical, this group has to be finitely generated and abelian. The category contains infinitely many non-isomorphic simple objects, but only finitely many for any fixed monodromy.

Similar work for the category of bounded weight modules has been done by D. Grantcharov and V. Serganova [GrS1], [GrS2]: they have found a quiver with relations, whose category of modules is equivalent to the category of bounded weight modules (see also [SM] and [GVM]).

3. PRELIMINARIES

3.1. Symplectic geometry. By $\mathcal{T}X$ we denote the tangent bundle of a smooth variety X , by $T_x X$ the tangent space to X at a point $x \in X$; by $T_x^* X$ the dual to $T_x X$ space. For a smooth G -variety X we denote by $\tau_X : \mathfrak{g} \rightarrow \mathcal{T}X$ the canonical homomorphism. Let Y be a smooth subvariety of X . We denote by $N_{Y/X}$ and $N_{Y/X}^*$ the total spaces of the normal and conormal bundles to Y in X respectively.

Definition 3.1. Suppose that X is a smooth variety which admits a closed nondegenerate two-form ω . Such a pair (X, ω) is called a *symplectic variety*. If X is a G -variety and ω is G -invariant, (X, ω) is called a *symplectic G -variety*.

Example 3.2. Let X be a smooth G -variety. Then T^*X has a one-form α_X defined at a point $(l, x) (l \in T_x^* X)$ by the equality

$$\alpha_X(\xi) = l(\pi_* \xi)$$

for any $\xi \in T_{(l,x)}(T^*X)$, where $\pi : T^*X \rightarrow X$ is the projection. The differential $d\alpha_X$ is a nondegenerate G -invariant two-form on T^*X and therefore $(T^*X, d\alpha_X)$ is a symplectic G -variety.

Example 3.3. Let \mathcal{O} be a G -orbit in \mathfrak{g}^* . Then \mathcal{O} has a *Kostant-Kirillov-Souriau two-form* $\omega(\cdot, \cdot)$ defined at a point $x \in \mathfrak{g}^*$ by the equality

$$\omega_x(\tau_{\mathfrak{g}^*} p|_x, \tau_{\mathfrak{g}^*} q|_x) = x([p, q])$$

for $p, q \in \mathfrak{g}$.

Definition 3.4. Let (X, ω) be a symplectic variety. We call a subvariety $Y \subset X$

- a) *isotropic* if $\omega|_{T_y Y} = 0$ for the generic point $y \in Y$;
- b) *coisotropic* if $\omega|_{(T_y Y)^\perp} = 0$ for the generic point $y \in Y$;
- c) *Lagrangian* if $T_y Y = (T_y Y)^\perp$ for the generic point $y \in Y$ or equivalently if it is both isotropic and coisotropic.

Example 3.5. Let X be a smooth G -variety and $Y \subset X$ be a smooth G -subvariety. Then $N_{Y/X}^*$ is Lagrangian in T^*X .

Proposition 3.6 (see for example [NG, Lemma 1.3.27]). Any closed irreducible conical (i.e. \mathbb{C}^* -stable) Lagrangian G -subvariety of T^*X is the closure of the total space of the conormal bundle $N_{Y/X}^*$ to a G -subvariety $Y \subset X$.

Definition 3.7. Let (X, ω) be a K -symplectic variety. We say that (X, ω) is *K -isotropic* if any K -orbit of some open K -stable set $\tilde{X} \subset X$ is isotropic. We say that (X, ω) is *K -coisotropic* if any K -orbit of some open K -stable subset $\tilde{X} \subset X$ is coisotropic.

Let X be a G -variety. The map

$$\mathrm{T}^*X \times \mathfrak{g} \rightarrow \mathbb{C} \quad ((x, l), g) \mapsto l(\tau_X g|_x),$$

where $g \in \mathfrak{g}, x \in X, l \in \mathrm{T}_x^*X$, induces a map

$$\phi_X : \mathrm{T}^*X \rightarrow \mathfrak{g}^*$$

called the *moment map*. This map provides the following description of a nilpotent orbit $Gu \subset \mathfrak{g}^*$. Suppose that P is a parabolic subgroup of G .

Theorem 3.1 (R. W. Richardson [Rich]). The moment map

$$\phi_{G/P} : \mathrm{T}^*(G/P) \rightarrow \mathfrak{g}^*$$

is a proper morphism to the closure of some nilpotent orbit \overline{Gu} and is a finite morphism over Gu .

Example 3.8. Let $G = \mathrm{SL}_n$ and $G/P = \mathbb{P}(\mathbb{C}^n)$. Then $\phi_{G/P}(\mathrm{T}^*(G/P))$ (considered as a subset of \mathfrak{sl}_n) coincides with the set of nilpotent matrices of rank ≤ 1 . The open SL_n -orbit in $\phi_{G/P}(\mathrm{T}^*(G/P))$ is a set of SL_n -highest weight vectors and is contained in the closure of any nonzero nilpotent SL_n -orbit in \mathfrak{sl}_n^* .

For $G = \mathrm{SL}_n$, each moment map $\phi_{G/P}$ corresponding to a parabolic subgroup P is a birational isomorphism of $\mathrm{T}^*(G/P)$ with the image of $\phi_{G/P}$, and one can obtain the closure of any nilpotent orbit \overline{Gu} as the image of a suitable moment map $\phi_{G/P}$ (see [CM] and references therein). Assume $X = G/P$. The following computation shows that $\phi_X^*(\omega_{Gu}) = d\alpha_{\mathrm{T}^*X}|_{\phi_X^{-1}Gu}$:

$$\begin{aligned} d\alpha_X|_{(x,t)}(\tau_X p, \tau_X q) &= \\ (\tau_X p \cdot \alpha_X(\tau_X q))|_{(x,t)} - (\tau_X q \cdot \alpha_X(\tau_X p))|_{(x,t)} - \alpha_X([\tau_X p, \tau_X q])|_{(x,t)} &= \\ (\tau_X q \cdot \phi_X^* q)|_{(x,t)} - (\tau_X q \cdot \phi_X^* p) - (\phi_X^*[p, q])|_{(x,t)} &= \\ (\phi_X^*[p, q])|_{(x,t)} - (\phi_X^*[q, p])|_{(x,t)} - (\phi_X^*[p, q])|_{(x,t)} = x([p, q]) = \omega_x([p, q]) \end{aligned}$$

for any $p, q \in \mathfrak{g}$ and $(x, t) \in \mathrm{T}^*X \subset \mathfrak{g}^* \times X$.

In what follows we call quotients of $\mathrm{SL}(W)$ by parabolic subgroups 'partial W -flag varieties' (see Subsection 3.3).

3.2. Spherical varieties. Let V be a finite-dimensional K -module. We recall that B is a Borel subgroup of K .

Definition 3.9. Let X be an irreducible K -variety. Then X is called *K -spherical* if and only if there is an open orbit of B on X . A K -module W is called *K -spherical* if it is K -spherical as a K -variety.

It is well known that K -spherical varieties have many beautiful properties. In particular, the number of K -orbits on such varieties is finite [VK]. A subgroup $K' \subset K$ is called *spherical* if the quotient

K/K' is K -spherical. For example, any symmetric subgroup $K' \subset K$ is spherical.

The following lemma is a reformulation in the terms of the present thesis of a result of É. Vinberg and B. Kimelfeld [VK, Thm. 2].

Lemma 3.10. An irreducible quasiaffine algebraic K -variety X is K -spherical if and only if the space of regular functions $\mathbb{C}[X]$ is a bounded \mathfrak{k} -module. Moreover, if X is K -spherical, then $\mathbb{C}[X]$ is a multiplicity-free \mathfrak{k} -module.

Theorem 3.2 (D. Panyushev [Pan, Thm 2.1]). Let X be a smooth irreducible K -variety and M a smooth locally closed K -stable subvariety. Then the generic stabilizers of the actions of B on X , $N_{M/X}$ and $N_{M/X}^*$ are isomorphic.

Let X be an irreducible K -variety. Then there exist open subset $\tilde{X} \subset X$ such that the stabilizers B_x and B_y are conjugate for all $x, y \in \tilde{X}$ [Pan]. For all $x \in \tilde{X}$ there exists a unique connected reductive subgroup $L(x) \subset K_x$ such that B_x is a Borel subgroup of $L(x)$ [Pan] (see also [Gr]).

Let $\text{Gr}(r; V)$ be the variety of r -dimensional subspaces of V . For $r \in \{1, \dots, n_V - 1\}$ and $x \in \text{Gr}(r; V)$ we denote by $V^r(x) \subset V$ the r -dimensional subspace which corresponds to x . We apply the construction of [Pan] to the case $X := \text{Gr}(r; V)$. Then $L(x)$ stabilizes $V^r(x)$. Therefore the datum (K, V, r) determines the modules $(L(x), V^r(x))$. The type of $(L(x), V^r(x))$ does not depend on a point $x \in \tilde{X}$ and therefore the datum (K, V, r) determines the pair (L, V^r) . One can compute the subgroup L via a technique of doubled actions [Pan].

Definition 3.11. We denote by $c_K(X)$ the codimension in X of the generic orbit of B .

Remark 3.12. The variety X is K -spherical if and only if $c_K(X) = 0$.

Lemma 3.13. Let P_1, P_2 be parabolic subgroups of G and L_1, L_2 be Levi subgroups of P_1 and P_2 respectively. Then

$$c_G(G/P_1 \times G/P_2) = c_{L_1}(G/P_2) = c_{L_2}(G/P_1).$$

Proof. Let \tilde{B} be a Borel subgroup of G . The generic stabilizer for the action of \tilde{B} on G/L_1 is a Borel subgroup of L_1 . Therefore the generic stabilizer for the action of a Borel subgroup of L_1 on G/P_2 coincides with the generic stabilizer for the action of \tilde{B} on $G/P_1 \times G/P_2$. We denote this stabilizer S . Let r be the rank of G . Then

$$\begin{aligned} c_{L_1}(G/P_2) &= \dim G/P_2 - \left(\frac{\dim L_1 + r}{2} - \dim S\right) = \\ \dim(G/P_1 \times G/P_2) - \left(\frac{\dim G + r}{2} - \dim S\right) &= c_G(G/P_1 \times G/P_2). \end{aligned}$$

□

3.3. Grassmannians. Let V be a finite-dimensional vector space. The set of flags $V_1 \subset \dots \subset V_s \subset V$ with fixed dimensions (n_1, \dots, n_s) is a homogeneous space of the group $\mathrm{GL}(V)$, and we denote this variety by $\mathrm{Fl}(n_1, \dots, n_s; V)$. We call such a variety *partial flag variety*. The varieties $\mathbb{P}(V)$ and $\mathrm{Fl}(1; V)$ are naturally identified.

For any $r \in \{1, \dots, n_V - 1\}$ we denote $\mathrm{Fl}(r; V)$ by $\mathrm{Gr}(r; V)$. For $r = 1$, the variety $\mathrm{Gr}(r; V)$ coincides with $\mathbb{P}(V)$. We have

$$\mathrm{Gr}(r; V) \cong \mathrm{Gr}(n_V - r; V^*).$$

Let $n_1, \dots, n_s \in \{1, \dots, n_V - 1\}$ be numbers such that

$$n_1 < \dots < n_s$$

and $\mathrm{Fl}(n_1, \dots, n_s; V)$ be the corresponding partial flag variety. Let $P(x)$ be the stabilizer of a point $x \in \mathrm{Fl}(n_1, \dots, n_s; V)$ in $\mathrm{SL}(V)$ and $\mathfrak{n}(x)$ be a nilpotent radical of the Lie algebra of $P(x)$. Then $\mathfrak{n}(x) \subset \mathfrak{sl}(V)$ consists of nilpotent elements, and

$$\bigcup_{x \in \mathrm{Fl}(n_1, \dots, n_s; V)} \mathfrak{n}(x) \subset \mathfrak{sl}(V)^*$$

coincides with the image of the moment map

$$\phi : \mathrm{T}^*\mathrm{Fl}(n_1, \dots, n_s; V) \rightarrow \mathfrak{sl}(V)^*.$$

Moreover, the image of ϕ in $\mathfrak{sl}(V)^*$ is the closure of a unique nilpotent $\mathrm{SL}(V)$ -orbit. In this way, to any partial flag variety one assigns a unique nilpotent orbit. The condition 'to be in closure of' on the set of nilpotent orbits induces a partial order on the set of partial flag varieties. We will make essential use of this partial order. In what follows we refer to one partial flag variety as being higher or lower than another in terms of this partial order. Partial flag varieties which are equivalent in terms of this partial order are called *cotangent-equivalent* (see also [Kn2]).

We now describe this latter equivalence explicitly. Let $\mathrm{Fl}_1, \mathrm{Fl}_2$ be partial W -flag varieties and

$$(n_1, \dots, n_s), (n'_1, \dots, n'_{s'})$$

be the corresponding dimension vectors. These vectors define the following partitions

$$(n_1, n_2 - n_1, \dots, n_V - n_s) \text{ and } (n'_1, n'_2 - n'_1, \dots, n_V - n'_{s'}) \text{ of } n_V.$$

Lemma 3.14 ([CM, Ch. 6.2]). Two partial flag varieties are cotangent-equivalent if and only if their corresponding partitions coincide as sets.

For example, $\mathrm{Gr}(n_1; V)$ and $\mathrm{Gr}(n_V - n_1; V)$ are cotangent-equivalent.

The relation between partial flag varieties and nilpotent $\mathrm{SL}(V)$ -orbits of $\mathfrak{sl}(V)^*$ has been described in terms of partitions (see [CM]); the partial order on nilpotent $\mathrm{SL}(V)$ -orbits in $\mathfrak{sl}(V)^*$ has been also described in terms of partitions (see [CM]). Therefore in order to check that a given partial flag variety is higher than another one it suffices to check the corresponding condition on partitions. In this way we establish in particular the following statements.

- (1) Any partial flag variety is higher than or is cotangent-equivalent to $\mathbb{P}(V)$.
- (2) If $r_1, r_2 \in \{1, \dots, \lfloor \frac{nV}{2} \rfloor\}$ and $r_1 > r_2$, then $\mathrm{Gr}(r_1; V)$ is higher than $\mathrm{Gr}(r_2; V)$.
- (3) The subset of Grassmannians $\mathrm{Gr}(r; V)$ is totally ordered.
- (4) If Fl is a partial flag variety which is not cotangent-equivalent to $\mathbb{P}(V)$, then Fl is higher than or is cotangent-equivalent to $\mathrm{Gr}(2; V)$.
- (5) Any partial V -flag variety is cotangent-equivalent to one of the following

$$\mathrm{Gr}(r; V), \quad \mathrm{Fl}(1, 2; V), \quad \mathrm{Fl}(1, 3; V),$$

or is higher than $\mathrm{Fl}(1, 3; V)$.

- (6) Any partial V -flag variety is cotangent-equivalent to one of the following

$$\mathrm{Gr}(r; V), \quad \mathrm{Fl}(1, r; V), \quad \mathrm{Fl}(1, 2, 3; V), \quad \mathrm{Fl}(2, 4; V),$$

or is higher than $\mathrm{Fl}(2, 4; V)$.

Proposition 3.15. Suppose Fl_1 is a K -spherical variety and Fl_2 is lower than Fl_1 . Then Fl_2 is a K -spherical variety.

To prove the proposition we need to recall some results of I. Losev.

Theorem 3.3 (I. Losev [Lo]). Suppose X is a strongly equidefectinal [Lo, Def. 1.2.5] normal affine irreducible Hamiltonian K -variety. Then

$$\mathbb{C}(X)^K = \mathrm{Quot}(\mathbb{C}[X]^K),$$

where $\mathrm{Quot}(\mathbb{C}[X]^K)$ is the field of fractions of $\mathbb{C}[X]^K$.

Corollary 3.16. Suppose X is a strongly equidefectinal affine irreducible Hamiltonian K -variety. Then $\mathbb{C}(X)^K = \mathrm{Quot}(\mathbb{C}[X]^K)$.

Proof. Let \tilde{X} be the spectrum of the integral closure of $\mathbb{C}[X]$ in $\mathrm{Quot}(\mathbb{C}[X])$ and $\tilde{X} \rightarrow X$ be the canonical finite map. Then \tilde{X} is a normal affine irreducible Hamiltonian K -variety [Kd]. A straightforward check shows that \tilde{X} is strongly equidefectinal. As $\mathbb{C}[\tilde{X}]^K$ is a finite extension of $\mathbb{C}[X]^K$ of degree 1 and $\mathrm{Quot}(\mathbb{C}[\tilde{X}]^K) = \mathbb{C}(\tilde{X})^K = \mathbb{C}(X)^K$, we have

$$\mathrm{Quot}(\mathbb{C}[X]^K) = \mathbb{C}(X)^K.$$

□

The closure of any G -orbit in \mathfrak{g}^* is a strongly equidefectinal affine irreducible Hamiltonian K -variety [Lo, Corollary 3.4.1].

Theorem 3.4. Let $\mathcal{Z} \subset \mathfrak{g}^*$ and $\mathcal{Z}' \subset \mathfrak{g}^*$ be nilpotent G -orbits such that $\mathcal{Z}' \subset \overline{\mathcal{Z}}$. If \mathcal{Z} is K -coisotropic then \mathcal{Z}' is K -coisotropic.

Proof. As K -action on \mathcal{Z} is K -coisotropic, $\mathbb{C}(\mathcal{Z})^K$ is a Poisson-commutative subfield of $\mathbb{C}(\mathcal{Z})$ [Vi, Ch. II, Prop. 5]. By Theorem 3.3 this is equivalent to the Poisson-commutativity of $\mathbb{C}[\mathcal{Z}]^K$. As $\mathbb{C}[\mathcal{Z}']^K$ is a quotient of $\mathbb{C}[\mathcal{Z}]^K$, $\mathbb{C}[\mathcal{Z}']^K$ is Poisson-commutative. Then the field $\mathbb{C}(\mathcal{Z}')^K$ is Poisson-commutative too. Therefore the K -action on \mathcal{Z}' is K -coisotropic. □

Proof of Proposition 3.15. The sphericity of a K -variety X is equivalent to the K -coisotropicity of the K -variety T^*X . Therefore the statement follows from Theorem 3.4. □

Corollary 3.17. Suppose $\text{Fl}(n_1, \dots, n_s; V)$ is a K -spherical variety. Then the variety $\mathbb{P}(V)$ is K -spherical.

The following Lemma becomes a useful tool in Section 6 and is a trivial corollary of Proposition 3.15.

Lemma 3.18. If some partial W -flag variety, which is not cotangent-equivalent to $\mathbb{P}(W)$, is K -spherical, then the variety $\text{Gr}(2; W)$ is K -spherical.

3.4. D-modules versus \mathfrak{g} -modules. Let $\mathfrak{h}_{\mathfrak{g}} \subset \mathfrak{g}$ be a Cartan subalgebra of \mathfrak{g} ; $\Delta \subset \mathfrak{h}_{\mathfrak{g}}$ be a root system; $\Delta^+ \subset \Delta$ be a set of positive roots. Denote by $\hat{\mathfrak{h}}^*$ the set of weights λ such that $\alpha^\vee(\lambda)$ is not a strictly positive integer for any root α^\vee of the dual root system $\Delta^\vee \subset \mathfrak{h}_{\mathfrak{g}}^*$.

For a fixed λ we denote by $\mathcal{D}^\lambda(X)$ the sheaf of twisted differential operators on X and by $D^\lambda(X)$ its space of global sections. The algebras $D^\lambda(X)$ and $D^\mu(X)$ are naturally identified if λ and μ lie in a single shifted orbit of the Weyl group [HMSW]. Moreover, any such orbit intersects $\hat{\mathfrak{h}}^*$ [HMSW]. If $\lambda \in \hat{\mathfrak{h}}^*$, the isomorphism

$$\tau : U(\mathfrak{g})/(\text{Ker}\chi_\lambda) \xrightarrow{\sim} D^\lambda(X)$$

established in [BeBe2] enables us to identify the category of $D^\lambda(X)$ -modules with the category of \mathfrak{g} -modules affording the central character $\chi = \chi_\lambda$. We have the following functors:

$$\begin{array}{l|l} \text{GSec: Sheaves}_X^\lambda \longrightarrow \mathfrak{g}\text{-modules}^\chi & \text{Loc: Sheaves}_X^\lambda \longleftarrow \mathfrak{g}\text{-modules}^\chi \\ \mathcal{F} \rightarrow \Gamma(X, \mathcal{F}) & M \otimes_{(1 \otimes \tau)U(\mathfrak{g})} \mathcal{D}(X) \longleftarrow M \quad ; \end{array}$$

A. Beilinson and J. Bernstein [BeBe2] have proved that if $\lambda \in \hat{\mathfrak{h}}^*$ then $\text{GSec}(\text{Loc})$ equals the identity.

For $\lambda \in \mathfrak{h}_{\mathfrak{g}}^*$ the sheaf $\mathcal{D}^\lambda(X)$ has a natural filtration by degree

$$0 \subset \mathcal{O}(X) \subset \mathcal{D}_1 \subset \dots \subset \mathcal{D}(X) = \varinjlim_{i \in \mathbb{Z}_{\geq 0}} \mathcal{D}_i.$$

The relative spectrum of the associated graded sheaf of algebras

$$\text{gr } \mathcal{D}^\lambda(X) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} (\mathcal{D}_{i+1}/\mathcal{D}_i)$$

is isomorphic to T^*X .

Let \mathcal{M} be a quasicoherent $\mathcal{D}^\lambda(X)$ -module which admits $\mathcal{O}(X)$ -coherent generating subsheaf \mathcal{M}_{gen} of \mathcal{M} . The associated graded sheaf $\text{gr}\mathcal{M}$ with respect to the filtration

$$0 \subset \mathcal{M}_{gen} \subset \mathcal{D}_1\mathcal{M}_{gen} \subset \dots \subset \mathcal{M}$$

is a $\text{gr}\mathcal{D}^\lambda(X)$ -module. By definition, the *singular support* $\mathcal{V}(\mathcal{M})$ of \mathcal{M} is the support of $\text{gr}\mathcal{M}$ in $\text{Spec}_X \text{gr}\mathcal{D}^\lambda(X) \cong T^*X$.

Theorem 3.5 (O. Gabber [Gab]). The variety $\mathcal{V}(\mathcal{M})$ is coisotropic in T^*X . In particular

$$\dim \tilde{\mathcal{V}} \geq \dim X$$

for any irreducible component $\tilde{\mathcal{V}} \subset \mathcal{V}(\mathcal{M})$.

Definition 3.19. The $\mathcal{D}^\lambda(X)$ -module \mathcal{M} is called *holonomic* if

$$\dim \mathcal{V}(\mathcal{M}) = \dim X.$$

3.5. Associated varieties of \mathfrak{g} -modules. The algebra $U(\mathfrak{g})$ has a natural filtration by degree

$$0 \subset \mathbb{C} \subset U_1 \subset \dots \subset U(\mathfrak{g}) = \bigcup_{i \in \mathbb{Z}_{\geq 0}} U_i.$$

The associated graded algebra

$$\text{gr } U(\mathfrak{g}) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} (U_{i+1}/U_i)$$

is isomorphic to $S(\mathfrak{g})$. The filtration $\{U_i\}_{i \in \mathbb{Z}_{\geq 0}}$ induces the filtration on any ideal I of $U(\mathfrak{g})$, namely $\{I \cap U_i\}_{i \in \mathbb{Z}_{\geq 0}}$, hence $\text{gr}I \subset S(\mathfrak{g})$ is a well defined ideal. The ideal $\text{gr}I$ of the commutative algebra $S(\mathfrak{g})$ determines the variety

$$V(\text{gr}I) := \{x \in \mathfrak{g}^* \mid f(x) = 0 \text{ for all } f \in \text{gr}I\}.$$

In particular, if $I = \text{Ann}M$ for a \mathfrak{g} -module M , we set $\text{GV}(M) := V(\text{gr}I)$.

Theorem 3.6 ([Jo2]). For a simple \mathfrak{g} -module M the variety $\text{GV}(M)$ is the closure of an orbit Gu , and furthermore $0 \in \overline{Gu}$.

Suppose that M is a $U(\mathfrak{g})$ -module and that a filtration

$$0 \subset M_0 \subset M_1 \subset \dots \subset M = \bigcup_{i \in \mathbb{Z}_{\geq 0}} M_i$$

of vector spaces is given. We say that this filtration is *good* if

$$(1) U_i M_j = M_{i+j}; \quad (2) \dim M_i < \infty \text{ for all } i \in \mathbb{Z}_{\geq 0}.$$

Such a filtration arises from any finite-dimensional space of generators M_0 . The corresponding associated graded object $\text{gr} M = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} M_{i+1}/M_i$ is a module over $\text{gr} U(\mathfrak{g}) \cong S(\mathfrak{g})$, and we set

$$J_M := \{s \in S(\mathfrak{g}) \mid \text{there exists } k \in \mathbb{Z}_{\geq 0} \text{ such that } s^k m = 0 \text{ for all } m \in \text{gr} M\}.$$

In this way we associate to any \mathfrak{g} -module M the variety

$$V(M) := \{x \in \mathfrak{g}^* \mid f(x) = 0 \text{ for all } f \in J_M\}.$$

It is easy to check that the module $\text{gr} M$ depends on the choice of good filtration, but the ideal J_M and the variety $V(M)$ does not. Indeed, let M_0, M'_0 be different generating spaces of M and $\{M_i\}_{i \in \mathbb{Z}_{\geq 0}}, \{M'_i\}_{i \in \mathbb{Z}_{\geq 0}}$ be the corresponding filtrations of M . Then there exist $r, s \in \mathbb{Z}_{\geq 0}$ such that $M_0 \in M'_r$ and $M'_0 \in M_s$. Let $f \in S(\mathfrak{g})$ be an element of the degree d such that

$$f M_i \subset M_{i+d-1} \text{ for all } i \in \mathbb{Z}_{\geq 0}.$$

Then

$$\begin{aligned} f^{r+s+1} M'_0 &\subset f^{r+s+1} M_s \subset \\ M_{s+(d-1)(r+s+1)} &\subset M'_{s+r+(d-1)(r+s+1)} = M'_{(s+r+1)d-1}. \end{aligned}$$

Therefore $J_M^{s+r+1} \subset J'_M$. In the same way $J'_M^{s+r+1} \subset J_M$, and therefore

$$V(M) = V(M').$$

Theorem 3.7 (Bernstein's theorem [KL, p. 118]). Let M be a finitely generated \mathfrak{g} -module. Then $\dim V(M) \geq \frac{1}{2} \dim \text{GV}(M)$.

Let M be a finitely generated \mathfrak{g} -module which affords a generalized central character.

Theorem 3.8 (O. Gabber [Gab]). Let \tilde{V} be an irreducible component of $V(M)$ and \mathcal{Z} be the unique open G -orbit of

$$G\tilde{V} := \{x \in \mathfrak{g}^* \mid x = gv \text{ for some } g \in G \text{ and } v \in \tilde{V}\}.$$

Then $\tilde{V} \cap \mathcal{Z}$ is a coisotropic subvariety of \mathcal{Z} . In particular

$$\dim \tilde{V} \geq \frac{1}{2} \dim G\tilde{V}.$$

Definition 3.20. A simple \mathfrak{g} -module M is called *holonomic* if

$$\dim \tilde{V} = \frac{1}{2} \dim G\tilde{V}$$

for any irreducible component \tilde{V} of $V(M)$.

Corollary 3.21 (S. Fernando [F]). The vector space

$$V(M)^\perp = \{g \in \mathfrak{g} \mid v(g) = 0 \text{ for all } v \in V(M)\}$$

is a Lie algebra and $V(M)$ is a $V(M)^\perp$ -variety.

Theorem 3.9 (S. Fernando [F, Cor. 2.7], V. Kac [Kac2]). Set

$$\mathfrak{g}[M] := \{g \in \mathfrak{g} \mid \dim(\text{span}_{i \in \mathbb{Z}_{\geq 0}} \{g^i m\}) < \infty \text{ for all } m \in M\}.$$

Then $\mathfrak{g}[M]$ is a Lie algebra and $\mathfrak{g}[M] \subset V(M)^\perp$.

Corollary 3.22. For a $(\mathfrak{g}, \mathfrak{k})$ -module M , $V(M) \subset \mathfrak{k}^\perp$ and $V(M)$ is a K -variety.

Let M be a $(\mathfrak{g}, \mathfrak{k})$ -module and M_0 be a \mathfrak{k} -stable finite-dimensional space of generators of M ; $J_M, \text{gr}M$ be the corresponding objects constructed as above. Consider the $S(\mathfrak{g})$ -modules

$$J_M^{-i}\{0\} := \{m \in \text{gr}M \mid j_1 \dots j_i m = 0 \text{ for all } j_1, \dots, j_i \in J_M\}.$$

One can easily see that these modules form an ascending filtration

$$0 \subset J_M^{-1} \subset \dots \subset \text{gr}M$$

such that

$$\bigcup_{i=1}^{\infty} J_M^{-i}\{0\} = \text{gr}M.$$

Since $S(\mathfrak{g})$ is a Nötherian ring, the filtration stabilizes, i.e. $J_M^{-i}\{0\} = \text{gr}M$ for some i . By $\overline{\text{gr}}M$ we denote the corresponding graded object. By definition, $\overline{\text{gr}}M$ is an $S(\mathfrak{g})/J_M$ -module. Suppose that $f\overline{\text{gr}}M = 0$ for some $f \in S(\mathfrak{g})$. Then $f^i \text{gr}M = 0$ and hence $f \in J_M$. This proves that the annihilator of $\overline{\text{gr}}M$ in $S(\mathfrak{g})/J_M$ equals zero.

As M is a finitely generated \mathfrak{g} -module, the $S(\mathfrak{g})$ -modules $\text{gr}M$ and $\overline{\text{gr}}M$ are finitely generated. Let \tilde{M}_0 be a \mathfrak{k} -stable finite-dimensional space of generators of $\overline{\text{gr}}M$. Then there is a surjective homomorphism

$$\psi : \tilde{M}_0 \otimes_{\mathbb{C}} (S(\mathfrak{g})/J_M) \rightarrow \overline{\text{gr}}M.$$

Set

$$\text{Rad}M := \{m \in \overline{\text{gr}}M \mid \text{there exists } f \in S(\mathfrak{g})/J_M \text{ such that } fm = 0 \text{ and } f \neq 0\}.$$

The space $\text{Rad}M$ is a \mathfrak{k} -stable $S(\mathfrak{g})$ -submodule of $\overline{\text{gr}}M$ and $\tilde{M}_0 \not\subset \text{Rad}M$. The homomorphism ψ determines an injective homomorphism

$$\hat{\psi} : S(\mathfrak{g})/J_M \rightarrow \tilde{M}_0^* \otimes_{\mathbb{C}} \overline{\text{gr}}M.$$

Proposition 3.23. a) The module M is bounded if and only if all irreducible components of $V(M)$ are K -spherical.

b) If the equivalent conditions of a) are satisfied, then any irreducible component \tilde{V} of $V(M)$ is a conical Lagrangian subvariety of $G\tilde{V}$.

Proof. a) Suppose that all irreducible components of $V(M)$ are K -spherical. Then

$$\tilde{M}_0 \otimes_{\mathbb{C}} (S(\mathfrak{g})/J_M)$$

is a bounded \mathfrak{k} -module. Therefore $\overline{\text{gr}}M$ is bounded, which implies that M is a bounded \mathfrak{k} -module too.

Assume now that a \mathfrak{g} -module M is \mathfrak{k} -bounded. Then $S(\mathfrak{g})/J_M$ is a \mathfrak{k} -bounded module and all irreducible components of $V(M)$ are K -spherical. This completes the proof of a).

b) Let $\tilde{V} \subset V(M)$ be an irreducible component and $x \in \tilde{V}$ be a generic point. As $x \in \mathfrak{k}^\perp$, we have

$$x([k_1, k_2]) = \omega_x(\tau_{\mathfrak{g}^*} k_1|_x, \tau_{\mathfrak{g}^*} k_2|_x) = 0$$

for all $k_1, k_2 \in \mathfrak{k}$. Therefore any K -orbit in \mathfrak{k}^\perp is isotropic. As \tilde{V} is a spherical variety, \tilde{V} has an open K -orbit. Therefore \tilde{V} is Lagrangian in $G\tilde{V}$ and this completes the proof of b). \square

Corollary 3.24 ([PS]). Let M be a finitely generated bounded $(\mathfrak{g}, \mathfrak{k})$ -module. Then $\dim \mathfrak{b}_{\mathfrak{k}} \geq \frac{1}{2} \dim \text{GV}(M)$.

Proof. As M is bounded, $V(M)$ is K -spherical and therefore $\dim \mathfrak{b}_{\mathfrak{k}} \geq \dim V(M)$. We have $\dim V(M) \geq \frac{1}{2} \dim \text{GV}(M)$. \square

Lemma 3.25 ([VP]). Let \hat{X} be an affine K -variety. Then $\mathbb{C}[X]$ has finite type as a \mathfrak{k} -module if and only if X contains only finite number of the closed K -orbits. In the latter case any irreducible component of X contains precisely one closed K -orbit.

Lemma 3.26. A finitely generated $(\mathfrak{g}, \mathfrak{k})$ -module M has finite type over \mathfrak{k} if and only if the variety $V(M)$ contains only finitely many closed K -orbits. In this case the unique closed orbit of $V(M)$ is the point 0.

Proof. If $V(M)$ contains the unique closed K -orbit then $\mathbb{C}[V(M)]$ is a \mathfrak{k} -module of finite type and therefore $\tilde{M}_0 \otimes_{\mathbb{C}} (S(\mathfrak{g})/J_M)$ and $\overline{\text{gr}}M$ are \mathfrak{k} -modules of finite type.

Assume that M is a \mathfrak{k} -module of finite type. Then $S(\mathfrak{g})/J_M$ is a \mathfrak{k} -module of finite type and $V(M)$ contains only finite number of the closed K -orbits. On the other hand, $V(M)$ is \mathbb{C}^* -stable and hence any irreducible component of $V(M)$ contains 0. \square

3.6. Other faces of the support variety. In this section X is a variety of Borel subalgebras of \mathfrak{g} . We recall the singular support $\mathcal{V}(\mathcal{M}) \subset T^*X$ of any coherent $\mathcal{D}^\lambda(X)$ -module \mathcal{M} is defined. The correspondence between \mathfrak{g} -modules and $\mathcal{D}^\lambda(X)$ -modules allows us to assign interesting geometric objects to a \mathfrak{g} -module.

Let M be a finitely generated \mathfrak{g} -module which affords a central character χ and $\lambda \in \hat{\mathfrak{h}}^*$ (see Subsection 3.4) be a weight such that $\chi = \chi_\lambda$.

Definition 3.27. The *singular support* of M is the variety $\mathcal{V}(M) := \mathcal{V}(\text{Loc}M)$.

Definition 3.28. The *support variety* $L(M)$ of M is the projection of $\mathcal{V}(M)$ to X .

Let $\phi_X : T^*X \rightarrow \mathfrak{g}^*$ be the moment map. D. Barlet and M. Kashiwara [BK], have proved that

$$V(M) = \phi_X(\mathcal{V}(\text{Loc}M))$$

(see also [BoBry]). Therefore we have a diagram

$$\begin{array}{ccc} & \mathcal{V}(M) \subset T^*X & \\ \swarrow \phi_X & & \searrow \text{pr} \\ V(M) \subset \mathfrak{g}^* & & L(M) \subset X \end{array} .$$

3.7. Hilbert-Mumford criterion. Let X be an affine K -variety, V be a K -module.

Theorem 3.10 (Hilbert-Mumford [VP]). The closure of any K -orbit $\overline{Kx} \subset X$ contains a unique closed orbit $K\bar{x} \subset X$. There exists a group homomorphism $\mu : \mathbb{C}^* \rightarrow K$ such that $\lim_{t \rightarrow 0} \mu(t)x = \bar{x} \in K\bar{x}$.

The *null cone* $N_K(V) := \{x \in V \mid 0 \in \overline{Kx}\}$ is a closed algebraic subvariety of V [VP].

Theorem 3.11 ([VP]). Fix $x \in V$. Then $0 \in \overline{Kx}$ if and only if there exists a rational semisimple element $h \in \mathfrak{k}$ such that $x \in V_h^{>0}$; here $V_h^{>0}$ is the direct sum of h -eigenspaces in V with positive eigenvalues.

Corollary 3.29 ([VP]). There exists a finite set $H \subset \mathfrak{k}$ of rational semisimple elements such that $N_K(V) := \cup_{h \in H} KV_h^{>0}$, where

$$KV_h^{>0} := \{v \in V \mid v = kv_h \text{ for some } k \in K \text{ and } v_h \in V_h^{>0}\}.$$

3.8. A monoid of projective functors. Let $\mathfrak{b}_{\mathfrak{g}}$ be a Borel subalgebra of \mathfrak{g} and let $\mathfrak{h}_{\mathfrak{g}}$ be a Cartan subalgebra of $\mathfrak{b}_{\mathfrak{g}}$. Let $\Delta \subset \mathfrak{h}_{\mathfrak{g}}^*$ be the root system of \mathfrak{g} , $\Delta^+ \subset \Delta$ be the set of positive roots of $\mathfrak{b}_{\mathfrak{g}}$, $\Pi \subset \Delta^+$ be the set of simple roots. We denote by s_{α} the reflection of $\mathfrak{h}_{\mathfrak{g}}^*$ with respect to α for $\alpha \in \Delta$, and $W^{\mathfrak{g}}$ is the group generated by $\{s_{\alpha}\}_{\alpha \in \Delta}$, i.e. the Weyl group of \mathfrak{g} . Let $\alpha^{\vee} \in \mathfrak{h}_{\mathfrak{g}}$ be the coroot of $\alpha \in \Delta$, and

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

We introduce a partial order on $\mathfrak{h}_{\mathfrak{g}}^*$ (cf. [BeG, 1.5]). If $\phi, \psi \in \mathfrak{h}_{\mathfrak{g}}^*$, $\gamma \in \Delta^+$ we put $\phi <^{\gamma} \psi$ whenever $\phi = s_{\gamma} \psi$ and $\gamma^{\vee}(\psi) \in \mathbb{Z}_{>0}$. We put $\phi < \psi$ whenever there exists sequences of weights ϕ_0, \dots, ϕ_n and of roots $\gamma_0, \dots, \gamma_n$ such that

$$\phi <^{\gamma_0} \phi_0 < \dots <^{\gamma_n} \phi_n = \psi.$$

We write $\phi \leq \psi$ whenever $\phi = \psi \vee \phi < \psi$. A weight is called *dominant* if it is maximal with respect to partial order $<$.

We denote by $W_\phi^{\mathfrak{g}}$ the stabilizer in $W^{\mathfrak{g}}$ of $\phi \in \mathfrak{h}_{\mathfrak{g}}^*$. By definition, $\phi \in \mathfrak{h}_{\mathfrak{g}}^*$ *integral* if $\alpha^\vee(\phi) \in \mathbb{Z}$ for any $\alpha \in \Delta$. Furthermore, $\phi \in \mathfrak{h}_{\mathfrak{g}}^*$ is *regular* if $W_\phi^{\mathfrak{g}} = \{e\}$. We call a pair of weights $(\phi, \psi) \in \mathfrak{h}_{\mathfrak{g}}^* \times \mathfrak{h}_{\mathfrak{g}}^*$ *correctly ordered* if ϕ is dominant, $\phi - \psi$ is an integral weight and $\psi \leq w\psi$ for all $w \in W_\phi^{\mathfrak{g}}$. In addition, we define two weights ϕ_1, ϕ_2 to be *equivalent* if $\phi_1 - \phi_2$ is integral and $W_{\phi_1}^{\mathfrak{g}} = W_{\phi_2}^{\mathfrak{g}}$.

Let \mathcal{O} be the category of finitely generated \mathfrak{g} -modules such that the action of $\mathfrak{b}_{\mathfrak{g}}$ is locally finite and the action of $\mathfrak{h}_{\mathfrak{g}}$ is semisimple. Let $\lambda \in \mathfrak{h}_{\mathfrak{g}}^*$ be a weight. We denote by M_λ the Verma module with the highest weight $\lambda - \rho$, and by L_λ the unique simple quotient of M_λ . Both M_λ and L_λ are objects of the category \mathcal{O} . We denote by P_λ a minimal projective cover of L_λ in \mathcal{O} . If λ is dominant, then $M_\lambda \simeq P_\lambda$.

The modules $M_\lambda, L_\lambda, P_\lambda$ afford the same generalized central character and we denote this central character by χ_λ . Let I be a primitive ideal of $U(\mathfrak{g})$. By a famous theorem of Duflo [Dix], $I = \text{Ann } L_\lambda$ for some weight $\lambda \in \mathfrak{h}_{\mathfrak{g}}^*$.

Let χ be a central character. The set of weights λ such that M_λ are annihilated by $\text{Ker } \chi$ is nonempty and we denote this set $W^{\mathfrak{g}}(\chi)$. The set $W^{\mathfrak{g}}(\chi)$ is an orbit of $W^{\mathfrak{g}}$. Therefore we can identify the set of central characters with the set of $W^{\mathfrak{g}}$ -orbits in $\mathfrak{h}_{\mathfrak{g}}^*$.

The Weyl group $W^{\mathfrak{g}}$ acts on $\mathfrak{h}_{\mathfrak{g}}^* \times \mathfrak{h}_{\mathfrak{g}}^*$:

$$w((\lambda_1, \lambda_2)) \mapsto (w(\lambda_1), w(\lambda_2)).$$

Any $W^{\mathfrak{g}}$ -orbit in $\mathfrak{h}_{\mathfrak{g}}^* \times \mathfrak{h}_{\mathfrak{g}}^*$ contains a correctly ordered representative.

Let E be a finite-dimensional \mathfrak{g} -module. We have the functor

$$F_E : \mathfrak{g}\text{-mod} \rightarrow \mathfrak{g}\text{-mod}, \quad M \mapsto E \otimes M.$$

The restriction of F_E to the category of \mathfrak{g} -modules affording a given generalized central character χ is a direct sum of a finite number of indecomposable exact functors. Following [BeG], we call a direct summand of F_E a *projective functor*.

The set of indecomposable projective functors arising from all finite-dimensional \mathfrak{g} -modules E is naturally identified with the set of $W^{\mathfrak{g}}$ -orbits in $\mathfrak{h}_{\mathfrak{g}}^* \times \mathfrak{h}_{\mathfrak{g}}^*$ [BeG, 3.3]. Let $(\phi, \psi) \in \mathfrak{h}_{\mathfrak{g}}^* \times \mathfrak{h}_{\mathfrak{g}}^*$ be a correctly ordered pair and \mathcal{H}_ϕ^ψ be the corresponding projective functor. Then

$$\mathcal{H}_\phi^\psi M_\phi = P_\psi$$

and the functors \mathcal{H}_ϕ^ψ and \mathcal{H}_ψ^ϕ are adjoint one to each other. Assume that $\phi - \psi$ is integral and $W_\phi^{\mathfrak{g}} = W_\psi^{\mathfrak{g}}$. Then

$$(\mathcal{H}_\phi^\psi, \mathcal{H}_\psi^\phi)$$

is a pair of mutually inverse equivalences of the respective categories of \mathfrak{g} -modules affording the generalized central characters χ_ϕ and χ_ψ .

Let $\text{PFunc}(\chi_\lambda)$ be the set of projective functors as above which preserve the category of \mathfrak{g} -modules affording the generalized central character χ_λ . Obviously $\text{PFunc}(\chi_\lambda)$ carries an additive structure and is a monoid with respect to it. We denote by $\text{PF}\overline{\text{unc}}(\chi_\lambda)$ the minimal abelian group which contains $\text{PFunc}(\chi_\lambda)$. As the composition of projective functors is a projective functor, $\text{PF}\overline{\text{unc}}(\chi_\lambda)$ carries a multiplicative structure. Therefore $\text{PF}\overline{\text{unc}}(\chi_\lambda)$ is a unitary ring.

Proposition 3.30. Let λ be a regular dominant integral weight. Then the ring $\text{PF}\overline{\text{unc}}(\chi_\lambda)$ is isomorphic to the ring $\mathbb{Z}[\mathbb{W}^\mathfrak{g}]$.

Proof. Let \mathcal{O}_λ be the subcategory of \mathcal{O} consisting of \mathfrak{g} -modules which afford the generalized central character χ_λ . Let $[\mathcal{O}]_\mathfrak{g}$ be the free abelian group generated by the isomorphism classes of simple \mathfrak{g} -modules of the category \mathcal{O} , and let $[\mathcal{O}_\lambda]_\mathfrak{g}$ be the subgroup generated by the simple objects of \mathcal{O}_λ . Any object M of \mathcal{O} has finite length and therefore determines $[M] \in [\mathcal{O}]_\mathfrak{g}$.

Let $\mathcal{H}_1, \mathcal{H}_2 \in \text{PFunc}(\chi_\lambda)$. Then $\mathcal{H}_1 = \mathcal{H}_2$ if and only if

$$[\mathcal{H}_1(M_\lambda)] = [\mathcal{H}_2(M_\lambda)].$$

For any $w \in \mathbb{W}^\mathfrak{g}$, we have $\mathcal{H}_\lambda^{w\lambda} \in \text{PFunc}(\chi_\lambda)$ and $\mathcal{H}_\lambda^{w\lambda} M_\lambda = P_{w\lambda}$. The elements $\{[M_{w\lambda}]\}_{w \in \mathbb{W}^\mathfrak{g}}$ form a basis of the free abelian group $[\mathcal{O}_\lambda]_\mathfrak{g}$, and therefore $[\mathcal{O}_\lambda]_\mathfrak{g}$ is identified with $\mathbb{Z}^{\mathbb{W}^\mathfrak{g}}$ as a set. We introduce an action of $\mathbb{W}^\mathfrak{g}$ on $[\mathcal{O}_\lambda]_\mathfrak{g}$ via the formula

$$(w, [M_{w'\lambda}]) \mapsto [M_{ww'\lambda}] \text{ for any } w', w \in \mathbb{W}^\mathfrak{g}.$$

Any functor $\mathcal{H} \in \text{PFunc}(\chi_\lambda)$ commutes with this action. As $\{[P_{w\lambda}]\}_{w \in \mathbb{W}^\mathfrak{g}}$ is a basis of the free abelian group $[\mathcal{O}_\lambda]_\mathfrak{g}$, the map

$$\text{PF}\overline{\text{unc}}(\chi_\lambda) \rightarrow \mathbb{Z}[\mathbb{W}^\mathfrak{g}] = \mathbb{Z}^{\mathbb{W}^\mathfrak{g}},$$

given by $\mathcal{H} \mapsto [\mathcal{H}(M_\lambda)]$ on generators, is an isomorphism. \square

Theorem 3.12 ([BeG]). Let λ_1, λ_2 be dominant weights such that the difference $\lambda_1 - \lambda_2$ is integral and $\mathbb{W}_{\lambda_1}^\mathfrak{g} = \mathbb{W}_{\lambda_2}^\mathfrak{g}$, i.e. λ_1 is equivalent to λ_2 . Then

- a) the categories of \mathfrak{g} -modules which afford the generalized central characters χ_{λ_1} and χ_{λ_2} are equivalent;
- b) for a given $w \in \mathbb{W}^\mathfrak{g}$, the categories of \mathfrak{g} -modules annihilated by $\text{Ann } L_{w\lambda_1}$ and $\text{Ann } L_{w\lambda_2}$ are equivalent.

The subcategories of locally finite \mathfrak{k} -modules and bounded \mathfrak{k} -modules are stable under this equivalence.

3.9. Bounded weight modules. In the joint work [PS] I. Penkov and V. Serganova proved that 'boundness' is not a property of a module M but of the ideal $\text{Ann}M$. More precisely, this means the following.

Theorem 3.13. Let M and N be simple $(\mathfrak{g}, \mathfrak{k})$ -modules such that $\text{Ann}M = \text{Ann}N$. Suppose that the function $[M : \cdot]_{\mathfrak{k}}$ is uniformly bounded by a constant C_M . Then the function $[N : \cdot]_{\mathfrak{k}}$ is uniformly bounded by C_M (see also Theorem 2.5).

To classify bounded modules M we should first classify 'bounded ideals' $\text{Ann}M$, i.e. ideals for which exist at least one bounded module. It is well known that the two-sided ideals of $U(\mathfrak{g})$ containing a fixed maximal ideal in $Z(\mathfrak{g})$ are closely related to category \mathcal{O} [BeG]. Therefore one may expect that a classification of bounded ideals has something in common with the classification of Verma modules L_λ whose weight multiplicities are bounded.

We fix the following notation:

- $\{e_i\}_{i \leq n_W} \subset W$ is a basis of W , $\{e_i^*\}_{i \leq n_W} \subset W^*$ is the dual basis;
- $\{e_{i,j}\}_{i \leq n_W, j \leq n_W} \subset \mathfrak{gl}(W)$ stands for the elementary matrix $e_i \otimes e_j^*$;
- $\varepsilon_i := e_{i,i} - \frac{1}{n_W}(e_{1,1} + e_{2,2} + \dots + e_{n_W, n_W})$ for $i \leq n_W$;
- $\mathfrak{h}_W := \text{span}\langle e_{i,i} \rangle_{i \leq n_W}$.

3.9.1. *The case $\mathfrak{g} = \mathfrak{sl}(W)$.* We identify all weights of $\mathfrak{h}_W^{\mathfrak{sl}} := \mathfrak{h}_W \cap \mathfrak{sl}(W)$ with the set of n_W -tuples modulo the equivalence relation

$$(\lambda_1, \lambda_2, \dots, \lambda_{n_W}) \leftrightarrow (\lambda_1 + k, \lambda_2 + k, \dots, \lambda_{n_W} + k).$$

We fix the set of positive roots of $\mathfrak{sl}(W)$ in $\mathfrak{h}_W^{\mathfrak{sl}}$ for which ρ is identified with $(n_W, \dots, 1)$ and ε_i is identified with the n_W -tuple (i zeros, 1 , $n_W - i - 1$ zeros). Let $\bar{\lambda}$ be an n_W -tuple and λ be the corresponding weight. We put $I(\lambda) := \text{Ann} L_\lambda$. Duflo's theorem [Dix] claims now that, if I is any primitive ideal of $U(\mathfrak{sl}(W))$, then

$$I = I(\lambda)$$

for some n_W -tuple $\bar{\lambda}$.

Theorem 3.14 ([M]). The $(\mathfrak{sl}(W), \mathfrak{h}_W^{\mathfrak{sl}})$ -module L_λ is $\mathfrak{h}_W^{\mathfrak{sl}}$ -bounded if and only if λ is a semi-decreasing tuple.

The bounded $(\mathfrak{sl}(W), \mathfrak{h}_W^{\mathfrak{sl}})$ -modules split into coherent families [M], and all simple modules from one coherent family share an annihilator. Let $\bar{\lambda}$ be a semi-decreasing tuple. Assume that $\bar{\lambda}$ is not regular integral, i.e. it is singular integral or semi-integral. Then there exists precisely one coherent family annihilated by $\text{Ker} \chi_\lambda$ [M]. Therefore all simple bounded $(\mathfrak{sl}(W), \mathfrak{h}_W^{\mathfrak{sl}})$ -modules which afford the central character χ_λ have the same annihilator.

V. Serganova and D. Grantcharov [GrS1] note that the lemma below follows from the results of O. Mathieu [M]. See also Theorem 3.12.

Lemma 3.31. Assume $\bar{\lambda}$ is a semi-decreasing tuple and that $\bar{\lambda}$ is not regular integral. Then there exists $s \in \mathbb{C}$ such that

$$e^{2\pi i s} = m(\lambda)$$

and the categories of $\mathfrak{sl}(W)$ -modules annihilated by

$$I(\lambda) \text{ and } I(s, n_W - 1, \dots, 1)$$

are equivalent.

The subcategories of locally finite \mathfrak{k} -modules and bounded \mathfrak{k} -modules are stable under this equivalence.

Assume that $\bar{\lambda}$ is regular integral. Let $\text{ord}(\bar{\lambda})$ be a decreasing n_W -tuple which coincides as a set with $\bar{\lambda}$. For $k \in \{1, \dots, n_W - 1\}$ we denote by s_k the permutation $(k, k + 1) \in S_{n_W}$. There exist precisely $n_W - 1$ different coherent families annihilated by $\text{Ker} \chi_\lambda$ and any such family contains $L_{s_k \text{ord}(\lambda)}$ for some $k \in \{1, \dots, n_W - 1\}$. Therefore

$$I(\lambda) = I(s_k \text{ord}(\lambda))$$

for some $k \in \{1, \dots, n_W - 1\}$ [M].

Consider the Weyl algebra of differential operators $D(W)$ on W . This algebra is generated by e_i and ∂_{e_i} for $i \leq n_W$. Set

$$E := e_1 \partial_{e_1} + \dots + e_{n_W} \partial_{e_{n_W}}.$$

The operator $[E, \cdot]: D(W) \rightarrow D(W)$ is semisimple, and $D(W)$ splits into a direct sum of eigenspaces

$$\{D^i(W)\}_{i \in \mathbb{Z}}$$

with respect to this operator.

The homomorphism of Lie algebras

$$\mathfrak{gl}(W) \rightarrow D^0(W), \quad e_{i,j} \mapsto e_i \partial_{e_j}$$

induces a surjective homomorphism

$$\phi : U(\mathfrak{gl}(W)) \rightarrow D^0(W).$$

The element $E - t$ generates a two-sided ideal in $D^0(W)$. We denote the corresponding quotient by $D^t \mathbb{P}(W)$. Furthermore, ϕ induces a surjective homomorphism

$$\phi_t : U(\mathfrak{sl}(W)) \rightarrow D^t \mathbb{P}(W)$$

and

$$\text{Ker} \phi_t = I(t, n_W - 1, \dots, 1),$$

see [M].

3.9.2. *The case $\mathfrak{g} = \mathfrak{sp}(W \oplus W^*)$.* We note that \mathfrak{h}_W is a Cartan subalgebra of $\mathfrak{sp}(W \oplus W^*)$. We identify the weights \mathfrak{h}_W^* with n_W -tuples. We fix the set of positive roots of $\mathfrak{sp}(W \oplus W^*)$ in \mathfrak{h}_W for which ρ is identified with $(n_W, \dots, 1)$. Let $\bar{\mu}$ be an n_W -tuple and μ be the corresponding weight. We denote $\text{Ann } L_\mu$ by $I_{\mathfrak{sp}}(\mu)$. Let I be a primitive ideal of $U(\mathfrak{sp}(W \oplus W^*))$. Then $I = I_{\mathfrak{sp}}(\mu)$ for some n_W -tuple $\bar{\mu}$.

Theorem 3.15 ([M]). The $(\mathfrak{sp}(W \oplus W^*), \mathfrak{h}_W)$ -module L_μ is \mathfrak{h}_W -bounded if and only if $\bar{\mu}$ is a Shale-Weil tuple.

The bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{h}_W)$ -modules split into coherent families [M] and all simple modules from one coherent family share an annihilator. Let $\bar{\mu}$ be a Shale-Weil tuple. There exists precisely one coherent family annihilated by $\text{Ker } \chi_\mu$ [M]. Therefore all simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{h}_W)$ -modules affording the central character χ_μ have the same annihilator. The modules L_μ and $L_{\sigma\mu}$ belong to the same coherent family. Recall that μ_0 is the n_W -tuple $(n_W - \frac{1}{2}, n_W - \frac{3}{2}, \dots, \frac{1}{2})$.

Lemma 3.32. Let $\bar{\mu}$ be a Shale-Weil n_W -tuple. Then the categories of $U(\mathfrak{sp}(W \oplus W^*))$ -modules annihilated by

$$I_{\mathfrak{sp}}(\mu) \text{ and } I_{\mathfrak{sp}}(\mu_0)$$

are equivalent.

Proof. This is a simple corollary of Theorem 3.12 b). \square

The subcategories of locally finite \mathfrak{k} -modules and bounded \mathfrak{k} -modules are stable under the equivalence of Lemma 3.32.

The spaces $\{D^i(W)\}_{i \in \mathbb{Z}}$ define a \mathbb{Z} -grading of an algebra $D(W)$. Therefore the spaces

$$\{\oplus_{i \in \mathbb{Z}} D^{2i}(W), \oplus_{i \in \mathbb{Z}} D^{2i+1}(W)\}$$

define a \mathbb{Z}_2 -grading of the algebra $D(W)$. We set

$$D^{\bar{0}}(W) := \oplus_{i \in \mathbb{Z}} D^{2i}(W), \quad D^{\bar{1}}(W) := \oplus_{i \in \mathbb{Z}} D^{2i+1}(W).$$

The space $\text{span} \langle e_i e_j, e_i \partial_{e_j}, \partial_{e_i} \partial_{e_j}, 1 \rangle_{i, j \leq n_W}$ is a Lie algebra with respect to commutator. This Lie algebra is isomorphic to $\mathfrak{sp}(W \oplus W^*) \oplus \mathbb{C}$. We note that $\text{span} \langle e_i \partial_{e_j} \rangle_{i, j \leq n_W}$ is a Lie subalgebra and is isomorphic to $\mathfrak{gl}(W)$. This defines a homomorphism of associative algebras

$$\phi_{\mathfrak{sp}} : U(\mathfrak{sp}(W \oplus W^*)) \rightarrow D(W),$$

and $I(\mu_0) = \text{Ker } \phi_{\mathfrak{sp}}$, $D^{\bar{0}}(W) = \text{Im } \phi_{\mathfrak{sp}}$.

3.10. A lemma on S_{n_W} -modules. The set of isomorphism classes of simple S_{n_W} -modules is naturally identified with the set of partitions

$$m_1 \geq m_2 \geq \dots \geq m_s (m_1 + m_2 + \dots + m_s = n_W)$$

of n_W . Let \mathbb{C}^{n_W-1} be the natural $(n_W - 1)$ -dimensional module of S_{n_W} and \mathbb{C} be the trivial one-dimensional module. Recall that s_i is the simple transposition $(i, i + 1)$. Let R be an S_{n_W} -module. For any $i \in \{1, \dots, n_W - 1\}$ we denote by $R_{\bar{i}}$ the vector space

$$\{v \in R \mid s_j v = v \text{ for all } j \neq i\}.$$

Lemma 3.33. Suppose that R is an S_{n_W} -module such that

$$R = +_{i \leq n_W-1} R_{\bar{i}}.$$

Then R is a direct sum of several copies of \mathbb{C}^{n_W-1} and \mathbb{C} .

Proof. Without loss of generality we assume that R is irreducible. If $R_{\bar{1}}$ equals zero, the sum $+_{i \leq n_W-1} R_{\bar{i}}$ is s_1 -invariant and therefore $R \cong \mathbb{C}$. If $R_{\bar{1}} \neq 0$, the restriction of R to S_{n_W-1} contains \mathbb{C} as a simple submodule. Therefore R is

$$\mathbb{C}^{n_W-1}, \text{ or } \mathbb{C},$$

see [FH]. □

3.11. Regular singularities. Let C be a smooth connected affine curve. We recall that $\mathbb{C}[C]$ is the algebra of regular functions of C and $D(C)$ is the algebra of regular differential operators on C . Let $(x) \subset \mathbb{C}[C]$ be a maximal ideal and $x \in C$ be the corresponding point. Set

$$D_x^{\geq 0}(C) := \{D \in D(C) \mid Df \subset (x) \text{ for all } f \in (x)\}.$$

We say that a $D(C)$ -module F has *regular singularities at x* if for any finite-dimensional space $F_0 \subset F$, $D_x^{\geq 0}(C)F_0$ is a finitely generated $\mathbb{C}[C]$ -module.

Let C^+ be a unique compact connected curve which contains C as an open subset. We say that a $D(C)$ -module F has *regular singularities* if for any point $x \in C^+$ there exists an affine open set $C(x) \subset C^+$ such that $x \in C(x)$ and $F \otimes_{\mathbb{C}[C]} \mathbb{C}[C \cap C(x)]$ considered as a $D(C(x))$ -module by extension from $C \cap C(x)$ to $C(x)$ has regular singularities at x .

Definition 3.34. Let X be a smooth variety and \mathcal{F} be a coherent $\mathcal{D}(X)$ -module. We say that \mathcal{F} has *regular singularities* if \mathcal{F} has regular singularities after restriction to any smooth connected affine curve.

4. HOLONOMICITY OF $(\mathfrak{g}, \mathfrak{k})$ -MODULES OF FINITE TYPE

We are now going to prove Theorem 2.2 stated in Section 2.

Proof of Theorem 2.2. Without loss of generality we assume that \mathfrak{g} is semisimple. We identify \mathfrak{g} and \mathfrak{g}^* by use of the Killing form. Let $h \in \mathfrak{k}$ be a rational semisimple element. We denote by \mathfrak{g}_h the direct sum of h -eigenspaces of \mathfrak{g} with nonnegative eigenvalues; by $G_h \subset G$ the parabolic subgroup with the Lie algebra \mathfrak{g}_h . Set $Z_G := G/G_h$. In a similar way we define \mathfrak{k}_h, K_h, Z_K . We denote by \mathfrak{n}_h the nilpotent radical of \mathfrak{g}_h . Let $e \in K$ be the unit element. The orbit $Ke \subset Z_G$ is isomorphic to Z_K . Put

- $G\mathfrak{n}_h := \{x \in \mathfrak{g} \mid x = gn \text{ for some } n \in \mathfrak{n}_h, g \in G\}$,
- $K\mathfrak{n}_h := \{x \in \mathfrak{g} \mid x = kn \text{ for some } n \in \mathfrak{n}_h, k \in K\}$,
- $K\mathfrak{n}_h \cap \mathfrak{k}^\perp := \{x \in \mathfrak{g} \mid x = kn \text{ for some } n \in \mathfrak{n}_h \cap \mathfrak{k}^\perp, k \in K\}$.

Let $\phi : T^*Z_G \rightarrow \mathfrak{g}^*$ be the moment map. It is easy to see that $G\mathfrak{n}_h$ coincides with $\phi(T^*Z_G)$, $K\mathfrak{n}_h$ coincides with $\phi(T^*Z_G|_{Z_K})$, $K\mathfrak{n}_h \cap \mathfrak{k}^\perp$ coincides with $\phi(N_{Z_K/Z_G}^*)$:

$$\begin{array}{ccccc} T^*Z_G & \longleftarrow & T^*Z_G|_{Z_K} & \longleftarrow & N_{Z_K/Z_G}^* \\ \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ G\mathfrak{n}_h & \longleftarrow & K\mathfrak{n}_h & \longleftarrow & K\mathfrak{n}_h \cap \mathfrak{k}^\perp \end{array} .$$

As the variety N_{Z_K/Z_G}^* is isotropic in T^*Z_G , the image $\phi(N_{Z_K/Z_G}^*)$ is isotropic in $G(\mathfrak{n}_h \cap \mathfrak{k}^\perp)$ and any subvariety $\tilde{V} \subset K\mathfrak{n}_h \cap \mathfrak{k}^\perp$ is isotropic in $G\tilde{V}$. Therefore by Corollary 3.29 any subvariety $\tilde{V} \subset \mathfrak{k}^\perp \cap N_K(\mathfrak{g}^*)$ is isotropic in $G\tilde{V}$. \square

We introduce the following notation:

- $V_{\mathfrak{g}, \mathfrak{k}}$ is the set of all irreducible components of intersections of $N_K(\mathfrak{k}^\perp)$ with all possible G -orbits of $N_G(\mathfrak{g}^*)$.
- $\mathcal{V}_{\mathfrak{g}, \mathfrak{k}}$ is the set of all possible irreducible components of the preimages of elements of $V_{\mathfrak{g}, \mathfrak{k}}$ under the moment map $T^*X \rightarrow \mathfrak{g}^*$.
- $L_{\mathfrak{g}, \mathfrak{k}}$ is the set of images in X of all elements of $\mathcal{V}_{\mathfrak{g}, \mathfrak{k}}$.

Let M be a finitely generated \mathfrak{g} -module which affords a central character χ and $\lambda \in \hat{\mathfrak{h}}^*$ (see Subsection 3.4) be a weight such that $\chi = \chi_\lambda$.

Theorem 4.1. If M is a $(\mathfrak{g}, \mathfrak{k})$ -module of finite type, then

- a) the irreducible components of $V(M)$ are contained in $V_{\mathfrak{g}, \mathfrak{k}}$; the irreducible components of $\mathcal{V}(M)$ are contained in $\mathcal{V}_{\mathfrak{g}, \mathfrak{k}}$, the irreducible components of $L(M)$ are contained in $L_{\mathfrak{g}, \mathfrak{k}}$.
- b) The module $\text{Loc}M$ is holonomic. If M is simple, M is holonomic.

Proof. Let $\tilde{V} \subset V(M)$ be an irreducible component and \mathcal{Z} be the closure of $G\tilde{V}$ in \mathfrak{g}^* . By Theorem 3.8 the variety \tilde{V} is coisotropic. On the other hand

$$\tilde{V} \subset N_K(\mathfrak{k}^\perp) \cap \mathcal{Z},$$

and therefore \tilde{V} is isotropic. Hence \tilde{V} is Lagrangian and is an irreducible component of $\mathcal{Z} \cap N_K(\mathfrak{g}^*) \cap \mathfrak{k}^\perp$.

As intersections of $V(M)$ with any irreducible component of M are isotropic,

$$\dim \mathcal{V}(\text{Loc}M) \leq \dim X.$$

Therefore $\text{Loc}M$ is holonomic.

Assume that M is simple. As all irreducible components of $V(M)$ are Lagrangian, M is holonomic. \square

Corollary 4.1. The statements of Theorem 2.1 and Theorem 2.3 follow from Theorem 4.1.

Corollary 4.2. Let M be a simple $(\mathfrak{g}, \mathfrak{k})$ -module of finite type and \tilde{V} be an irreducible component of $V(M)$. Then

$$\dim \tilde{V} = \frac{1}{2} \dim \text{GV}(M) \text{ and } G\tilde{V} = \text{GV}(M).$$

Proof. As \tilde{V} is Lagrangian in $G\tilde{V}$,

$$\dim \tilde{V} = \frac{1}{2} \dim G\tilde{V}$$

(see also [GL]). On the other hand $G\tilde{V} \subset \text{GV}(M)$, and

$$\dim \tilde{V} \geq \frac{1}{2} \dim \text{GV}(M).$$

As $\text{GV}(M)$ is irreducible, $\text{GV}(M)$ coincides with $G\tilde{V}$. \square

5. BOUNDED SUBALGEBRAS OF $\mathfrak{sl}(W)$

We now prove Theorem 2.4 stated in the Introduction. In this section we assume that $\mathfrak{g} = \mathfrak{sl}(W)$ for some finite-dimensional vector space W . First we prove Theorem 2.5.

Proof of Theorem 2.5. Let \mathcal{Z} be an $\mathrm{SL}(V)$ -orbit open in $\mathrm{GV}(M)$. Assume M is a bounded $(\mathfrak{sl}(W), \mathfrak{k})$ -module. Let \tilde{V} be an irreducible component of $V(M)$. Then $\tilde{V} \cap \mathcal{Z}$ is an open subset of \tilde{V} and is a conical Lagrangian subvariety of \mathcal{Z} . By the discussion following Example 3.8, \mathcal{Z} is K -birationally isomorphic to $T^*\mathrm{Fl}$ for some partial W -flag variety Fl . As \tilde{V} is a conical Lagrangian subvariety of \mathcal{Z} , \tilde{V} is birationally isomorphic to $N_{Z/\mathrm{Fl}}^*$ for some smooth subvariety $Z \subset \mathrm{Fl}$ (Proposition 3.6). As \tilde{V} is K -spherical, $N_{Z/\mathrm{Fl}}^*$ is K -spherical. Therefore Fl is K -spherical (Theorem 3.2) and $T^*\mathrm{Fl}$ is K -coisotropic. Hence \mathcal{Z} is K -coisotropic.

Assume $\mathrm{GV}(M)$ is K -coisotropic. Let \tilde{V} be an irreducible component of $V(M)$ and $\tilde{\mathcal{Z}}$ be a G -orbit open in $G\tilde{V}$. By the discussion following Example 3.8, $\tilde{\mathcal{Z}}$ is K -birationally isomorphic to $T^*\mathrm{Fl}$ for some partial W -flag variety Fl . As $\tilde{\mathcal{Z}} \subset \mathcal{Z}$, the variety $\tilde{\mathcal{Z}}$ is K -coisotropic by Theorem 3.4. Therefore the variety Fl is K -spherical and in particular has finitely many K -orbits. As $\tilde{V} \cap \tilde{\mathcal{Z}} \subset \mathfrak{k}^\perp$, $\tilde{V} \cap \tilde{\mathcal{Z}}$ is isomorphic to an irreducible subvariety of the total space of the conormal bundle to a K -orbit in Fl . Therefore

$$\dim \tilde{V} \leq \dim \mathrm{Fl}.$$

On the other hand, $\tilde{V} \cap \tilde{\mathcal{Z}}$ is coisotropic in $\tilde{\mathcal{Z}}$ and therefore

$$\dim \tilde{V} \geq \dim \mathrm{Fl}.$$

Therefore $\dim \tilde{V} = \dim \mathrm{Fl}$ and \tilde{V} is birationally isomorphic to the total space of the conormal bundle to a K -orbit in Fl . Hence \tilde{V} is K -spherical (Theorem 3.2).

As all irreducible components of $V(M)$ are K -spherical, M is a bounded K -module. □

Theorem 5.1. If there exists a simple infinite-dimensional bounded $(\mathfrak{sl}(W), \mathfrak{k})$ -module M , then $\mathrm{Gr}(r; W)$ is a spherical K -variety for some $r \in \{1, \dots, n_W - 1\}$.

Proof. By Theorem 2.5 the variety $\mathrm{GV}(M)$ is K -coisotropic. By the discussion following Example 3.8 the variety $\mathrm{GV}(M)$ is K -birationally isomorphic to $T^*\mathrm{Fl}$ for some partial flag variety Fl . Hence Fl is K -spherical and $\mathrm{Gr}(r; W)$ is a K -spherical variety for some r . □

Theorem 5.2. Let Fl be a partial W -flag variety. Assume that Fl is K -spherical. Then there exists a simple infinite-dimensional multiplicity-free $(\mathfrak{sl}(W), \mathfrak{k})$ -module.

Proof. Theorem 6.3 in [PS] proves the existence of a simple infinite-dimensional multiplicity-free $(\mathfrak{sl}(W), \mathfrak{k})$ -module under the assumption that there exists a partial W -flag variety for which K has a proper closed orbit on Fl such that the total space of its conormal bundle is K -spherical. By Theorem 3.2 this latter condition is equivalent to the K -sphericity of Fl . It remains to consider the case when Fl has no proper closed K -orbits on Fl , i.e. K has only one orbit on Fl . If K has an open orbit on Fl then $[\mathfrak{k}, \mathfrak{k}] \cong \mathfrak{sp}(W)$ [On]. However, in this last case $\text{Gr}(2; W)$ has a proper closed K -orbit and is K -spherical. \square

We have thus proved the following weaker version of Theorem 2.4.

Corollary 5.1. A pair $(\mathfrak{sl}(W), \mathfrak{k})$ admits an infinite-dimensional bounded simple $(\mathfrak{sl}(W), \mathfrak{k})$ -module if and only if $\text{Gr}(r; W)$ is a spherical K -variety for some r .

Proof. The statement follows directly from Theorems 5.2 and 5.1. \square

Corollary 5.2 (see also [PS], Conjecture 6.6). If there exists a bounded simple infinite-dimensional $(\mathfrak{sl}(W), \mathfrak{k})$ -module, then there exists a multiplicity-free simple infinite-dimensional $(\mathfrak{sl}(W), \mathfrak{k})$ -module.

Proof. The statement follows directly from Corollary 5.1. \square

As $\mathbb{P}(W)$ is lower than or cotangent-equivalent to any other Grassmannian (see discussion following Lemma 3.14), r in Corollary 5.1 can be chosen to equal 1 (see Proposition 3.15). This completes the proof of Theorem 2.4.

All finite-dimensional \mathfrak{k} -modules W such that $\mathbb{P}(W)$ is a K -spherical variety are known from the work of C. Benson and G. Ratcliff [BR] (see also [Kac1] and [Le]). The list of respective pairs (\mathfrak{k}, W) is reproduced in the Appendix. Theorem 2.4 implies the following.

Corollary 5.3. The list of pairs $(\mathfrak{sl}(W), \mathfrak{k})$ for which \mathfrak{k} is reductive and bounded in $\mathfrak{sl}(W)$ coincides with the list of C. Benson and G. Ratcliff reproduced in the Appendix.

6. SPHERICAL PARTIAL FLAG VARIETIES

We recall that K is a reductive Lie group. Let V be a finite-dimensional K -module. We denote by \mathfrak{k}_V the image of \mathfrak{k} in $\text{End}(V)$.

Definition 6.1. A K -module V is called *weakly irreducible* if it is not a proper direct sum $V_1 \oplus V_2$ of two \mathfrak{k} -submodules such that

$$[\mathfrak{k}_{V_1}, \mathfrak{k}_{V_1}] \oplus [\mathfrak{k}_{V_2}, \mathfrak{k}_{V_2}] = [\mathfrak{k}_V, \mathfrak{k}_V].$$

All spherical representations are classified in the work [BR] (see also [Le] and [Kac1]) and we now recall this classification. According to [BR], a K -module V is K -spherical if and only if the pair $([\mathfrak{k}_V, \mathfrak{k}_V], V)$ is a direct sum of pairs (\mathfrak{k}_i, V_i) listed in the Appendix (cf. [BR]) and in addition

$$(\mathfrak{k}_V +_i \mathfrak{c}_i) = N_{\mathfrak{gl}(V)}(\mathfrak{k}_V +_i \mathfrak{c}_i)$$

for certain abelian Lie algebras \mathfrak{c}_i attached to (\mathfrak{k}_i, V_i) where

$$N_{\mathfrak{gl}(V)}(\mathfrak{k}_V +_i \mathfrak{c}_i)$$

is the normalizer of $\mathfrak{k}_V +_i \mathfrak{c}_i$ inside $\mathfrak{gl}(V)$.

Assume that V is a direct sum of two simple modules, $V = V_1 \oplus V_2$. For $a, b \in \mathbb{C}$ we denote by $h_{a,b}$ the rational semisimple element such that

$$h_{a,b}|_{V_1} = a\text{Id} \text{ and } h_{a,b}|_{V_2} = b\text{Id},$$

where Id is the identity map. We use similar notation if V is semisimple of length 1 or 3.

Theorem 6.1. a) Assume that a partial flag variety $\text{Fl}(n_1, \dots, n_s; V)$ is not cotangent-equivalent to $\mathbb{P}(V)$. If the partial flag variety $\text{Fl}(n_1, \dots, n_s; V)$ is K -spherical, then $\text{Fl}(n_1, \dots, n_s; V)$ is cotangent-equivalent to $\text{Fl}(n'_1, \dots, n'_s; V)$ for some datum

$$(n'_1, \dots, n'_s; [\mathfrak{k}_V, \mathfrak{k}_V], V)$$

which appears in the following list.

I) Case $s = 1$ ('Grassmannians').

I-1) $(r; \mathfrak{sl}_n, \mathbb{C}^n); (r; \mathfrak{so}_n, \mathbb{C}^n)(n \geq 3); (r; \mathfrak{sp}_n, \mathbb{C}^n)$.

I-2-1-1) $(2; \mathfrak{sp}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(m \geq 1);$

I-2-1-2) $(2; \mathfrak{sp}_n \oplus \mathfrak{sp}_m, \mathbb{C}^n \oplus \mathbb{C}^m);$

I-2-2) $(3; \mathfrak{sl}_n \oplus \mathfrak{sp}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(n \geq 1);$

I-2-3) $(r; \mathfrak{sp}_n, \mathbb{C}^n \oplus \mathbb{C});$

I-2-4) $(r; \mathfrak{sl}_n \oplus \mathfrak{sp}_4, \mathbb{C}^n \oplus \mathbb{C}^4)(n \geq 1);$

I-2-5) $(r; \mathfrak{sl}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(n, m \geq 1);$

I-3-1-1) $(2; \mathfrak{sl}_n \oplus \mathfrak{sl}_m \oplus \mathfrak{sl}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(m, n, q \geq 1);$

I-3-1-2) $(2; \mathfrak{sl}_n \oplus \mathfrak{sl}_m \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(m \geq 1, n \geq 1);$

I-3-1-3) $(2; \mathfrak{sl}_n \oplus \mathfrak{sp}_m \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(n \geq 1);$

I-3-1-4) $(2; \mathfrak{sp}_n \oplus \mathfrak{sp}_m \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)$;

I-3-2) $(r; \mathfrak{sl}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C})(n, m \geq 1)$.

II) Case $s \geq 2$.

II-1-1) $(n_1, \dots, n_s; \mathfrak{sl}_n, \mathbb{C}^n)$;

II-1-2) $(1, 2, 3; \mathfrak{sp}_n, \mathbb{C}^n)$;

II-1-3) $(1, r; \mathfrak{sp}_n, \mathbb{C}^n)$;

II-2-1) $(n_1, \dots, n_s; \mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C})$;

II-2-2) $(1, r; \mathfrak{sl}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(m, n \geq 1)$;

II-2-3) $(r_1, r_2; \mathfrak{sl}_2 \oplus \mathfrak{sl}_n, \mathbb{C}^2 \oplus \mathbb{C}^n)(n \geq 1)$;

II-2-4) $(1, 2; \mathfrak{sl}_n \oplus \mathfrak{sp}_m, \mathbb{F}^n \oplus \mathbb{C}^m)(n \geq 1)$;

II-2-5) $(1, 2; \mathfrak{sp}_n \oplus \mathfrak{sp}_m, \mathbb{C}^n \oplus \mathbb{C}^m)$.

b) For all data $(n_1, \dots, n_s; \mathfrak{k}', V)$ from this list there exists a reductive subgroup $K \subset \mathrm{GL}(V)$ with Lie algebra \mathfrak{k} such that $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$ and $\mathrm{Fl}(n_1, \dots, n_s; V)$ is a K -spherical variety.

The rest of this section is devoted to the proof of Theorem 6.1.

6.1. Simplifications. Let V_s and V_b be K -modules of dimensions n_s, n_b such that $n_s, n_b \geq 1$. We start with the following remark. The varieties

$$\mathrm{Hom}(V_s, V_b) \text{ and } \mathrm{Gr}(n_s; V_s \oplus V_b)$$

are birationally isomorphic. This shows that the K -module $\mathrm{Hom}(V_s, V_b)$ is K -spherical if and only if $\mathrm{Gr}(n_s; V_s \oplus V_b)$ is K -spherical.

Theorem 6.2. Fix $r \in \{1, \dots, n_b\}$. Let (L, V_b^r) be the pair determined by the datum (K, V_b, r) (see the discussion following Theorem 3.2). Then the variety $\mathrm{Gr}(r; V)$ is K -spherical if and only if the module $\mathrm{Hom}(V_b^r, V_s)$ is L -spherical and the variety $\mathrm{Gr}(r; V_b)$ is K -spherical.

Proof. Assume that $\mathrm{Gr}(r; V)$ is K -spherical. Then $\mathrm{Gr}(r; V_b)$ is K -spherical and $\mathrm{Gr}(r; V_b^r \oplus V_s)$ is L -spherical. The variety $\mathrm{Gr}(r; V_b^r \oplus V_s)$ is L -spherical if and only if $\mathrm{Hom}(V_b^r, V_s)$ is an L -spherical module.

Assume that the module $\mathrm{Hom}(V_b^r, V_s)$ is L -spherical and the variety $\mathrm{Gr}(r; V_b)$ is K -spherical. Then $\mathrm{Gr}(r; V_b^r \oplus V_s)$ is an L -spherical variety. Therefore the variety $\mathrm{Gr}(r; V)$ is K -spherical. \square

Corollary 6.2. Suppose that the variety $\mathrm{Gr}(r; V)$ is K -spherical for some $r \in \{2, \dots, n_b\}$. Then

- a) $\mathrm{Hom}(\mathbb{C}^r, V_s)$ is a spherical $\mathrm{GL}_r \times K$ -module;
- b) if $n_b \neq 2$, then the variety $\mathrm{Fl}(1, 2; V_b)$ is K -spherical.

Proof. The first statement is obvious. By Lemma 3.18 we can assume that $r = 2$. Let (L, V_b^2) be the pair determined by the datum $(K, V_b, 2)$. As $\mathrm{Gr}(2; V_b^2 \oplus V_s)$ is an L -spherical variety, $\mathbb{P}(V_b^2 \oplus V_s)$ is an L -spherical variety by Proposition 3.15. Therefore $\mathrm{Fl}(1, 2; V_b)$ is a K -spherical variety. \square

The following lemma is a first approximation to Theorem 6.1.

Lemma 6.3. Let W be a K -module. Suppose that $W \otimes \mathbb{C}^r$ is a spherical $K \times \mathrm{GL}_r$ -module for some $r \in \mathbb{Z}_{\geq 2}$. Then one of the following possibilities holds.

- 1) $r = 2$ and the datum $([\mathfrak{k}_W, \mathfrak{k}_W], W)$ appears in the following list:
 $(\mathfrak{sl}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(n, m \geq 1)$, $(\mathfrak{sl}_n \oplus \mathfrak{sp}_{2m}, \mathbb{C}^n \oplus \mathbb{C}^{2m})(n \geq 1)$,
 $(\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2m}, \mathbb{C}^{2n} \oplus \mathbb{C}^{2m})$, $(\mathfrak{sp}_{2n}, \mathbb{C}^{2n})$, $(\mathfrak{sl}_n, \mathbb{C}^n)$.
- 2) $r = 3$ and $(\mathfrak{k}_W \cap \mathfrak{sl}(W), W) \cong (\mathfrak{sp}_{2n}, \mathbb{C}^{2n})$.
- 3) $r \geq 3$ and the datum $([\mathfrak{k}_W, \mathfrak{k}_W], W)$ appears in the following list:
 $(\mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C})(n \geq 1)$, $(\mathfrak{sl}_n, \mathbb{C}^n)(n \geq 1)$, $(\mathfrak{sp}_4, \mathbb{C}^4)$.

Proof. The result follows directly from the tables of [BR]. \square

Let $0 < n_1 < \dots < n_s < n_W$ and $0 < n'_1 < \dots < n'_{s'} < n_W$ be sequences of integers. By Lemma 3.13

$$\mathrm{Fl}(n_1, \dots, n_s; W) \times \mathrm{Fl}(n'_1, \dots, n'_{s'}; W)$$

is $\mathrm{GL}(W)$ -spherical if and only if the variety $\mathrm{Fl}(n'_1, \dots, n'_{s'}; W)$ is K -spherical, where

$$K = \mathrm{GL}_{n_W - n_s} \times \dots \times \mathrm{GL}_{n_1}$$

is a Levi subgroup of a parabolic subgroup of $\mathrm{GL}(W)$. Let

$$(d_1, \dots, d_{s+1}) \text{ and } (d'_1, \dots, d'_{s'+1})$$

be the corresponding to (n_1, \dots, n_s) and $(n'_1, \dots, n'_{s'})$ partitions of n_W .

The following theorem is a particular case of a result of E. Ponomareva [Po]; she classified G -spherical products of two compact G -homogeneous spaces for arbitrary algebraic groups G .

Theorem 6.3 (see also [Pan3]). The variety

$$\mathrm{Fl}(n_1, \dots, n_s; W) \times \mathrm{Fl}(n'_1, \dots, n'_{s'}; W)$$

is $\mathrm{GL}(W)$ -spherical if and only if the (unordered) pair of sets

$$(\{d_1, \dots, d_{s+1}\}, \{d'_1, \dots, d'_{s'+1}\})$$

appears in the following list: $(\{p_1, p_2\}, \{q_1, q_2\})$; $(\{p_1, p_2\}, \{1, q_1, q_2\})$;
 $(\{2, p_1\}, \{q_1, q_2, q_3\})$; $(\{1, p_1\}, \{q_1, \dots, q_{s'+1}\})$.

Proof. A straightforward computation shows that the pair

$$(\{1, 1, n\}, \{1, 1, n\})$$

does not yield a spherical variety for $n \geq 1$ (see also Theorem 6.2). A straightforward computation shows that the pair

$$(\{3, n+1\}, \{2, 2, n\})$$

does not yield a spherical variety for $n \geq 2$ (see also Theorem 6.2). The rest of the proof is an exercise to Proposition 3.15. \square

6.2. Simple spherical modules. Let W be a simple K -module. We recall that by Lemma 3.18 if some partial W -flag variety, which is not cotangent-equivalent to $\mathbb{P}(W)$, is K -spherical, then the variety $\text{Gr}(2; W)$ is K -spherical too.

Lemma 6.4. Let $\text{Gr}(2; \mathbb{C}^m \otimes \mathbb{C}^n)$ be an $\text{SL}_n \times \text{SL}_m$ -spherical variety for some $m, n \in \mathbb{Z}_{\geq 2}$. Then $m = n = 2$.

Proof. Without loss of generality we assume that $m \geq n$. Assume that $n = 2$. Then

$$\dim(\mathfrak{b}_{\mathfrak{sl}_m} \oplus \mathfrak{b}_{\mathfrak{sl}_2}) \geq \dim \text{Gr}(2; \mathbb{C}^m \otimes \mathbb{C}^2).$$

We have $\frac{m(m+1)}{2} + 1 \geq 2(2m - 2)$ and therefore $(m - 2)(m - 5) \geq 0$. Hence $m \in \{2, 5, 6, 7, \dots\}$.

Assume $m \geq 5$. As $\text{Gr}(2; \mathbb{C}^2 \otimes \mathbb{C}^m)$ is $\text{SL}_2 \times \text{SL}_m$ -spherical, it must be that $\text{Gr}(2; \mathbb{C}^2 \otimes \mathbb{C}^4)$ is $\text{SL}_2 \times \text{SL}_4$ -spherical. This is false by dimension reasons.

Assume $n \geq 3$. Then

$$\frac{14}{18}(m^2 + n^2) \geq \frac{m^2+m}{2} - 1 + \frac{n^2+n}{2} - 1 + 4 \quad .$$

As $\dim(\mathfrak{b}_{\mathfrak{sl}_m} \oplus \mathfrak{b}_{\mathfrak{sl}_n}) \geq \dim \text{Gr}(2; \mathbb{C}^m \otimes \mathbb{C}^n)$, we have

$$\begin{aligned} \frac{m^2+m}{2} - 1 + \frac{n^2+n}{2} - 1 + 4 &\geq 2mn, \\ \frac{14}{18}(m^2 + n^2) &\geq 2mn \text{ and } m > 2n. \end{aligned}$$

As $\text{Gr}(2; \mathbb{C}^n \otimes \mathbb{C}^m)$ is $\text{SL}_n \times \text{SL}_m$ -spherical, we have $\text{Gr}(2; \mathbb{C}^n \otimes \mathbb{C}^{2n})$ is $\text{SL}_n \times \text{SL}_{2n}$ -spherical. This implies that $\text{SL}_n \times \text{SL}_2$ is a spherical subgroup of SL_{2n} . Hence $n = 2$ [Kr]. \square

Theorem 6.4. Suppose the variety $\text{Gr}(2; W)$ is K -spherical. Then the pair $(\mathfrak{k}_W \cap \mathfrak{sl}(W), W)$ is isomorphic to one of the following 3 pairs:

$$(\mathfrak{sl}(W), W), (\mathfrak{so}(W), W), (\mathfrak{sp}(W), W).$$

Proof. Without loss of generality we assume that $\mathfrak{k} = \mathfrak{k}_W \subset \mathfrak{gl}(W)$ and that \mathfrak{k} is a semisimple Lie algebra. We recall that B is a Borel subgroup of K .

A straightforward computation shows that $\text{Gr}(2; W)$ is $\text{SL}(W)$ -spherical, and also $\text{SO}(W)$ -spherical and $\text{SP}(W)$ -spherical. In the rest of the proof we assume that (\mathfrak{k}_W, W) is not isomorphic to

$$(\mathfrak{sl}(W), W), (\mathfrak{so}(W), W) \text{ and } (\mathfrak{sp}(W), W).$$

1) Assume that \mathfrak{k} is a simple Lie algebra. If there is an open orbit of B on $\text{Gr}(2; W)$ then there is an open orbit of $B \times \text{GL}_2$ on $W \otimes \mathbb{C}^2$. The simple modules with an open orbit of a reductive group are classified in [SK]. In Table 1 below we reproduce, following [SK], all simple

$K \times \mathrm{GL}_2$ -modules with an open orbit of $K \times \mathrm{GL}_2$ such that W is a K -spherical module.

Table 1. The pairs $([\mathfrak{k}_W, \mathfrak{k}_W], W)$ such that $W \otimes \mathbb{C}^2$ has an open orbit of $K \times \mathrm{GL}_2$ and W is a spherical K -module.

No	The pair (\mathfrak{k}_W, W)	$\dim(\mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{gl}_2)$	$2n_W$
1	$(\mathfrak{sl}_n, \mathbb{C}^n)(n \geq 2)$	$\frac{n(n+1)}{2} + 3$	$2n$
2	$(\mathfrak{so}_n, \mathbb{C}^n)(n \geq 3)$	$\frac{1}{2}(\frac{n(n-1)}{2} + [\frac{n}{2}])$	$2n$
3	$(\mathfrak{sp}_{2n}, \mathbb{C}^{2n})(n \geq 2)$	$n^2 + n$	$4n$
4	$(\mathfrak{sl}_3, \mathbb{S}^2 \mathbb{C}^3)$	9	12
5	$(\mathfrak{sl}_{2n+1}, \Lambda^2 \mathbb{C}^{2n+1})$	$2n^2 + 3n + 4$	$4n^2 + 2n$
6	$(\mathfrak{sl}_6, \Lambda^2 \mathbb{C}^6)$	24	30
7	$(\mathfrak{so}_7, \mathrm{Spin}(\mathbb{C}^7))$	16	16
8	$(\mathfrak{so}_{10}, \mathrm{Spin}(\mathbb{C}^{10}))$	29	32
9	$(\mathrm{G}_2, \mathbb{C}^7)$	12	14
10	$(\mathrm{E}_6, \mathbb{C}^{27})$	42	54

Here $\mathrm{Spin}(F)$ is any spinor module of $\mathrm{SO}(F)$, and \mathbb{C}^7 in case **9** is the unique faithful G_2 -module of minimal dimension, \mathbb{C}^{27} in case **10** is a faithful E_6 -module of minimal dimension. If $\mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{gl}_2$ has an open orbit on $W \otimes \mathbb{C}^2$, then $\dim(\mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{gl}_2) \geq 2n_W$.

Under the assumption that \mathfrak{k} is simple we complete the proof by the following case-by-case considerations.

Case **5**. The computation $4n^2 + 2n - (2n^2 + 3n + 4) = 2n^2 - n - 4 = n^2 - 4 + n(n-1) > 0 (n \geq 2)$ shows that $\dim(\mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{gl}_2) < 2n_W$.

Case **7**. A generic isotropy subalgebra for the action of $\mathfrak{so}_7 \oplus \mathfrak{gl}_2$ on $\mathrm{Spin}(\mathbb{C}^7) \otimes \mathbb{C}^2$ is isomorphic to \mathfrak{gl}_3 [SK]. All spherical subgroups of SO_7 are not quotients of $\mathrm{SL}_3 \times \mathbb{C}^*$ [Kr]. Therefore $\mathrm{Spin}(\mathbb{C}^7) \otimes \mathbb{C}^2$ has no open $\mathrm{B}(\mathrm{SO}_7) \times \mathrm{GL}_2$ -orbit.

For the cases **4, 6, 8, 9, 10** from Table 1 we have $\dim(\mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{gl}_2) < 2n_W$ and therefore $\mathrm{Gr}(2; W)$ has no open B -orbit.

2) Let \mathfrak{k} be a direct sum $\mathfrak{k}_1 \oplus \mathfrak{k}_2$ of 2 noncommutative ideals. Since W is a simple K -module, W is isomorphic to $V_1 \otimes V_2$, where $V_i (i = 1, 2)$ are simple \mathfrak{k}_i -modules such that $\mathfrak{k}_{V_i} = \mathfrak{k}_i$. As $\mathrm{Gr}(2; V_1 \otimes V_2)$ is G -spherical, $\mathrm{Gr}(2; V_1 \otimes V_2)$ is $\mathrm{SL}(V_1) \times \mathrm{SL}(V_2)$ -spherical too. Since $n_{V_{1,2}} \geq 2$, the only possibility for (\mathfrak{k}, W) is $(\mathfrak{so}_4, \mathbb{C}^4)$ by Lemma 6.4. In this case $\mathrm{Gr}(2; W)$ is a spherical K -variety. \square

Remark 6.5. Fix $r \in \{1, \dots, n_W - 1\}$. Then

- a) the variety $\mathrm{Gr}(r; W)$ is $\mathrm{SL}(W)$ -spherical;
- b) the variety $\mathrm{Gr}(r; W)$ is $\mathrm{SO}(W)$ -spherical;
- c) the variety $\mathrm{Gr}(r; W)$ is $\mathrm{SP}(W)$ -spherical.

Lemma 6.6. Suppose that $\text{Fl}(1, 2; W)$ is a K -spherical variety. Then the pair $(\mathfrak{k}_W \cap \mathfrak{sl}(W), W)$ is isomorphic to one of the following pairs:

$$(\mathfrak{sl}(W), W), (\mathfrak{sp}(W), W).$$

Proof. As $\text{Fl}(1, 2; W)$ is a K -spherical variety, the variety $\text{Gr}(2; W)$ is K -spherical. Therefore $(\mathfrak{k} \cap \mathfrak{sl}(W), W)$ is isomorphic to

$$(\mathfrak{sl}(W), W), \text{ to } (\mathfrak{so}(W), W), \text{ or to } (\mathfrak{sp}(W), W).$$

A generic isotropy subgroup for the action of SO_n on $\text{Fl}(1, 2; \mathbb{C}^n)$ is isomorphic to SO_{n-2} . Such a subgroup is not spherical in SO_n for all $n \geq 3$ [Kr]. \square

Corollary 6.7. The only partial flag varieties which are $\text{SO}(W)$ -spherical are $\text{Gr}(r; W)$ for all $r \in \{1, \dots, n-1\}$.

Proof. This is a simple corollary of Proposition 3.15 and of the discussion preceding this proposition. \square

Remark 6.8. All partial W -flag varieties are $\text{SL}(W)$ -spherical.

Lemma 6.9. Assume that $2|n_W$ and $n_W \geq 6$. Then the variety $\text{Fl}(2, 4; W)$ is not $\text{SP}(W)$ -spherical.

Proof. To the datum $(\text{SP}(W), W, 4)$ one assigns the datum $(\text{SL}_2 \times \text{SL}_2, \mathbb{C}^2 \oplus \mathbb{C}^2)$. Therefore the variety $\text{Fl}(2, 4; W)$ is $\text{SP}(W)$ -spherical if and only if the variety $\text{Gr}(2; \mathbb{C}^2 \oplus \mathbb{C}^2)$ is $\text{SL}_2 \times \text{SL}_2$ -spherical. However, $\text{Gr}(2; \mathbb{C}^2 \oplus \mathbb{C}^2)$ is not $\text{SL}_2 \times \text{SL}_2$ -spherical. \square

We recall that any partial W -flag variety is cotangent-equivalent to one of the following

$$\text{Gr}(r; W), \text{ Fl}(1, r; W), \text{ Fl}(1, 2, 3; W), \text{ Fl}(2, 4; W),$$

or is higher than $\text{Fl}(2, 4; W)$, see subsection 3.3.

Corollary 6.10. The only $\text{SP}(W)$ -spherical partial flag varieties, up to cotangent-equivalence, are

$$\text{Gr}(r; W), \text{ Fl}(1, r; W), \text{ Fl}(1, 2, 3; W).$$

In particular, all partial \mathbb{C}^4 -flag varieties are SP_4 -spherical.

6.3. Weakly irreducible spherical modules. Let W be a weakly irreducible, but not irreducible, K -spherical module. Then W is a direct sum of two nonzero simple K -modules $W_s \oplus W_b$ of dimensions n_s, n_b . Without loss of generality we assume that $n_b \geq n_s$.

Lemma 6.11. The variety $\text{Gr}(2; W)$ is not K -spherical.

Proof. Assume $n_b = 2$. Then the pair $([\mathfrak{k}, \mathfrak{k}], W)$ is isomorphic to the pair $(\mathfrak{sl}_2, \mathbb{C}^2 \oplus \mathbb{C}^2)$. The variety $\text{Gr}(2; \mathbb{C}^2 \oplus \mathbb{C}^2)$ is not GL_2 -spherical for dimension reasons.

Assume $n_b > 2$. Then the variety $\text{Fl}(1, 2; W_b)$ is K -spherical by Corollary 6.2. Therefore $(\mathfrak{k}_{W_b} \cap \mathfrak{sl}(W_b), W_b)$ is isomorphic to $(\mathfrak{sl}(W_b), W_b)$ or to $(\mathfrak{sp}(W_b), W_b)$. Then the pair $([\mathfrak{k}, \mathfrak{k}], W)$ is isomorphic to

$$(\mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C}^n), \text{ to } (\mathfrak{sl}_n, \mathbb{C}^n \oplus (\mathbb{C}^n)^*), \text{ or to } (\mathfrak{sp}_{2n}, \mathbb{C}^{2n} \oplus \mathbb{C}^{2n})$$

(see the tables in [BR]). Note, that $n_s = n_b$ in all three cases.

Consider the pair $(\mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C}^n)$. If the variety $\text{Gr}(2; W)$ is K -spherical, $\text{Gr}(2; \mathbb{C}^n \otimes \mathbb{C}^2)$ must be $\text{GL}_n \times \text{GL}_2$ -spherical. This is not the case by Lemma 6.4.

Next we consider the pair $(\mathfrak{sp}_{2n}, \mathbb{C}^{2n} \oplus \mathbb{C}^{2n})$. If the variety $\text{Gr}(2; W)$ is K -spherical, $\text{Gr}(2; \mathbb{C}^{2n} \otimes \mathbb{C}^2)$ must be $\text{GL}_{2n} \times \text{GL}_2$ -spherical. This is not the case by Lemma 6.4.

Finally consider the pair $(\mathfrak{sl}_n, \mathbb{C}^n \oplus (\mathbb{C}^n)^*) (n \geq 3)$. If the variety $\text{Gr}(2; W)$ is GL_n -spherical, the variety $\mathbb{C}^2 \otimes \mathbb{C}^n$ is GL_2 -spherical by Theorem 6.2. This is not the case for dimension reasons. \square

Corollary 6.12. The only K -spherical partial W -flag varieties for a weakly irreducible, but not irreducible, K -spherical module W are

$$\text{Gr}(1; W) \text{ and } \text{Gr}(n_W - 1; W).$$

6.4. On the length of certain modules. Let V be a spherical K -module and W be a proper nonzero submodule of V . We recall that by Lemma 3.18, if some partial V -flag variety, which is not cotangent-equivalent to $\mathbb{P}(V)$, is K -spherical, then the variety $\text{Gr}(2; V)$ is K -spherical too.

Lemma 6.13. Assume that W is a weakly irreducible submodule of V and $\text{Gr}(2; V)$ is a K -spherical variety. Then the pair $(\mathfrak{sl}(W) \cap \mathfrak{k}_W, W)$ is isomorphic

$$\text{to } (\mathfrak{sl}(W), W) \text{ or to } (\mathfrak{sp}(W), W).$$

Proof. Assume $n_W = 2$. Then the pair $(\mathfrak{k}_W \cap \mathfrak{sl}(W), W)$ is isomorphic to $(\mathfrak{sl}_2, \mathbb{C}^2)$.

Assume $n_W \geq 3$. The variety $\text{Fl}(1, 2; W)$ is K -spherical by Corollary 6.2. Therefore the pair $(\mathfrak{k}_W \cap \mathfrak{sl}(W), W)$ is isomorphic to $(\mathfrak{sl}(W), W)$, or to $(\mathfrak{sp}(W), W)$ by Corollary 6.12 and Lemma 6.6. \square

Lemma 6.14. Let W be a K -module such that $n_W > 3$ and $\text{Gr}(2; W)$ is K -spherical. Then the length of W as a \mathfrak{k} -module equals at most 3.

Proof. See Theorem 6.3. \square

6.5. Concluding case-by-case considerations. The following theorem is a second approximation to Theorem 6.1.

Theorem 6.5. Let W_b, W_s be K -modules of the dimensions n_b, n_s such that $n_b \geq n_s \geq 2$. Suppose that $\text{Hom}(W_s, W_b)$ is a K -spherical module. Then one of the following holds.

- 1) $[\mathfrak{k}_{W_s}, \mathfrak{k}_{W_s}] = 0$, $n_s = 2$ and $([\mathfrak{k}_{W_b}, \mathfrak{k}_{W_b}], W_b)$ is either $(\mathfrak{sl}(W_b), W_b)$, or $(\mathfrak{sp}(W_b), W_b)$;
- 2) $([\mathfrak{k}_{W_s}, \mathfrak{k}_{W_s}], W_s) = (\mathfrak{sl}_2, \mathbb{C}^2)$ and $([\mathfrak{k}_{W_b}, \mathfrak{k}_{W_b}], W_b)$ appears in the following list: $(\mathfrak{sl}(W_b), W_b)$, $(\mathfrak{sp}(W_b), W_b)$, $(\mathfrak{sl}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)$ ($m, n \geq 1$), $(\mathfrak{sl}_n \oplus \mathfrak{sp}_{2m}, \mathbb{C}^n \oplus \mathbb{C}^{2m})$ ($n \geq 1$), $(\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2m}, \mathbb{C}^{2n} \oplus \mathbb{C}^{2m})$;
- 3) $([\mathfrak{k}_{W_s}, \mathfrak{k}_{W_s}], W_s) = (\mathfrak{sl}_n, \mathbb{C}^n)$ ($n \geq 3$) and $([\mathfrak{k}_{W_b}, \mathfrak{k}_{W_b}], W_b)$ appears in the following list: $(\mathfrak{sl}(W_b), W_b)$, $(\mathfrak{sl}_m, \mathbb{C}^m \oplus \mathbb{C})$, $(\mathfrak{sp}_4, \mathbb{C}^4)$;
- 3') $([\mathfrak{k}_{W_s}, \mathfrak{k}_{W_s}], W_s) = (\mathfrak{sl}_n, \mathbb{C}^n \oplus \mathbb{C})$ ($n \geq 2$) and $([\mathfrak{k}_{W_b}, \mathfrak{k}_{W_b}], W_b) = (\mathfrak{sl}(W_b), W_b)$;
- 4) $([\mathfrak{k}_{W_s}, \mathfrak{k}_{W_s}], W_s) = (\mathfrak{sl}_3, \mathbb{C}^3)$ and $([\mathfrak{k}_{W_b}, \mathfrak{k}_{W_b}], W_b) = (\mathfrak{sp}(W_b), W_b)$;
- 5) $([\mathfrak{k}_{W_s}, \mathfrak{k}_{W_s}], W_s) = (\mathfrak{sp}_4, \mathbb{C}^4)$ and $([\mathfrak{k}_{W_b}, \mathfrak{k}_{W_b}], W_b) = (\mathfrak{sl}(W_b), W_b)$.

Proof. As the variety $\text{Hom}(W_s, W_b)$ is K -spherical, the variety $\text{Gr}(n_s; W_s \oplus W_b)$ is K -spherical. Therefore the variety $\mathbb{P}(W_s \oplus W_b)$ is K -spherical and the length of the K -module $W_s \oplus W_b$ is not more than 3. Using this condition and the results of [BR] it is straightforward to derive the required list. We also use Lemma 6.13. \square

In the remainder of this section we collect the additional information needed to prove Theorem 6.1.

Let W be a nonzero K -module. Denote by $\mathbb{C}_{a,b}^*$ the one-dimensional subgroup of $\text{GL}(W)$ which corresponds to $\mathcal{C}h_{a,b}$ (see Section 2). We use similar notation if W has length 1 or 3.

Fix $r \in \{1, \dots, n_W - 1\}$. Recall that the datum (K, W, r) determines the datum (L, W^r) (see the discussion following Theorem 3.2).

Lemma 6.15. The following data (K, W, r) determine the pairs (L, W^r) as follows:

- 1) $(\text{SL}(W), W, r) \rightarrow (\text{GL}_r, \mathbb{C}^r)$;
- 2) $(\text{SP}(W), W, 2) \rightarrow (\text{SL}_2, \mathbb{C}^2)$;
- 3) $(\text{SP}(W), W, 3) \rightarrow (\text{SL}_2 \times \mathbb{C}_{0,1}^*, \mathbb{C}^2 \oplus \mathbb{C})$;
- 4) $(\text{SP}(W), W, 4) \rightarrow (\text{SL}_2 \times \text{SL}_2, \mathbb{C}^2 \oplus \mathbb{C}^2)$.

Proof. We omit the proof. \square

Lemma 6.16. For any $q \in \mathbb{Z}_{\geq 2}$ the following statements are equivalent.

- 1) The variety $\text{Gr}(2; W \oplus \mathbb{C}^{2q})$ is $K \times \text{SP}_{2q}$ -spherical.
- 2) The variety $\text{Gr}(2; W \oplus \mathbb{C}^2)$ is $K \times \text{SL}_2$ -spherical.
- 3) The variety $\text{Hom}(\mathbb{C}^2, W)$ is $\text{SL}_2 \times K$ -spherical.

Proof. As the datum $(\mathrm{SP}_{2q}, \mathbb{C}^{2q}, 2)$ determines the pair $(\mathrm{SL}_2, \mathbb{C}^2)$, part 1) is equivalent to part 2) by Theorem 6.2. Part 3) is a reformulation of part 2). \square

Lemma 6.17. For any $q \in \mathbb{Z}_{\geq 2}$ the following statements are equivalent.

- 1) The variety $\mathrm{Gr}(3; W \oplus \mathbb{C}^{2q})$ is $K \times \mathrm{SP}_{2q}$ -spherical.
- 2) The variety $\mathrm{Gr}(3; W \oplus \mathbb{C}^2 \oplus \mathbb{C})$ is $K \times \mathrm{SL}_2 \times \mathbb{C}_{0,0,1}^*$ -spherical.
- 3) The variety $\mathrm{Hom}(\mathbb{C}^2 \oplus \mathbb{C}, W)$ is $\mathrm{SL}_2 \times \mathbb{C}_{0,1}^* \times K$ -spherical.

Proof. As the datum $(\mathrm{SP}_{2q}, \mathbb{C}^{2q}, 3)$ determines the pair $(\mathrm{SL}_2 \times \mathbb{C}_{0,1}^*, \mathbb{C}^2 \oplus \mathbb{C})$, part 1) is equivalent to part 2) by Theorem 6.2. Part 3) is a reformulation of part 2). \square

Corollary 6.18. Suppose that $\mathrm{Gr}(3; W \oplus \mathbb{C}^{2q})$ is a $K \times \mathrm{SP}_{2q}$ -spherical variety for some $q \in \mathbb{Z}_{\geq 2}$. Then $\mathfrak{sl}(W) \subset \mathfrak{k}_W$.

Corollary 6.19. The variety $\mathrm{Fl}(1, 3; W \oplus \mathbb{C}^{2q})$ is not a $K \times \mathrm{SP}_{2q} \times \mathbb{C}_{0,1}^*$ -spherical variety for any $q \in \mathbb{Z}_{\geq 2}$.

Proof. The variety $\mathrm{Fl}(1, 3; W \oplus \mathbb{C}^{2q})$ is $K \times \mathrm{SP}_{2q} \times \mathbb{C}_{0,1}^*$ -spherical if and only if the variety $\mathrm{Fl}(1, 3; W \oplus \mathbb{C}^2 \oplus \mathbb{C})$ is $K \times \mathrm{GL}_2 \times \mathbb{C}_{0,0,1}^*$ -spherical (Lemma 6.17). The latter is false by Theorem 6.3. \square

Lemma 6.20. For any $q \in \mathbb{Z}_{\geq 3}$ the following statements are equivalent.

- 1) The variety $\mathrm{Gr}(4; W \oplus \mathbb{C}^{2q})$ is $K \times \mathrm{SP}_{2q} \times \mathbb{C}_{0,1}^*$ -spherical.
- 2) The variety $\mathrm{Gr}(4; W \oplus \mathbb{C}^2 \oplus \mathbb{C}^2)$ is $K \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathbb{C}_{0,1,1}^*$ -spherical.
- 3) The variety $\mathrm{Hom}(\mathbb{C}^2 \oplus \mathbb{C}^2, W)$ is $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathbb{C}_{1,1}^* \times K$ -spherical.

Proof. As the datum $(\mathrm{SP}_{2q}, \mathbb{C}^{2q}, 4)$ determines the pair $(\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathbb{C}_{1,1}^*, \mathbb{C}^2 \oplus \mathbb{C}^2)$, part 1) is equivalent to part 2) by Theorem 6.2. Part 3) is a reformulation of part 2). \square

Corollary 6.21. Suppose that $\mathrm{Gr}(4; W \oplus \mathbb{C}^{2q})$ is a $K \times \mathrm{SP}_{2q}$ -spherical variety for some $q \in \mathbb{Z}_{\geq 3}$. Then either $n_W = 2$ and $\mathfrak{sl}(W) \subset \mathfrak{k}_W$, or $n_W = 1$.

Proposition 6.22. Let V be a K -module.

a) Suppose that $\mathrm{Gr}(r; V)$ is a K -spherical variety for some $r \in \{2, \dots, \lfloor \frac{n_V}{2} \rfloor\}$. Then the datum

$$(r; [\mathfrak{k}_V, \mathfrak{k}_V], V) \tag{3}$$

appears in the following list:

- 1) $(r; \mathfrak{sl}_n, \mathbb{C}^n)$, $(r; \mathfrak{so}_n, \mathbb{C}^n)$ ($n \geq 3$), $(r; \mathfrak{sp}_n, \mathbb{C}^n)$;
- 2-1-1) $(2; \mathfrak{sp}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)$ ($m \geq 1$);
- 2-1-2) $(2; \mathfrak{sp}_n \oplus \mathfrak{sp}_m, \mathbb{C}^n \oplus \mathbb{C}^m)$;
- 2-2) $(3; \mathfrak{sl}_n \oplus \mathfrak{sp}_m, \mathbb{C}^n \oplus \mathbb{C}^m)$ ($n \geq 1$);
- 2-3) $(r; \mathfrak{sp}_n, \mathbb{C}^n \oplus \mathbb{C})$;

- 2-4) $(r; \mathfrak{sl}_n \oplus \mathfrak{sp}_4, \mathbb{C}^n \oplus \mathbb{C}^4)(n \geq 1)$;
 2-5) $(r; \mathfrak{sl}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m)(n, m \geq 1)$;
 3-1-1) $(2; \mathfrak{sl}_n \oplus \mathfrak{sl}_m \oplus \mathfrak{sl}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(m, n, q \geq 1)$;
 3-1-2) $(2; \mathfrak{sl}_n \oplus \mathfrak{sl}_m \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(m \geq 1, n \geq 1)$;
 3-1-3) $(2; \mathfrak{sl}_n \oplus \mathfrak{sp}_m \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)(n \geq 1)$;
 3-1-4) $(2; \mathfrak{sp}_n \oplus \mathfrak{sp}_m \oplus \mathfrak{sp}_q, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C}^q)$;
 3-2) $(r; \mathfrak{sl}_n \oplus \mathfrak{sl}_m, \mathbb{C}^n \oplus \mathbb{C}^m \oplus \mathbb{C})(n, m \geq 1)$.

b) For all data $(r; \mathfrak{k}', V)$ from the list there exists a reductive subgroup $K \subset \mathrm{GL}(V)$ with Lie algebra \mathfrak{k} such that $\mathrm{Gr}(r; V)$ is K -spherical and $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$.

Proof. Let V be a direct sum of nonzero simple \mathfrak{k} -modules

$$V_1 \oplus V_2 \oplus \dots \oplus V_l.$$

By Lemma 3.18 the variety $\mathrm{Gr}(2; V)$ is K -spherical and therefore $l \leq 3$ by Lemma 6.14.

If $l = 1$, the K -module V is simple. This case is considered in Subsection 6.2 (Theorem 6.4 and Remark 6.5).

In the rest of the proof we assume that $l \geq 2$, i.e. $l = 2, 3$. By Lemma 6.13, for all i ,

$$[\mathfrak{k}_{V_i}, \mathfrak{k}_{V_i}] = \mathfrak{sl}(V_i), \text{ or } [\mathfrak{k}_{V_i}, \mathfrak{k}_{V_i}] = \mathfrak{sp}(V_i) \text{ and } n_{V_i} \geq 4.$$

If the variety $\mathrm{Gr}(r; V)$ is K -spherical then the pair of sets

$$(\{r, n_V - r\}, \{n_{V_1}, \dots, n_{V_l}\})$$

appears in the list of Theorem 6.3. All entries of the required list with $r = 2$ are recovered by the list of Theorem 6.3.

In the rest of the proof we assume that $r \geq 3$. It remains to show that the datum (3) is as in cases 2-2, 2-3, 2-4, 2-5 or 3-2. If $\mathfrak{sl}(V_i) \subset \mathfrak{k}_{V_i}$ for all i , then Theorem 6.3 implies that we are in cases 2-5 or 3-2.

In the rest of the proof we assume that

$$[\mathfrak{k}_{V_1}, \mathfrak{k}_{V_1}] = \mathfrak{sp}(V_1) \text{ and } n_{V_1} \geq 4.$$

As $r \geq 3$, $\mathrm{Gr}(3; V)$ is a K -spherical variety by Proposition 3.15. Then

$$\mathfrak{sl}(V_2 \oplus \dots \oplus V_s) \subset \mathfrak{k}_{V_2 \oplus \dots \oplus V_s},$$

in particular, $l = 2$. For $r = 3$, this is case 2-2 from the list.

In the rest of the proof we assume that $r \geq 4$. Case 2-4 corresponds to $n_{V_1} = 4$. Assume now that $n_{V_1} \geq 6$. As $\mathrm{Gr}(r; V)$ is a K -spherical variety, $\mathrm{Gr}(4; V)$ is a K -spherical variety. Therefore $n_{V_2} \leq 2$ by Corollary 6.21. A straightforward computation shows that $\mathrm{Gr}(4; V_1 \oplus \mathbb{C}^2)$

is not an $\mathrm{SP}(V_1) \times \mathrm{GL}_2$ -spherical variety for all n_{V_1} such that $n_{V_1} \geq 6$. Therefore $n_{V_2} = 1$ and we are in case 2-3.

This proves a). Part b) is straightforward. \square

We are now ready to prove Theorem 6.1.

Proof of theorem 6.1. Let V be a direct sum of simple nonzero \mathfrak{k} -modules

$$V_1 \oplus V_2 \oplus \dots \oplus V_l.$$

If $l = 1$, the \mathfrak{k} -module V is simple. This case is considered in Subsection 6.2 (Remark 6.8, Corollary 6.7, Corollary 6.10).

In the rest of the proof we assume that $l \geq 2$. The claim of Theorem 6.1 for $s = 1$ is established in Proposition 6.22.

In the rest of the proof we also assume that $s \geq 2$. By Lemma 3.18 the variety $\mathrm{Gr}(2; V)$ is K -spherical and $l \leq 3$ by Lemma 6.14. Lemma 6.13 implies, for all i , that

$$[\mathfrak{k}_{V_i}, \mathfrak{k}_{V_i}] = \mathfrak{sl}(V_i), \text{ or } [\mathfrak{k}_{V_i}, \mathfrak{k}_{V_i}] = \mathfrak{sp}(V_i) \text{ and } n_{V_i} \geq 4.$$

If the variety $\mathrm{Fl}(n_1, \dots, n_s; V)$ is K -spherical then the pair of sets

$$(\{n_1, \dots, n_V - n_s\}, \{n_{V_1}, \dots, n_{V_l}\})$$

appears in the list of Theorem 6.3. If $\mathfrak{sl}(V_i) \subset \mathfrak{k}_{V_i}$ for all i , then Theorem 6.3 implies that we are in cases II-1-1, II-2-1, II-2-2, II-2-3.

In the rest of the proof we assume that $[\mathfrak{k}_{V_1}, \mathfrak{k}_{V_1}] = \mathfrak{sp}(V_1)$ and $n_{V_1} \geq 4$. It remains to show that the datum (1) is as in cases II-1-2, II-1-3, II-2-4, II-2-5.

We have now $l \leq 2$ by Theorem 6.3 and therefore $l = 2$. The variety $\mathrm{Fl}(n_1, \dots, n_s; V)$ is cotangent-equivalent to

$$\mathrm{Fl}(1, 2; V), \text{ or to } \mathrm{Fl}(1, 3; V),$$

or is higher than $\mathrm{Fl}(1, 3; V)$ (see Subsection 3.3). As the variety $\mathrm{Fl}(1, 3; V)$ is not K -spherical by Corollary 6.19, $\mathrm{Fl}(n_1, \dots, n_s; V)$ is cotangent-equivalent to $\mathrm{Fl}(1, 2; V)$ and we are in cases II-2-4, II-2-5.

This proves a). Part b) is straightforward. \square

7. JOSEPH IDEALS

In this section \mathfrak{g} is semisimple. Let $\lambda \in \mathfrak{h}_{\mathfrak{g}}^*$ be a dominant weight. A two-sided ideal I of $U(\mathfrak{g})$ determines the submodule IM_{λ} of M_{λ} . This defines a map α from the lattice of ideals of $U(\mathfrak{g})$ to the lattice of submodules of M_{λ} .

Theorem 7.1 (A. Joseph [Jo1, Prop. 4.3]). The map α is an embedding of the lattice of two-sided ideals $I \subset U(\mathfrak{g})$ such that $\text{Ker}\chi_{\lambda} \subset I$ to the lattice of submodules of M_{λ} . If λ is regular, α is an isomorphism.

Remark 7.1. The two-sided ideals of $U(\mathfrak{g})$ have been studied extensively in the last 30 years. This section adapts some known results to the setup of the dissertation. We thank Anthony Joseph and Anna Melnikov for useful comments and references.

7.1. The case $\mathfrak{g} = \mathfrak{sl}(W)$. Let $\mathcal{Q} \subset \mathfrak{sl}(W)^*$ be the non-trivial nilpotent orbit of minimal dimension. This nilpotent orbit consists of matrices of rank 1. We represent such matrices as $l \otimes v$, for some $v \in W$ and $l \in W^*$ with $l(v) = 0$.

Definition 7.2. Let I be a primitive ideal of $U(\mathfrak{sl}(W))$. We call I a *Joseph ideal* whenever $V(I) = \bar{\mathcal{Q}}$.

Lemma 7.3. Let I be a Joseph ideal and let \tilde{L} be a finitely generated $(\mathfrak{sl}(W), \mathfrak{b}_W^{\mathfrak{sl}})$ -module such that $I \subset \text{Ann}\tilde{L}$. Then \tilde{L} is an $(\mathfrak{sl}(W), \mathfrak{h}_W^{\mathfrak{sl}})$ -bounded module.

Proof. As \tilde{L} is an $(\mathfrak{sl}(W), \mathfrak{b}_W^{\mathfrak{sl}})$ -module, $V(\tilde{L}) \subset \bar{\mathcal{Q}} \cap (\mathfrak{b}_W^{\mathfrak{sl}})^{\perp}$. We have

$$\bar{\mathcal{Q}} \cap (\mathfrak{b}_W^{\mathfrak{sl}})^{\perp} = \{x \in \mathfrak{sl}(W)^* \mid x = l \otimes v \text{ for some } v \in W \text{ and } l \in W^* \text{ such that } l(bv) = 0 \text{ for all } b \in \mathfrak{b}_W^{\mathfrak{sl}}\}.$$

The variety $\bar{\mathcal{Q}} \cap (\mathfrak{b}_W^{\mathfrak{sl}})^{\perp}$ has irreducible components $\{\mathcal{Q}_i\}_{i \leq n_W - 1}$, where

$$\mathcal{Q}_i := \{x \in \mathfrak{sl}(W)^* \mid x = l \otimes v \text{ for some } v \in \text{span}\langle e_1, \dots, e_i \rangle \text{ and } l \in \text{span}\langle e_{i+1}^*, \dots, e_{n_W}^* \rangle\}.$$

The dimension of \mathcal{Q}_i equals $n_W - 1$ for any $i \leq n_W - 1$. Let $H_W^{\mathfrak{sl}}$ be the connected subgroup of $SL(W)$ with Lie algebra $\mathfrak{h}_W^{\mathfrak{sl}}$. The varieties \mathcal{Q}_i are $H_W^{\mathfrak{sl}}$ -stable and are $H_W^{\mathfrak{sl}}$ -spherical for all $i \leq n_W - 1$. Therefore \tilde{L} is a bounded $(\mathfrak{sl}(W), \mathfrak{h}_W^{\mathfrak{sl}})$ -module by Proposition 3.23. \square

Corollary 7.4. a) Let I be a Joseph ideal and let $\lambda \in \mathfrak{h}_W^{\mathfrak{sl}*}$ be a weight such that $I = \text{Ann} L_{\lambda}$. Then L_{λ} is a bounded $\mathfrak{h}_W^{\mathfrak{sl}}$ -module. In particular, $\bar{\lambda}$ is a semi-decreasing tuple.

b) If $\bar{\lambda}$ is a semi-decreasing n_W -tuple then $I(\lambda)$ is a Joseph ideal.

Proof. Part a) is trivial. We have

$$\dim \text{GV}(M) \leq 2\dim V(M)$$

by Theorem 3.7. As $\bar{\lambda}$ is semi-decreasing, L_λ is bounded and therefore $V(L_\lambda)$ is $H_W^{\mathfrak{sl}}$ -spherical. Hence $\dim \text{GV}(M) \leq 2(n_W - 1)$. The only nilpotent orbit which satisfies this inequality is \mathcal{Q} . \square

Corollary 7.5 ([Jo4, Table 3]). Let $I \subset U(\mathfrak{sl}(W))$ be a two-sided ideal. Then I is a Joseph ideal if and only if one of the following conditions holds:

- a) there exists a semi-integral semi-decreasing tuple $\bar{\lambda}$ such that $I = I(\lambda)$;
- b) there exists a singular integral semi-decreasing tuple $\bar{\lambda}$ such that $I = I(\lambda)$;
- c) there exists a regular integral semi-decreasing tuple $\bar{\lambda}$ and a number $k \in \{1, \dots, n_W - 1\}$ such that $I = I(s_k \lambda)$.

Proof. Follows from Corollary 7.4 and the discussion about coherent families in Subsection 3.9. \square

Lemma 7.6. Let $k \in \{1, 2, \dots, n_W - 1\}$ and λ be a regular integral dominant weight. Then $\text{Ann } L_{s_k \lambda}$ is a submaximal ideal of $U(\mathfrak{sl}(W))$, i.e. for any two ideals I_1, I_2 such that $\text{Ann } L_{s_k \lambda} \subsetneq I_1, I_2$ we have $I_1 = I_2$.

Proof. The Verma module M_λ has a unique maximal submodule $\text{rad}^1 M_\lambda$ such that $M_\lambda / \text{rad}^1 M_\lambda$ is a simple module L_λ . Let $\text{rad}^2 M_\lambda$ be the minimal submodule of $\text{rad}^1 M_\lambda$ such that $\text{rad}^1 M_\lambda / \text{rad}^2 M_\lambda$ is a semisimple $\mathfrak{sl}(W)$ -module. The module $M_\lambda / \text{rad}^1 M_\lambda$ is finite-dimensional and $\text{rad}^1 M_\lambda / \text{rad}^2 M_\lambda$ has a unique submodule isomorphic to $L_{s_k \lambda}$. Let $\text{rad}^{1+\varepsilon_k} M_\lambda$ be a the submodule of $\text{rad}^1 M_\lambda$ such that $\text{rad}^1 M_\lambda / \text{rad}^{1+\varepsilon_k} M_\lambda = L_{s_k \lambda}$. Let I be the corresponding to $\text{rad}^{1+\varepsilon_k} M_\lambda$ two-sided ideal. Then I is submaximal and $I \subset \text{Ann } L_{s_k \lambda}$. As the unique ideal which is larger then I is of finite codimension, we have $\text{Ann } L_{s_k \lambda} = I$. Therefore the ideal $\text{Ann } L_{s_k \lambda}$ is submaximal. \square

Let $\bar{\lambda}$ be an n_W -tuple. We can consider the algebra $U(\mathfrak{sl}(W))/I(\lambda)$ as an $(\mathfrak{sl}(W)_l \oplus \mathfrak{sl}(W)_r)$ -module, where both $\mathfrak{sl}(W)_l$ and $\mathfrak{sl}(W)_r$ are isomorphic to $\mathfrak{sl}(W)$ and the action is given by

$$(g_1, g_2)m = g_1 m - m g_2$$

for $g_1, g_2 \in \mathfrak{sl}(W)$ and $m \in U(\mathfrak{sl}(W))/I(\lambda)$. Fix $i \in \{1, \dots, n_W - 1\}$. The action of

$$\mathfrak{sl}(W)_d := \{(g, g)\}_{g \in \mathfrak{sl}(W)} \subset \mathfrak{sl}(W)_l \oplus \mathfrak{sl}(W)_r$$

on $U(\mathfrak{sl}(W))/I(s_i \rho)$ is locally finite. Therefore $U(\mathfrak{sl}(W))/I(s_i \rho)$ is an $(\mathfrak{sl}(W)_l \oplus \mathfrak{sl}(W)_r, \mathfrak{sl}(W)_d)$ -module. This module affords the central character (χ_ρ, χ_ρ) . The category of $(\mathfrak{sl}(W)_l \oplus \mathfrak{sl}(W)_r, \mathfrak{sl}(W)_d)$ -modules

which afford the central character (χ_ρ, χ_ρ) is equivalent to the category of $(\mathfrak{sl}(W), \mathfrak{b}_W^{\mathfrak{sl}})$ -modules which afford the central character χ_ρ . The equivalence is given by the functor

$$\begin{aligned} \mathbb{H}_{M_\rho} : \mathfrak{sl}(W)_l \oplus \mathfrak{sl}(W)_r\text{-mod} &\rightarrow \mathfrak{sl}(W)\text{-mod}, \\ M &\mapsto M \otimes_{\mathbb{U}(\mathfrak{sl}(W)_r)} M_\rho, \end{aligned}$$

see [BeG, Prop. 5.9]. The action of the $\mathfrak{sl}(W)_l$ -projective functors on the categories of $(\mathfrak{sl}(W)_l \oplus \mathfrak{sl}(W)_r, \mathfrak{sl}(W)_d)$ -modules and $(\mathfrak{sl}(W), \mathfrak{b}_W^{\mathfrak{sl}})$ -modules commutes with the functor \mathbb{H}_{M_ρ} .

Fix $i, j \in \{1, \dots, n_W - 1\}$. Let P_j be the maximal parabolic subgroup of $\mathrm{SL}(V)$ for which the space $\mathrm{span}\langle e_r \rangle_{r \leq j}$ is stable and \mathfrak{p}_j be the Lie algebra of P_j . We have

$$\mathbb{F}_W \mathcal{H}_\rho^\rho = \sum_{i \leq n_W} \mathcal{H}_\rho^{\rho + \varepsilon_i}$$

and

$$\mathcal{H}_\rho^{\mathfrak{s}_i \rho} = \mathcal{H}_{\rho + \varepsilon_{i+1}}^\rho \mathcal{H}_\rho^{\rho + \varepsilon_{i+1}} = \mathcal{H}_{\rho + \varepsilon_{i+1}}^{\mathfrak{s}_i \rho} \mathcal{H}_\rho^{\rho + \varepsilon_{i+1}}$$

for all $i \leq n_W - 1$.

Lemma 7.7. Assume $i \neq j$. Then

$$\mathcal{H}_\rho^{\rho + \varepsilon_{i+1}}(M_\rho / \mathrm{rad}^{1 + \varepsilon_j} M_\rho) = 0 \text{ and } \mathcal{H}_\rho^{\mathfrak{s}_i \rho}(M_\rho / \mathrm{rad}^{1 + \varepsilon_j} M_\rho) = 0.$$

Proof. The module $M_\rho / \mathrm{rad}^{1 + \varepsilon_j} M_\rho$ is an $(\mathfrak{sl}(W), \mathfrak{p}_j)$ -module. Therefore

$$\mathcal{H}_\rho^{\rho + \varepsilon_{i+1}}(M_\rho / \mathrm{rad}^{1 + \varepsilon_j} M_\rho)$$

is an $(\mathfrak{sl}(W), \mathfrak{p}_j)$ -module. If $\mathcal{H}_\rho^{\rho + \varepsilon_{i+1}}(M_\rho / \mathrm{rad}^{1 + \varepsilon_j} M_\rho) \neq 0$, then some simple quotient of $\mathcal{H}_\rho^{\rho + \varepsilon_{i+1}} M_\rho$ is an $(\mathfrak{sl}(W), \mathfrak{p}_j)$ -module. If $i \neq j$, a unique simple quotient of

$$\mathcal{H}_\rho^{\rho + \varepsilon_{i+1}} M_\rho = M_{\rho + \varepsilon_{i+1}}$$

is not an $(\mathfrak{sl}(W), \mathfrak{p}_j)$ -module, i.e. the action of \mathfrak{p}_j on $L_{\rho + \varepsilon_{i+1}}$ is not locally finite. Therefore

$$\mathcal{H}_\rho^{\rho + \varepsilon_{i+1}}(M_\rho / \mathrm{rad}^{1 + \varepsilon_j} M_\rho) = 0 \text{ and } \mathcal{H}_\rho^{\mathfrak{s}_i \rho}(M_\rho / \mathrm{rad}^{1 + \varepsilon_j} M_\rho) = 0.$$

□

As $[P_{\mathfrak{s}_i \rho}] = [M_\rho] + [M_{\mathfrak{s}_i \rho}]$, the projective functor $\mathcal{H}_\rho^{\mathfrak{s}_i \rho}$ is identified with $\mathfrak{s}_i + 1$ under the identification of $\mathrm{PF}\overline{\mathrm{unc}}(\chi_\rho)$ with $\mathbb{Z}[S_{n_W}]$. As the set $\{\mathfrak{s}_i + 1\}_{i \leq n_W - 1}$ generates $\mathbb{Z}[S_{n_W}]$, the set $\{\mathcal{H}_\rho^{\mathfrak{s}_i \rho}\}_{i \leq n_W - 1}$ generates the algebra $\mathrm{PF}\overline{\mathrm{unc}}(\chi_\rho)$.

Corollary 7.8. Let M be an $\mathfrak{sl}(W)$ -module. Suppose that M is annihilated by a submaximal ideal $I(\mathfrak{s}_j \rho)$ for some $j \leq n_W - 1$. Then

$$\mathcal{H}_\rho^{\rho + \varepsilon_{i+1}} M = 0 \text{ for all } i \neq j.$$

Remark 7.9. The notion of left cell [Jo6] (see also [Vo2]) relates nilpotent orbits with Weyl group modules. Moreover, the ordering of primitive ideals is related to left cells via tensoring by finite-dimensional representations. As the projective functors are building blocks of the functors of tensoring with a finite-dimensional representation, the action of the projective functors on a primitive ideal I is closely related [Jo5] with the nilpotent orbit $V(I)$ (see also [Jo3]). Corollary 7.8 is an explicit example of this relationship.

Proposition 7.10. Let M be an infinite-dimensional finitely generated $\mathfrak{sl}(W)$ -module which affords the generalized central character χ_ρ . Then there exists $i \leq n_W - 1$ such that $\mathcal{H}_\rho^{\rho+\varepsilon_{i+1}} M \neq 0$.

Proof. Without loss of generality we assume that $M = \mathfrak{sl}(W)M$. In [BeBe] the authors mention that there exists a Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}(W)$ with nilpotent radical \mathfrak{n} such that $(M/\mathfrak{n}M)^\mathfrak{b} \neq 0$. We have

$$(M/\mathfrak{n}M)^\mathfrak{b} = (M_\rho \otimes_{\mathbb{C}} M)/(\mathfrak{sl}(W)(M_\rho \otimes_{\mathbb{C}} M)) \neq 0 \quad (4).$$

As the sequence

$$\bigoplus_{i \leq n_W - 1} P_{s_i \rho} \rightarrow M_\rho \rightarrow L_\rho \rightarrow 0$$

is exact, the sequence

$$\frac{\bigoplus_{i \leq n_W - 1} P_{s_i \rho} \otimes_{\mathbb{C}} M}{\mathfrak{sl}(W) \bigoplus_{i \leq n_W - 1} P_{s_i \rho} \otimes_{\mathbb{C}} M} \rightarrow \frac{M_\rho \otimes_{\mathbb{C}} M}{\mathfrak{sl}(W)(M_\rho \otimes_{\mathbb{C}} M)} \rightarrow \frac{L_\rho \otimes_{\mathbb{C}} M}{\mathfrak{sl}(W)(L_\rho \otimes_{\mathbb{C}} M)} \rightarrow 0$$

is exact. As

$$(L_\rho \otimes_{\mathbb{C}} M)/\mathfrak{sl}(W)(L_\rho \otimes_{\mathbb{C}} M) = M/\mathfrak{sl}(W)M = 0,$$

formula (4) implies

$$(P_{s_i \rho} \otimes_{\mathbb{C}} M)/\mathfrak{sl}(W)(P_{s_i \rho} \otimes_{\mathbb{C}} M) \neq 0$$

for some $i \leq n_W - 1$. A straightforward computation shows that

$$\frac{\frac{P_{s_i \rho} \otimes_{\mathbb{C}} M}{\mathfrak{sl}(W)(P_{s_i \rho} \otimes_{\mathbb{C}} M)}}{\frac{\mathcal{H}_\rho^{\rho+\varepsilon_{i+1}} M_\rho \otimes_{\mathbb{C}} \mathcal{H}_\rho^{\rho+\varepsilon_{n_W+1-i}} M}{\mathfrak{sl}(W)(\mathcal{H}_\rho^{\rho+\varepsilon_{i+1}} M_\rho \otimes_{\mathbb{C}} \mathcal{H}_\rho^{\rho+\varepsilon_{n_W+1-i}} M)}} = \frac{M_\rho \otimes_{\mathbb{C}} \mathcal{H}_\rho^{s_{n_W} W^{-i} \rho} M}{\mathfrak{sl}(W)(M_\rho \otimes_{\mathbb{C}} \mathcal{H}_\rho^{s_{n_W} W^{-i} \rho} M)}.$$

Therefore $\mathcal{H}_\rho^{s_{n_W} W^{-i} \rho} M \neq 0$. \square

7.2. The case $\mathfrak{g} = \mathfrak{sp}(W \oplus W^*)$. Let $\mathcal{Q}_{\mathfrak{sp}} \subset \mathfrak{sp}(W \oplus W^*)^*$ be the non-trivial nilpotent orbit of minimal dimension. This nilpotent orbit consists of matrices of rank 1. We represent such matrices as $v \otimes v$ with $v \in W \oplus W^*$.

Definition 7.11. Let I be a primitive ideal of $U(\mathfrak{sp}(W \oplus W^*))$. We call I a *Joseph ideal* whenever $V(I) = \bar{\mathcal{Q}}_{\mathfrak{sp}}$.

Lemma 7.12. Let I be a Joseph ideal and let \tilde{L} be a simple $(\mathfrak{sp}(W \oplus W^*), \mathfrak{b}_W^{\mathfrak{sp}})$ -module. Then \tilde{L} is an $(\mathfrak{sp}(W \oplus W^*), \mathfrak{h}_W)$ -bounded module.

Proof. As \tilde{L} is an $(\mathfrak{sp}(W \oplus W^*), \mathfrak{b}_W^{\mathfrak{sp}})$ -module, we have $V(\tilde{L}) \subset \bar{\mathcal{Q}}_{\mathfrak{sp}} \cap (\mathfrak{b}_W^{\mathfrak{sp}})^\perp$. Furthermore,

$$(\mathfrak{b}_W^{\mathfrak{sp}})^\perp \cap \bar{\mathcal{Q}}_{\mathfrak{sp}} = \{x \in \mathfrak{sp}(W \oplus W^*)^* \mid x = v \otimes v \text{ for some } v \in W \oplus W^* \text{ such that } (v, bv) = 0 \text{ for all } b \in \mathfrak{b}_W^{\mathfrak{sp}}\}.$$

The variety $(\mathfrak{b}_W^{\mathfrak{sp}})^\perp \cap \bar{\mathcal{Q}}_{\mathfrak{sp}}$ has the unique irreducible component \mathcal{Q}_1 , where

$$\mathcal{Q}_1 := \{x \in \mathfrak{sp}(W \oplus W^*)^* \mid x = v \otimes v \text{ for some } v \in \text{span}\langle e_1, \dots, e_{n_W} \rangle\}.$$

The dimension of \mathcal{Q}_1 equals to n_W . Let H_W be an irreducible subgroup of $\text{SP}(W \oplus W^*)$ with Lie algebra \mathfrak{h}_W . The variety \mathcal{Q}_1 is H_W -stable and is H_W -spherical. Therefore \tilde{L} is a bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{h}_W)$ -module by Proposition 3.23. \square

Corollary 7.13. a) Let I be a Joseph ideal and let $\bar{\mu}$ be an n_W -tuple such that $I = \text{Ann } L_\mu$. Then L_μ is a bounded \mathfrak{h}_W -module. In particular, $\bar{\mu}$ is a Shale-Weil tuple.

b) Let $\bar{\mu}$ be a Shale-Weil n_W -tuple then $I_{\mathfrak{sp}}(\mu)$ is a Joseph ideal.

Proof. Part a) is trivial. We have $\dim \text{GV}(M) \leq 2\dim V(M)$ by Theorem 3.7. As $\bar{\mu}$ is Shale-Weil, L_μ is bounded and therefore $V(L_\mu)$ is H_W -spherical. Hence $\dim \text{GV}(M) \leq 2n_W$. The only nilpotent orbit which satisfies this inequality is $\mathcal{Q}_{\mathfrak{sp}}$. \square

We recall that σ stands to the reflection with respect to the simple long root of $\mathfrak{sp}(W \oplus W^*)$ for our fixed choice of a Borel subalgebra, see Subsection 3.9.

Lemma 7.14. Let $\bar{\mu}$ be a Shale-Weil tuple. Then $\text{Ann } L_\mu$ is a maximal ideal of $U(\mathfrak{sp}(W \oplus W^*))$.

Proof. Without loss of generality we assume that $\bar{\mu}$ is positive, in particular μ is dominant. Let I be an ideal of $U(\mathfrak{sp}(W \oplus W^*))$ such that $I(\mu) \subsetneq I$. Then $I(\mu)M_\mu \subsetneq IM_\mu$. As I annihilates M_μ/IM_μ , we have $I \subset \text{Ann } L_{\mu'}$ for some simple \mathfrak{h}_W -bounded module $L_{\mu'}$. Therefore either $\mu' = \mu$ or $\mu' = \sigma\mu$. Hence $I \subset I(\mu)$ and $I = I(\mu)$. \square

Let $\bar{\mu}$ be a Shale-Weil n_W -tuple. Let $\bar{\mu}'$ be an n_W -tuple and let $\mu' \in \mathfrak{h}_W^*$ be the corresponding weight.

Lemma 7.15. Suppose $\mathcal{H}_\mu^{\mu'} L_\mu \neq 0$. Then μ' is a Shale-Weil tuple.

Proof. The $\mathfrak{sp}(W \oplus W^*)$ -module L_μ is \mathfrak{h}_W -bounded and therefore $\mathcal{H}_\mu^{\mu'} L_\mu$ is \mathfrak{h}_W -bounded. As L_μ is a quotient of M_μ , $\mathcal{H}_\mu^{\mu'} L_\mu$ is a quotient of $\mathcal{H}_\mu^{\mu'} M_\mu$. As $L_{\mu'}$ is the unique simple quotient of $\mathcal{H}_\mu^{\mu'} M_\mu$, $L_{\mu'}$ is an \mathfrak{h}_W -bounded $\mathfrak{sp}(W \oplus W^*)$ -module. Therefore μ' is a Shale-Weil tuple by Theorem 3.15. \square

Corollary 7.16. Let M be an $\mathfrak{sp}(W \oplus W^*)$ -module such that $I_{\mathfrak{sp}}(\mu) = \text{Ann} M$. Suppose $\mathcal{H}_\mu^{\mu'} M \neq 0$. Then μ' is a Shale-Weil n_W -tuple.

8. MODULES OF SMALL GROWTH

Definition 8.1. Let M be a \mathfrak{g} -module. We say that M is $Z(\mathfrak{g})$ -finite if $\dim Z(\mathfrak{g})m < \infty$ for all $m \in M$, where

$$Z(\mathfrak{g})m := \{m' \in M \mid m' = zm \text{ for some } m \in M \text{ and } z \in Z(\mathfrak{g})\}.$$

8.1. **The case $\mathfrak{g} = \mathfrak{sl}(W)$.**

Proposition 8.2. If M is an infinite-dimensional finitely generated $\mathfrak{sl}(W)$ -module then $\dim V(M) \geq n_W - 1$.

Proof. The variety $GV(M)$ is a union of several nilpotent $SL(W)$ -orbits in $\mathfrak{sl}(W)^*$. There is a unique nonzero nilpotent orbit of minimal dimension equal to $2(n_W - 1)$. As M is infinite-dimensional, $\dim V(M) \geq \frac{1}{2}\dim GV(M) \geq n_W - 1$ by Theorem 3.7. \square

Definition 8.3. Let M be a finitely generated $\mathfrak{sl}(W)$ -module which is $Z(\mathfrak{sl}(W))$ -finite. We say that M is of *small growth* if $\dim V(M) \leq n_W - 1$, i.e. if $\dim V(M)$ equals either $n_W - 1$ or 0.

The $\mathfrak{sl}(W)$ -modules of small growth form a full subcategory of the category of $\mathfrak{sl}(W)$ -modules. This subcategory is stable under the projective functors.

We recall that a finitely generated \mathfrak{g} -module M has a graded version $\text{gr}M$ which is a finitely generated $S(\mathfrak{g})$ -module. The sum of ranks² of $\text{gr}M$ over the function rings of all irreducible components of $V(M)$ of maximal dimension is called *the Bernstein number of M* (see also [KL, p. 78]), and we denote this number $b(M)$.

Proposition 8.4. Any $\mathfrak{sl}(W)$ -module M of small growth is of finite length.

Proof. As M is finitely generated, it is enough to check the descending chain condition for M . Let

$$\dots \subset M_{-i} \subset \dots \subset M_{-0} = M$$

be a strictly descending chain of $\mathfrak{sl}(W)$ -submodules. These $\mathfrak{sl}(W)$ -submodules M_{-i} are finitely generated. We have $\dim V(M_{-i}) = n_W - 1$ for all $i \in \mathbb{Z}_{\geq 0}$ or $\dim V(M_{-i}) = 0$ for some $i \in \mathbb{Z}_{\geq 0}$. In the second case $\dim M_{-i} < \infty$ and therefore the chain $\dots \subset M_{-i}$ stabilizes.

Assume that $\dim V(M_{-i}) = n_W - 1$ for all $i \in \mathbb{Z}_{\geq 0}$. If $i \geq j$ then $b(M_{-i}) \leq b(M_{-j})$, i.e. the sequence $\{b(M_{-i})\}_{i \in \mathbb{Z}_{\geq 0}}$ is decreasing. Therefore there exists $i \in \mathbb{Z}_{\geq 0}$ such that $b(M_{-i}) = b(M_{-j})$ for all $j \geq$

²The rank of finitely generated module over a commutative ring is the rank of a maximal free submodule.

i . Therefore $\dim V(M_{-i}/M_{-j}) < n_W - 1$ and M_{-i}/M_{-j} is a finite-dimensional $\mathfrak{sl}(W)$ -module for all $j \geq i$. We have an inclusion

$$M_{-i}/\bigcap_{j \geq i} M_{-j} \hookrightarrow \bigoplus_{j \geq i} M_{-i}/M_{-j}.$$

The right-hand side is a direct sum of a finite-dimensional $\mathfrak{sl}(W)$ -modules. As M_{-i} is $Z(\mathfrak{sl}(W))$ -finite and finitely generated, only a finite number of simple $\mathfrak{sl}(W)$ -modules appear in this direct sum. Therefore $M_{-i}/\bigcap_{j \geq i} M_{-j}$ is a finite-dimensional $\mathfrak{sl}(W)$ -module. \square

Proposition 8.5. Let I be the annihilator of a simple infinite-dimensional $\mathfrak{sl}(W)$ -module M of small growth. Then I is a Joseph ideal.

Proof. As $\dim \text{GV}(M) \leq 2\dim V(M)$ (Theorem 3.7),

$$\dim \text{GV}(M) \leq 2(n_W - 1).$$

As M is infinite-dimensional, I is a Joseph ideal. \square

Let \mathcal{J} be the set of simple $\mathfrak{sl}(W)$ -modules of small growth, and $\langle \mathcal{J} \rangle$ be the vector space generated by the set \mathcal{J} . The action of the projective functors on the category of modules of small growth defines an action of the projective functors on $\langle \mathcal{J} \rangle$ by linear operators. As any $\mathfrak{sl}(W)$ -module M of small growth has finite length, M defines the vector $[M] \in \langle \mathcal{J} \rangle$ which is a sum of the simple M -subquotients with their multiplicities in M .

Let $\bar{\lambda}$ be a decreasing n_W -tuple and \mathcal{J}_λ be the set of simple $\mathfrak{sl}(W)$ -modules of small growth annihilated by $\text{Ker} \chi_\lambda$, and $\langle \mathcal{J}_\lambda \rangle$ be the free vector space generated by \mathcal{J}_λ . The action of $\text{PFunc}(\chi_\lambda)$ defines an action of S_{n_W} on $\langle \mathcal{J}_\lambda \rangle$, see Proposition 3.30. We recall that \mathbb{C}^{sgn} is the sign representation of S_{n_W} . Set $\langle \mathcal{J}_\lambda \rangle^{sgn} := \langle \mathcal{J}_\lambda \rangle \otimes_{\mathbb{C}} \mathbb{C}^{sgn}$. As vector spaces $\langle \mathcal{J}_\lambda \rangle$ and $\langle \mathcal{J}_\lambda \rangle^{sgn}$ are identical.

Lemma 8.6. An S_{n_W} -module $\langle \mathcal{J}_\lambda \rangle^{sgn}$ is a direct sum of copies of \mathbb{C}^{n_W-1} and \mathbb{C} .

Proof. Without loss of generality we assume that $\lambda = \rho$ (ρ is defined in Subsection 3.8). The action of $\mathcal{H}_\rho^{s_k \rho}$ on \mathcal{J}_λ corresponds to multiplication by $(s_k - 1)$ on $\langle \mathcal{J}_\lambda \rangle^{sgn}$ (cf. Corollary 7.8). Let M be a simple $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -module of small growth and k be a number such that $\text{Ann} M = I(s_k \rho)$. Then $\mathcal{H}_\rho^{s_l \rho} M = 0$ for all $k \neq l$ (Corollary 7.8). As the simple modules generate $\langle \mathcal{J}_\lambda \rangle^{sgn}$,

$$\langle \mathcal{J}_\lambda \rangle^{sgn} = \sum_{i \leq n_W-1} \langle \mathcal{J}_\lambda \rangle_i^{sgn}.$$

Therefore as an S_{n_W} -module $\langle \mathcal{J}_\lambda \rangle^{sgn}$ is a direct sum of copies of \mathbb{C}^{n_W-1} and \mathbb{C} (Lemma 3.33). \square

Lemma 8.7. Let M be a simple $\mathfrak{sl}(W)$ -module of small growth annihilated by $\text{Ker}\chi_\rho$ and $k \in \{1, \dots, n_W - 1\}$ be a number. Then, for some $j \in \{1, 2\}$,

$$[\mathcal{H}_\rho^{\rho+\varepsilon_k} M] = j[M']$$

for a simple $\mathfrak{sl}(W)$ -module M' of small growth.

Proof. Assume that $\mathcal{H}_\rho^{\rho+\varepsilon_k} M$ is nonzero. The identity homomorphism

$$\text{Id}: \mathcal{H}_\rho^{\rho+\varepsilon_k} M \rightarrow \mathcal{H}_\rho^{\rho+\varepsilon_k} M$$

gives rise to a nonzero homomorphism

$$M \rightarrow \mathcal{H}_{\rho+\varepsilon_k}^\rho \mathcal{H}_\rho^{\rho+\varepsilon_k} M = \mathcal{H}_\rho^{\text{sk}\rho} M.$$

Let M' be a simple subquotient of $\mathcal{H}_\rho^{\rho+\varepsilon_k} M$ such that M is a subquotient of $\mathcal{H}_{\rho+\varepsilon_k}^\rho M'$. As $\mathcal{H}_\rho^{\rho+\varepsilon_k} \mathcal{H}_{\rho+\varepsilon_k}^\rho = 2\text{Id}$, we have $\mathcal{H}_\rho^{\rho+\varepsilon_k} \mathcal{H}_{\rho+\varepsilon_k}^\rho M' = M' \oplus M'$. Therefore $\mathcal{H}_\rho^{\rho+\varepsilon_k} M$ is semisimple of length 2 or less and M' is, up to isomorphism, the only simple constituent of M . \square

Lemma 8.8. Let M be a simple $\mathfrak{sl}(W)$ -module of small growth annihilated by $I(\rho + \varepsilon_k)$. Then, for some $j \in \{1, 2\}$,

$$[\mathcal{H}_{\rho+\varepsilon_k}^\rho M] = j[M'] + [T],$$

where M' is a simple $\mathfrak{sl}(W)$ -module of small growth and T is some $\mathfrak{sl}(W)$ -module of small growth such that

$$\mathcal{H}_\rho^{\rho+\varepsilon_k} T = 0, \mathcal{H}_\rho^{\rho+\varepsilon_k} M' = \frac{2}{j} M.$$

Proof. Let $0 \subset M^0 \subset \dots \subset M^i = M$ be a composition series of $\mathcal{H}_{\rho+\varepsilon_k}^\rho M$. As $\mathcal{H}_\rho^{\rho+\varepsilon_k} \mathcal{H}_{\rho+\varepsilon_k}^\rho M = 2\text{Id}$, the number of simple subquotients M^{l+1}/M^l such that $\mathcal{H}_\rho^{\rho+\varepsilon_k}(M^{l+1}/M^l)$ is nonzero does not exceed 2. Assume that there is a unique such subquotient M' . Then there exists a semisimple $\mathfrak{sl}(W)$ -module T such that

$$\mathcal{H}_{\rho+\varepsilon_k}^\rho [M] = [M'] + [T],$$

and

$$[\mathcal{H}_\rho^{\rho+\varepsilon_k} M'] = 2[M], \mathcal{H}_\rho^{\rho+\varepsilon_k} [T] = 0.$$

Suppose that such a subquotient is not unique. Then there exist two simple $\mathfrak{sl}(W)$ -modules \hat{M}_1 and \hat{M}_2 and a semisimple $\mathfrak{sl}(W)$ -module T such that

$$[\mathcal{H}_{\rho+\varepsilon_k}^\rho M] = [\hat{M}_1] + [\hat{M}_2] + [T] \text{ and } [\mathcal{H}_\rho^{\rho+\varepsilon_k} \hat{M}_1] = [\mathcal{H}_\rho^{\rho+\varepsilon_k} \hat{M}_2] = [M].$$

We have $\text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_\rho^{\rho+\varepsilon_k} \hat{M}_1, \mathcal{H}_\rho^{\rho+\varepsilon_k} \hat{M}_1) =$

$$\text{Hom}_{\mathfrak{sl}(W)}(\hat{M}_1, \mathcal{H}_{\rho+\varepsilon_k}^\rho \mathcal{H}_\rho^{\rho+\varepsilon_k} \hat{M}_1) = \text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_{\rho+\varepsilon_k}^\rho \mathcal{H}_\rho^{\rho+\varepsilon_k} \hat{M}_1, \hat{M}_1).$$

Then there exists a sequence of nonzero $\mathfrak{sl}(W)$ -homomorphisms

$$\mathcal{H}_\rho^{s_k \rho} \hat{M}_1 \rightarrow \hat{M}_1 \rightarrow \mathcal{H}_\rho^{s_k \rho} \hat{M}_1.$$

Therefore there exists an $\mathfrak{sl}(W)$ -homomorphism $\phi_1 : \mathcal{H}_{\rho+\varepsilon_k}^\rho M \rightarrow \mathcal{H}_{\rho+\varepsilon_k}^\rho M$ whose image is \hat{M}_1 . In the same way there exists an $\mathfrak{sl}(W)$ -homomorphism

$$\phi_2 : \mathcal{H}_{\rho+\varepsilon_k}^\rho M \rightarrow \mathcal{H}_{\rho+\varepsilon_k}^\rho M$$

with image \hat{M}_2 . Assume that $\hat{M}_1 \not\cong \hat{M}_2$. Then both compositions $\phi_1 \circ \phi_2$ and $\phi_2 \circ \phi_1$ equal zero. We have

$$\begin{aligned} & \text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_{\rho+\varepsilon_k}^\rho M, \mathcal{H}_{\rho+\varepsilon_k}^\rho M) = \\ & \text{Hom}_{\mathfrak{sl}(W)}(M, \mathcal{H}_\rho^{\rho+\varepsilon_k} \mathcal{H}_{\rho+\varepsilon_k}^\rho M) = \text{Hom}_{\mathfrak{sl}(W)}(M, \mathcal{H}_{\rho+\varepsilon_k}^{\rho+\varepsilon_k} M). \end{aligned}$$

As M is simple, $\text{Hom}_{\mathfrak{sl}(W)}(M, \mathcal{H}_{\rho+\varepsilon_k}^{\rho+\varepsilon_k} M) = \text{Hom}_{\mathfrak{sl}(W)}(\text{Id}, \mathcal{H}_{\rho+\varepsilon_k}^{\rho+\varepsilon_k})$, where the right-hand side refers to homomorphisms of functors. Therefore

$$\text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_{\rho+\varepsilon_k}^\rho M, \mathcal{H}_{\rho+\varepsilon_k}^\rho M) = \text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_{\rho+\varepsilon_k}^\rho, \mathcal{H}_{\rho+\varepsilon_k}^\rho).$$

However,

$$\text{Hom}_{\mathfrak{sl}(W)}(\mathcal{H}_{\rho+\varepsilon_k}^\rho, \mathcal{H}_{\rho+\varepsilon_k}^\rho) = \text{Hom}_{\mathfrak{sl}(W)}(\mathbb{P}_{s_k \rho}, \mathbb{P}_{s_k \rho}) = \mathbb{C}[x]/(x^2).$$

As $\phi_1 \circ \phi_2 = 0$, ϕ_1 and ϕ_2 are identified with $\alpha_1 x$ and $\alpha_2 x$ for some constants $\alpha_1, \alpha_2 \in \mathbb{C}$. Then α_1 is proportional to α_2 and therefore

$$\hat{M}_1 \cong \hat{M}_2.$$

□

Corollary 8.9. There is a natural bijection between the set of simple infinite-dimensional $\mathfrak{sl}(W)$ -modules of small growth annihilated by $I(s_k \rho)$ and the set of simple $\mathfrak{sl}(W)$ -modules of small growth annihilated by $I(\rho + \varepsilon_k)$.

Lemma 8.10. The subspace of S_{n_W} -invariants in $\langle \mathcal{J}_\lambda \rangle$ is one-dimensional and is generated by the class of the simple finite-dimensional module.

Proof. Without loss of generality we assume that $\lambda = \rho$. Let $\{\alpha_i\}$ be numbers such that $\sum \alpha_i [M_i]$ is non-zero and S_{n_W} -invariant. Then $\mathcal{H}_\rho^{\rho+\varepsilon_k} \sum \alpha_i [M_i] = 0$, i.e. $\sum \alpha_i \mathcal{H}_\rho^{\rho+\varepsilon_k} M_i = 0$. Therefore $\alpha_i = 0$ for all infinite-dimensional simple modules annihilated by $I(s_k \rho)$. Hence $\alpha_i \neq 0$ implies that M_i is finite-dimensional. □

Corollary 8.11. For any $i, j \in \{1, \dots, n_W - 1\}$ there is a natural bijection between the set of simple infinite-dimensional $\mathfrak{sl}(W)$ -modules of small growth annihilated by $I(s_i \rho)$ and the set of simple infinite-dimensional $\mathfrak{sl}(W)$ -modules of small growth annihilated by $I(s_j \rho)$.

The following statement can be considered as a weak analogue of Theorem 3.12.

Corollary 8.12. Let $\bar{\lambda}_1, \bar{\lambda}_2$ be semi-decreasing n_W -tuples such that

$$m(\lambda_1) = m(\lambda_2).$$

There is a natural bijection between the set of simple infinite-dimensional $\mathfrak{sl}(W)$ -modules of small growth annihilated by $I(\lambda_1)$ and the set of simple infinite-dimensional $\mathfrak{sl}(W)$ -modules of small growth annihilated by $I(\lambda_1)$ and $I(\lambda_2)$.

Proof. This follows from Lemma 3.31, Corollary 8.9 and Theorem 3.12. \square

8.2. The case $\mathfrak{g} = \mathfrak{sp}(W \oplus W^*)$. All proofs for this subsection are similar to their counterparts in the previous subsection. We leave them to the reader.

Proposition 8.13. If M is an infinite-dimensional finitely generated $\mathfrak{sp}(W \oplus W^*)$ -module, then $\dim V(M) \geq n_W$.

Definition 8.14. Let M be a finitely generated $\mathfrak{sp}(W \oplus W^*)$ -module which is $Z(\mathfrak{sp}(W \oplus W^*))$ -finite. We say that M is of *small growth* if $\dim V(M) \leq n_W$, i.e. if $\dim V(M)$ equals either n_W or 0.

The $\mathfrak{sl}(W)$ -modules of small growth form a full subcategory of the category of $\mathfrak{sp}(W \oplus W^*)$ -modules. This subcategory is stable under the projective functors.

Proposition 8.15. Any $\mathfrak{sp}(W \oplus W^*)$ -module M of small growth is of finite length.

Proposition 8.16. Let I be an annihilator of a simple infinite-dimensional $\mathfrak{sl}(W)$ -module M of small growth. Then I is a Joseph ideal.

Let μ be a Shale-Weil tuple. The set $\text{PFunc}(\chi_\mu)$ is generated by one involutive element $\mathcal{H}_\mu^{\sigma\mu}$ which preserve the set of simple modules. We recall that $\bar{\mu}_0$ is an n_W -tuple $(n_W - \frac{1}{2}, n_W - \frac{3}{2}, \dots, \frac{1}{2})$.

Proposition 8.17. The categories of $\mathfrak{sp}(W \oplus W^*)$ -modules of small growth annihilated by $I(\mu_0)$ and $I(\mu)$ are equivalent.

9. CATEGORIES OF $(\mathfrak{sl}(S^2V), \mathfrak{sl}(V))$ AND $(\mathfrak{sl}(\Lambda^2V), \mathfrak{sl}(V))$ -MODULES

9.1. Construction of $(D(W), \mathfrak{k})$ -modules. We recall that K is a connected reductive Lie group with Lie algebra \mathfrak{k} and a Borel subgroup B . Let \mathcal{G} be an associative algebra and $\psi: U(\mathfrak{k}) \rightarrow \mathcal{G}$ be a homomorphism of associative algebras injective on \mathfrak{k} . We identify \mathfrak{k} with its image. Any \mathcal{G} -module can be considered as a \mathfrak{k} -module.

Definition 9.1. (cf. Def. 1.1) A $(\mathcal{G}, \mathfrak{k})$ -*module* is a \mathcal{G} -module which is locally finite as a \mathfrak{k} -module.

Definition 9.2. (cf. Def. 1.5) A *bounded* $(\mathcal{G}, \mathfrak{k})$ -*module* is a $(\mathcal{G}, \mathfrak{k})$ -module bounded as a \mathfrak{k} -module.

Let W be a spherical K -module. There is an obvious homomorphism $U(\mathfrak{k}) \rightarrow D(W)$ (see the discussion at the end of Subsection 3.9.1).

The algebra of differential operators $D(W)$ has a natural filtration by degree

$$0 \subset \mathbb{C}[W] \subset D_1 \subset \dots \subset D(W) = \bigcup_{i \in \mathbb{Z}_{\geq 0}} D_i.$$

The associated graded algebra

$$\text{gr } D(W) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} (D_{i+1}/D_i)$$

is isomorphic to $\mathbb{C}[T^*W]$. Let M be $D(W)$ -module with a finite-dimensional space of generators M_{gen} . This defines a filtration

$$\{D_i M_{gen}\}_{i \in \mathbb{Z}_{\geq 0}}$$

of M . The associated graded space

$$\text{gr } M := \bigoplus_{i \in \mathbb{Z}_{\geq 0}} (D_{i+1} M_{gen}/D_i M_{gen})$$

is a finitely generated $\mathbb{C}[T^*W]$ -module. We denote the support of this $\mathbb{C}[T^*W]$ -module as $\mathcal{V}(\text{Loc}M)$. As finitely generated $D(W)$ -modules are in a natural one-to-one correspondence with coherent $\mathcal{D}(W)$ -modules, M corresponds to the $\mathcal{D}(W)$ -module $\text{Loc}M$. The variety $\mathcal{V}(\text{Loc}M)$ coincides with the singular support of $\text{Loc}M$ (see Section 1) and is conical and coisotropic in T^*W . In what follows we assume that M is a finitely generated $(D(W), \mathfrak{k})$ -module.

Proposition 9.3. a) The module M is bounded if and only if all irreducible components of $\mathcal{V}(\text{Loc}M)$ are K -spherical.

b) If the equivalent conditions of a) are satisfied, any irreducible component \tilde{V} of $\mathcal{V}(\text{Loc}M)$ is a conical Lagrangian subvariety of T^*W .

Proof. The proof repeats the proof of Proposition 3.23 verbatim. \square

As explained in Section 2, to a holonomic $\mathcal{D}(W)$ -module \mathcal{M} one assigns a pair (S, Y) , where $S \subset W$ is an irreducible subvariety and Y

is an $\mathcal{O}(S)$ -coherent $\mathcal{D}(S)$ -module. Let K_{ss} be the connected simply-connected semisimple group with Lie algebra $[\mathfrak{k}, \mathfrak{k}]$, $A(K)$ be the center of K and $\mathfrak{a} \subset \mathfrak{k}$ be the Lie algebra of $A(K)$.

Lemma 9.4. The module M is \mathfrak{k} -bounded. If M is simple then M is \mathfrak{k} -multiplicity free. The $\mathcal{D}(W)$ -module $\text{Loc}M$ is holonomic and has regular singularities with respect to the stratification by K -orbits on W .

Proof. Let $\phi_{\mathfrak{k}}^{-1}(0) \subset T^*W$ be the union of conormal bundles to all K -orbits in W . As \mathfrak{k} acts locally finitely on M , $\mathcal{V}(M) \subset \phi_{\mathfrak{k}}^{-1}(0)$. Since the dimension of any irreducible component of $\phi_{\mathfrak{k}}^{-1}(0)$ is n_W and $\dim \mathcal{V}(M) \geq n_W$ (Theorem 3.5), $\mathcal{V}(M)$ is the union of irreducible components of $\phi_{\mathfrak{k}}^{-1}(0)$. Therefore $\text{Loc}M$ is holonomic.

Let \tilde{V} be an irreducible component of $\mathcal{V}(M)$. It is the total space of the conormal bundle to a K -orbit $S \subset W$. As W is a K -spherical variety, \tilde{V} is a spherical variety [Pan]. Therefore M is a \mathfrak{k} -bounded module by Proposition 9.3. Assume M is simple. Since the algebra $D(W)^{\mathfrak{k}}$ is commutative [Kn1], M is multiplicity free by [PS2, Corollary 5.8].

We have to prove that $\text{Loc}M$ has regular singularities. Without loss of generality we assume that M is simple. Let (S, Y) be the pair corresponding to $\text{Loc}M$. Then S is K -stable. As W is K -spherical, S is a K -orbit. Since $\text{Loc}M$ is K_{ss} -equivariant, Y is also K_{ss} -equivariant. As \mathfrak{a} acts locally finitely on M , \mathfrak{a} acts locally finitely on $\Gamma(A(K)_s, M|_{A(K)_s})$ for any point $s \in S$. Since K is a quotient of $K_{ss} \times A(K)$, Y has regular singularities. Therefore $\text{Loc}M$ has regular singularities (cf. [Bo, 12.11]). \square

Lemma 9.5. If a holonomic $D(W)$ -module $\text{Loc}M$ has regular singularities with respect to the stratification of W by K -orbits, then M is \mathfrak{k} -locally finite.

Proof. Without loss of generality assume that M is simple. Let (S, Y) be the pair corresponding to $\text{Loc}M$. Then S is a K -orbit. As S is a homogeneous $K_{ss} \times A(K)$ -space, the sheaf Y is K_{ss} -equivariant [Bo, 12.11]. Hence $\text{Loc}M$ is K_{ss} -equivariant and therefore $[\mathfrak{k}, \mathfrak{k}]$ acts locally finitely on M . As $Y|_{A(K)_s}$ has regular singularities for any point $s \in S$, \mathfrak{a} acts locally finitely on $\Gamma(A(K)_s, M|_{A(K)_s})$ for any point $s \in S$. Therefore \mathfrak{a} acts locally finitely on M . As $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}] \oplus \mathfrak{a}$, \mathfrak{k} acts locally finitely on M . \square

By a famous theorem of M. Kashiwara [Ka], the category of holonomic $D(W)$ -modules with regular singularities with respect to a given stratification is equivalent to the category of perverse sheaves on W with respect to the same stratification.

Corollary 9.6. The category of bounded $(D(W), \mathfrak{k})$ -modules is equivalent to the category of perverse sheaves on W with respect to the stratification of W by K -orbits.

The categories of such perverse sheaves has been described explicitly for the stratifications of S^2V and Λ^2V by the $GL(V)$ -orbits [BG]. In particular, in [BG] the authors have described the simple objects of this categories.

The following theorem provides a checkable condition under which the functions $[M_{1,2} : \cdot]_{\mathfrak{k}}$ are pairwise disjoint (see Section 2) for non-isomorphic simple $(D(W), \mathfrak{k})$ modules M_1 and M_2 . The proof, which we present below, follows essentially the proof of [PS2, Corollary 5.8].

Theorem 9.1. Assume $D(W)^{\mathfrak{k}}$ is the image of $Z(\mathfrak{k})$. Let E be a simple \mathfrak{k} -module. Then there exists at most one simple $(D(W), \mathfrak{k})$ -module M such that $\text{Hom}_{\mathfrak{k}}(E, M) \neq 0$.

To prove Theorem 9.1 we need the following preparatory lemma.

Lemma 9.7. Let E be a simple finite-dimensional \mathfrak{k} -module. Then for any $n \in \mathbb{Z}_{\geq 1}$ and $\phi \in \text{Hom}(E, E \otimes \mathbb{C}^n)$ there exist $\phi_1, \dots, \phi_s \in \text{Hom}_{\mathfrak{k}}(E, E \otimes \mathbb{C}^n)$ such that $\phi_i \in U(\mathfrak{k})\phi U(\mathfrak{k})$ and $\phi \in +_i U(\mathfrak{k})\phi_i U(\mathfrak{k})$.

Proof. We omit the proof. □

Proof of Theorem 9.1. Let R be the isotypic component of E in $D(W) \otimes_{U(\mathfrak{k})} E$ and pr denote the projection $D(W) \otimes_{U(\mathfrak{k})} E \rightarrow R$. For any $g \in D(W)$ there exists $g' \in D(W)$ such that $\text{pr}(gm) = g'm$ for any $m \in E$. Therefore g' induces a homomorphism ϕ from E to R . Let ϕ_i be as in Lemma 9.7. Then for any i there exist $g_i \in D(W)$ such that $g_i m = \phi_i m$ for any $m \in E$. We have $[g_i, k]m = 0$ for any $k \in \mathfrak{k}$ and $m \in E$. As the adjoint action of \mathfrak{k} on $D(W)$ is semisimple, $g_i = c_i + a_i$ for some $c_i, a_i \in D(W)$ such that $c_i \in D(W)^{\mathfrak{k}}$ and $a_i m = 0$ for all $m \in E$. Hence $g_i E \subset U(\mathfrak{k})E$ and therefore $R = E$.

Let \tilde{M} be the unique maximal submodule of $D(W) \otimes_{U(\mathfrak{k})} E$ which does not contain $R = E$. Then the quotient $D(W) \otimes_{U(\mathfrak{k})} E / \tilde{M}$ is simple and, up to isomorphism, is the unique simple $(D(W), \mathfrak{k})$ -module for which $\dim \text{Hom}_{\mathfrak{k}}(E, \cdot) \neq 0$. □

Let $\mathfrak{k} = \mathfrak{gl}(V)$ and let W equal Λ^2V or S^2V . Then $D(W)^{\mathfrak{k}}$ is the image of $Z(\mathfrak{k})$ [HU]. Therefore, for non-isomorphic simple $(D(W), \mathfrak{k})$ -modules $M_{1,2}$, the functions $[M_{1,2} : \cdot]_{\mathfrak{k}}$ are pairwise disjoint, i.e. the product of these functions is the zero function (see Section 2).

9.2. A useful algebraic trick. We recall that E is the Euler operator and that $\{D^i(W)\}_{i \in \mathbb{Z}}$ are the E -eigenspaces of $D(W)$ (see the discussion at the end of Subsection 3.9.1). In this section we relate the category of $(D(W), E)$ -modules to the categories of $D^0(W)$ and $D^{\bar{0}}(W)$ -modules (see also [PS2]). Let M be a locally finite E -module. Fix $t \in \mathbb{C}$. Set

$$M_t := \{m \in M \mid (E - t)^n m = 0 \text{ for some } n \in \mathbb{Z}_{\geq 0}\}.$$

If M is $D(W)$ -module which is semisimple an an E -module, then M_t is a $D^t\mathbb{P}(W)$ -module.

Definition 9.8. We say that a $D(W)$ -module M is a $D(W)$ -module with monodromy $e^{2\pi it}$ if $M = \bigoplus_{j \in \mathbb{Z}} M_{t+j}$.

Any simple $D(W)$ -module with a locally finite action of E is a module with a monodromy. Let δ_0 be the delta function at $0 \in W$. The functions 1 and δ_0 generate $D(W)$ -modules which we denote $D(W)1$ and $D(W)\delta_0$.

Lemma 9.9. Let M be a simple $D(W)$ -module with monodromy $e^{2\pi it}$. Assume that

$$\dim M_t < \infty \text{ and } n_W \geq 2.$$

Then M is isomorphic to one of the following modules:

$$0, D(W)1, D(W)\delta_0.$$

Proof. Let $\dim M_t \neq 0$. Then the action of $\mathfrak{sl}(W)$ on M_t is locally finite and therefore M is a locally finite $\mathfrak{sl}(W)$ -module. As W is a spherical $SL(W)$ -module, M is a holonomic $D(W)$ -module with regular singularities by Lemma 9.4. Let Sh be the corresponding to M simple perverse sheaf. Then Sh is constructible with respect to the stratification $\{\{0\}, W - \{0\}\}$ of W . As

$$\pi_1(\{0\}) = \pi_1(W - \{0\}) = 0,$$

Sh is the simple perverse sheaf corresponding to $\{0\}$ or $W - \{0\}$, in both cases the local system being trivial. The latter two perverse sheaves correspond to the simple $D(W)$ -modules $D(W)\delta_0$ and $D(W)1$, respectively. Therefore M is isomorphic to 0 , to $D(W)1$, or to $D(W)\delta_0$.

Assume that $M_t = 0$. Then, for all $i \leq n_W$, the action of e_i on M is locally finite, or the action of ∂_{e_i} is locally finite for all i . In the first case M is isomorphic to $D(W)\delta_0$ by Kashiwara's theorem. In the second case $D(W)$ is isomorphic to $D(W)1$ by Kashiwara's theorem (we can interchange the roles of e_i and ∂_{e_i}). \square

Lemma 9.10. a) Let M be a nonzero simple $D(W)$ -module with monodromy $e^{2\pi it}$ which is not isomorphic to $D(W)1$ and $D(W)\delta_0$. Then M_t is a simple infinite-dimensional $D^t\mathbb{P}(W)$ -module.

b) Let M_{res} be a simple infinite-dimensional $D^t\mathbb{P}(W)$ -module. Then the $D(W)$ -module

$$D(W) \otimes_{D^0(W)} M_{res}$$

has a unique simple quotient \tilde{M} . This quotient is not isomorphic to $D(W)1$ or to $D(W)\delta_0$ and has monodromy $e^{2\pi it}$. Moreover, there is an isomorphism $\tilde{M}_t = M_{res}$ of $D^t\mathbb{P}(W)$ -modules.

Proof. a) By Lemma 9.9 we have $\dim M_t = \infty$. Let $\tilde{M}_t \subset M_t$ be a $D^t\mathbb{P}(W)$ -submodule of M_t . The inclusion $\tilde{M}_t \rightarrow M_t$ induces a homomorphism

$$\text{inc}: D(W) \otimes_{D^0(W)} \tilde{M}_t \rightarrow M$$

of \mathbb{Z} -graded $D(W)$ -modules. We have

$$(D(W) \otimes_{D^0(W)} \tilde{M}_t)_t = \tilde{M}_t,$$

and the map

$$\text{inc}_t: (D(W) \otimes_{D^0(W)} \tilde{M}_t)_t \rightarrow M_t$$

is the inclusion map. As M is simple, inc equals 0 or is surjective and therefore \tilde{M}_t is isomorphic to 0 or to M_t . Therefore a) follows.

Any submodule \tilde{M} of $D(W) \otimes_{D^0(W)} M_{res}$ either intersects M_{res} trivially, or coincides with $D(W) \otimes_{D^0(W)} M_{res}$. Therefore $D(W) \otimes_{D^0(W)} M_{res}$ has a unique maximal submodule \tilde{M} and this submodule is isomorphic to a direct sum of copies of $D(W)1$ and $D(W)\delta_0$. The quotient

$$\tilde{M} := (D(W) \otimes_{D^0(W)} M_{res}) / \tilde{M}$$

is simple and $\tilde{M}_t = M_{res}$. As M_{res} is infinite-dimensional, \tilde{M} is not isomorphic to $D(W)1$ or to $D(W)\delta_0$. \square

This reduces the study of (simple) $D^t\mathbb{P}(W)$ -modules to a study of some (simple) modules over the Weyl algebra. Sometimes the latter is much easier.

We recall that $D(W) = D^{\bar{0}}(W) \oplus D^{\bar{1}}(W)$ is a \mathbb{Z}_2 -graded algebra. We say that a $D^{\bar{0}}(W)$ -module *has half-monodromy* $e^{\pi it}$ if M is a direct sum $\bigoplus_{j \in \mathbb{Z}} M_{t+2j}$. We denote by

$$(D(W), E)\text{-mod}$$

the category of $D(W)$ -modules with a locally finite action of E and by

$$(D(W), E)\text{-mod}^{e^{2\pi it}}$$

the subcategory of $(D(W), E)\text{-mod}$ of modules with monodromy $e^{2\pi it}$ for any $t \in \mathbb{C}$. Similarly we define

$$(D^{\bar{0}}(W), E)\text{-mod} \text{ and } (D^{\bar{0}}(W), E)\text{-mod}^{e^{\pi it}}.$$

The sign ' \cong ' stands for equivalence of categories. For any $t \in \mathbb{C}$ we have two functors:

$$\begin{aligned} \text{Res}_{e^{2\pi it}}^{e^{\pi it}} : D(W)\text{-mod}^{e^{2\pi it}} &\rightarrow D^{\bar{0}}(W)\text{-mod}^{e^{\pi it}}, \\ M &\mapsto \bigoplus_{j \in \mathbb{Z}} M_{t+2j}; \\ \text{Ind}_{e^{\pi it}}^{e^{2\pi it}} : D^{\bar{0}}(W)\text{-mod}^{e^{\pi it}} &\rightarrow D(W)\text{-mod}^{e^{2\pi it}}, \\ M &\mapsto D(W) \otimes_{D^{\bar{0}}(W)} M. \end{aligned}$$

Theorem 9.2. The functors Ind and Res are mutually inverse equivalences of the categories of $(D(W), E)\text{-mod}^{e^{2\pi it}}$ and $(D^{\bar{0}}(W), E)\text{-mod}^{e^{\pi it}}$.

Proof. For a $D(W)$ -module M with monodromy $e^{2\pi it}$, set

$$M_{\bar{t}} := \bigoplus_{j \in \mathbb{Z}} M_{t+2j+1}.$$

Then $M = M_{\bar{t}} \oplus M_{\bar{t}+1}$, and this is a \mathbb{Z}_2 -grading on M considered as a $D(W)$ -module.

Let $M_{\bar{t}}$ be a $D^{\bar{0}}(W)$ -module with half-monodromy $e^{\pi it}$. The $D(W)$ -module

$$D(W) \otimes_{D^{\bar{0}}(W)} M_{\bar{t}}$$

is \mathbb{Z}_2 -graded and has monodromy $e^{2\pi it}$. Moreover,

$$(D(W) \otimes_{D^{\bar{0}}(W)} M_{\bar{t}})_{\bar{t}} = D^{\bar{0}}(W) \otimes_{D^{\bar{0}}(W)} M_{\bar{t}} = M_{\bar{t}}.$$

Therefore $\text{Res}_{e^{2\pi it}}^{e^{\pi it}} \circ \text{Ind}_{e^{\pi it}}^{e^{2\pi it}}$ is the identity functor.

Let M be a $D(W)$ -module with monodromy $e^{2\pi it}$. If $M_{\bar{t}} = 0$, then M has finite length and, up to isomorphism, the only simple constituents of it are $D(W)1$ and $D(W)\delta_0$. In both cases $M_{\bar{t}} \neq 0$. There is a natural homomorphism

$$\psi : D(W) \otimes_{D^{\bar{0}}(W)} M_{\bar{t}} \rightarrow M.$$

We have

$$(D(W) \otimes_{D^{\bar{0}}(W)} M_{\bar{t}})_{\bar{t}} = D(W)^{\bar{0}} \otimes_{D^{\bar{0}}(W)} M_{\bar{t}} = M_{\bar{t}}.$$

Moreover,

$$\psi_{\bar{t}} : (D(W) \otimes_{D^{\bar{0}}(W)} M_{\bar{t}})_{\bar{t}} \rightarrow M_{\bar{t}}$$

is an isomorphism. Therefore $(M/\text{Im}\psi)_{\bar{t}} = 0$ and hence $M/\text{Im}\psi = 0$. This shows that $\text{Ind}_{e^{\pi it}}^{e^{2\pi it}} \circ \text{Res}_{e^{2\pi it}}^{e^{\pi it}}$ is the identity functor. \square

Corollary 9.11. We have

$$(D^{\bar{0}}(W), E)\text{-mod} \cong (D(W), E)\text{-mod} \oplus (D(W), E)\text{-mod}.$$

Proof. We have

$$\begin{aligned} (\mathbb{D}^{\bar{0}}(W), E)\text{-mod} &\cong \bigoplus_{t \in \mathbb{C}/\mathbb{Z}} (\mathbb{D}^{\bar{0}}(W), E)\text{-mod}^{e^{2\pi it}}, \\ (\mathbb{D}(W), E)\text{-mod} &\cong \bigoplus_{t \in \mathbb{C}/\mathbb{Z}} (\mathbb{D}(W), E)\text{-mod}^{e^{2\pi it}}. \end{aligned}$$

On the other hand $(\mathbb{D}^{\bar{0}}(W), E)\text{-mod}^{e^{\pi it}} \cong (\mathbb{D}(W), E)\text{-mod}^{e^{2\pi it}}$. The two half-monodromies $e^{\pi it}$, $-e^{\pi it} = e^{\pi i(t+1)}$ combine into monodromy $e^{2\pi it}$, and

$$(\mathbb{D}^{\bar{0}}(W), E)\text{-mod} \cong (\mathbb{D}(W), E)\text{-mod} \oplus (\mathbb{D}(W), E)\text{-mod}.$$

□

9.3. A description of $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -modules. In this section

$$K := \mathrm{SL}(V), \mathfrak{g} := \mathfrak{sl}(W)$$

where $W := \Lambda^2 V (n_V = 2k, n_V \geq 5)$ or $W = S^2 V (n_V \geq 3)$.

Lemma 9.12. Let M be a simple bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -module. Then $\mathrm{Ann}M$ is a Joseph ideal.

Proof. By Theorem 2.5 the variety $\mathrm{GV}(M)$ is $\mathrm{SL}(V)$ -coisotropic. According to the discussion following Example 3.8, $\mathrm{GV}(M)$ is birationally isomorphic to $T^*\mathrm{Fl}$ for some partial W -flag variety Fl . Hence Fl is an $\mathrm{SL}(V)$ -spherical variety. As the only partial W -flag varieties which are $\mathrm{SL}(V)$ -spherical are $\mathbb{P}(W)$ and $\mathbb{P}(W^*)$, $\mathrm{GV}(M)$ coincides with \mathcal{Q} (cf. Example 3.8). □

We are now ready to prove Theorems 2.6-2.8.

Proof of Theorem 2.6. As $\mathrm{Ann}M$ is a Joseph ideal, $V(M) \subset \mathcal{Q}$. As M has finite type over \mathfrak{k} , we have $\dim V(M) = \frac{1}{2} \dim \mathcal{Q} = n_W - 1$ and M is of small growth. □

Proof of Theorem 2.7. By Lemma 9.12 the ideal $\mathrm{Ann}M$ is a Joseph ideal. Therefore $\mathrm{Ann}M = I(\lambda)$ for some semi-decreasing n_W -tuple $\bar{\lambda}$ by Corollary 7.5. □

Proof of Theorem 2.8. As $\mathrm{Ann}M = I(\lambda)$ for some semi-decreasing tuple $\bar{\lambda}$, then $V(\mathrm{Ann}M) = \mathrm{GV}(M) = \mathcal{Q}$ (Corollary 7.5). The variety $\mathbb{P}(W)$ is $\mathrm{SL}(V)$ -spherical, and hence the variety $T^*\mathbb{P}(W)$ is $\mathrm{SL}(V)$ -coisotropic. As \mathcal{Q} is birationally isomorphic to $T^*\mathbb{P}(W)$ (Example 3.8), \mathcal{Q} is $\mathrm{SL}(V)$ -coisotropic. Therefore M is a bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -module by Theorem 2.5. □

Let $B_t(W)$ denote the cardinality of the set of isomorphism classes of simple infinite-dimensional bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -modules annihilated by $I(t, n_W - 1, \dots, 1)$. Let $P_{e^{2\pi it}}(W)$ be the cardinality of the set of isomorphism classes of simple perverse sheaves on W (with respect

to the stratification by $\mathrm{GL}(V)$ -orbits) such that they have fixed monodromy $e^{2\pi it}$ and are neither supported at 0 nor are smooth on W . The monodromies of the simple perverse sheaves on W with respect to the stratification by $\mathrm{GL}(V)$ -orbits are described in quiver terms in the Appendix.

Lemma 9.13. We have $B_t(W) = P_{e^{2\pi it}}(W) < \infty$ for all $t \in \mathbb{C}$.

Proof. Let $\lambda_t \in \mathfrak{h}_W^{\mathrm{sl}*}$ be the weight corresponding to $(t, n_W - 1, \dots, 1)$. The quotient $U(\mathfrak{sl}(W))/\mathrm{Ann} L_{\lambda_t}$ is isomorphic to $D^t\mathbb{P}(W)$ (see the discussion at the end of Subsection 3.9.1). Therefore the isomorphism classes of simple infinite-dimensional $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -modules annihilated by $I(\lambda_t)$ are in one-to-one correspondence with the simple $(D(W), \mathfrak{gl}(V))$ -modules with monodromy $e^{2\pi it}$, which are not isomorphic to $D(W)1$ or $D(W)\delta_0$ (see Lemma 9.10). \square

For a semi-decreasing n_W -tuple $\bar{\lambda}$ we denote by $B_{\bar{\lambda}}(W)$ the cardinality of the set of isomorphism classes of simple infinite-dimensional bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -modules annihilated by $I(\bar{\lambda})$. Then for some $t \in \mathbb{C}$ such that $e^{2\pi it} = m(\bar{\lambda})$, we have

$$B_{\bar{\lambda}}(W) = B_t(W) = P_{m(\bar{\lambda})}(W) < \infty,$$

by Lemma 3.31 and Corollary 8.12. This proves Theorem 2.9.

We are now ready to prove Theorem 2.10.

Proof of Theorem 2.10. a) Let M be a bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -module. We can consider M as an $(\mathfrak{gl}(W), \mathfrak{gl}(V))$ -module. Let E be a generator of the center of $\mathfrak{gl}(W)$. Then M is a direct sum of E -eigenspaces $\bigoplus_{t \in \mathbb{C}} M_t$. As M is a bounded $(\mathfrak{gl}(W), \mathfrak{gl}(V))$ -module, M_t is a bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -module.

b) Any simple bounded $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -module annihilated by $I(s_1\rho)$ is an $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -submodule of an $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -module. Any other simple bounded module is isomorphic to a subquotient of a tensor product $F \otimes M$ where F is a finite-dimensional $\mathrm{SL}(W)$ -module and M is a simple $(\mathfrak{sl}(W), \mathfrak{sl}(V))$ -module annihilated by $I(\rho + \varepsilon_k)$. Let F_{sp} be an $\mathrm{SP}(W \oplus W^*)$ -module which contains F as an $\mathrm{SL}(W)$ -submodule (such a module always exists, see [Gr]). Then $F_{\mathrm{sp}} \otimes M$ is a bounded $\mathfrak{gl}(V)$ -module and contains M as an $\mathfrak{gl}(W)$ -submodule. \square

9.4. Categories of $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules. In this section

$$K := \mathrm{GL}(V), \quad \mathfrak{g} := \mathfrak{sp}(W \oplus W^*)$$

where $W := \Lambda^2 V (n_V = 2k, n_V \geq 5)$ or $W = \mathrm{S}^2 V (n_V \geq 3)$.

Lemma 9.14. Let M be a simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -module. Then $\mathrm{Ann} M$ is a Joseph ideal.

Proof. Let $\tilde{\mathcal{Q}}$ be the image of the moment map $\phi : T^*\mathbb{P}(W \oplus W^*) \rightarrow \mathfrak{sp}(W \oplus W^*)^*$. Assume that $\mathrm{GV}(M)$ does not equal to 0 or to $\bar{\mathcal{Q}}_{\mathrm{sp}}$. Then [CM, 6.2] $\tilde{\mathcal{Q}} \subset \mathrm{GV}(M)$ and $\dim \mathrm{GV}(M) \geq 2(2n_W - 1)$. Therefore $\dim \mathfrak{b}_{\mathrm{sl}(V)} \geq 2n_W - 1$ by Corollary 3.24. This inequality is false. \square

We are now ready to prove Theorems 2.11— 2.14 and Theorem 2.16.

Proof of Theorem 2.11. As $\mathrm{Ann}M$ is a Joseph ideal, we have $V(M) \subset \bar{\mathcal{Q}}_{\mathrm{sp}} \cap \mathfrak{k}^\perp$. As $\dim \bar{\mathcal{Q}}_{\mathrm{sp}} \cap \mathfrak{k}^\perp = n_W$, M is of small growth. \square

Proof of Theorem 2.12. As M is a bounded module, $\mathrm{Ann}M$ is a Joseph ideal. Therefore $\mathrm{Ann}M = I(\bar{\mu})$ for some Shale-Weil tuple $\bar{\lambda}$ by Corollary 7.13. \square

Proof of Theorem 2.13. As M is an $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -module, we have $V(M) \subset \mathfrak{gl}(V)^\perp \cap \mathcal{Q}_{\mathrm{sp}}$. Moreover, $\mathcal{Q}_{\mathrm{sp}}$ is a quotient of $W \oplus W^*$ by \mathbb{Z}_2 (see Subsection 7.2), hence $\mathfrak{gl}(V)^\perp \cap \mathcal{Q}_{\mathrm{sp}}$ is a quotient by \mathbb{Z}_2 of the union of conormal bundles to $\mathrm{GL}(V)$ -orbits. Since W is $\mathrm{GL}(V)$ -spherical, these conormal bundles are $\mathrm{GL}(V)$ -spherical. Therefore any irreducible component of $V(M)$ is $\mathrm{GL}(V)$ -spherical, and M is $\mathfrak{gl}(V)$ -bounded by Proposition 3.23. \square

We recall that $\bar{\mu}_0$ is an n_W -tuple $(n_W - \frac{1}{2}, n_W - \frac{3}{2}, \dots, \frac{1}{2})$ and $\mu_0 \in \mathfrak{h}_W^*$ is the corresponding weight.

Proof of Theorem 2.14. Let $\bar{\mu}_1, \bar{\mu}_2$ be Shale-Weil n_W -tuples. The categories of bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules annihilated by $I(\mu_1)$ and $I(\mu_2)$ are equivalent by Theorem 3.12. \square

Proof of Theorem 2.16. The category of $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules annihilated by $I(\mu)$ is equivalent, via the functors $\mathcal{H}_{\mu_0}^\mu$ and $\mathcal{H}_{\mu'}^{\mu_0}$, to the category of $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules annihilated by $I(\mu_0)$. The second category is equivalent to the direct sum of two copies of the category of perverse sheaves on W with respect to the stratification by $\mathrm{GL}(V)$ -orbits (see Subsection 9.2). \square

Let $\langle \mathcal{J}_{\mu_0} \rangle$ be the free vector space generated by the isomorphism classes of simple bounded $(\mathfrak{sp}(W \oplus W^*), \mathfrak{gl}(V))$ -modules annihilated by $\mathrm{Ker} \chi_{\mu_0}$. We recall that $\mathrm{PF}\bar{\mathrm{un}}\bar{\mathrm{c}}(\chi_{\mu_0})$ acts on $\langle \mathcal{J}_{\mu_0} \rangle$ by linear operators (Section 3.8). Let $\bar{\mu}'$ be an n_W -tuple such that $\mathcal{H}_{\mu_0}^{\mu'}|_{\langle \mathcal{J}_{\mu_0} \rangle} \neq 0$. Then $\bar{\mu}'$ is a Shale-Weil tuple by Corollary 7.16. Therefore the action of $\mathrm{PF}\bar{\mathrm{un}}\bar{\mathrm{c}}(\chi_{\mu_0})$ collapses to an action of $\mathcal{H}_{\mu_0}^{\mu_0}$ and $\mathcal{H}_{\mu_0}^{\sigma\mu_0}$. The functor $\mathcal{H}_{\mu_0}^{\sigma\mu_0}$ is involutive and we call it Inv (see also Section 2).

Let M be a simple $U(\mathfrak{sp}(W \oplus W^*))$ -module of small growth annihilated by $I(\mu_0)$. Let E be a finite-dimensional simple $\mathfrak{sp}(W \oplus W^*)$ -module. We recall that $F_E := E \otimes \cdot$. Then $F_E \mathcal{H}_{\mu_0}^{\mu_0}$ is a direct sum $\bigoplus_i \mathcal{H}_{\mu_0}^{\mu_i}$ of indecomposable projective functors. Therefore

$$E \otimes M = \bigoplus_i \mathcal{H}_{\mu_0}^{\mu_i} M.$$

If μ_i is not a Shale-Weil tuple for some i , then $\mathcal{H}_{\mu_0}^{\mu_i} M = 0$ (Corollary 7.16). If μ_i is a negative Shale Weil tuple for some i , then $\mathcal{H}_{\mu_0}^{\mu_i} = \mathcal{H}_{\mu_0}^{\sigma \mu_i} \text{Inv}$. Therefore

$$F_E M = \bigoplus_{i \leq s} \mathcal{H}_{\mu_0}^{\mu'_i} M \oplus (\bigoplus_{i \leq s'} \mathcal{H}_{\mu_0}^{\mu''_i} \text{Inv} M)$$

for some positive Shale-Weil tuples $\bar{\mu}'_1, \dots, \bar{\mu}'_s$ and $\bar{\mu}''_1, \dots, \bar{\mu}''_{s'}$.

Let \mathcal{J} be the free vector space generated by the simple $\mathfrak{sp}(W \oplus W^*)$ -modules of small growth and $\langle \mathcal{J} \rangle$ be the free vector space generated by \mathcal{J} (cf. Section 8). One can express the projective functors $\mathcal{H}_{\mu_0}^{\mu}$ in terms of the functors F_E . More precisely, for any Shale-Weil tuple $\bar{\mu}$ we have

$$\mathcal{H}_{\mu_0}^{\mu} = (\bigoplus_{i \leq s} c_i F_{E_i} \mathcal{H}_{\mu_0}^{\mu_0}) + (\bigoplus_{i \leq s'} c'_i F_{E'_i} \text{Inv}) \quad (5)$$

for some finite-dimensional $\mathfrak{sp}(W \oplus W^*)$ -modules $E_1, \dots, E_s, E'_1, \dots, E'_s$ and some integers $c_1, \dots, c_s, c'_1, \dots, c'_s$; this is an equality for linear operators on $\langle \mathcal{J} \rangle$. Therefore

$$\mathcal{H}_{\mu_0}^{\mu} ([M] + [\text{Inv}M]) = ([M] + \text{Inv}[M]) \otimes_{\mathbb{C}} (\bigoplus_i c_i [E_i] + \bigoplus_i c'_i [E'_i])$$

and

$$\mathcal{H}_{\mu_0}^{\mu} ([M] - [\text{Inv}M]) = ([M] - \text{Inv}[M]) \otimes_{\mathbb{C}} (\bigoplus_i c_i [E_i] - \bigoplus_i c'_i [E'_i]).$$

This shows that if we are given the functions $[M : \cdot]_{\mathfrak{k}}$ and $[\text{Inv}M : \cdot]_{\mathfrak{k}}$, the coefficients c_i, c'_i , and the functions $[E_i : \cdot]_{\mathfrak{k}}, [E'_i : \cdot]_{\mathfrak{k}}$, we can find the functions $[\mathcal{H}_{\mu_0}^{\mu} M : \cdot]_{\mathfrak{k}}$ and $[\mathcal{H}_{\mu_0}^{\mu} \text{Inv}M : \cdot]_{\mathfrak{k}}$ as formal power series.

Moreover, it is enough to determine the coefficients c_i, c'_i and the modules E_i, E'_i for one simple module M of small growth annihilated by $I(\mu_0)$. This work has been done by O. Mathieu [M] for a simple module L_{μ_0} , and we now explain the result.

Let $\bar{\mu}$ be a positive Shale-Weil tuple and be the coefficients c_i, c'_i , the modules E_i, E'_i be as in formula (5). Let L and L^{σ} , $L^{\bar{\mu}}$ and $L^{\sigma \bar{\mu}}$ be finite-dimensional Spin_{2n_W} -modules as in Section 2. Then

$$\begin{aligned} & ([L^{\bar{\mu}} : \cdot]_{\mathfrak{h}_W} + [L^{\sigma \bar{\mu}} : \cdot]_{\mathfrak{h}_W}) = \\ & (\bigoplus_i c_i [E_i : \cdot]_{\mathfrak{h}_W} + \bigoplus_i c'_i [E'_i : \cdot]_{\mathfrak{h}_W}) \otimes ([L : \cdot]_{\mathfrak{h}_W} + [L^{\sigma} : \cdot]_{\mathfrak{h}_W}), \\ & ([L^{\bar{\mu}} : \cdot]_{\mathfrak{h}_W} - [L^{\sigma \bar{\mu}} : \cdot]_{\mathfrak{h}_W}) = \\ & (\bigoplus_i c_i [E_i : \cdot]_{\mathfrak{h}_W} - \bigoplus_i c'_i [E'_i : \cdot]_{\mathfrak{h}_W}) \otimes ([L : \cdot]_{\mathfrak{h}_W} - [L^{\sigma} : \cdot]_{\mathfrak{h}_W}). \end{aligned}$$

As the functions

$$[L^{\bar{\mu}} : \cdot]_{\mathfrak{h}_W}, [L^{\sigma \bar{\mu}} : \cdot]_{\mathfrak{h}_W}, [L : \cdot]_{\mathfrak{h}_W}, [L^{\sigma} : \cdot]_{\mathfrak{h}_W}$$

are computed by the Weyl character formula, we consider the above mentioned formula as an explicit description of the coefficients c_i, c'_i and the $\mathfrak{sp}(W \oplus W^*)$ -modules E_i, E'_i .

The mystery of this formula is that the function $[L : \cdot]_{\mathfrak{h}_W}$ comes from the universe of Spin_{2n_W} -modules and the functions $[E_i : \cdot]_{\mathfrak{h}_W}$ come from the universe of SP_{2n_W} -modules.

10. APPENDIX

10.1. Results of C. Benson, G. Ratcliff, V. Kac. The classification of spherical modules has been worked out in several steps. V. Kac has classified simple spherical modules in [Kac1], C. Benson and G. Ratcliff have classified all spherical modules in [BR]. The classification is contained also in paper [Le] of A. Leahy. Below we reproduce their list.

Let W be a K -module. Then W is a spherical \mathfrak{k} -variety if and only if the pair $([\mathfrak{k}, \mathfrak{k}], W)$ is a direct sum of pairs (\mathfrak{k}_i, W_i) listed below and in addition $\mathfrak{k} + \bigoplus_i c_i = N_{\mathfrak{gl}(W)}(\mathfrak{k} + \bigoplus_i c_i)$ for certain abelian Lie algebras c_i attached to (\mathfrak{k}_i, W_i) . Here $N_{\mathfrak{gl}(W)}\mathfrak{k}$ stands for the normalizer of \mathfrak{k} in $\mathfrak{gl}(W)$ and c_i is a 0-, 1- or 2-dimensional Lie algebra listed in square brackets after the pair (\mathfrak{k}_i, W_i) . This subalgebra is generated by linear operators h_1 and $h_{m,n}$. By definition, $h_1 = \text{Id}$. The notation $h_{m,n}$ is used only when $W = W_1 \oplus W_2$: in this case $h_{m,n}|_{W_1} = m \cdot \text{Id}$ and $h_{m,n}|_{W_2} = n \cdot \text{Id}$. The notation $'(\mathfrak{k}_i, \{W_i, W_i'\})'$ is shorthand for $'(\mathfrak{k}_i, W_i)'$ and $'(\mathfrak{k}_i, W_i)'$. Finally, ω_i stands for the i -th fundamental weight and the corresponding fundamental representation. We follow the enumeration convention for fundamental weights of [VO].

Table 10.1: Weakly irreducible spherical pairs.

- 0) $(0, \mathbb{C})[0]$.
- i) Irreducible representations of simple Lie algebras:
 - 1) $(\mathfrak{sl}_n, \{\omega_1, \omega_{n-1}\})[\text{Ch}_1](n \geq 2)$;
 - 2) $(\mathfrak{so}_n, \omega_1)[0](n \geq 3)$;
 - 3) $(\mathfrak{sp}_{2n}, \omega_1)[\text{Ch}_1](n \geq 2)$;
 - 4) $(\mathfrak{sl}_n, \{2\omega_1, 2\omega_{n-1}\})[0](n \geq 3)$;
 - 5) $(\mathfrak{sl}_{2n+1}, \{\omega_2, \omega_{n-2}\})[\text{Ch}_1](n \geq 2)$;
 - 6) $(\mathfrak{sl}_{2n}, \{\omega_2, \omega_{n-2}\})[0](n \geq 3)$;
 - 7) $(\mathfrak{so}_7, \omega_3)[0]$;
 - 8) $(\mathfrak{so}_8, \{\omega_3, \omega_4\})[0]$;
 - 9) $(\mathfrak{so}_9, \omega_4)[0]$;
 - 10) $(\mathfrak{so}_{10}, \{\omega_4, \omega_5\})[\text{Ch}_1]$;
 - 11) $(\text{E}_6, \omega_1)[0]$;
 - 12) $(\text{G}_2, \omega_1)[0]$.
- ii) Irreducible representations of nonsimple Lie algebras:
 - 1) $(\mathfrak{sl}_n \oplus \mathfrak{sl}_m, \{\omega_1, \omega_{n-1}\} \otimes \{\omega_1, \omega_{m-1}\})[\text{Ch}_1](m > n \geq 2)$;
 - 2) $(\mathfrak{sl}_n \oplus \mathfrak{sl}_n, \{\omega_1, \omega_{n-1}\} \otimes \{\omega_1, \omega_{n-1}\})[0](n \geq 2)$;
 - 3) $(\mathfrak{sl}_2 \oplus \mathfrak{sp}_{2n}, \omega_1 \otimes \omega_1)[0](n \geq 2)$;
 - 4) $(\mathfrak{sl}_3 \oplus \mathfrak{sp}_{2n}, \{\omega_1, \omega_2\} \otimes \omega_1)[0](n \geq 2)$;
 - 5) $(\mathfrak{sl}_n \oplus \mathfrak{sp}_4, \{\omega_1, \omega_{n-1}\} \otimes \omega_1)[\text{Ch}_1](n \geq 5)$;
 - 6) $(\mathfrak{sl}_4 \oplus \mathfrak{sp}_4, \{\omega_1, \omega_3\} \otimes \omega_1)[0]$.

iii) Reducible representations of Lie algebras:

- 1) $(\mathfrak{sl}_n \oplus \mathfrak{sl}_m \oplus \mathfrak{sl}_2; (\{\omega_1, \omega_{n-1}\} \oplus \{\omega_1, \omega_{n-1}\}) \otimes \omega_1)[\mathcal{Ch}_{1,0} \oplus \mathcal{Ch}_{0,1}](n, m \geq 3)$;
- 2) $(\mathfrak{sl}_n; \{\omega_1 \oplus \omega_1, \omega_{n-1} \oplus \omega_{n-1}\})[\mathcal{Ch}_{1,1}](n \geq 3)$;
- 3) $(\mathfrak{sl}_n; \omega_1 \oplus \omega_{n-1})[\mathcal{Ch}_{1,-1}](n \geq 3)$;
- 4) $(\mathfrak{sl}_{2n}; \{\omega_1, \omega_{n-1}\} \oplus \{\omega_2, \omega_{n-2}\})[\mathcal{Ch}_{0,1}](n \geq 2)$;
- 5) $(\mathfrak{sl}_{2n+1}; \omega_1 \oplus \omega_2)[\mathcal{Ch}_{1,-m}](n \geq 2)$;
- 6) $(\mathfrak{sl}_{2n+1}; \omega_{n-1} \oplus \omega_2)[\mathcal{Ch}_{1,m}](n \geq 2)$;
- 7) $(\mathfrak{sl}_n \oplus \mathfrak{sl}_m; \{\omega_1, \omega_{n-1}\} \otimes (\mathbb{C} \oplus \{\omega_1, \omega_{m-1}\}))[\mathcal{Ch}_{1,0}](2 \leq n < m)$;
- 8) $(\mathfrak{sl}_n \oplus \mathfrak{sl}_m; \{\omega_1, \omega_{n-1}\} \otimes (\mathbb{C} \oplus \{\omega_1, \omega_{m-1}\}))[\mathcal{Ch}_{1,1}](m \geq 2, n \geq m+2)$;
- 9) $(\mathfrak{sl}_n \oplus \mathfrak{sl}_m; \{\omega_1, \omega_{n-1}\} \oplus \{\omega_1^*(= \omega_{n-1}), \omega_{n-1}^*(= \omega_1)\} \otimes \{\omega_1, \omega_{m-1}\})[\mathcal{Ch}_{1,0}](2 \leq n < m)$;
- 10) $(\mathfrak{sl}_n \oplus \mathfrak{sl}_m; \{\omega_1, \omega_{n-1}\} \oplus \{\omega_1^*(= \omega_{n-1}), \omega_{n-1}^*(= \omega_1)\} \otimes \{\omega_1, \omega_{m-1}\})[\mathcal{Ch}_{1,-1}](m \geq 2, n \geq m+2)$;
- 11) $(\mathfrak{sl}_n \oplus \mathfrak{sp}_{2m} \oplus \mathfrak{sl}_2; (\{\omega_1, \omega_{n-1}\} \oplus \omega_1) \otimes \omega_1)[\mathcal{Ch}_{0,1}](n \geq 3, m \geq 1)$;
- 12) $(\mathfrak{sl}_2; \{\omega_1 \oplus \omega_1\})[0]$;
- 13) $(\mathfrak{sl}_n \oplus \mathfrak{sl}_n; \{\omega_1, \omega_{n-1}\} \oplus \{\omega_1^{(*)}, \omega_{n-1}^{(*)}\} \otimes \{\omega_1, \omega_{n-1}\})[0](n \geq 2)$;
- 14) $(\mathfrak{sl}_{n+1} \oplus \mathfrak{sl}_n; \{\omega_1, \omega_n\} \oplus \{\omega_1^{(*)}, \omega_n^{(*)}\} \otimes \{\omega_1, \omega_{n-1}\})[0](n \geq 2)$;
- 15) $(\mathfrak{sl}_2 \oplus \mathfrak{sp}_{2n}; \omega_1 \otimes (\mathbb{C} \oplus \omega_1))[0](n \geq 2)$;
- 16) $(\mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2m} \oplus \mathfrak{sl}_2; (\omega_1 \oplus \omega_1) \otimes \omega_1)[0](m, n \geq 2)$;
- 17) $(\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2; (\omega_1 \oplus \omega_1) \otimes \omega_1)[0]$;
- 18) $(\mathfrak{so}_8; \{\omega_1 \oplus \omega_3, \omega_1 \oplus \omega_4, \omega_3 \oplus \omega_4\})[0]$.

10.2. Results of T. Braden and M. Grinberg. Let n be a positive integer and V be a \mathbb{C} -vector space of dimension $2n$. Then the category of perverse sheaves on $\Lambda^2 V$ with respect to the stratification by $\mathrm{GL}(V)$ -orbits is equivalent to the category of representations of the following quiver A with relations:

$$A_0 \begin{array}{c} \xleftarrow{p_0} \\ \xrightarrow{q_0} \end{array} A_1 \begin{array}{c} \xleftarrow{p_1} \\ \xrightarrow{q_1} \end{array} \cdots \begin{array}{c} \xleftarrow{p_{n-2}} \\ \xrightarrow{q_{n-2}} \end{array} A_{n-1} \begin{array}{c} \xleftarrow{p_{n-1}} \\ \xrightarrow{q_{n-1}} \end{array} A_n,$$

$$\xi_i \text{ and } \nu_i \text{ are invertible for all } i, \xi_i = \nu_i \text{ for } i \in \{1, \dots, n-1\},$$

where $\xi_i := 1 + q_{i-1}p_{i-1}$ for $i \in \{1, \dots, n\}$, and $\nu_i := 1 + p_iq_i$ for $i \in \{0, \dots, n-1\}$. Let R be a representation of the quiver A . If R is simple, the invertible operators $\{\nu_0^n, \nu_1^{n-1}\xi^1, \dots, \xi_n^n\}$ are proportional to the identity map with a fixed constant $c \in \mathbb{C}^*$. We call c the *monodromy* of R .

The set of eigenvalues of $1 + p_iq_i$ is independent from i , and we call them *eigenvalues* of R . If R is simple, this set consists of one element λ . For a given eigenvalue $\lambda \neq 1$, there exists precisely one simple representation of A with eigenvalue λ . The simple representations of A with eigenvalue 1 are enumerated by the vertices of the quiver.

By definition, the *support of R* is the set of vertices corresponding to non-zero vector spaces. The corresponding to R perverse sheaf is supported at 0 if and only if R is supported at A_0 . The corresponding to R perverse sheaf is smooth along W if and only if R is supported at A_n .

Let now V be a \mathbb{C} -vector space of dimension n . Then the category of perverse sheaves on S^2V with respect to the stratification by $GL(V)$ -orbits is equivalent to the category of representations of the following quiver B with relations:

$$B_0 \xleftarrow{p_0} B_1 \xleftarrow{q_1} \cdots \xleftarrow{q_{n-2}} B_{n-1} \xleftarrow{q_{n-1}} B_n,$$

ξ_i and ν_i are invertible for all i , $\xi_i^2 = \nu_i^2$ for $i \in \{1, \dots, n-1\}$,

$$p_j \nu_{j+1} = -\nu_j p_j, \quad q_j \nu_j = -\nu_{j+1} q_j, \quad p_j \xi_{j+1} = -\xi_j p_j, \quad q_j \xi_j = -\xi_{j+1} q_j$$

whenever both sides of equalities are well-defined. Here

$$\xi_i := 1 + q_{i-1} p_{i-1} \text{ for } i \in \{1, \dots, n\}, \quad \nu_i := 1 + p_i q_i \text{ for } i \in \{0, \dots, n-1\}.$$

Let R be a representation of the quiver B . Assume R is simple. Then the invertible operators $\{\nu_0^n, \nu_1^{n-1} \xi^1, \dots, \xi_n^n\}$ are proportional to the identity map with a fixed constant $c \in \mathbb{C}^*$. We call c *the monodromy of R* . The set of eigenvalues of $(-1)^i \xi_i$ consists of one element $\bar{\xi}$; the set of eigenvalues of $(-1)^i \nu_i$ consists of one element $\bar{\nu}$. We call the pair $(\bar{\xi}, \bar{\nu})$ *the spectrum of R* . We have $\bar{\xi} = \bar{\nu} = 1$ or $\bar{\xi} = -\bar{\nu}$. The simple representations of B with spectrum $(1, 1)$ are enumerated by the inner vertices of the B -quiver. For $\lambda \neq \pm 1$ there exists precisely one simple representation of B with the spectrum $(\lambda, -\lambda)$. The simple representations of B with spectra $(1, -1)$ and $(-1, 1)$ are enumerated by the vertices of the quiver.

By definition, the *support of R* is the set of vertices corresponding to non-zero vector spaces. The corresponding to R perverse sheaf is supported at 0 if and only if R is supported at B_0 . The corresponding to R perverse sheaf is smooth along W if and only if R is supported at B_n .

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REFERENCES

- [BK] Daniel Barlet, Masaki Kashiwara, Duality of D -modules on Flag manifolds, IMRN(2000), 1243–1257.
- [Be] Alexander Beilinson, Localization of representations of reductive Lie algebras, Proc. of the IMC (Warsaw, 1983), PWN, Warsaw, 1984, 699–710.
- [BeBe] Alexander Beilinson, Joseph Bernstein, A generalization of Casselman’s submodule theorem, Repr. th. of red. gr. (Park City, Utah, 1982), Progr. Math., **40** (1983), Birkhäuser Boston, Boston, MA, 1983, 35–52.
- [BeBe2] Alexander Beilinson, Joseph Bernstein, Localisation de \mathfrak{g} -modules, C.R.Acad. Sci. Paris **292**(1981), 15–18.
- [BR] Chal Benson, Gail Ratcliff, A Classification of Multiplicity Free Actions, J. of Alg. **181**(1996), 152–186.
- [BeG] Joseph Bernstein, Sergey Gelfand, Tensor products of finite and infinite-dimensional representations of semisimple Lie algebras, Comp. Math. **41**(1980), 245–285.
- [Bo] Armand Borel et. all, Algebraic D -modules. Perspectives in Mathematics, **2**, Acad. Press, Inc., Boston, MA, 1987.
- [BoBry] Walter Borho, Jean-Luc Brylinski, Differential operators on homogeneous spaces. III, Invent. Math. **80**(1985), 1–68.
- [BG] Tom Braden, Mikhail Grinberg, Perverse sheaves on rank stratifications, Duke Math. J. **96**(1999), 317–362.
- [CM] David Collingwood, William McGovern, Nilpotent orbits in semisimple Lie algebras, Van Nostrand Reinhold Math. Ser., Van Nostrand Reinhold Co., New York, 1993.
- [Dix] Jacques Dixmier, Algebres Enveloppantes, Gauthier-Villars, 1974.
- [F] Suren Fernando, Lie algebra modules with finite dimensional weight spaces. I, Trans. Amer. Math. Soc. **322**(1990), 757–781.
- [FH] William Fulton, Joe Harris, Representation theory. A first course, Grad. Texts in Math. **129**, Springer-Verlag, New York, 1991.
- [Gab] Ofer Gabber, The integrability of the characteristic variety, Amer. J. Math. **103**(1981), 445–468.
- [GL] Ofer Gabber, Equidimensionalité de la variété caractéristique, Exposé de O. Gabber rédigé par T. Levasseur, Paris, 1982.
- [GVM] Sergei Gelfand, Robert MacPherson, Kari Vilonen, Perverse sheaves and quivers, Duke Math. J. **83**(1996), 621–643.
- [GrS2] Dimitar Grantcharov, Vera Serganova, Category of $\mathfrak{sp}(2n)$ -modules with bounded weight multiplicities, Mosc. Math. J. **6**(2006), 119–134, 222.
- [GrS1] Dimitar Grantcharov, Vera Serganova, Cuspidal representations of $\mathfrak{sl}(n+1)$, Adv. Math. **224**(2010), 1517–1547.
- [Gr] Frank Grosshans, Algebraic homogeneous spaces and invariant theory, Lect. Notes in Math. **1673**, Springer-Verlag, Berlin, 1997.
- [HC] Harish-Chandra, Representations of semi-simple Lie groups I, Trans. Amer. Math. Soc. **76**(1953), 26–65.
- [HMSW] Henryk Hecht, Dragan Milicic, Wilfried Schmidt, Joseph Wolf, Localization and standard modules for real semisimple Lie groups I: The duality theorem, Inv. Math. **90**(1987), 297–332.
- [HU] Roger Howe, Toru Umeda, The Capelli identity, the double commutant theorem, and multiplicity-free actions, Math. Ann. **290**(1991), 565–619.

- [Jo1] Anthony Joseph, Dixmier's problem for Verma and principal series submodules, *J. London Math. Soc.* **20**(1979), 193–204.
- [Jo2] Anthony Joseph, On the associated variety of the primitive ideal, *J. of Alg.* **88**(1984), 238–278.
- [Jo3] Anthony Joseph, On the cyclicity of vectors associated with Duflo involutions. *Noncommutative harmonic analysis and Lie groups (Marseille-Luminy, 1985)*, *Lect. Notes in Math.* **1243**(1987), Springer, Berlin, 144–188.
- [Jo4] Anthony Joseph, Orbital varieties of the minimal orbit, *Ann. Sci. École Norm. Sup.* **31**(1998), 17–45.
- [Jo5] Anthony Joseph, The primitive spectrum of an enveloping algebra. *Orbites unipotentes et représentations, III. Astérisque* **173–174** (1989), 9, 13–53.
- [Jo6] Anthony Joseph, W -module structure in the primitive spectrum of the enveloping algebra of a semisimple Lie algebra, *Noncommutative harmonic analysis (Proc. Third Colloq., Marseille-Luminy, 1978)*, *Lect. Notes in Math.* **728**(1979) Springer, Berlin, 116–135.
- [Kac1] Victor Kac, Some remarks on nilpotent orbits, *J. of Alg.* **64**(1980), 190–213.
- [Kac2] Victor Kac, Constructing groups associated to infinite dimensional Lie algebras, *Infinite dimensional groups with applications (Berkeley, Calif., 1984)*, in: *MSRI Publ. 4*, Springer-Verlag, New York, 1985.
- [Ka] Masaki Kashiwara, The Riemann-Hilbert problem for holonomic systems, *Publ. Res. Inst. Math. Sci.* **20**(1984), 319–365.
- [Ki] Donald King, Spherical nilpotent orbits and the Kostant-Sekiguchi correspondence, *Trans. Amer. Math. Soc.* **354**(2002), 4909–4920.
- [Kd] Dmitriy Kaledin, Normalization of a Poisson algebra is Poisson, *Proc. Steklov Inst. Math.* **264**, no **1**(2009), 70–73.
- [Kn1] Friedrich Knop, Classification of multiplicity free symplectic representations, *J. of Alg.* **301**(2006), 531–553.
- [Kn2] Friedrich Knop, Weylgruppe und Momentabbildung (German), *Invent. Math.* **99**(1990), 1–23.
- [KL] Günter Krause, Thomas Lenagan, Growth of algebras and Gelfand-Kirillov dimension, *AMS Grad. Studies in Math.* **22**, 2000.
- [KV] Antony Knapp, David Vogan, Cohomological induction and unitary representations, *Princ. Math. Ser.* **45**, Princeton University Press, 1995.
- [Kr] Manfred Krämer, Spärische Untergruppen in kompakten zusammenhängenden Liegruppen, *Comp. Mat.* **38**(1979), 129–153.
- [Le] Andrew Leahy, A classification of multiplicity free representations, *J. Lie Theory* **8**(1998), 376–391.
- [Lo] Ivan Losev, Algebraic Hamiltonian actions, *Math. Z.* **263**(2009), 685–723.
- [M] Olivier Mathieu, Classification of irreducible weight modules, *Ann. de l'Inst. Fourier* **50**(2000), 537–592.
- [Mi] Todor Milev, Root Fernando-Kac subalgebras of finite type, to appear in *J. of Alg.*, arXiv:1009.5260.
- [NG] Neil Chriss, Victor Ginzburg, Representation theory and complex geometry, Birkhäuser Boston, 1997.
- [On] Arkadiy Onishchik, Topology of transitive transformation groups, Johann Ambrosius Barth Verlag, Leipzig, 1994.
- [Pan] Dmitri Panyushev, On the conormal bundles of a G -stable subvariety, *Man. Math.* **99**(1999), 185–202.

- [Pan2] Dmitri Panyushev, Some amazing properties of spherical nilpotent orbits, *Math. Z.* **245**(2003), 557–580.
- [Pan3] Dmitri Panyushev, Complexity and rank of double cones and tensor product decompositions, *Comment. Math. Helv.* **68**(1993), 455–468.
- [PS] Ivan Penkov, Vera Serganova, Bounded generalized Harish-Chandra modules, to appear in *Ann. de l’Inst. Fourier*
- [PS2] Ivan Penkov, Vera Serganova, Bounded simple $(\mathfrak{g}, \mathfrak{sl}(2))$ -modules for $\mathrm{rk}\mathfrak{g} = 2$, *J. Lie Theory* **20**(2010), 581–615.
- [PSZ] Ivan Penkov, Vera Serganova, Gregg Zuckerman, On the existence of $(\mathfrak{g}, \mathfrak{k})$ -modules of finite type, *Duke Math. J.* **125**(2004), 329–349.
- [PZ] Ivan Penkov, Gregg Zuckerman, Generalized Harish-Chandra modules with generic minimal \mathfrak{k} -type, *Asian J. Math.* **8**(2004), 795–812.
- [Pe] Alexey Petukhov, Bounded reductive subalgebras of \mathfrak{sl}_n , to appear in *Transform. Groups*
- [Po] Elizaveta Ponomareva, in preparation
- [Rich] Roger Richardson, Conjugacy classes of parabolic subgroups in semisimple algebraic groups, *Bull. London Math. Soc.* **6**(1974), 21–24.
- [SK] Mikio Sato, Tatsuo Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, *Nagoya Math. J.* **65**(1977), 1–155.
- [SM] Catharina Stroppel, Volodymyr Mazorchuk, Blocks of the category of cuspidal $\mathrm{sp}(2n)$ -modules, to appear in *Pacif. J. Math.*
- [Vi] Érnest Vinberg, Commutative homogeneous spaces and co-isotropic symplectic actions, *Russian Math. Surv.* **56**(2001), 1–60.
- [VK] Érnest Vinberg, Boris Kimelfeld, Homogeneous domains on flag manifolds and spherical subsets of semisimple Lie groups (Russian), *Funktsional. Anal. i Prilozhen.* **12 no 3**(1978), 12–19.
- [VP] Érnest Vinberg, Vladimir Popov, Invariant theory (Russian) Algebraic geometry, **4** (Russian), *Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989*, 137–315.
- [VO] Érnest Vinberg, Arkadiy Onishchik, Lie groups and algebraic groups, Springer series in Soviet mathematics, Springer-Verlag, Berlin, 1990.
- [Vo2] David Vogan, Ordering of the primitive spectrum of a semisimple Lie algebra, *Math. Ann.* **248**(1980), 195–203.
- [ZT] Vladimir Zhgoon, Dmitri Timashev, in preparation