The Formalism of Left and Right Connections on Supermanifolds

Yu. I. Manin, I. B. Penkov
Steklov Mathematical Institute, USSR Academy of Sciences
117956 Moscow GSP1, USSR

Introduction

This paper originated in the authors' attempts to understand characteristic classes in supergeometry.

We start with recalling one of the constructions of characteristic classes in the category of C^∞- manifolds, [BC], [MS], [W]. Let X be a C^∞- manifold, $\mathcal{F}$ a locally free sheaf (the sheaf of sections of a vector bundle) on it. The characteristic classes of $\mathcal{F}$ in the de Rham cohomology can be constructed as follows. Choose a connection ∇ on $\mathcal{F}$ and consider its curvature $F_\nabla \in \Gamma(X, End\mathcal{F} \otimes \Omega^2 X)$. The forms $trF^i_\nabla \in \Gamma(X, \Omega^{2i} X)$ are closed and their cohomology classes do not depend on the choice of ∇. If the structure group of $\mathcal{F}$ is reduced, instead of the traces of F^i_∇ one should consider all invariant polynomials on the Lie algebra of the structure group. In the complex-analytic situation a global connection may not exist, instead one should first construct the Atiyah-class F^{an}_∇, which is the obstruction to the existence of a holomorphic connection on $\mathcal{F}, F^{an}_\nabla \in H^1(X, End\mathcal{F} \otimes \Omega^1 X)$, and then consider $tr[(F^{an}_\nabla)^i] \in H^i(X, \Omega^i X)$.

In supergeometry one may carry out these constructions too. However, we were not satisfied with both of them because the cohomology classes one obtains in this way have "pure even dimension", being possible to integrate differential forms only along chains of odd dimension zero. But in [BL1] Bernstein and Leites introduced objects, which can be integrated along chains of odd codimension zero. These are the integral forms. It was natural to conjecture that the complex of integral forms is connected with some "integral sequence of a sheaf", in the same manner as the de Rham complex leads to the de Rham sequence of a sheaf with connection.

The aim of our paper is to present this new construction. We introduce the notion of a right connection on a vector bundle and discuss its integral sequence (we call it the Spencer sequence of the right connection).

Although the curvature of a right connection is again a differential form, we are able to produce characteristic classes with values in integral forms by a trick explained in §4.

A way out of the purely even construction is suggested by another remark of Bernstein and Leites. One may integrate the so called pseudodifferential forms, fastly decreasing in the differentials of the odd coordinates, [BL2]. In our case they can be constructed as power series of the form $\sum_0^\infty a_i trF^i_\nabla$.

The problem of characteristic classes in supergeometry is closely connected with the problem of the (co)homology theory, which carries them. The approach via classifying spaces gives a possible definition of "Schubert supercells" with varying odd dimension.

For example in the projective space $\mathbf{P}^{n|m}$ lie $\mathbf{P}^{a|b}$ with all possible a and b: $0 \le a \le n$, $0 \le b \le m$, which may be given with $n-a$ even and $m-b$ odd equations. However it is not clear in what homology theory there are classes of these cells, with fewer relations between them, than in the case $m=0$.

An idea of A. S. Schwarz to realize characteristic classes not by forms, but by densities, seems quite interesting too. Densities are objects which can be integrated along chains of arbitrary codimension, but they do not have a natural structure of a complex, and have some unusual properties.

§1. Preliminaries

1.1. Let $X = (\mathcal{K}, O_x)$ be a supermanifold over k, where by k we denote one of the fields $\mathbb{R}$ or $\mathbb{C}$. This means that $\mathcal{K}$ is a topological space and $O_x = O_0 \oplus O_1$ - a sheaf of supercommutative rings with the following properties:

1. If we denote by $N \hookrightarrow O_x$ the ideal of nilpotents, then the ringed space $X_{red} \stackrel{\text{def}}{=} (\mathcal{K}, O_x/N)$ is a C^∞- real or analytic (real or complex) manifold.

2. The O_x/N -module N/N^2 is locally free, $rkN/N^2 < \infty$ and O_x is locally isomorphic to the Grassman algebra $\Lambda^{\cdot}(N/N^2)$. In this paper we consider parallelly C^∞- or analytic supermanifolds over k. When it is necessary, we restrict ourselves with the smooth or analytic category.

Given a supermanifold X, there is always an embedding $g_X : X_{red} \hookrightarrow X$, but in the analytic category there may be no inverse projection. However the following is true:

Theorem (Batchelor, [B]): In the C^∞ -case there always exists an isomorphism $X \simeq (\mathcal{K}, \Lambda^{\cdot}(N/N^2))$.

If $\mathcal{F}$ is a locally free O_x -module of rank $p \mid q$, the modules $\prod \mathcal{F}$ and $\mathcal{F}^*$, of ranks $q \mid p$ and $p \mid q$, are defined. $\prod \mathcal{F}$ is "$\mathcal{F}$ with the opposite parity", and $\mathcal{F}^*$ is the dual of $\mathcal{F}$, [L].

By $\mathcal{T}X$ we denote the tangent sheaf of X, which is locally free of rank $dimX$. In this paper we set $dimX = n \mid m$ where $n \stackrel{\text{def}}{=} dimX_{red}$ and m is by definition equal to $rk_{O/N}N/N^2$.

1.2. The algebra of differential forms on X is $\Omega^{\cdot}X \stackrel{\text{def}}{=} \mathbf{S}(\prod \mathcal{T}X^*)$, where by $\mathbf{S}$ we denote the (super)symmetric algebra. Its homogeneous components $\Omega^i X$ are locally free sheaves on X, and there is defined the de Rham complex $(\Omega^{\cdot}X, d)$, where $d : \Omega^i X \to \Omega^{i+1} X$ acts by the formula: $d = \sum_i du_i \frac{\partial}{\partial u_i}$, u_i being homogeneous coordinates on X, $u_1, \cdots, u_n$ are even and $u_{n+1}, \cdots, u_{n+m}$ odd. Further if a is an homogeneous element of a $\mathbb{Z}_2$ -graded object, by $\overline{a}$ we denote its parity: $\overline{a}$ is 0 or 1.

The analogue of the canonical sheaf on a supermanifold is the sheaf of volume forms $BerX$, which we call the Berezinian. This is an invertible sheaf of O_x -modules of parity $\frac{1}{2}(1-(-1)^{n+m})$. It is convenient to write its sections in the form:

$$f(\kappa, \xi).D(d\kappa_1 \cdots d\kappa_n d\xi_1 \cdots d\xi_n)$$

where $f(\kappa, \xi)$ is a local section of O_x, $u = (\kappa, \xi)$, $u_i = \kappa_i$, $i \le n$, $u_j = \xi_{j-n}$, $n+1 \le j \le n+m$, [L], [BL1]; D is multiplied by the Berezinian of the Jacobian matrix under the change of coordinates.

Iff $m \ne 0$, $BerX$ does not appear among the Ω^i -s (if $m=0$, $BerX = \Omega^n X$). In this case $BerX$ is included in another complex - the complex of integral forms $\sum_{\cdot} X$, [BL1],

[BL3], the members of which are defined by:

$$\sum_{n-i} X = BerX \otimes_{O_x} \mathbf{S}^i(\prod \mathcal{T}X),\ i \geq 0.$$

Its differential $\partial : \sum_{n-i} X \to \sum_{n-i+1} X$ acts in the following way:

$$\partial(D(du) \otimes Q) = (-1)^{n+m} D(du) \otimes \sum_j \frac{\partial^2 Q}{\partial(\prod \frac{\partial}{\partial u_j})\partial u_j}$$

(here Q is a local section of $\mathbf{S}^i(\prod \mathcal{T}X)$, and $\prod \frac{\partial}{\partial u_j}$ are local sections of $\prod \mathcal{T}X$, $\overline{\prod \frac{\partial}{\partial u_j}} = \overline{\frac{\partial}{\partial u_j}} + 1 = \overline{u_j} + 1$).

We note that d and ∂ are odd k -linear maps.

Iff $m = 0$, there is a canonical isomorphism between $\Omega^{\cdot} X$ and $\sum_{\cdot} X$.

Let ϕ be a vector field. We shall use also the Lie derivative

$$L_\phi : BerX \to BerX$$

defined in [BL3] by the formula $L_\phi = \partial \circ i_\phi$, where i_ϕ is the operator of left tensor multiplication by $\prod \phi$:

$$i_\phi : BerX \to BerX \otimes \prod \mathcal{T}X,$$

$$(\prod \phi = \sum (-1)^{\overline{\phi_i}} \phi_i \prod \frac{\partial}{\partial u_i},\ \text{ if } \phi = \sum \phi_i \frac{\partial}{\partial u_i}).$$

§2. Left and Right Connections

2.1. Denote by $\mathcal{D}_{\leq 1}$ the sheaf of differential operators on X of order ≤ 1, and by $\mathcal{F}$ a locally free sheaf of O_x -modules.

Definition. A left connection on $\mathcal{F}$ is an even k -linear map: $\Delta_l : \mathcal{D}_{\leq 1} \otimes_k \mathcal{F} \to \mathcal{F}$, with the conditions:

$$\begin{aligned} &\text{L0.} \quad \Delta_l(a \otimes f) &&= af \\ &\text{L1.} \quad \Delta_l(\phi \otimes af) &&= \Delta_l(\phi a \otimes f) \\ &\text{L2.} \quad \Delta_l(a\phi \otimes f) &&= a\Delta_l(\phi \otimes f) \end{aligned}$$

where ϕ is a vector field - a local section of $\mathcal{T}X$, and a is a function - a local section of O_x.

A right connection on $\mathcal{F}$ is an even k -linear map $\Delta_r : \mathcal{F} \otimes_k \mathcal{D}_{\leq 1} \to \mathcal{F}$, with the conditions:

$$\begin{aligned} &\text{R0.} \quad \Delta_r(f \otimes a) &&= fa \\ &\text{R1.} \quad \Delta_r(f \otimes \phi a) &&= \Delta_r(f \otimes \phi)a \\ &\text{R2.} \quad \Delta_r(fa \otimes \phi) &&= \Delta_r(f \otimes a\phi). \end{aligned}$$

Note that R1 and L1 imply that connections are not O_x -linear maps, for example:

$$\begin{aligned} a\Delta_r(f \otimes \phi) &= (-1)^{\overline{a}(\overline{f}+\overline{\phi})} \Delta_r(f \otimes \phi) \cdot a \\ &= (-1)^{\overline{a}(\overline{f}+\overline{\phi})} \Delta_r(f \otimes \phi a) \\ &= (-1)^{\overline{a}(\overline{f}+\overline{\phi})} \Delta_r(f \otimes \phi(a)) + (-1)^{\overline{a}\overline{f}} \Delta_r(f \otimes a\phi) \\ &= \Delta_r(af \otimes \phi) + (-1)^{(\overline{a}+\overline{f})\overline{\phi}} \phi(a) f. \end{aligned}$$

If we fix a vector field ϕ, we obtain operators $\Delta_l(\phi)$ and $\Delta_r(\phi)$:

$$\begin{aligned}\Delta_l(\phi)(f) &= \Delta_l(\phi\otimes f)\\ \Delta_r(\phi)(f) &= (-1)^{\overline{f}\overline{\phi}}\Delta_r(f\otimes\phi)\end{aligned}$$

which we call as usual covariant derivatives along ϕ.

2.2. The sequences of de Rham and Spencer. For every sheaf $\mathcal{F}$ with a left connection Δ_l we can define its de Rham sequence $DR^{\cdot}(\mathcal{F})$ and for every sheaf $\mathcal{F}$ with right connection Δ_r we can define its Spencer sequence $S_{\cdot}(\mathcal{F})$:

Definition. To a pair $(\mathcal{F},\Delta_l)$ we associate $DR^{\cdot}(\mathcal{F})$, where $DR^i(\mathcal{F})=\Omega^iX\otimes_{O_x}\mathcal{F}$, and for every $i\geq 0$ we have a differential operator $\nabla_l(i):\Omega^iX\otimes_{O_x}\mathcal{F}\to\Omega^{i+1}X\otimes\mathcal{F}$ given by the formula:

$$\nabla_l(i)(du_{K_1}\cdots du_{K_i}\otimes f)=\sum_j(-1)^{\overline{u}_j(\sum_1^i\overline{du_{K_r}})}du_j\cdot du_{K_1}\cdots du_{K_i}\otimes\Delta_l(\frac{\partial}{\partial u_j}\otimes f).$$

Definition. To a pair $(\mathcal{F},\Delta_r)$ we associate its Spencer sequence $S_{\cdot}(\mathcal{F})$, where $S_{n-i}(\mathcal{F})=\mathcal{F}\otimes_{O_x}\mathrm{S}^i(\prod\mathcal{T}X)$ ($\mathrm{S}(\prod\mathcal{T}X)$ is the supersymmetric algebra of $\prod\mathcal{T}X$), and for every $i>0$ we have a differential operator $\nabla_r(-i+n)\ :\mathcal{F}\otimes_{O_x}\mathrm{S}^i(\prod\mathcal{T}X)\to\mathcal{F}\otimes_{O_x}\mathrm{S}^{i-1}(\prod\mathcal{T}X)$, given by the formula:

$$\nabla_r(-i+n)(f\otimes\pi\frac{\partial}{\partial u_{l_1}}\cdots\frac{\partial}{\partial u_{l_i}})=$$

$$\sum_k(-1)^{(\sum_{j=1}^{k-1}\overline{u}_{l_j}+k-1)(\overline{u}_{l_k}+1)+\overline{f}+1}\Delta_r(f\otimes\frac{\partial}{\partial u_{l_k}})\otimes(\pi\frac{\partial}{\partial u_{l_1}}\cdots\widehat{\frac{\partial}{\partial u_{l_k}}}\cdots\frac{\partial}{\partial u_{l_i}}).$$

It is easy to check that $\nabla_l(i)$ and $\nabla_r(-i+n)$ are defined invariantly by Δ_l and Δ_r and are odd maps. We call them the i-th covariant differentials of Δ_l or Δ_r. Of course in general $DR^{\cdot}(\mathcal{F})$ and $S_{\cdot}(\mathcal{F})$ are not complexes.

Remark. The connections Δ_l and Δ_r are uniquely determined by their covariant differentials $\nabla_l(1)$ and $\nabla_r(n-1)$ because the action of $\phi\cdot a$ is uniquely determined by the action of $a\phi$ (Leibniz rule), so it is sufficient to give explicitly only the action of vector fields, or equivalently the covariant differentials $\nabla_l(1)$ or $\nabla_r(n-1)$.

2.3. The curvature of a connection. As usual, in order to define the curvature of a connection we have to consider the operators $\nabla_l(i+1)\cdot\nabla_l(i)$ and $\nabla_r(-i+n)\cdot\nabla_r(-i+n-1)$ in more detail.

2.3.1.**Lemma:** The operators $\nabla_l(i+1)\cdot\nabla_l(i)$ and $\nabla_r(-i+n)\cdot\nabla_r(-i+n-1)$ are O_x -linear, $\forall i\geq 0$.

Proof: The statement about $\nabla_l(i+1)\cdot\nabla_l(i)$ may be checked exactly as in the case $m=0$, thus we omit this computation and turn to the case of a right connection Δ_r. Consider first $\nabla_r(n-1)\cdot\nabla_r(n-2):\mathcal{F}\otimes\mathrm{S}^2(\prod\mathcal{T}X)\to\mathcal{F}$. We have:

$$\nabla_r(n-1)\cdot\nabla_r(n-2)(f\otimes\pi\frac{\partial}{\partial u_i}\pi\frac{\partial}{\partial u_j})=$$

$$(-1)^{\overline{u}_i}[\Delta_r(\Delta_r(f\otimes\frac{\partial}{\partial u_i})\otimes\frac{\partial}{\partial u_j})-(-1)^{\overline{u}_i\overline{u}_j}\Delta_r(\Delta_r(f\otimes\frac{\partial}{\partial u_j})\otimes\frac{\partial}{\partial u_i})].$$

The linearity of this map may be checked directly - let us show for example, that:

$$\nabla_r(n-1)\cdot\nabla_r(n-2)(af\otimes\pi\frac{\partial}{\partial u_i}\pi\frac{\partial}{\partial u_j}) = a\nabla_r(n-1)\cdot\nabla_r(n-2)(f\otimes\pi\frac{\partial}{\partial u_i}\pi\frac{\partial}{\partial u_j}).$$

We have :

$$\nabla_r(n-1)\cdot\nabla_r(n-2)(af\otimes\pi\frac{\partial}{\partial u_i}\pi\frac{\partial}{\partial u_j}) =$$

$$= (-1)^{\bar{a}\bar{f}+\bar{u}_i}[\Delta_r(\Delta_r(f\otimes a\frac{\partial}{\partial u_i})\otimes\frac{\partial}{\partial u_j}) - (-1)^{\bar{u}_i\bar{u}_j}\Delta_r(\Delta_r(f\otimes a\frac{\partial}{\partial u_j})\otimes\frac{\partial}{\partial u_i})] =$$

$$= (-1)^{\bar{a}\bar{f}+\bar{u}_i}\{(-1)^{\bar{a}\bar{u}_i}[\Delta_r(\Delta_r(f\otimes\frac{\partial}{\partial u_i}\cdot a)\otimes\frac{\partial}{\partial u_j}) - \Delta_r(f\otimes\frac{\partial}{\partial u_i}(a)\cdot\frac{\partial}{\partial u_j})]-$$

$$-(-1)^{\bar{u}_i\bar{u}_j+\bar{a}\bar{u}_j}[\Delta_r(\Delta_r(f\otimes\frac{\partial}{\partial u_j}\cdot a)\otimes\frac{\partial}{\partial u_i}) - \Delta_r(f\otimes\frac{\partial}{\partial u_j}(a)\cdot\frac{\partial}{\partial u_i})]\} =$$

$$= (-1)^{\bar{a}\bar{f}+\bar{u}_i}\{[(-1)^{\bar{a}\bar{u}_i+\bar{a}\bar{u}_j}\Delta_r(\Delta_r(f\otimes\frac{\partial}{\partial u_i})\otimes\frac{\partial}{\partial u_j}\cdot a)-$$

$$-(-1)^{\bar{a}(\bar{u}_i+\bar{u}_j)}\Delta_r(f\otimes\frac{\partial}{\partial u_i})\frac{\partial}{\partial u_j}(a) - (-1)^{\bar{a}\bar{u}_i+(\bar{u}_i+\bar{a})\bar{u}_j}\Delta_r(f\otimes\frac{\partial}{\partial u_j})\frac{\partial}{\partial u_i}(a)+$$

$$+(-1)^{\bar{a}\bar{u}_i+(\bar{u}_i+\bar{a})\bar{u}_j}f\frac{\partial}{\partial u_j}(\frac{\partial}{\partial u_i}(a))]-$$

$$-(-1)^{\bar{u}_i\bar{u}_j}[(-1)^{\bar{a}\bar{u}_j+\bar{a}\bar{u}_i}\Delta_r(\Delta_r(f\otimes\frac{\partial}{\partial u_j})\frac{\partial}{\partial u_i}\cdot a)-$$

$$-(-1)^{\bar{a}(\bar{u}_i+\bar{u}_j)}\Delta_r(f\otimes\frac{\partial}{\partial u_j})\frac{\partial}{\partial u_i}(a) - (-1)^{\bar{a}(\bar{u}_i+\bar{u}_j)+\bar{u}_i\bar{u}_j}\cdot$$

$$\cdot\Delta_r(f\otimes\frac{\partial}{\partial u_i})\frac{\partial}{\partial u_j}(a) + (-1)^{\bar{a}(\bar{u}_i+\bar{u}_j)+\bar{u}_i\bar{u}_j}f\frac{\partial}{\partial u_i}(\frac{\partial}{\partial u_j}(a))]\} =$$

$$= (-1)^{(\bar{a}\bar{f}+\bar{u}_i+\bar{a}(\bar{u}_i+\bar{u}_j))}[\Delta_r(\Delta_r(f\otimes\frac{\partial}{\partial u_i})\otimes\frac{\partial}{\partial u_j}\cdot a)-$$

$$-(-1)^{\bar{u}_i\bar{u}_j}\Delta_r(\Delta_r(f\otimes\frac{\partial}{\partial u_j})\otimes\frac{\partial}{\partial u_i}\cdot a)] =$$

$$= (-1)^{\bar{a}\bar{f}+\bar{a}(\bar{u}_i+\bar{u}_j)}\nabla_r(n-1)\cdot\nabla_r(n-2)(f\otimes\pi\frac{\partial}{\partial u_i}\pi\frac{\partial}{\partial u_j})\cdot a =$$

$$= a\nabla_r(n-1)\cdot\nabla_r(n-2)(f\otimes\pi\frac{\partial}{\partial u_i}\pi\frac{\partial}{\partial u_j}).$$

We have now an even O_x -linear map:

$$\nabla_r(n-1)\cdot\nabla_r(n-2):\mathcal{F}\otimes S^2(\prod\mathcal{T}X)\to\mathcal{F}$$

which may be identified with a section of $(End\mathcal{F}\otimes\Omega^2X)_0$. Let's denote it by F_{∇_r}. It is easy to see that all the maps $\nabla_r(-i+n)\cdot\nabla_r(-i+n-1)$ coincide with the inner multiplication by F_{∇_r}. We call F_{∇_r} the curvature of Δ_r.

Considering left connections we get F_{∇_l}, which is a section of $(End\mathcal{F} \otimes \Omega^2 X)_0$ again, but the maps $\nabla_l(i+1) \cdot \nabla_l(i)$ are just multiplication by F_{∇_l} in the algebra $End\mathcal{F} \otimes \Omega^{\cdot} X$. □

2.3.2.**Definition.** A connection is called integrable iff its curvature is zero.

Remark. Let $\mathcal{D}$ be the ring of differential operators on X. It is standard to check that endowing a sheaf $\mathcal{F}$ with a left integrable connection is equivalent of endowing $\mathcal{F}$ with the structure of a left $\mathcal{D}$ -module; the integrability condition allows us to construct a left action of $\mathcal{D}$, coinciding with Δ_l on the differential operators of order ≤ 1. Similarly to give a right connection on $\mathcal{F}$ means to give a right $\mathcal{D}$ -module structure on $\mathcal{F}$.

2.3.3.**Remark.** For pairs $(\mathcal{F}, \Delta_l)$ with $F_{\nabla_l} = 0$, the de Rham sequence $DR^{\cdot}(\mathcal{F})$ is a complex and the Poincaré lemma is true:

$$\begin{aligned} H^i(DR^{\cdot}(\mathcal{F})) &= 0, \qquad i \neq 0 \\ H^0(DR^{\cdot}(\mathcal{F})) &= \mathcal{F}_{hor} \end{aligned}$$

where $\mathcal{F}_{hor}$ is the local system of horizontal sections of $\mathcal{F}$. For the case $(0, d)$ this is proved in [BL3]. The general case is similar. Given a pair $(\mathcal{F}, \Delta_r)$, with Δ_r integrable, the Spencer sequence is by definition also a complex. Here we have the following analogue of the Poincaré lemma:

$$\begin{aligned} H^i(S.(\mathcal{F})) &= 0, \qquad i \neq 0 \\ h^0(S.(\mathcal{F})) &= \mathcal{F}^{hor} \end{aligned}$$

where $\mathcal{F}^{hor}$ is also a local system, which is a subquotient of $\mathcal{F} \otimes \mathbf{S}(\prod \mathcal{T}X)$. We shall explain this later in more detail.

§3. The Equivalence Theorem

In this section we describe a procedure, allowing given a left connection on $\mathcal{F}$, to construct a right connection on $BerX \otimes_{O_x} \mathcal{F}$. This gives rise to an equivalence of the categories of pairs $(\mathcal{F}, \Delta_l)$ and (G, Δ_r).

3.1. Let us first establish some properties of the complexes of differential and integral forms:

3.1.1.**Proposition.** Let η be a local section of $\Omega^i X$. Then the following Leibniz formulae are true:

$$\begin{aligned} \partial(\eta \cdot s) &= -\partial\eta \cdot s + (-1)^{\overline{\eta}}\eta\partial s && \text{(means inner multiplication)} \\ d(\eta \cdot \zeta) &= d\eta \cdot \zeta + (-1)^{\overline{\eta}}\eta \cdot d\zeta && \text{(means multiplication in } \mathbf{S}(\Omega^1 X)) \end{aligned}$$

where s is an integral form, and ζ - a differential form.

The proof is a straightforward local computation.

3.1.2. **Proposition.** The sheaf $BerX$ has a canonical right integrable connection, whose Spencer complex can be identified with $\Sigma.X$.

Proof: We define the map Δ_r by:

$$\Delta_r(\omega \otimes \phi) = -(-1)^{\overline{\omega}\overline{\phi}} L_\phi \omega,$$

where ω is a local section of $BerX$, and L_ϕ - the Lie derivative, see 1.2.

An easy computation shows that Δ_r is really a connection and that $\nabla_r(-i+n) : BerX \otimes \mathbf{S}^i(\prod \mathcal{T}X) \to BerX \otimes \mathbf{S}^{i-1}(\prod \mathcal{T}X)$ is identified with ∂. This means that $BerX$ has a canonical structure of a right $\mathcal{D}$ -module, [P]. □

3.2. The functors B and B^{-1}.

Given a pair $(\mathcal{F}, \Delta_l)$, consider now the pair $(BerX \otimes_{O_x} \mathcal{F}, B_{\Delta_l})$, where B_{Δ_l} acts on $BerX \otimes_{O_x} \mathcal{F}$ as follows:

$$B_{\Delta_l}(\omega \otimes f \otimes \phi) = -((-1)^{\overline{\phi}(\overline{f}+\overline{\omega})} L_\phi \omega \otimes f + (-1)^{\overline{\phi}\,\overline{f}} \omega \otimes \Delta_l(\phi \otimes f)).$$

3.2.1.**Proposition.** a) B_{Δ_l} is a right connection.

b) $F_{B_{\Delta_l}} = 0 \Longleftrightarrow F_{\Delta_l} = 0$.

Proof: This can also be checked directly. We say only that b) means that the functor B transforms left $\mathcal{D}$ -modules into right $\mathcal{D}$ -modules. This statement is well known, cf. [P]□.

3.2.2. The inverse functor B^{-1} transforms right connections on $\mathcal{F}$ into left connections on the sheaf $Hom_{O_x}(BerX, \mathcal{F}) = (BerX)^* \otimes \mathcal{F}$. By definition $b^{-1}\Delta_r$ acts as follows:

$$B^{-1}\Delta_r(\phi \otimes g)(\omega) = (-1)^{\overline{\phi}(\overline{g}+\overline{\omega})}\{g(\Delta_r(\omega \otimes \phi)) - \Delta_r(f(\omega) \otimes \phi)\},$$

(here, g is a local section of $Hom_{O_x}(BerX, \mathcal{F})$).

B^{-1} is really the inverse functor to B. Let us check for example that $B^{-1}(BerX) = Hom(BerX, BerX)$ is just O_x with its canonical left connection $\Delta_l(\phi \otimes f) = \phi(f)$, ($f$ being a local section of $Hom(BerX, BerX) = O_x$). We have:

$$B^{-1}\Delta_r(\phi \otimes f)\omega =$$

$$= (-1)^{\overline{\phi}(\overline{f}+\overline{\omega})}\{f \cdot \Delta_r(\omega \otimes \phi) - \Delta_r(f \cdot \omega \otimes \phi)\} =$$

$$= (-1)^{\overline{\phi}(\overline{f}+\overline{\omega})+\overline{\omega}\overline{\phi}} f \cdot L_\phi \omega + L_\phi(f \cdot \omega) =$$

$$(-1)^{\overline{\phi}\,\overline{f}} f \cdot L_\phi \omega + L_\phi(f) \cdot \omega + (-1)^{\overline{\phi}\,\overline{f}} f \cdot L_\phi(\omega) = \phi(f).$$

The general case is similar. Now one can state:

3.2.3. **Proposition.** B and B^{-1} are inverse equivalences of the categories of pairs $(\mathcal{F}, \Delta_l)$ and (G, Δ_r), in which the morphisms are sheaf morphisms, commuting with the connections.

Remark. The notion of a right connection is of course essential only in the case $m \neq 0$, because for manifolds the Spencer sequence $S.(\mathcal{F})$ of $(\mathcal{F}, \Delta_r)$ is canonically isomorphic to $DR^{\cdot}(B^{-1}(\mathcal{F}))$.

We explain now the notion of horizontal sections for a right integrable connection. It is easy to check that for a left integrable connection its sheaf of horizontal sections $\mathcal{F}_{hor}$ is canonically identified with the sheaf of morphisms of O_x into $\mathcal{F}$, as sheaves with left connections or as$\mathcal{D}$ -modules: $\mathcal{F}_{hor} = Hom_{\mathcal{D}}(O_x, \mathcal{F})$. Similarly we can define $\mathcal{F}^{hor} = Hom_{\mathcal{D}}(BerX, \mathcal{F})$, where $\mathcal{F}^{hor}$ is now equipped with a right integrable connection. $\mathcal{F}^{hor}$ is a local system, and $rk\mathcal{F}^{hor} = rk\mathcal{F}$. One may check that $\mathcal{F}^{hor}$ is the only cohomology of the complex of sheaves $S.(\mathcal{F})$. The equivalence theorem implies that $B(\mathcal{F}^{hor}) = \mathcal{F}_{hor}$.

3.3. Coordinate computations. Now we want to give an explicit coordinate description of left and right connections.

As usual, a left connection Δ_l is determined by its connection form $\chi \in \Gamma(\Omega^1 X \otimes End\mathcal{F})_1$, which is an odd matrix of differential forms (in the even case all matrices from $\Omega^1 X \otimes End\mathcal{F}$ are automatically odd, because $\Omega^1 X$ is odd). The connection form depends, of course, on the choice of a trivialization of $\mathcal{F}$ and acts by the formula

$$\nabla_l(i)(f) = df + \chi f$$

where df means differentiation of the components of f (which is a local section of $\mathcal{F}\otimes\Omega^i X$) in this trivialization. (Note that $\nabla_l(i) = d + \chi$ is odd, as required). Invariantly this means that if the set of all left connections on $\mathcal{F}$ – $Conn_l(\mathcal{F})$ is not empty, then it is a principal homogeneous space over $\Gamma(X, End\mathcal{F} \otimes \Omega^1 X)_1$.

Naturally the set of all right connections on $\mathcal{F}$ – $Conn(\mathcal{F})$ is also either empty or a principal homogeneous space over $\Gamma(X, End\mathcal{F} \otimes \Omega^1 X)_1$, because of the equivalence theorem. (In the category of C^∞ -manifolds, $Conn_l(\mathcal{F})$ and $Conn_r(\mathcal{F})$ are never empty; this can happen only in the analytic category).

Proposition. Let $(\mathcal{F}, \Delta_l)$ have a connection form χ in some trivialization of $\mathcal{F}$. Then

$$B(\nabla_l)(-i+n) : \mathcal{F} \otimes BerX \otimes \mathbf{S}^i(\textstyle\prod TX) \to \mathcal{F} \otimes BerX \otimes \mathbf{S}^{i-1}(\textstyle\prod TX)$$

acts by the formulas $\nabla_r(s) = \partial s - \chi s$, where ∂ acts on the components of s (they are sections of $BerX \otimes \mathbf{S}^i(\prod TX)$ and χ acts by inner multiplication).

The proof is a direct computation using the formula from 3.2.1.

Consider now the curvature of a left connection. As usual one may compute it in the following way:

$$F_{\nabla_l} = (d+\xi)(d+\xi) = d^2 + \xi\cdot d + d\cdot\xi + \xi\cdot\xi =$$

$$= 0 + \xi\cdot d + d(\xi) + (-1)^{\bar{\xi}\bar{d}}\xi\cdot d + \xi\cdot\xi = d(\xi) + \xi\cdot\xi$$

Similarly one may compute the curvature of the right connection $B(\mathcal{F}, \Delta_l)$:

$$F_{\nabla_r} = (\partial-\xi)(\partial-\xi) = \partial^2 - \xi\cdot\partial - \partial\cdot\xi + \xi\cdot\xi =$$

$$= 0 - \xi\cdot\partial + d(\xi) - (-1)^{\bar{\partial}\bar{\xi}}\xi\cdot d + \xi\cdot\xi = d(\xi) + \xi\cdot\xi$$

(we use here Leibniz rule from 3.1. and the last proposition).

Corollary: $F_{\nabla_l} = F_{B(\nabla_l)}$.□

3.4. Tensor operations. There are several canonical tensor operations on left connections, for example, if $\mathcal{F}_1$ and $\mathcal{F}_2$ have left connections Δ_{l1} and Δ_{l2}, then Δ_{l1} and Δ_{l2} induce the left connections $\Delta_{l1} \otimes \Delta_{l2}$ on $\mathcal{F}_1 \otimes_{O_x} \mathcal{F}_2$ and $\Delta_{l1}^* \otimes \Delta_{l2}$ on $Hom(\mathcal{F}_1, \mathcal{F}_2)$:

$$\Delta_{l1}\otimes\Delta_{l2}(\phi\otimes f_1\otimes f_2) = \Delta_{l1}(\phi\otimes f_1)\otimes f_2 + (-1)^{\bar{f_1}\bar{f_2}}\Delta_{l2}(\phi\otimes f_2)\otimes f_1$$

$$\Delta_{l1}^*\otimes\Delta_{l2}(\phi\otimes\tilde{f})(f_1) = \Delta_{l2}(\phi\otimes\tilde{f}(f_1)) - (-1)^{\bar{\tilde{f}}\bar{\phi}}\tilde{f}(\Delta_{l1}(\phi\otimes f_1))$$

where f_i are local sections on $\mathcal{F}_i$ and $\tilde{f}$ is a local section on $Hom(\mathcal{F}_1, \mathcal{F}_2)$.

The tensor operations on right connections are not so evident, for example, if $\mathcal{F}_1$ and $\mathcal{F}_2$ have right connections in the holomorphic category it can happen that the sets $Conn_r(\mathcal{F}_1 \otimes \mathcal{F}_2)$ and $Conn_r(\mathcal{F}_1^* \otimes \mathcal{F}_2)$ are empty. Indeed, let X -be the projective space $\mathbf{P}^n$ and let $\mathcal{F}_1 = \mathcal{F}_2 = BerX = \Omega^n X$ (the sheaf of holomorphic n -forms). Then it is standard to check that the sheaves $(\Omega^n X)^2$ and O_x have no right holomorphic connections at all. The reason of this "asymmetry" is that the sheaf O_x, over which we tensor, has a canonical left connection, but no canonical right connection.

The right way of obtaining correct tensor operations on right connections is to transform them into left connections using the functor B^{-1}, apply there the already known tensor operations and then "come back" by the functor B. For example given two pairs $(\mathcal{F}_1, \Delta_{r1}), (\mathcal{F}_2, \Delta_{r2})$ the sheaf $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes (BerX)^*$ has a canonical right connection.

3.5. Bianchi identities.

3.5.1. **Proposition.** (Bianchi identity for left connections). Let $\mathcal{F}$ be equipped with a left connection Δ_l. Denote by $\tilde{\Delta}_l$ the corresponding left connection on $End\mathcal{F}$. Then

$$\tilde{\nabla}_l(2)(F_{\nabla_l}) = 0.$$

Exactly the same computation as in the case $m = 0$ proves this statement.

3.5.2. Let $\mathcal{F}$ be now equipped with a right connection Δ_r. The sheaf $End\mathcal{F} \otimes (BerX)^*$ is equipped with the connection $\tilde{\Delta}_r$, where the pair $(End\mathcal{F} \otimes (BerX)^*, \tilde{\Delta}_r)$ is by definition equal to the pair $B(End\mathcal{F}', \widetilde{B^{-1}\Delta_r})$,

$$\mathcal{F}' \otimes BerX = \mathcal{F},$$

and its Spencer sequence has the form:

$$\to End\mathcal{F}' \otimes BerX \otimes \mathbf{S}^n(\prod \mathcal{T}X) \to \cdots \to End\mathcal{F}' \otimes BerX \otimes \mathbf{S}^{n-2}(\prod \mathcal{T}X) \to \quad .$$

Consider also the complex of integral forms $S_.(BerX)$. In $S_0(BerX) = BerX \otimes \mathbf{S}^n(\prod \mathcal{T}X)$ we have a canonical (up to a constant) cohomology class, corresponding to $(BerX)^{hor}$. $(BerX)^{hor}$ is equal to $(B(O_x))^{hor}$ which is the constant sheaf k. Let us denote this class with η. Then $\tilde{\eta} = id \otimes \eta$ lies in $End\mathcal{F} \otimes BerX \otimes \mathbf{S}^n(\prod \mathcal{T}X)$ and we can apply to $F_{\nabla_r} \cdot \tilde{\eta}$ the operator $\tilde{\Delta}_r$.

Proposition (Bianchi identity for right connections):

$$\tilde{\nabla}_r(2)(\mathcal{F}_{\nabla_r} \cdot \tilde{\eta}) = 0.$$

Proof: Let θ be the connection form of $B^{-1}\Delta_r$ in some trivialization of $(BerX)^* \otimes \mathcal{F}$. Then $\tilde{\nabla}_r(\kappa)$ acts as $d + ad\theta$, where $ad\theta(\xi) = \theta\xi - (-1)^{\overline{\theta\xi}}\xi\theta$. We have:

$$(\partial - ad\theta)(F_{\nabla_r} \cdot \tilde{\eta}) = (\partial - ad\theta)(F_{B^{-1}(\nabla_r)} \cdot \tilde{\eta}) =$$

$$= -dF_{B^{-1}(\nabla_r)} \cdot \tilde{\eta} + F_{B^{-1}(\nabla_r)}\partial(\tilde{\eta}) - ad\theta(F_{B^{-1}(\nabla_r)}) \cdot \tilde{\eta} =$$

$$= -\widetilde{B^{-1}(\nabla_r)}(2)[F_{B^{-1}(\nabla_r)}] \cdot \eta + F_{B^{-1}(\nabla_r)} \cdot \partial(\tilde{\eta}).$$

But the first term is equal to zero by the Bianchi identity for left connections, and the second term is zero because $\partial(\tilde{\eta}) = 0$ by definition. □

§4. Remarks on Characteristic classes

If φ is an even homogeneous invariant polynomial on the space of $p|q$ −matrices over a supercommutative ring A−

$$\varphi : M_{p|q}(A) \to A, \qquad \varphi(ZYZ^{-1}) = \varphi(Y), \ \overline{Z} = 0,$$

considering a locally free sheaf $\mathcal{F}, rk\mathcal{F} = p \mid q$, with (smooth) left or right connection Δ_l or Δ_r, we can apply φ to F_{∇_l} or F_{∇_r}, which are sections of $(End\mathcal{F} \otimes \Omega^2 X)_0$ and obtain

$\varphi(F_{\nabla_l})$ and $\varphi(F_{\nabla_r})$ (they are correctly defined global differential forms of degree $2deg\varphi$). As in case $m = 0$, we have the following fundamental result:

Theorem (Chern-Weil)

A) $d\varphi(F_{\nabla_l}) = 0; \qquad \partial(\varphi(F_{\nabla_r}) \cdot \eta) = 0.$

B) The cohomology classes $\varphi_l \in H^{2deg\varphi}(\Gamma(X, \Omega^{\cdot} X))$ (corresponding to $\varphi(F_{\nabla_l})$) and $\varphi_r \in H^{2deg\varphi}(\Gamma(X, \Sigma_{\cdot} X))$ (coresponding to $\varphi(F_{\nabla_r})$) do not depend on the choice of Δ_l or Δ_r.

The statements about left connections can be proved exactly as in the case $m = 0$, using minimum supercommutative algebra, and we turn to the statements about right connections:

A) $\partial(\varphi(F_{\nabla_r}) \cdot \eta) = \partial(\varphi(F_{B^{-1}(\nabla_r)} \cdot \eta) = -(d\varphi(F_{B^{-1}(\nabla_r)})) \cdot \eta + \varphi(F_{B^{-1}(\nabla_r)}) \cdot \partial\eta = 0.$

The first term is zero because $B^{-1}(\nabla_r)$ is a left connection, and the second term is zero because $\partial\eta = 0$.

B) Let Δ_r' and Δ_r'' be two right connections. We have:

$$\varphi(F_{\nabla_r'}) \cdot \eta - \varphi(F_{\nabla_r''}) \cdot \eta = (\varphi(F_{B^{-1}(\nabla_r')}) - \varphi(F_{B^{-1}(\nabla_r'')})) \cdot \eta =$$

$$= (d\alpha) \cdot \eta = \partial(-\alpha \cdot \eta).$$

Here we use again the fact, that $\partial\eta = 0$, and that $\varphi(F_{B^{-1}(\nabla_r')}) - \varphi(F_{B^{-1}(\nabla_r'')}) = d\alpha$, where α is a differential form.

This allows us to construct the Chern classes of a locally free sheaf $\mathcal{F}$ using the invariant forms $\varphi_r(F_\nabla) = str(F_\nabla^r)$ by two ways. Unfortunately these do not give new interesting invariants of $\mathcal{F}$, because the previous construction uses smooth connections, and it is well known that in the C^∞ -case the category of locally free sheaves on X is equivalent to the category of $\mathbf{Z}_2$ -graded locally free sheaves on the reduced manifold X_{red}. This is a variant of the theorem of Batchelor [B], but may be proved directly too. Thus, the classes we get by the Chern-Weil construction are nothing but the Chern classes of the sheaf $\mathcal{F}_{red}$ (respectively $(\Omega^n X_{red})^* \otimes \mathcal{F}_{red}$).

Finally, we note that the Atiyah-style classes in the holomorphic situation are more interesting because the cohomology ring $\oplus_i H^i(X, \Omega^i X)$ even in the simplest case of $\mathbf{P}^{1|2}$ (the projectivization of a linear space of dimension $2 \mid 2$) does not coincide with the ring $\oplus_i H^i(X_{red}, \Omega^i X_{red})$.

References

[1] [B] Batchelor M., The structure of supermanifolds, Transactions of the AMS, 253, 329-338 (1979).

[2] [BC] Bott R., Chern S. S., Hermitean vector bundles and the equidistribution of the zeroes of their holomorphic sectionsm, Acta Math., 114, 71-112 (1965).

[3] [Bj] Björk J. E., Rings of differential operators, North-Holland, Amsterdam. Oxford. New York, 375p. (1979).

[4] [BL1] Bernstein J. N., Leites D. A., Integrable forms and the Stoke's formula on supermanifolds, Funct. Anal. Appl., 11, 45 (1977).

[5] [BL2] Bernstein J. N., Leites D. A., Integration of differential forms on supermanifolds, Funct. Anal. Appl., **11**, 45 (1977).

[6] [BL3] Bernstein J. N., Leites D. A., Invariant differential operators and irreducible representations of Lie superalgebras of vector fields, Serdika, 7, 320-384, (1981) (in Russian).

[7] [L] Leites D. A., Introduction to the theory of supermanifolds, Russian Mathematical Surveys, **35**, 1 (1980).

[8] [MS] Milnor J., Stasheff J., Characteristic classes, Annals of mathematical studies, **76**, Princeton, New Jersey, (1974).

[9] [P] Penkov I., $\mathcal{D}$ -modules on supermanifolds, Inv. Math., **71**, 501-512, (1983).

[10] [W] Wells R. O., Differential analysis on complex manifolds, Prentice-Hall Inc, Englewood Cliffs, N. J., (1973).