PRIMITIVE IDEALS OF $\mathrm{U}(\mathfrak{sl}(\infty))$

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Abstract. We provide an explicit description of the primitive ideals of the enveloping algebra $\mathrm{U}(\mathfrak{sl}(\infty))$ of the infinite-dimensional finitary Lie algebra $\mathfrak{sl}(\infty)$ over an uncountable algebraically closed field of characteristic 0. Our main new result is that any primitive ideal of $\mathrm{U}(\mathfrak{sl}(\infty))$ is integrable. A classification of integrable primitive ideals has been known previously, and relies on the pioneering work of A. Zhilinskii. We also present an inclusion criterion for primitive ideals of $\mathrm{U}(\mathfrak{sl}(\infty))$.

Key words: Primitive ideals, finitary Lie algebras, highest weight modules.

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1. Introduction

A two-sided ideal I of an associative algebra A is called *primitive* if I is the annihilator of a simple A-module. Given an infinite-dimensional associative algebra A, it may be too hard to classify simple A-modules (this problem seems to be open for the algebra of differential operators in two variables) but it may still be possible to provide an explicit description of the primitive ideals of A. This is precisely the situation when $A = \mathrm{U}(\mathfrak{g})$ is the enveloping algebra of a semisimple finite-dimensional Lie algebra over a field of characteristic 0. Here, a rough description of primitive ideals is given by the celebrated Duflo Theorem [6]: it claims that every primitive ideal of $A = \mathrm{U}(\mathfrak{g})$ is the annihilator of some (in general not unique) simple highest weight module. This reduces the problem of classifying primitive ideals to a combinatorial problem. The latter problem has been solved due to the efforts of many mathematicians, in particular D. Barbasch, D. Vogan [1, 2], W. Borho, J.-C. Jantzen [3], A. Joseph [8], D. Kazhdan, G. Lusztig [10], and others, see, for example, [20]. In [14] we have made an attempt to summarize the combinatorial description of the primitive ideals of $\mathrm{U}(\mathfrak{g})$ for classical simple Lie algebras $\mathfrak{g}$ in a language suitable for studying the case when $\mathrm{rk}\,\mathfrak{g}$ tends to ∞.

We consider the Lie algebra $\mathfrak{sl}(\infty)$ which consists of traceless finitary infinite matrices, i.e. traceless infinite matrices each of which has only finitely many nonzero entries. An understanding of the primitive ideals of the enveloping algebra $\mathrm{U}(\mathfrak{sl}(\infty))$ is needed for the current development of the representation theory of $\mathfrak{sl}(\infty)$. The main result of the present paper is an explicit description of all primitive ideals of $\mathrm{U}(\mathfrak{sl}(\infty))$. While the problem of classifying irreducible representations of the Lie algebra $\mathfrak{sl}(\infty)$ appears to be intractable, we show that the problem of classifying primitive ideals in $\mathrm{U}(\mathfrak{sl}(\infty))$ admits a beautifully simple answer. In contrast with the case of a finite-dimensional simple Lie algebra, $\mathrm{U}(\mathfrak{sl}(\infty))$ has only countably many primitive ideals. This is related to the circumstance that a generic irreducible highest weight $\mathrm{U}(\mathfrak{sl}(\infty))$-module has zero annihilator, see [14].

Moreover, in the recent review [14] we presented the classification of primitive ideals of $\mathrm{U}(\mathfrak{sl}(\infty))$ subject to the condition that they are integrable. This classification relies on the work of A. Zhilinskii [17, 18, 19]. We recall that a two-sided ideal I is *integrable* if $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} M$ for an integrable, not necessarily simple, $\mathrm{U}(\mathfrak{sl}(\infty))$-module M, i.e., a $\mathrm{U}(\mathfrak{sl}(\infty))$-module M which becomes a sum of finite-dimensional $\mathrm{U}(\mathfrak{sl}(n))$-modules after being restricted to $\mathrm{U}(\mathfrak{sl}(n))$ for each $n \geq 2$. The main result of the present paper is that every primitive ideal of $\mathrm{U}(\mathfrak{sl}(\infty))$ is integrable, and hence the integrable primitive ideals described in [14] are all primitive ideals.

In an appendix we give also an explicit inclusion criterion for primitive ideals of $\mathrm{U}(\mathfrak{sl}(\infty))$.

2. Main result

Fix an uncountable algebraically closed field $\mathbb{F}$ of characteristic 0. All vector spaces (in particular, Lie algebras) are assumed to be defined over $\mathbb{F}$. If W is a vector space, then $W^* := \mathrm{Hom}_{\mathbb{F}}(W, \mathbb{F})$.

One can define the Lie algebra $\mathfrak{sl}(\infty)$ as the direct limit of (arbitrary) inclusions of the form

$$\mathfrak{sl}(2) \hookrightarrow \mathfrak{sl}(3) \hookrightarrow \ldots \hookrightarrow \mathfrak{sl}(n) \hookrightarrow \ldots.$$

It is well known that this property determines $\mathfrak{sl}(\infty)$ up to isomorphism.

If we fix a simple n-dimensional $\mathfrak{sl}(n)$-module $V(n)$ for some $n \geq 3$, then for any $m > n$ there is a unique, up to isomorphism, simple m-dimensional $\mathfrak{sl}(m)$-module $V(m)$ whose restriction to $\mathfrak{sl}(n)$ is isomorphic to $V(n) \oplus \mathbb{F}^{m-n}$ where $\mathbb{F}^{m-n}$ is a trivial $\mathfrak{sl}(n)$-module. Therefore, the direct limit $\varinjlim_m V(m)$ is well defined, and we denote it by V. Similarly, we define V_* as the direct limit $\varinjlim_m V(m)^*$. In what follows we consider also the symmetric and exterior algebras $\mathrm{S}^{\cdot}(V) := \oplus_{k\geq 0} \mathrm{S}^k(V)$ and $\Lambda^{\cdot}(V) := \oplus_{k\geq 0} \Lambda^k(V)$, as well as $\mathrm{S}^{\cdot}(V_*)$ and $\Lambda^{\cdot}(V_*)$.

Next, for any (possibly empty) Young diagram Y whose row lengths form a sequence

$$l_1 \geq l_2 \geq ... \geq l_s > 0$$

(the empty sequence for $Y = \emptyset$), we define the $\mathfrak{sl}(\infty)$-module V_Y as a direct limit $\varinjlim_{n\geq s} V_Y(n)$: here $V_Y(n)$ denotes a simple finite-dimensional $\mathfrak{sl}(n)$-module with highest weight

$$l_1 \geq l_2 \geq ... \geq l_s > 0 \geq 0 \geq ... \geq 0$$

having n entries (for $Y = \emptyset$ the highest weight of $V_Y(n)$ equals 0). The $\mathfrak{sl}(n)$-module $V_Y(n)$ is isomorphic to a simple direct summand of the tensor product

$$\mathrm{S}^{l_1}(V(n)) \otimes \mathrm{S}^{l_2}(V(n)) \otimes .... \otimes \mathrm{S}^{l_s}(V(n)),$$

and the direct limit $\varinjlim_{n\geq s} V_Y(n)$ is clearly well defined up to isomorphism. Similarly, we define $\mathfrak{sl}(\infty)$-module $(V_Y)_*$ as the direct limit $\varinjlim_{n\geq s}(V_Y(n))^*$.

Finally, we set

$$I(x, y, Y_l, Y_r) := \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))}\,(V_{Y_l} \otimes (\mathrm{S}^{\cdot}(V))^{\otimes x} \otimes (\Lambda^{\cdot}(V))^{\otimes y} \otimes (V_{Y_r})_*)$$

where $x, y \in \mathbb{Z}_{\geq 0}$, Y_l and Y_r are Young diagrams with respective row lengths $l_1, ..., l_s$ and $r_1, ..., r_t$.

The classification of primitive ideals of $\mathrm{U}(\mathfrak{sl}(\infty))$ can now be stated as follows.

Theorem 2.1. *All ideals $I(x, y, Y_l, Y_r)$ are primitive and nonzero, and any nonzero primitive ideal I of $\mathrm{U}(\mathfrak{sl}(\infty))$ equals exactly one of these ideals.*

Since Proposition 4.8 in [13] asserts that the primitive ideals $I(x, y, Y_l, Y_r)$ are precisely the integrable primitive ideals of $\mathrm{U}(\mathfrak{sl}(\infty))$, in order to prove Theorem 2.1 it is enough to prove the following.

Theorem 2.2. *Every nonzero primitive ideal of $\mathrm{U}(\mathfrak{sl}(\infty))$ is integrable.*

It was pointed out by the reviewer that the lattice of two-sided ideals has several symmetries and that one of them arises from the fact that we can interchange V and V_*. This defines an involution $I \to I_*$ on the lattice of integrable ideals, and

$$I(x, y, Y_l, Y_r)_* = I(x, y, Y_r, Y_l).$$

3. Corollaries, examples and further results

A brief review of basic facts concerning splitting Cartan and Borel subalgebras of $\mathfrak{sl}(\infty)$ (including the definitions), as well as the roots of $\mathfrak{sl}(\infty)$, see in [14] and Examples below. For any splitting Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}(\infty)$ with fixed Cartan subalgebra $\mathfrak{h}$, there is a well-defined notion of simple $\mathfrak{b}$-highest weight module $L_{\mathfrak{b}}(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$. Given a weight $\lambda \in \mathfrak{h}^*$, by definition $L_{\mathfrak{b}}(\lambda)$ is the unique simple quotient of the induced module $\mathrm{U}(\mathfrak{sl}(\infty)) \otimes_{\mathrm{U}(\mathfrak{b})} \mathbb{F}_\lambda$, where $\mathbb{F}_\lambda$ is a one-dimensional $\mathfrak{b}$-module on which $\mathfrak{h}$ acts through the weight λ.

There is a class of Borel subalgebras of $\mathfrak{sl}(\infty)$, which we call *ideal*. A quick definition of an ideal Borel subalgebra $\mathfrak{b}$ of $\mathfrak{sl}(\infty)$ is as follows: a splitting Borel subalgebra of $\mathfrak{sl}(\infty)$ is ideal if and only if any simple object in the category of tensor modules $\mathbb{T}_{\mathfrak{sl}(\infty)}$ defined in [5] is a $\mathfrak{b}$-highest weight $\mathfrak{sl}(\infty)$-module.

Here is an equivalent definition in terms of roots. Recall that the roots of $\mathfrak{sl}(\infty)$ have the form $\{\varepsilon_i - \varepsilon_j\}_{i\neq j \in \mathbb{Z}_{>0}}$, where ε_i are certain standard vectors, see [14, Appendix A]. The splitting Borel subalgebras $\mathfrak{b} \subset \mathfrak{sl}(\infty)$ which contain $\mathfrak{h}$ are in bijections with total orders on $\mathbb{Z}_{>0}$: given a total order $\succ$ on $\mathbb{Z}_{>0}$, the positive roots of the corresponding Borel subalgebra are

$$\{\varepsilon_i - \varepsilon_j \mid i \succ j\}.$$

A splitting Borel subalgebra is ideal if the total order $\succ$ satisfies the condition: for every nonnegative integer n there exist i_1, i_2 such that

$$|\{i \in \mathbb{Z}_{>0} \mid i_1 \succ i\}| = n, \quad |\{i \in \mathbb{Z}_{>0} \mid i \succ i_2\}| = n.$$

The following is an analogue of Duflo's Theorem.

Theorem 3.1. *Let $\mathfrak{b}$ be an ideal Borel subalgebra of $\mathfrak{sl}(\infty)$ with a fixed splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$.*
a)For any primitive ideal I of $\mathrm{U}(\mathfrak{sl}(\infty))$ there exists $\lambda \in \mathfrak{h}^*$ such that $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(\lambda)$.
b) If $I = I(0, 0, Y_l, Y_r)$ then the weight $\lambda \in \mathfrak{h}^*$, such that $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(\lambda)$, is unique.

Proof. Part a) is implied by Theorem 2.1 and [13, Theorem 3.1]. Part b) follows directly from a more general uniqueness result of A. Sava [16], see also [14]. □

Examples. *Consider the splitting Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}(\infty)$ corresponding to the total order*

$$1 \succ 3 \succ 5 \succ ... \quad ... \succ 6 \succ 4 \succ 2.$$

This Borel subalgebra is ideal. Given a primitive ideal $I(x,y,Y_l,Y_r)$ with

$$Y_l = (l_1 \geq ... \geq l_s > 0), \quad Y_r = (r_1 \geq ... \geq r_t > 0),$$

the weight λ in Theorem 3.1a) can be chosen as

$$\sum_{1\leq i\leq x} i\alpha\varepsilon_{2i-1} + \sum_{1\leq j\leq s} l_i\varepsilon_{2i+2x-1} + y(\sum_{k\geq 1}\varepsilon_{2k-1}) - \sum_{1\leq j\leq t} r_j\varepsilon_{2j}$$

for an arbitrary $\alpha \in \mathbb{F}\backslash\mathbb{Q}$. Moreover, one can show that, if $x=0$, then the above weight λ is unique with the property $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(\lambda)$.

For other ideal Borel subalgebras, a primitive ideal may be the annihilator of several nonisomorphic simple highest weight modules. Indeed, consider the Borel subalgebra $\mathfrak{b}'$ of $\mathfrak{sl}(\infty)$ defined by the total order

$$3 \succ 5 \succ 7 \succ ... \quad \succ 1 \succ \quad ... \succ 6 \succ 4 \succ 2.$$

Then $\mathfrak{b}'$ is also ideal, but one can check that $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}'}(\lambda) = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}'}(\lambda')$ where

$$\lambda = \sum_{i\geq 0}\varepsilon_{2i+1}, \quad \lambda' = \sum_{i\geq 1}\varepsilon_{2i+1}.$$

Finally, if $x=y=0$ and $\mathfrak{b}''$ is any ideal Borel subalgebra then a simple highest weight module $L_{\mathfrak{b}''}(\lambda)$ with

$$\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}''}(\lambda) = I(0,0,Y_l,Y_r),$$

is isomorphic to the socle of the tensor product $V_{Y_l} \otimes (V_{Y_r})_*$ (see in [15] *the proof that this tensor product has simple socle).*

4. Proof of Theorem 2.2

The proof consists of several reduction steps which we will go through one-by-one.

By an ideal of an associative algebra we always mean a two-sided ideal. We set

$$\mathrm{U} := \mathrm{U}(\mathfrak{sl}(\infty)), \quad \mathrm{U}_n := \mathrm{U}(\mathfrak{sl}(n)) \subset \mathrm{U}\,.$$

For an ideal $I \in \mathrm{U}$, we put $I_n := I \cap \mathrm{U}_n$ for $n \geq 2$.

Let A be an associative algebra and M be an A-module. We say that M is *integrable* if, for any finitely generated subalgebra $A' \subset A$ and any $m \in M$, we have

$$\dim(A' \cdot m) < \infty.$$

Define an ideal $I \subset A$ to be *integrable* if $I = \mathrm{Ann}_A M$ for an integrable (not necessarily simple) A-module M. An ideal $I \subset A$ is *locally integrable* if, for any finitely generated subalgebra $A' \subset A$, the ideal $I \cap A'$ is an integrable ideal of A'.

It is easy to see that an ideal $I \subset \mathrm{U}$ is locally integrable if, for every $n \geq 2$, the ideal I_n is an intersection of ideals of finite codimension in U_n.

Theorem 2.2 is a direct corollary of the following two statements:

Theorem 4.1. *If $I \subset \mathrm{U}$ is a primitive ideal then I is locally integrable.*

Theorem 4.2. *If I is a locally integrable ideal then I is integrable.*

A stronger version of Theorem 4.1 is proved in Subsection 4.1, and in Subsections 4.2-4.3 we prove a chain of statements which imply Theorem 4.2.

4.1. **Proof of Theorem 4.1.** Let I be an ideal of an associative algebra A. We denote by $\sqrt{I}$ the intersection of all primitive ideals of A containing I. Note that $\sqrt{I}$ is the pullback in A of the Jacobson radical of the ring A/I. If I is a primitive ideal then $I = \sqrt{I}$.

It is clear that Theorem 4.1 is a consequence of the following statement.

Proposition 4.3. *Let I be an ideal of* U. *Then $\sqrt{I}$ is a locally integrable ideal.*

To prove this proposition, we first need the following alternative description of $\sqrt{I}$.

Lemma 4.4. *Let $I \subset A$ be an ideal and let the cardinality of $\mathbb{F}$ exceed the $\mathbb{F}$-dimension of A. Then the following conditions on an element $z \in A$ are equivalent:*

1) $z \in \sqrt{I}$,

2) for every $a \in A$ there is $k \in \mathbb{Z}_{>0}$, such that $(az)^k \in I$.

Proof. The fact that 1) implies 2) follows from [11, p. 344, Corollary 1.8]. We now show that 2) implies 1).

Let $z \in A$ satisfy 2), and let $\bar{z}$ be the image of z in A/I. Assume to the contrary that there exists a simple A/I-module M such that $\bar{z} \cdot M \neq 0$. Pick $m \in M$ with $\bar{z} \cdot m \neq 0$. There exists $\bar{a} \in A$ such that $\bar{a} \cdot (\bar{z} \cdot m) = m$. Let $k \in \mathbb{Z}_{>0}$ satisfy $(\bar{a}\bar{z})^k = 0$. Then

$$0 = (\bar{z}(\bar{a}\bar{z})^k) \cdot m = \bar{z} \cdot m \neq 0.$$

This contradicts our assumption that $\bar{z} \cdot M \neq 0$. Hence $\bar{z} \in \sqrt{I}$. □

The proof of Proposition 4.3 is based on the following proposition.

Proposition 4.5. *Let I be an ideal of* U. *Then there exists $r \in \mathbb{Z}_{>0}$ such that, for any $n \gg 0$ and any primitive ideal $J(n) \subset \mathrm{U}_n$ containing I_n, the intersection $J(n) \cap \mathrm{U}_{n-r}$ is an integrable ideal in U_{n-r}.*

To prove Proposition 4.5 we need several different ingredients. In particular, we need a description of primitive ideals of $\mathrm{U}(\mathfrak{sl}(n))$, and the notion of associated variety of an ideal.

Associated variety. Let $J(n) \subset \mathrm{U}_n$ be an ideal. Recall that U_n has a standard filtration by the degrees of elements and therefore we can attach to J the graded ideal

$$\mathrm{gr}\, J(n) \subset \mathrm{S}^{\cdot}(\mathfrak{sl}(n)).$$

Denote by $\mathrm{Var}(J(n))$ the set of points of $\mathfrak{sl}(n)^*$ annihilated by $\mathrm{gr}\, J(n)$. It is clear that

$$\text{if } J_1 \subset J_2 \text{ then } \mathrm{Var}(J_2) \subset \mathrm{Var}(J_1).$$

If I is an ideal of U then the intersection $I_n = I \cap \mathrm{U}_n$ determines a sequence of $\mathrm{SL}(n)$-stable varieties $\mathrm{Var}(I_n) \subset \mathfrak{sl}(n)^*$, and we have

$$\phi_{m,n}(\mathrm{Var}(I_m)) \subset \mathrm{Var}(I_n) \tag{1}$$

for the map $\phi_{m,n} : \mathfrak{sl}(m)^* \to \mathfrak{sl}(n)^*$ induced by the inclusion $\mathfrak{sl}(n) \hookrightarrow \mathfrak{sl}(m)$.

For any $n \geq 2$ and any $r' \in \mathbb{Z}_{\geq 0}$ we put

$$\mathfrak{sl}(n)^{\leq r'} := \{x \in \mathfrak{sl}(n) \mid \exists \lambda \in \mathbb{F} : \mathrm{rk}(x - \lambda \mathbf{1}_n) \leq r'\},$$

where rk refers to the rank of a matrix, and $\mathbf{1}_n$ is the identity $n \times n$-matrix, cf. [12]. We identify $\mathfrak{sl}(n)$ and $\mathfrak{sl}(n)^*$ via the Killing form and so we consider $\mathfrak{sl}(n)^{\leq r'}$ as a subset of $\mathfrak{sl}(n)^*$.

Lemma 4.6. *Let I be a nonzero ideal of* U. *Then there exists $r \in \mathbb{Z}_{\geq 0}$ such that*

$$\mathrm{Var}(I_n) \subset \mathfrak{sl}(n)^{\leq r}$$

for all $n \gg 0$.

Proof. If I is nonzero then $\mathrm{Var}(I_m) \neq \mathfrak{sl}(m)^*$ for some $m \geq 2$. For every $n \geq m$ and every $X \in \mathrm{Var}(I_n)$ formula (1) shows that

$$\phi_{n,m}(\mathrm{SL}(n)X) \subset \mathrm{Var}(I_m) \neq \mathfrak{sl}(m)^*$$

where $\mathrm{SL}(n)X$ is the coadjoint orbit of X in $\mathfrak{sl}(n)^*$. Hence $\phi_{n,m}(\mathrm{SL}(n)X)$ is not dense in $\mathfrak{sl}(n)^*$. This together with [12, Lemma 4.12] implies the desired result for $r = m$ under the assumption that $n > 3m$. □

A well known theorem of A. Joseph implies that the associated variety of a primitive ideal $J(n) \subset \mathrm{U}_n$ equals the closure of a nilpotent coadjoint orbit, see [8]. The nilpotent coadjoint orbits of $\mathfrak{sl}(n)$ are identified with the conjugacy classes of nilpotent $n \times n$-matrices. These conjugacy classes are in 1-1 correspondence with the partitions of n: the partition attached to a conjugacy class comes from the Jordan normal form of a representative of this class. In this way we attach a partition of n to $J(n)$. By $p(n)$ we denote the partition conjugate to that partition. Let $r(n)$ to be the difference between n and the maximal element of $p(n)$. It is easy to check that $r(n)$ equals the rank of every element of the orbit defined by $p(n)$.

Lemma 4.7. *Let $X \in \mathfrak{sl}(n)^{\leq r}$ be a nilpotent matrix and $p(n)$ be the partition attached to the conjugacy class of X. Then $r(n) \leq r$.*

Proof. We have $\mathrm{rk}(X - \lambda \mathbf{1}_n) \leq r$ for $\lambda \in \mathbb{F}$. If $\lambda \neq 0$ then $\mathrm{rk}(X - \lambda \mathbf{1}_n) = n$, and therefore

$$r(n) = \mathrm{rk} X \leq n \leq r.$$

If $\lambda = 0$ then $r(n) = \mathrm{rk} X = \mathrm{rk}(X - \lambda \mathbf{1}_n) \leq r$. □

Description of primitive ideals of $\mathrm{U}(\mathfrak{sl}(n))$. We use a version of the classification of primitive ideals of U_n given in [13]. Namely, a primitive ideal $J(n)$ of U_n is determined by its intersection with the centre of U_n, together with the left cell in the integral Weyl group attached to this intersection, see for example [9, Section 6]. The intersection of $J(n)$ with the centre of U_n can be encoded by an unordered n-tuple $a'_1, ..., a'_n \in \mathbb{F}$. The integral Weyl group is isomorphic to a direct product of symmetric groups, and the factors of this direct product are parametrized by the equivalence classes of elements of $\{1, ..., n\}$ with respect to the equivalence relation

$$i \sim j \Leftrightarrow a'_i - a'_j \in \mathbb{Z}.$$

The left cells of the integral Weyl group of J are in 1-1 correspondence with collections of Young tableaux such that the entries of the ith tableau are precisely the elements of the ith equivalence class in $\{1, ..., n\}$, see [1, p. 172].

Inserting a'_i instead of i in all these semistandard tableaux, we attach to any primitive ideal J the datum

$$\cup_{t\in\mathbb{F}/\mathbb{Z}}\{a^t_{1,1}, a^t_{2,1}, ..., a^t_{l^t_1,1}; a^t_{1,2}, a^t_{2,2}, ..., a^t_{l^t_2,2}; ...; a^t_{1,h_t}, ..., a^t_{l^t_{h_t},h_t}\} \tag{2}$$

where

1) $h_t \neq 0$ only for a finite subset of $\mathbb{F}/\mathbb{Z}$ and $\Sigma_{t,j} l^t_j = n$,
2) $a^t_{i,j} \in \mathbb{F}$ and the image of $a^t_{i,j}$ in $\mathbb{F}/\mathbb{Z}$ equals t,
3) $a^t_{i,j} - a^t_{i',j} \in \mathbb{Z}_{>0}$ for all $t \in \mathbb{F}/\mathbb{Z}, 1 \leq j \leq h_t, 1 \leq i < i' \leq l^t_j$,
4) $l^t_j \leq l^t_{j'}$ for all $t \in \mathbb{F}/\mathbb{Z}, 1 \leq j < j' \leq h_j$.
5) $a^t_{i,j} - a^t_{i,j'} \in \mathbb{Z}_{\geq 0}$ for all $t \in \mathbb{F}/\mathbb{Z}, 1 \leq j \leq h_t, 1 \leq i \leq l^t_j, 1 \leq j' \leq j$.

Here h_t is the height of the tth tableau, $l^t_1, ..., l^t_{h_t}$ are the row lengths of the tth tableau, and $a^t_{i,j}$ is the ith entry of the jth row of the tth tableau.

We now assume that the above datum corresponds to a primitive ideal $J(n)$ of U_n, and let $t_1, ..., t_s$ be the elements of $\mathbb{F}/\mathbb{Z}$ for which $h_t \neq 0$. Then the parts of $p(n)$ (that is the partition attached to the associated variety of $J(n)$ defined above) are all nonzero elements in the sequence

$$l^{t_1}_1, l^{t_1}_2, ..., l^{t_1}_{h_{t_1}}, l^{t_2}_1, l^{t_2}_2, ..., l^{t_2}_{h_{t_2}}, ..., l^{t_s}_1, ..., l^{t_s}_{h_{t_s}}$$

(repetitions are possible). Therefore,

$$r(n) = n - \max_{t\in\mathbb{F}/\mathbb{Z},\ 1\leq j\leq h_t} l^t_j. \tag{3}$$

Denote by $\lambda^*(n)$ the $\mathfrak{sl}(n)$-weight corresponding to the sequence

$$\begin{array}{lll} a^{t_1}_{1,1}, a^{t_1}_{2,1}, ..., a^{t_1}_{l^{t_1}_1,1}, & a^{t_1}_{1,2}, a^{t_1}_{2,2}, ..., a^{t_1}_{l^{t_1}_2,2}, & ..., \\ a^{t_2}_{1,1}, a^{t_2}_{2,1}, ..., a^{t_2}_{l^{t_2}_1,1}, & a^{t_2}_{1,2}, a^{t_2}_{2,2}, ..., a^{t_2}_{l^{t_2}_2,2}, & ..., \\ & ... & \\ a^{t_s}_{1,1}, a^{t_s}_{2,1}, ..., a^{t_s}_{l^{t_s}_1,1}, & a^{t_s}_{1,2}, a^{t_s}_{2,2}, ..., a^{t_s}_{l^{t_s}_2,2}, & ..., \end{array}$$

by ρ the weight corresponding to the sequence

$$0, 1, 2, ..., n-1,$$

and set $\lambda(n) := \lambda^*(n) + \rho$.

Then $J(n) = \mathrm{Ann}_{\mathrm{U}_n} L(\lambda(n))$ where $L(\lambda(n))$ is a simple $\mathfrak{sl}(n)$-module with highest weight $\lambda(n)$. This can be seen for instance by following the algorithm in [13, Subsection 4.2]. It is clear that $L(\lambda(n))|_{\mathfrak{sl}(l^{t_i}_j)}$ is an integrable module for each root subalgebra $\mathfrak{sl}(l^{t_i}_j)$ of $\mathfrak{sl}(n)$ corresponding to the subsequence $a^{t_i}_{1,j}, ..., a^{t_i}_{l^{t_i}_j,j}$. This implies the following.

Corollary 4.8. *In the above notation, the ideal* $J(n) \cap \mathrm{U}(\mathfrak{sl}(l^t_j))$ *is an integrable ideal of* $\mathrm{U}(\mathfrak{sl}(l^t_j))$.

Proof of Proposition 4.5. According to Lemma 4.6, there exists $r \in \mathbb{Z}_{\geq 0}$ such that $\mathrm{Var}(I_n) \subset \mathfrak{sl}(n)^{\leq r}$ for $n \gg 0$. Since $J(n) \supset I_n$, we have

$$\mathrm{Var}(J(n)) \subset \mathrm{Var}(I_n) \subset \mathfrak{sl}(n)^{\leq r}. \tag{4}$$

Recall that $\mathrm{Var}(J(n))$ is the closure of the coadjoint orbit $\mathrm{SL}(n)X$ of a nilpotent matrix

$$X \in \mathfrak{sl}(n) \cong \mathfrak{sl}(n)^*$$

of rank $r(n)$. The inclusion (4) implies that $X \in \mathfrak{sl}(n)^{\leq r}$. Therefore Lemma 4.7 yields $r(n) \leq r$.

We attach to $J(n)$ the datum (2) as above. Corollary 4.8 shows that $J(n) \cap \mathrm{U}(\mathfrak{sl}(l^t_j))$ is an integrable ideal of $\mathrm{U}(\mathfrak{sl}(l^t_j))$. Finally, (3) together with the inequality $r \geq r(n)$ implies that $J(n) \cap \mathrm{U}(\mathfrak{sl}(n-r))$ is an integrable ideal of $\mathrm{U}(\mathfrak{sl}(n-r))$. □

Proof of Proposition 4.3. Proposition 4.5 implies the existence of $r \geq 0$ such that $\sqrt{I_{n+r}} \cap \mathrm{U}_n$ is an integrable ideal of U_n for $n \gg 0$. Next, Lemma 4.4 shows that $(\sqrt{I})_n = \cap_{n' \geq n} \sqrt{I_{n'}}$ for all $n \geq 2$. However,

$$\cap_{n' \geq n} \sqrt{I_{n'}} = (\cap_{n' \geq n+r} \sqrt{I_{n'}}) \cap \mathrm{U}_n\,.$$

Being integrable in $\mathrm{U}_{n'-r}$, the ideal $\sqrt{I_{n'}} \cap \mathrm{U}_{n'-r}$ is an intersection of ideals of finite codimension in $\mathrm{U}_{n'-r}$, hence $(\sqrt{I})_n = (\cap_{n' \geq n+r} \sqrt{I_{n'}}) \cap \mathrm{U}_n$ is an intersection of ideals of finite codimension in U_n. This means that the ideal $(\sqrt{I})_n$ is integrable for $n \gg 0$. A very similar argument shows that $(\sqrt{I})_n$ is integrable for all $n \geq 2$. □

4.2. **Locally integrable ideals and p.l.s..** Let I be a locally integrable ideal of U. As we pointed out, for every $n \geq 2$, $I_n \subset \mathrm{U}_n$ is an intersection of ideals of finite codimension in U_n. Therefore, I_n is the intersection of annihilators of finite-dimensional $\mathfrak{sl}(n)$-modules. Since any finite-dimensional $\mathfrak{sl}(n)$-module is semisimple, it follows that I_n is an intersection of annihilators of simple finite-dimensional U_n-modules.

Let Irr_n denote the set of isomorphism classes of simple finite-dimensional $\mathfrak{sl}(n)$-modules. We put

$$Q(I)_n := \{[M] \in \mathrm{Irr}_n \mid (I \cap \mathrm{U}_n) \subset \mathrm{Ann}_{\mathrm{U}_n} M\}$$

where M stands for a simple finite-dimensional $\mathfrak{sl}(n)$-module and $[M]$ denotes the isomorphism class of M. For any $n' \geq n$ and any subset $Q_{n'} \subset \mathrm{Irr}_{n'}$ we denote by $(Q_{n'})|_{\mathfrak{sl}(n)}$ the set of isomorphism classes of all simple $\mathfrak{sl}(n)$-submodules of the $\mathfrak{sl}(n')$-modules M with $[M] \in Q_{n'}$. It is clear that $Q(I)_{n'}|_{\mathfrak{sl}(n)} \subset Q(I)_n$. This leads to the following definition.

Let $Q = \{Q_2, Q_3, ..., Q_n, ...\}$ be a collection of subsets $Q_2 \subset \mathrm{Irr}_2, Q_3 \subset \mathrm{Irr}_3, ..., Q_n \subset \mathrm{Irr}_n, ...$ We call Q a *precoherent local system* (*p.l.s.* for short) if $Q_{n'}|_{\mathfrak{sl}(n)} \subset Q_n$ for all $n' \geq n \geq 2$. By definition, Q is a *coherent local system* (*c.l.s.* for short) if $Q_{n'}|_{\mathfrak{sl}(n)} = Q_n$ for all $n' \geq n \geq 2$. The notion of c.l.s. has been introduced by A. Zhilinskii [17].

The collection $\{Q_n(I)\}_{n\geq 2}$ defined above is immediately seen to be a p.l.s.. We denote this p.l.s. by $Q(I)$. Conversely, given a p.l.s. Q, we assign to Q the ideal

$$I(Q) := \cup_{n' \geq n}(\cap_{[M]\in Q_{n'}} \mathrm{Ann}_{\mathrm{U}_{n'}} M) \subset \mathrm{U}\,.$$

Clearly, $I(Q)$ is a locally integrable ideal I of U. Moreover, $I(Q(I)) = I$ for any locally integrable ideal $I \subset \mathrm{U}$. This reduces Theorem 4.2 to the following statement.

Proposition 4.9. *If Q is a p.l.s. then $I(Q)$ is an integrable ideal.*

Remark 4.10. *One can show that $Q(I)$ is a c.l.s. whenever I is an integrable ideal. Therefore, Proposition 4.9 implies that $Q(I)$ is in fact a c.l.s. under the weaker assumption that I is a locally integrable ideal of* U.

We say that two p.l.s. Q, Q' are *equivalent* if there exists $n \geq 2$ such that $Q_{n'} = Q'_{n'}$ for all $n' \geq n$. It is clear that if Q and Q' are equivalent, then $I(Q) = I(Q')$. It is known that if Q is a c.l.s., then $I(Q)$ is an integrable ideal [12]. Thus, in order to prove Proposition 4.9, it suffices to prove the following.

Proposition 4.11. *For any p.l.s. Q there exists a c.l.s. Q' such that Q and Q' are equivalent.*

Next, we reduce Proposition 4.11 to a purely combinatorial statement. We call a nonincreasing sequence $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ of integers a *$\mathbb{Z}$-partition* of *width* $\sharp\lambda := n$ ($\mathbb{Z}$-partitions of width n are precisely the integral dominant weights of $\mathfrak{gl}(n)$). We then identify Irr_n with the set of $\mathbb{Z}$-partitions of width n modulo the equivalence relation

$$(\lambda_1 \geq ... \geq \lambda_n) \sim (\lambda_1 + D \geq ... \geq \lambda_n + D), \quad D \in \mathbb{Z}.$$

By V_λ we denote a simple finite-dimensional $\mathfrak{sl}(n)$-module corresponding to the $\mathbb{Z}$-partition λ with $\sharp\lambda = n$. By a slight abuse of notation we write $\lambda \in \mathrm{Irr}_{\sharp\lambda}$.

The classical Gelfand-Tsetlin rule claims that, for $\mathbb{Z}$-partitions λ and μ with $\sharp\lambda = n$, $\sharp\mu = n-1$, the following conditions are equivalent:

- $\mathrm{Hom}_{\mathfrak{sl}(n-1)}(V_\mu, V_\lambda|_{\mathfrak{sl}(n-1)}) \neq 0$,
- there exists $D \in \mathbb{Z}$ such that $\lambda_1 \geq \mu_1 + D \geq \lambda_2 \geq \mu_2 + D \geq ... \geq \lambda_{n-1} \geq \mu_{n-1} + D \geq \lambda_n$.

We write $\lambda > \mu$ whenever these conditions hold.

For general $\mathbb{Z}$-partitions λ and μ with $\sharp\lambda \geq \sharp\mu$, the Gelfand-Tsetlin rule implies that the following conditions are equivalent:

- $\mathrm{Hom}_{\mathfrak{sl}(\sharp\mu)}(V_\mu, V_\lambda|_{\mathfrak{sl}(\sharp\mu)}) \neq 0$,
- there exists a sequence of $\mathbb{Z}$-partitions $\lambda = \lambda^0, \lambda^1, ..., \lambda^{n-m} = \mu$ such that

$$\lambda = \lambda^0 > \lambda^1 > ... > \lambda^{n-m} = \mu$$

and $\sharp\lambda^i = \sharp\lambda - i$. We write $\lambda \succ \mu$ whenever these latter conditions hold, and say that λ *dominates* μ.

We can now restate the definitions of c.l.s. and p.l.s. as follows. Let $Q = \{Q_2, Q_3, ..., Q_n\}$ be a collection of subsets $Q_2 \subset \mathrm{Irr}_2, Q_3 \subset \mathrm{Irr}_3, ..., Q_n \subset \mathrm{Irr}_n, ...$ Then
a) the following conditions are equivalent:
- Q is a p.l.s.,
- for all λ, μ such that $\lambda \succ \mu$ and $\lambda \in Q_{\sharp\lambda}$, we have $\mu \in Q_{\sharp\mu}$;

b) the following conditions are equivalent:
- Q is a c.l.s.,
- Q is a p.l.s. and for every $\mu \in Q_{\sharp\mu}$ there is $\lambda \in Q_{\sharp\mu+1}$ such that $\lambda \succ \mu$.

We denote by $Q^\vee(\lambda)$ the largest p.l.s. Q which does not contain (the equivalence class of) a given $\mathbb{Z}$-partition λ. It is clear that
$Q^\vee(\lambda)_n$ consists of all $\mathbb{Z}$-partitions of width n for $n < \sharp\lambda$,
$Q^\vee(\lambda)_{\sharp\lambda}$ consists of all $\mathbb{Z}$-partitions of width $\sharp\lambda$ except λ,
$Q^\vee(\lambda)_n$ consists of all $\mathbb{Z}$-partitions μ of width n such that $\mu \not\succ \lambda$ for all $n > \sharp\lambda$.
We are now ready to state

Proposition 4.12. *For any $\mathbb{Z}$-partition λ, the p.l.s. $Q^\vee(\lambda)$ is equivalent to the c.l.s.*
$$Q(\lambda) := \cup_{1\le k<l\le\sharp\lambda} Q(k, l, \lambda_k - \lambda_l)$$
where $Q(k, l, \lambda_k - \lambda_l)$ is the c.l.s. defined by the formula
$$Q(k, l, \lambda_k - \lambda_l)_m := \{\mu \in \mathrm{Irr}_m \mid \mu_k - \mu_{m-\sharp\lambda+l} < \lambda_k - \lambda_l\}.$$

The next subsection is devoted to the proof of Proposition 4.12. We conclude this subsection by showing how Proposition 4.12 implies Proposition 4.11, and therefore ultimately Proposition 4.3.

Proof of Proposition 4.11. Let Q be a p.l.s. Then, clearly
$$Q = \cap_{\lambda\notin Q} Q^\vee(\lambda).$$
According to Proposition 4.12, a p.l.s. of the form $Q^\vee(\lambda)$ is equivalent to the c.l.s. $Q(\lambda)$. The lattice of c.l.s. is artinian [17], and therefore we conclude that Q is equivalent to a c.l.s.
$$Q(\lambda_1) \cap .... \cap Q(\lambda_s)$$
for some finite set of elements $\lambda_1, ..., \lambda_s \notin Q$. □

4.3. **Combinatorics of $\mathbb{Z}$-partitions.** It is clear that Proposition 4.12 is implied by the following.

Proposition 4.13. *Let λ and μ be $\mathbb{Z}$-partitions such that $\sharp\mu \ge 4\sharp\lambda$. Then the following conditions are equivalent:*
1) $\mu \succ \lambda$,
2) $\mu_k - \mu_{\sharp\mu-\sharp\lambda+l} \ge \lambda_k - \lambda_l$ for any $1 \le k < l \le \sharp\lambda$.

In the proof of Proposition 4.13 we need the following three lemmas.

Lemma 4.14. *Let λ and μ be $\mathbb{Z}$-partitions such that $\sharp\mu \ge \sharp\lambda$, $\lambda_1 = \mu_1, \lambda_{\sharp\lambda} = \mu_{\sharp\mu}$. Then*
$$\mu_i \ge \lambda_i \ge \mu_{\sharp\mu-\sharp\lambda+i} \text{ for } 1 \le i \le \sharp\lambda$$
implies $\mu \succ \lambda$.

Proof. If $\sharp\lambda = \sharp\mu$ then clearly $\lambda = \mu$. For $\sharp\mu > \sharp\lambda$, arguing by induction, it clearly suffices to show the existence of a $\mathbb{Z}$-partition μ' such that
- $\mu > \mu', \sharp\mu' = \sharp\mu - 1$,
- $\mu'_1 = \mu_1, \mu'_{\sharp\mu'} = \mu_{\sharp\mu}$,
- $\mu'_i \ge \lambda_i \ge \mu'_{\sharp\mu'-\sharp\lambda+i}$ for $1 \le i \le \sharp\lambda$,

This is straightforward and we leave the details to the reader. □

Lemma 4.15. *Let λ and μ be $\mathbb{Z}$-partitions such that $\sharp\mu \ge 2\sharp\lambda$. Then the conditions*
a) for every $k, l \in \mathbb{Z}_{\ge1}$ such that $1 \le k < l \le \sharp\lambda$ we have $\mu_k - \mu_{\sharp\mu-\sharp\lambda+l} \ge \lambda_k - \lambda_l$,
b) there are $k, l \in \mathbb{Z}_{\ge1}$ such that $1 \le k < l \le \sharp\lambda$ and $\mu_k - \mu_{\sharp\mu-\sharp\lambda+l} = \lambda_k - \lambda_l$
imply $\mu \succ \lambda$.

Proof. Condition a) and b) implies via Lemma 4.14 the existence of $\mathbb{Z}$-partitions $\mu^0, \mu^1, ..., \mu^{m-n}$ such that
- $\mu^i \succ \mu^{i+1}$, $\sharp\mu^i = l - k + 1 + (\sharp\mu - \sharp\lambda - i)$,
- $\mu^0 = (\mu_k \ge ... \ge \mu_{\sharp\mu-\sharp\lambda+l})$,
- $\mu^{\sharp\mu-\sharp\lambda} = (\lambda_k \ge ... \ge \lambda_l)$,
- $\lambda_k = \mu^0_1 = \mu^1_1 = \mu^2_1 = ... = \mu^{\sharp\mu-\sharp\lambda}_0$, and $\lambda_l = \mu^0_{\sharp\mu^0} = \mu^1_{\sharp\mu^1} = ... = \mu^{\sharp\mu-\sharp\lambda}_{l-k+1}$.

We set

$$\hat{\mu^i}_j := \begin{cases} \mu_j \text{ for } j < k-i, \\ \lambda_j + (\mu_k - \lambda_k) \text{ for } k-i \le j \le k, \\ \mu^i_{j-k} \text{ for } k < j \le (\sharp\mu - i) - \sharp\lambda + l, \\ \lambda_{j-(\sharp\mu-\sharp\lambda)} + (\mu_{\sharp\mu-\sharp\lambda+l} - \lambda_l) \text{ for } (\sharp\mu - i) - \sharp\lambda + l < j \le \sharp\mu - \sharp\lambda + l, \\ \mu_j \text{ for } j > \sharp\mu - \sharp\lambda + l. \end{cases}$$

One can easily check that
- $\hat{\mu}_{i+1} \succ \hat{\mu}_i$,
- $\hat{\mu}^0 = \mu$ (here it is crucial that $\sharp\mu - \sharp\lambda \ge \sharp\lambda \ge \max(k, \sharp\lambda - l + 1)$),
- $\mu^{\sharp\mu-\sharp\lambda} = \lambda$.

Thus $\mu \succ \lambda$. □

Lemma 4.16. *Let λ, μ be $\mathbb{Z}$-partitions, and $i \in \mathbb{Z}_{>0}$ be a positive integer such that*

$$\sharp\lambda \le i \le \sharp\mu - i, \quad \mu_i - \mu_{\sharp\mu-i+1} \ge \lambda_1 - \lambda_{\sharp\lambda}.$$

Then $\mu \succ \lambda$.

Proof. Put $\mu' := (\mu_1, ..., \mu_i, \mu_{\sharp\mu-i+1}, ..., \mu_{\sharp\mu})$. It is clear that $\mu \succ \mu'$, and that μ' and λ satisfy the conditions of Lemma 4.16 as well. Therefore without loss of generality we can assume that $\mu = \mu'$, and thus that $i = \sharp\mu - i$. We can also assume that $\lambda_1 = \mu_i$. This implies that $\lambda_{\sharp\lambda} \ge \mu_{\sharp\mu-i+1} = \mu_{i+1}$.

Next, we observe that the following sequence of $\mathbb{Z}$-partitions each element dominates the next:

$$\mu_1 \ge ... \ge \mu_i \ge \mu_{i+1} \ge ... \ge \mu_{\sharp\mu}$$

$$\mu_1 \ge ... \ge \mu_{i-1} \ge \lambda_{\sharp\lambda} \ge \mu_{i+1} \ge ... \ge \mu_{\sharp\mu-1}$$

$$...$$

$$\mu_1 \ge ... \ge \mu_{i-k} \ge \lambda_{\sharp\lambda-k+1} \ge ... \ge \lambda_{\sharp\lambda} \ge \mu_{i+1} \ge ... \ge \mu_{\sharp\mu-k}$$

$$...$$

$$\mu_1 \ge ... \ge \mu_{i-\sharp\lambda} \ge \lambda_1 \ge ... \ge \lambda_{\sharp\lambda} \ge \mu_{i+1} \ge ... \ge \mu_{\sharp\mu-\sharp\lambda}.$$

The last $\mathbb{Z}$-partition dominates λ, hence $\mu \succ \lambda$. □

Proof of Proposition 4.13. It is clear that 1) implies 2). We show now that 2) implies 1). To do this, we assume to the contrary that 2) holds and $\mu \not\succ \lambda$. We claim that this contradicts Lemma 4.15.

Indeed, consider the $\mathbb{Z}$-partitions $\mu^{r,s}$, for $r, s \le \sharp\lambda$, where

$$\mu^{r,s}_i = \mu_{i+r}, \text{ for } 1 \le i \le \sharp\mu - s - r, \ \mu^{0,0} = \mu.$$

It is clear that $\mu \succ \mu^{r,s}$. Lemma 4.16 implies that $\mu_{\sharp\lambda} - \mu_{\sharp\mu-\sharp\lambda+1} < \lambda_1 - \lambda_{\sharp\lambda}$, and in particular that $\mu^{\sharp\lambda-1,\sharp\lambda-1}$ does not satisfy condition 2) of Proposition 4.13 considered as an abstract condition on a partition μ' instead of μ. Therefore, since $\mu = \mu^{0,0}$ satisfies this condition, there exist $r, s < \sharp\lambda$ such that $\mu^{r,s}$ satisfies this condition, and $\mu^{r+1,s}$ or $\mu^{r,s+1}$ does not satisfy this condition. These two cases are very similar, and we consider only the first one (leaving the second one to the reader).

We put $\mu' := \mu^{r,s}$ and assume that $\mu^{r+1,s} \not\succ \lambda$. Then there exist k, l, for $1 \le k < l \le \sharp\lambda$, such that

$$\mu'_k - \mu'_{\sharp\mu-\sharp\lambda+l} \ge \lambda_k - \lambda_l, \quad \mu'_{k+1} - \mu'_{\sharp\mu-\sharp\lambda+l} < \lambda_k - \lambda_l.$$

Without loss of generality we can assume that l is chosen so that the value of

$$\mu_{\sharp\mu-\sharp\lambda+l} + \lambda_k - \lambda_l$$

is maximal. This implies that, for

$$\mu'' := \mu'_1, \mu'_2, ..., \mu'_{k-1}, \mu'_k + \lambda_l - \lambda_k, \mu'_{k+1}, \mu'_{k+2}, ..., \mu'_{\sharp\mu-r-s-1},$$

we have $\mu \succ \mu''$, and all conditions of Lemma 4.15 are satisfied for the pair (λ, μ'') (here it is crucial that $\mu + 2 - (r+s) \ge 2\lambda$). Hence $\mu'' \succ \lambda$, and we have the desired contradiction. □

5. Appendix: The inclusion order on primitive ideals

As explained in [12, Subsection 7.3], a c.l.s. for $\mathfrak{sl}(\infty)$ can be encoded by a pair of nonincreasing sequences

$$p_1 \geq p_2 \geq p_3 \geq ... \text{ and } q_1 \geq q_2 \geq q_3 \geq ... \tag{5}$$

of elements of $\mathbb{Z}_{\geq 0} \sqcup \{+\infty\}$ with common limit

$$\lim_{i\to\infty} p_i = \lim_{i\to\infty} q_i = m,$$

see [17], see also [12, Proposition 7.5 and Subsection 7.3]. We denote by $\mathrm{cls}(p_1, p_2, ...; q_1, q_2, ...)$ the c.l.s. attached to the pair of sequences (5). The inclusion order on c.l.s. is described by the following theorem due to A. Zhilinskii.

Theorem 5.1 ([17, Subsection 2.5], see also [12, Subsection 7.3]). *Let $\{p_i, q_i\}_{i\geq 1}$ and $\{p'_i, q'_i\}_{i\geq 1}$ be pairs of nonincreasing sequences with respective limits m, m' as in (5). Then the following conditions are equivalent:*
a) $\mathrm{cls}(p'_1, p'_2, ...; q'_1, q'_2, ...) \subset \mathrm{cls}(p_1, p_2, ...; q_1, q_2, ...)$,
b) there exist $a, b \in \mathbb{Z}_{\geq 0}$ such that $a + b = m - m'$ and

$$p'_i \leq p_i - a, \quad q'_i \leq q_i - b.$$

Consider $I(x, y, Y_l, Y_r)$ for $Y_l = (l_1 \geq ... \geq l_s > 0), Y_r = (r_1 \geq ... \geq r_t > 0)$, and put

$$p_i^c := \begin{cases} +\infty & \text{if } 1 \leq i \leq c \\ y + l_{i-c} & \text{if } c+1 \leq i \leq c+s \\ y & \text{if } i > c+s+1 \end{cases}, \quad q_i^d := \begin{cases} +\infty & \text{if } 1 \leq i \leq d \\ y + r_{i-d} & \text{if } d+1 \leq i \leq d+t \\ y & \text{if } i > d+t+1 \end{cases} \tag{6}$$

for any $c, d \in \mathbb{Z}_{\geq 0}$.

Proposition 5.2. *We have*
a) $I = I(x, y, Y_l, Y_r) = I(\mathrm{cls}(p_1^x, p_2^x, ...; q_1^0, q_2^0, ...))$,
b) $I = I(x, y, Y_l, Y_r) = I(\mathrm{cls}(p_1^c, p_2^c, ...; q_1^d, q_2^d, ...))$ *for all c, d such that $c + d = x$,*
c) $Q(I) = \cup_{c+d=x}\mathrm{cls}(p_1^c, p_2^c, ...; q_1^d, q_2^d, ...)$.

Proof. Part a) follows from the definition of $I(x, y, Y_l, Y_r)$, see also [12, Theorem 7.9]. Part b) follows from the discussion in [12, Subsection 7.4], see formula

$$I(v, w, Q_f) = I(v + w, 0, Q_f)$$

in the notation of [12].

Part c) is implied by [12, Lemma 7.6c)]. □

Alltogether, this allows us to provide an explicit inclusion criterion for a pair of primitive ideals.

Theorem 5.3. *Let x, y, Y_l, Y_r be as above, and let $Y'_l = (l'_1 \geq ... \geq l'_{s'} > 0), Y'_r = (r'_1 \geq ... \geq r'_{t'} > 0)$ be Young diagrams and $x', y' \in \mathbb{Z}_{\geq 0}$. The following conditions are equivalent:*
a) there is a (not necessarily strict) inclusion

$$I(x, y, Y_l, Y_r) \subset I(x', y', Y'_l, Y'_r), \tag{7}$$

b) $x \geq x', y \geq y'$ and, for some $a, b, c, d \in \mathbb{Z}_{\geq 0}$ with $c + d = x - x', a + b = y - y'$, the inequalities

$$l_i + a \geq l'_{i+c}, \quad r_j + b \geq r'_{j+d},$$

where

$$l_i = 0 \text{ if } i > s, \quad l'_i = 0 \text{ if } i > s', \quad r_j = 0 \text{ if } j > t, \quad r'_j = 0 \text{ if } j > t'$$

are satisfied for all $i, j \geq 0$.

Proof. Let $\{p_i^c, q_i^d\}_{i\geq 1}$, $\{(p')_i^c, (q')_i^d\}$ be defined by (6).

We would like to show first that b) implies a). Condition b) yields

$$\mathrm{cls}((p')_1^{x'}, (p')_2^{x'}, ...; (q')_1^0, (q')_2^0, ...) \subset \mathrm{cls}(p_1^{x'+c}, p_2^{x'+c}, ...; q_1^d, q_2^d, ...)$$

by Theorem 5.1. This together with Proposition 5.2 a-b) implies a).

Now we show that a) implies b). The inclusion (7) yields

$$Q(I(x', y', Y'_l, Y'_r)) \subset Q(I(x, y, Y_l, Y_r)).$$

According to Proposition 5.2 c), this implies

$$\cup_{c+d=x'}\mathrm{cls}((p')_1^c, (p')_2^c, ...; (q')_1^d, (q')_2^d, ...) \subset \cup_{c+d=x}\mathrm{cls}(p_1^c, p_2^c, ...; q_1^d, q_2^d, ...).$$

In particular, we have

$$\mathrm{cls}((p')_1^{x'}, (p')_2^{x'}, ...; (q')_1^0, (q')_2^0, ...) \subset \cup_{c+d=x} \mathrm{cls}(p_1^c, p_2^c, ...; q_1^d, q_2^d, ...).$$

The c.l.s. $\mathrm{cls}((p')_1^{x'}, (p')_2^{x'}, ...; (q')_1^0, (q')_2^0, ...)$ is irreducible [17, Definition I.I.I], and therefore

$$\mathrm{cls}((p')_1^{x'}, (p')_2^{x'}, ...; (q')_1^0, (q')_2^0, ...) \subset \mathrm{cls}(p_1^c, p_2^c, ...; q_1^d, q_2^d, ...) \tag{8}$$

for some c, d such that $c + d = x$, see [17, Proposition I.I.2]. Next, by Theorem 5.1 the inclusion (8) implies that condition b) of Theorem 5.1 holds for $a, b \in \mathbb{Z}_{\geq 0}$ such that

$$a + b = \lim p_i^c - \lim (p')_i^c.$$

Since, by definition, $\lim p_i^c = y$ and $\lim(p')_i^c = y'$, we obtain that claim b) of Theorem 5.3 holds for

$$a, b\ (a + b = y - y') \text{ and } c - (x'), d\ ((c - x') + d = x - x').$$

□

Corollary 5.4. *The lattice of primitive ideals of* $\mathrm{U}(\mathfrak{sl}(\infty))$ *satisfies the ascending chain condition.*

Corollary 5.5. *The augmentation ideal* $I(0, 0, \emptyset, \emptyset)$ *is the only maximal ideal of* $\mathrm{U}(\mathfrak{sl}(\infty))$.

Proof. The statement is implied by Theorems 2.2 and 5.3. □

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