

ON IDEALS IN THE ENVELOPING ALGEBRA OF A LOCALLY SIMPLE LIE ALGEBRA

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ABSTRACT. We study (two-sided) ideals I in the enveloping algebra $U(\mathfrak{g}_\infty)$ of an infinite-dimensional Lie algebra \mathfrak{g}_∞ obtained as the union (equivalently, direct limit) of an arbitrary chain of embeddings of simple finite-dimensional Lie algebras

$$\mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \rightarrow \dots \rightarrow \mathfrak{g}_n \rightarrow \dots$$

with $\lim_{n \rightarrow \infty} \dim \mathfrak{g}_n = \infty$. Our main result is an explicit description of the zero-sets of the corresponding graded ideals $\text{gr } I$. We use this description and results of A. Zhilinskii to prove Baranov's conjecture that, if \mathfrak{g}_∞ is not diagonal in the sense of A. Baranov and A. Zhilinskii, then $U(\mathfrak{g}_\infty)$ has a single non-zero proper ideal: the augmentation ideal.

Our study is based on a complete description of the radical Poisson ideals in $\mathbf{S}(\mathfrak{g}_\infty)$ and their zero-sets. We then discuss in detail integrable ideals of $U(\mathfrak{g}_\infty)$. These are certain ideals $I \subset U(\mathfrak{g}_\infty)$ for which $I \cap U(\mathfrak{g}_n)$ is an intersection of ideals of finite codimension in $U(\mathfrak{g}_n)$ for any $n \geq 1$. We present a classification of prime integrable ideals based on work of A. Zhilinskii. For $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty$, all zero-sets of radical Poisson ideals of $\mathbf{S}(\mathfrak{g}_\infty)$ arise from prime integrable ideals of $U(\mathfrak{g}_\infty)$. For $\mathfrak{g}_\infty \cong \mathfrak{sp}_\infty$ only "half" of the zero-sets of Poisson ideals of $\mathbf{S}(\mathfrak{g}_\infty)$ arise from integrable ideals of $U(\mathfrak{g}_\infty)$.

Key words: associated variety, coherent local system, integrable ideal, locally finite Lie algebra, moment map, Poisson ideal.

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1. INTRODUCTION

We work over an algebraically closed field of characteristic zero. A *locally finite Lie algebra* is by definition a Lie algebra isomorphic to the limit of some direct system of finite-dimensional Lie algebras [BS]. In what follows we restrict ourselves to locally finite Lie algebras \mathfrak{g}_∞ defined as direct limits of respective sequences of embeddings of simple Lie algebras

$$\mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \rightarrow \dots \rightarrow \mathfrak{g}_n \rightarrow \dots \tag{1}$$

such that $\lim_{n \rightarrow \infty} \dim \mathfrak{g}_n = \infty$; for brevity we refer just to these Lie algebras as *locally simple* Lie algebras.

Locally simple Lie algebras are natural infinite-dimensional analogs of finite-dimensional simple Lie algebras, and their representation theory is interesting and challenging.

In this paper we study ideals in the universal enveloping algebra $U(\mathfrak{g}_\infty)$ of a locally simple Lie algebra \mathfrak{g}_∞ , in particular annihilators of simple \mathfrak{g}_∞ -modules, i.e. primitive ideals. The structure of ideals in $U(\mathfrak{g}_\infty)$ differs significantly from the structure of ideals in the enveloping algebra of a simple finite-dimensional Lie

algebra. This is not surprising once one observes that the center of $U(\mathfrak{g}_\infty)$ consists of constants only.

The theory of ideals in $U(\mathfrak{g}_\infty)$ has been initiated by A. Zhilinskii in [Zh1], [Zh2]. His method is to study “coherent local systems” of finite-dimensional modules of the sequence of Lie algebras \mathfrak{g}_n , and to construct ideals in $U(\mathfrak{g}_\infty)$ whose intersections with $U(\mathfrak{g}_n)$ are the joint annihilators of the modules at the n -th level of the respective local systems. We call the ideals of $U(\mathfrak{g}_\infty)$ arising in this way *integrable* ideals.

A central idea of the present paper is to gain information about ideals in $U(\mathfrak{g}_\infty)$ by studying their associated “varieties”. At first this leads to the study of radical Poisson ideals in the symmetric algebra $\mathbf{S}(\mathfrak{g}_\infty)$. Our first notable result is that $\mathbf{S}(\mathfrak{g}_\infty)$ admits a non-zero Poisson ideal J of locally infinite codimension (i.e. such that $J \cap \mathbf{S}(\mathfrak{g}_n)$ is of infinite codimension in $\mathbf{S}(\mathfrak{g}_n)$ for almost all n) if and only if \mathfrak{g}_∞ is isomorphic to one of the three locally simple Lie algebras $\mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

This result, together with results of A. Zhilinskii [Zh2], yields the following corollary: if \mathfrak{g} is not isomorphic to $\mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, and $U(\mathfrak{g}_\infty)$ has a non-zero proper ideal I which does not coincide with the augmentation ideal, then I is of locally finite codimension (i.e. $I \cap U(\mathfrak{g}_n)$ is of finite codimension in $U(\mathfrak{g}_n)$ for all n) and \mathfrak{g}_∞ is diagonal. Diagonal locally simple Lie algebras form a natural class of locally simple Lie algebras which has been introduced by A. Baranov and A. Zhilinskii; in fact, these authors have given an intricate and elegant classification of diagonal locally simple Lie algebras [BZh]. Moreover, the above corollary had been conjectured by A. Baranov.

As a second notable result on Poisson ideals we compute the set of zeros $\text{Var}(J)$ (i.e. the associated “variety”) of any Poisson ideal $J \subset \mathbf{S}(\mathfrak{g}_\infty)$ for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

We then pass to the study of ideals in $U(\mathfrak{g}_\infty)$ for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$. In section 7 we review Zhilinskii’s results and, most importantly, his classification of irreducible coherent local systems. Zhilinskii’s results yield an explicit description of all prime integrable ideals in $U(\mathfrak{g}_\infty)$, see Theorem 7.9.

We finally describe the associated “varieties” of integrable ideals. In particular, we show that if $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty$ and J is a radical Poisson ideal of $\mathbf{S}(\mathfrak{g}_\infty)$, there exists a prime integrable ideal I of $U(\mathfrak{g}_\infty)$ such that $\text{Var}(I) = \text{Var}(J)$. For $\mathfrak{g}_\infty = \mathfrak{sp}_\infty$ the situation is different: given an ideal J , there exists an ideal I of $U(\mathfrak{g}_\infty)$ such that $\text{Var}(I) = \text{Var}(J)$, but I is not necessarily integrable.

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2. PRELIMINARIES

We fix an algebraically closed field \mathbb{F} of characteristic zero. All vector spaces (including Lie algebras) are assumed to be defined over \mathbb{F} . If V is a vector space, V^* stands for the dual space $\text{Hom}_{\mathbb{F}}(V, \mathbb{F})$. All varieties we consider are algebraic varieties over \mathbb{F} (with Zariski topology). When considering locally finite Lie algebras or their enveloping algebras we assume that any given sequence (1) consists of inclusions, so we can freely interchange $\varinjlim \mathfrak{g}_n$ with $\cup_n \mathfrak{g}_n$, and $\varinjlim U(\mathfrak{g}_n)$ with $\cup_n U(\mathfrak{g}_n)$.

There is no classification of general locally simple Lie algebras: a classification is only available for the so called diagonal locally simple Lie algebras, see [BZh].

Among diagonal locally simple Lie algebras a prominent role is played by the three simple Lie algebras \mathfrak{sl}_∞ , \mathfrak{so}_∞ and \mathfrak{sp}_∞ which can be defined as unions of the respective chains of inclusions of classical finite-dimensional Lie algebras of types \mathfrak{sl} , \mathfrak{so} , or \mathfrak{sp} under the obvious “left upper-corner inclusions”. An important result, see [B] or [BS], states that, up to isomorphism, these three Lie algebras are the only locally simple finitary Lie algebras, i.e. locally simple Lie algebras which admit a countable-dimensional faithful module with a basis such that the endomorphism arising from each element of the Lie algebra is given by a matrix with finitely many non-zero entries.

Let G be a connected algebraic group with Lie algebra \mathfrak{g} and let $I \subset U(\mathfrak{g})$ be an ideal in the enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} (by an ideal in a ring we mean a proper two-sided ideal). The degree filtration $\{U(\mathfrak{g})^{\leq d}\}_{d \in \mathbb{Z}_{\geq 0}}$ on $U(\mathfrak{g})$ defines the filtration $\{I \cap U(\mathfrak{g})^{\leq d}\}_{d \in \mathbb{Z}_{\geq 0}}$ on I . The associated graded object

$$\mathrm{gr}I := \bigoplus_d ((I \cap U(\mathfrak{g})^{\leq d}) / (I \cap U(\mathfrak{g})^{\leq d-1}))$$

is a G -stable ideal of $\mathrm{gr}U(\mathfrak{g}) = \mathbf{S}(\mathfrak{g})$. We denote the set of zeros of $\mathrm{gr}I$ in \mathfrak{g}^* by $\mathrm{Var}(I) \subset \mathfrak{g}^*$. The variety $\mathrm{Var}(I)$ is a G -stable subvariety of \mathfrak{g}^* and is, by definition, the *associated variety* of I . The ideal I is of finite codimension in $U(\mathfrak{g})$ if and only if $\mathrm{gr}I$ is of finite codimension in $\mathbf{S}(\mathfrak{g})$. Moreover, $\mathrm{gr}I$ is of finite codimension in $\mathbf{S}(\mathfrak{g})$ if and only if $\mathrm{Var}(I) = 0$. We note that the radical $\mathrm{rad}(\mathrm{gr}I)$ of $\mathrm{gr}I$ is also G -stable.

It is well known that $\mathbf{S}(\mathfrak{g})$ is a Poisson algebra, i.e. $\mathbf{S}(\mathfrak{g})$ has an \mathbb{F} -bilinear operation

$$\{ \cdot, \cdot \}: \mathbf{S}(\mathfrak{g}) \times \mathbf{S}(\mathfrak{g}) \rightarrow \mathbf{S}(\mathfrak{g})$$

which is a Lie bracket and is compatible with multiplication. An ideal $J \subset \mathbf{S}(\mathfrak{g})$ is *Poisson* whenever $\{f, J\} \subset J$ for any $f \in \mathbf{S}(\mathfrak{g})$.

Assume that \mathfrak{g} is finite dimensional and semisimple and let G be the adjoint group of \mathfrak{g} . Then the radical Poisson ideals of $\mathbf{S}(\mathfrak{g})$ are in one-to-one correspondence with the G -stable Zariski-closed subsets of \mathfrak{g}^* . A description of such sets is presented in [Bor].

Let \mathfrak{g}_∞ be a locally simple Lie algebra. The same arguments as in the previous paragraph assign to any ideal $I \subset U(\mathfrak{g}_\infty)$ a radical ideal $\mathrm{rad}(\mathrm{gr}I) \subset \mathbf{S}(\mathfrak{g}_\infty)$, which is stable under the adjoint action of \mathfrak{g}_∞ on $\mathbf{S}(\mathfrak{g}_\infty)$; in what follows we call \mathfrak{g}_∞ -stable ideals of $\mathbf{S}(\mathfrak{g}_\infty)$ Poisson ideals. We denote the set of zeros of $\mathrm{gr}I$ in \mathfrak{g}_∞^* by $\mathrm{Var}(I)$ and refer to $\mathrm{Var}(I)$ as the associated “variety” of I .

Note that $\mathrm{Var}(I)$ is a pro-variety, i.e. an inverse limit of algebraic varieties. Indeed, fix a sequence (1) with $\mathfrak{g}_\infty = \varinjlim \mathfrak{g}_n$ and let $\overline{\mathrm{pr}_{\mathfrak{g}_n} \mathrm{Var}(I)} \subset \mathfrak{g}_n^*$ be the closure of the image of $\mathrm{Var}(I)$ under the natural projection $\mathrm{pr}_{\mathfrak{g}_n}: \mathfrak{g}_\infty^* \rightarrow \mathfrak{g}_n^*$; by definition $\overline{\mathrm{pr}_{\mathfrak{g}_n} \mathrm{Var}(I)} \subset \mathfrak{g}_n^*$ is the set of zeros of $(\mathrm{gr}I) \cap \mathbf{S}(\mathfrak{g}_n)$ in \mathfrak{g}_n^* . The space \mathfrak{g}_∞^* equals the inverse limit $\varprojlim \mathfrak{g}_n^*$, and therefore $\mathrm{Var}(I) \subset \mathfrak{g}_\infty^*$ is the inverse limit of the algebraic varieties $\overline{\mathrm{pr}_{\mathfrak{g}_n} \mathrm{Var}(I)}$.

An ideal I is of *locally finite codimension* in $U(\mathfrak{g}_\infty)$ (i.e. $I \cap U(\mathfrak{g}_n)$ is of finite codimension in $U(\mathfrak{g}_n)$ for all n) if and only if the ideal $\mathrm{gr}I$ is of *locally finite codimension* in $\mathbf{S}(\mathfrak{g}_\infty)$ (i.e. $(\mathrm{gr}I) \cap \mathbf{S}(\mathfrak{g}_n)$ is of finite codimension in $\mathbf{S}(\mathfrak{g}_n)$ for all n). Furthermore, $\mathrm{gr}I$ is of locally finite codimension in $\mathbf{S}(\mathfrak{g}_\infty)$ if and only if $\mathrm{Var}(I) = 0$. We say that an ideal I is of *locally infinite codimension* if I is not of locally finite codimension.

The above discussion implies in particular the following.

Proposition 2.1. Assume that \mathfrak{g}_∞ is a locally simple Lie algebra. If $U(\mathfrak{g}_\infty)$ admits a non-zero ideal of locally infinite codimension, then $\mathbf{S}(\mathfrak{g}_\infty)$ admits a non-zero Poisson ideal of locally infinite codimension.

3. POISSON IDEALS: STATEMENT OF RESULTS

Our first main result is the following theorem.

Theorem 3.1. If $\mathbf{S}(\mathfrak{g}_\infty)$ admits a non-zero Poisson ideal of locally infinite codimension, then $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

Corollary 3.2. If $U(\mathfrak{g}_\infty)$ admits a non-zero ideal of locally infinite codimension, then $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

Proof. Let $U(\mathfrak{g}_\infty)$ admit a non-zero ideal of locally infinite codimension. Then $\mathbf{S}(\mathfrak{g}_\infty)$ admits a non-zero Poisson ideal of locally infinite codimension by Proposition 2.1, hence $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ by Theorem 3.1. \square

Fix now a Lie algebra $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ together with a chain (1) such that $\varinjlim \mathfrak{g}_n = \mathfrak{g}_\infty$. Without loss of generality we assume that for $n \geq 3$ all \mathfrak{g}_n are simple and of the same type A, B, C, or D, and that $\text{rk } \mathfrak{g}_n = n$. By V_n we denote a natural representation of \mathfrak{g}_n (if \mathfrak{g}_n is of type A, there are two choices of V_n up to isomorphism). We further assume that, for $n \geq 3$, V_{n+1} considered as a \mathfrak{g}_n -module is isomorphic to V_n plus a trivial module.

Set

$$\mathfrak{g}_n^{\leq r} := \{X \in \mathfrak{g}_n \mid \text{there exists } \lambda \in \mathbb{F} \text{ such that } \text{rk}(X - \lambda \text{Id}_{V_n}) \leq r\}$$

where X is considered as a linear operator on V_n . Note that $\mathfrak{g}_n^{\leq r}$ is a Zariski closed subset of \mathfrak{g}_n . Choosing compatible identifications $\mathfrak{g}_n \cong \mathfrak{g}_n^*$, we can assume that $\mathfrak{g}_n^{\leq r} \subset \mathfrak{g}_n^*$. Furthermore, for $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ one can check directly that the projection $\mathfrak{g}_{n+1}^* \rightarrow \mathfrak{g}_n^*$ maps $\mathfrak{g}_{n+1}^{\leq r}$ surjectively onto $\mathfrak{g}_n^{\leq r}$. This yields a well-defined inverse limit of algebraic varieties $\varprojlim \mathfrak{g}_n^{\leq r}$. In what follows we denote $\varprojlim \mathfrak{g}_n^{\leq r}$ respectively by $\mathfrak{sl}_\infty^{\leq r}, \mathfrak{so}_\infty^{\leq r}, \mathfrak{sp}_\infty^{\leq r}$, or by $\mathfrak{s}_\infty^{\leq r}$ when we refer to all three Lie algebras $\mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

The radical ideals $J_n^{\leq r}$ of $\mathbf{S}(\mathfrak{g}_n)$ with respective zero-sets $\mathfrak{g}_n^{\leq r} \subset \mathfrak{g}_n^*$ form a chain whose union we denote by $J^{\leq r}$. The ideal $J^{\leq r}$ is a radical Poisson ideal of $\mathbf{S}(\mathfrak{g}_\infty)$. It turns out that any non-zero radical Poisson ideal of $\mathbf{S}(\mathfrak{g}_\infty)$ is of this form.

Theorem 3.3. Let $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ and $J \subset \mathbf{S}(\mathfrak{g}_\infty)$ be a non-zero radical Poisson ideal. Then $J = J^{\leq r}$ for some $r \in \mathbb{Z}_{\geq 0}$.

Theorems 3.1 and 3.3 are proved in section 5.

4. AUXILIARY RESULTS

Throughout the paper V denotes a finite-dimensional vector space of dimension d_V . In order to consider all simple classical groups simultaneously, we use S (respectively, \mathfrak{s}) as an abbreviation for SL, SO, Sp (respectively, $\mathfrak{sl}, \mathfrak{so}, \mathfrak{sp}$) and consider three different cases. In the case $S=SL$ we fix the zero bilinear form on the space V and set $S(V) := SL(V)$, $\mathfrak{s}(V) := \mathfrak{sl}(V)$. In the case $S=SO$ we fix a nondegenerate symmetric bilinear form on V and set $S(V) := SO(V)$, $\mathfrak{s}(V) := \mathfrak{so}(V)$. In the case $S=Sp$ we assume that d_V is even and fix a nondegenerate antisymmetric bilinear form on V . Then $S(V) := Sp(V)$, $\mathfrak{s}(V) := \mathfrak{sp}(V)$.

In this section G denotes a connected simple subgroup of $S(V)$ with Lie algebra $\mathfrak{g} \subset \mathfrak{s}(V)$. We start with some general statements about G -orbits in \mathfrak{g}^* (i.e. about coadjoint orbits). We identify \mathfrak{g} and \mathfrak{g}^* via the Cartan-Killing form. Fix $e \in \mathfrak{g}^*$. Denote the G -orbit of e in \mathfrak{g}^* by $\mathcal{O}[e]$. Let $\underline{\mathcal{O}}[e]$ be the unique closed G -orbit of the closure $\overline{\mathcal{O}[e]}$ in \mathfrak{g}^* . By the Luna Slice Theorem [VP], there exists a G -equivariant morphism $\mathcal{O}[e] \rightarrow \underline{\mathcal{O}}[e]$. Assume that $\underline{\mathcal{O}}[e] \neq 0$, i.e. e is not nilpotent. Let $h \in \underline{\mathcal{O}}[e]$ be a semisimple element. The centralizer of h in \mathfrak{g} is a Levi subalgebra¹ of some parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. Let $P \subset G$ be a parabolic subgroup with Lie algebra \mathfrak{p} . Then there exists a finite G -equivariant covering $\tilde{\mathcal{O}} \rightarrow \mathcal{O}[e]$ and a G -equivariant morphism $\tilde{\mathcal{O}} \rightarrow G/P$.

Assume now that $\underline{\mathcal{O}}[e] = \{0\}$, i.e. that e is nilpotent. By the Jacobson-Morozov Theorem there exist elements h, f such that

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h,$$

i.e. such that $\{e, h, f\}$ is an \mathfrak{sl}_2 -triple. The element h is rational semisimple. Hence \mathfrak{g} equals the direct sum $\bigoplus_{i \in \mathbb{Q}} \mathfrak{g}^i$ of $\text{ad}h$ -eigenspaces with rational eigenvalues. The direct sum $\mathfrak{g}_h^+ := \bigoplus_{0 \leq i \in \mathbb{Q}} \mathfrak{g}^i \subset \mathfrak{g}$ is a parabolic subalgebra of \mathfrak{g} and we denote it by \mathfrak{p}_e . The subalgebra \mathfrak{p}_e is determined by e . By $P_e \subset G$ we denote the parabolic subgroup with the Lie algebra \mathfrak{p}_e . There is a G -equivariant morphism $\mathcal{O}[e] \rightarrow G/P_e$.

The above discussion is summarized in the following lemma.

Lemma 4.1. A suitable finite covering of any coadjoint orbit admits a G -equivariant (and thus surjective) morphism to G/P , where P is a maximal parabolic subgroup.

For $\mathfrak{g} = \mathfrak{sl}(V)$, a quotient G/P is nothing but a grassmannian $\text{Gr}(r; V)$ for some $r < d_V$. For $\mathfrak{g} = \mathfrak{so}(V)$ and $\mathfrak{sp}(V)$ a quotient G/P is an irreducible component of the variety $\text{Gr}^{(0)r; V}$ of isotropic subspaces in V of dimension r for some $r \leq \frac{d_V}{2}$. The variety $\text{Gr}^{(0)r; V}$ is irreducible unless $\mathfrak{g} = \mathfrak{so}(V)$ and $r = \frac{d_V}{2}$. In this latter case $\text{Gr}^{(0)r; V}$ has two irreducible components which are isomorphic as varieties. More generally, for $k < r < d_V$, we denote by $\text{Gr}^{(k)r; V}$ the variety of subspaces of V of dimension r on which the restriction of the fixed form on V has rank k .

For $X \in \text{End}V$ we denote by $V_\lambda^X \subset V$ the generalized eigenspace of X with eigenvalue $\lambda \in \mathbb{F}$. Furthermore, $\mathfrak{g} \cdot V$ stands for the sum of the non-trivial simple \mathfrak{g} -submodules of V .

Proposition 4.2. Assume that there is an $S(V)$ -orbit \mathcal{O} in $\mathfrak{s}(V)^*$ such that its image in \mathfrak{g}^* under the canonical projection $\mathfrak{s}(V)^* \rightarrow \mathfrak{g}^*$ is not dense. Then

$$\dim(\mathfrak{g} \cdot V) < 2(\dim G - \text{rk } G)(\text{rk } G + 1) \text{ or } 2\dim G + 2 \geq d_V.$$

In order to prove Proposition 4.2 we need several preliminary statements. We start with the following lemma whose proof we leave to the reader.

Lemma 4.3. Let V' be any subspace in V of dimension $d_{V'}$.

- Assume that $\mathfrak{s} = \mathfrak{sl}$, \mathfrak{sp} . If $\dim(V^r \cap V') \geq 1$ for any isotropic subspace $V^r \subset V$ of dimension r , then $r + d_{V'} > d_V$.
- Assume that $\mathfrak{s} = \mathfrak{so}$ and $r < \frac{d_V}{2}$. If $\dim(V^r \cap V') \geq 1$ for any isotropic subspace $V^r \subset V$ of dimension r , then $r + d_{V'} > d_V$.
- Assume that $\mathfrak{s} = \mathfrak{so}$, d_V is even, and $r = \frac{d_V}{2}$. If $\dim(V^r \cap V') \geq 1$ for any

¹Under a Levi subalgebra of a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ we understand a maximal reductive in \mathfrak{g} subalgebra of \mathfrak{p} .

isotropic subspace $V^r \subset V$ of dimension r from some irreducible component of $\text{Gr}^{(0)r; V}$, then either $r + d_{V'} > d_V$, or $r + d_{V'} = d_V$ and V' is isotropic.

Let Z be a G -variety. We denote the maximal dimension of a G -orbit on Z by $m_G(Z)$. For a subvariety $Y \subset Z$ we set

$$GY := \{z \in Z \mid z = gy \text{ for some } y \in Y \text{ and } g \in G\}.$$

Lemma 4.4. a) Assume that $m_G(\mathbb{P}(V)) < \dim G$. Then

$$\dim(\mathfrak{g} \cdot V) < 2(\dim G - \text{rank} G)(\text{rank} G + 1).$$

b) Let $\mathfrak{s} = \mathfrak{so}$. Assume that

$$m_G(\mathbb{P}(V)) = \dim G \text{ and } m_G(\text{Gr}^{(0)1; V}) < \dim G.$$

Then

$$2\dim G + 2 \geq d_V.$$

Proof. The statement of part a) is the main result of [AnPo]. In part b) $\text{Gr}^{(0)1; V} = \mathbb{P}(\mathcal{V})$, where \mathcal{V} is the set of isotropic vectors in V . The inequality $m_G(\mathbb{P}(\mathcal{V})) < \dim G$ implies that the stabilizer of a generic point of $\mathbb{P}(\mathcal{V})$ under the action of G has positive dimension. Therefore there exists $A \in \mathfrak{g}$ such that $\mathcal{O}[A]$ intersects \mathfrak{g}_x for all x in an open subset of $\mathbb{P}(\mathcal{V})$. Then, for some eigenvalue λ of A , we have $\mathcal{V} \subset \overline{GV_\lambda^A}$ where

$$V_\lambda^A := \text{Ker}(A - \lambda \text{Id}_V)^{d_V}$$

and $\overline{GV_\lambda^A}$ is the closure of GV_λ^A in V . Furthermore, the equality $m_G(\mathbb{P}(V)) = \dim G$ implies that \mathcal{V} coincides with $\overline{GV_\lambda^A}$. Therefore $V_\lambda^A \subset \mathcal{V}$, and the space V_λ^A is isotropic. In particular, $\dim V_\lambda^A \leq \frac{d_V}{2}$, and consequently $\dim G \geq \frac{d_V}{2} - 1$. \square

Lemma 4.5. Let

$$2 \leq r \leq \frac{d_V}{2}, \quad m_G(\mathbb{P}(V)) = \dim G, \text{ and } m_G(\text{Gr}^{(0)r; V}) < \dim G$$

for some r . Then

$$2\dim G + 2 \geq d_V.$$

Proof. Let $x \in \text{Gr}^{(0)r; V}$ denote a point and $V(x)$ be the corresponding r -dimensional space. Let G_x be the stabilizer of x in G ; G_x acts on $V(x)$ and, if x is generic, we have

$$m_{G_x}(\mathbb{P}(V(x))) = \dim G_x.$$

Let $p \in V(x)$ be a non-zero vector, $\langle p \rangle \subset V(x)$ be the line generated by p , and $\mathfrak{g}p \subset V$ be the tangent space to Gp in p . Then $\dim((V(x)/\langle p \rangle) \cap \mathfrak{g}\langle p \rangle) \geq \dim G_x$, where $\mathfrak{g}\langle p \rangle$ is the image of $\mathfrak{g}p$ in $V/\langle p \rangle$. Hence, for a generic $\langle p \rangle \in \text{Gr}^{(0)1; V}$ and any

$$\tilde{V} \in \text{Gr}^{(0)(r-1); \langle p \rangle^\perp / \langle p \rangle},$$

we have $\dim(\tilde{V} \cap \mathfrak{g}\langle p \rangle) \geq \dim G_x$. In particular, $\dim(\tilde{V} \cap \mathfrak{g}\langle p \rangle) \geq 1$. Therefore, by Lemma 4.3 (applied to $V' = \mathfrak{g}\langle p \rangle$ and $V = \langle p \rangle^\perp / \langle p \rangle$) we obtain

$$r - 1 + \dim \mathfrak{g} \geq \dim(\langle p \rangle^\perp / \langle p \rangle).$$

As $\dim(\langle p \rangle^\perp / \langle p \rangle) \geq d_V - 2$ and $r \leq \frac{d_V}{2}$, we conclude that $2\dim G + 2 \geq d_V$. \square

Proposition 4.6 ([Vi, text after Prop. 3]). Let Z be an irreducible G -symplectic variety with a moment map $\Phi : Z \rightarrow \mathfrak{g}^*$. Then the dimension of $\Phi(Z)$ equals $m_G(Z)$.

Proof of Proposition 4.2. Since we can consider the morphism $\mathcal{O} \rightarrow \mathfrak{g}^*$ as a moment map, Proposition 4.6 implies that the image of \mathcal{O} in \mathfrak{g}^* is not dense if and only if

$$m_G(\mathcal{O}) < \dim G.$$

Furthermore, according to Lemma 4.1, for some r , $1 \leq r \leq \frac{d_V}{2}$, there exists an $S(V)$ -equivariant morphism from a finite covering of \mathcal{O} onto $\text{Gr}^{(0)r; V}$. Therefore

$$m_G(\text{Gr}^{(0)r; V}) < \dim G.$$

If $m_G(\text{Gr}^{(0)1; V}) < \dim G$, the claim of Proposition 4.2 follows from Lemma 4.4. Indeed, if also $m_G(\mathbb{P}(V)) < \dim G$, we apply Lemma 4.4 a), and if $m_G(\mathbb{P}(V)) = \dim G$, then $\mathfrak{s} = \mathfrak{so}$ (as $\mathbb{P}(V) \neq \text{Gr}^{(0)1; V}$) and we apply Lemma 4.4 b).

If

$$m_G(\text{Gr}^{(0)1; V}) = \dim G,$$

then necessarily $m_G(\mathbb{P}(V)) = \dim G$ and $r \geq 2$. In this case the claim of Proposition 4.2 follows from Lemma 4.5. \square

Proposition 4.2 is used in a crucial way in the proof of Theorem 3.1. The results in the remaining part of this section are necessary for the proof of Theorem 3.3 and Proposition 8.3.

In what follows W denotes a subspace of V of dimension d_W . If $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$, we assume that the restriction of the fixed form on V is nondegenerate on W . This yields embeddings $S(W) \rightarrow S(V)$ and $\mathfrak{s}(W) \rightarrow \mathfrak{s}(V)$. For $\mathfrak{s} = \mathfrak{sl}$ an embedding $S(W) \rightarrow S(V)$ is determined by a choice of complement to W in V . Moreover, in all three cases we have an orthogonal (with respect to the Cartan-Killing form on $\mathfrak{s}(V)$) projection $\phi : \mathfrak{s}(V) \rightarrow \mathfrak{s}(W)$. It is easy to see that $\phi(X) = \text{pr}_W \circ (X|_W)$, where $X \in \mathfrak{s}(V)$ is viewed as an element of $\text{End } V$ and $\text{pr}_W : V \rightarrow W$ is the orthogonal projection for $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$, and respectively the projection along the fixed complement of W in V for $\mathfrak{s} = \mathfrak{sl}$. We fix the embedding $\mathfrak{s}(W) \rightarrow \mathfrak{s}(V)$, and $\phi : \mathfrak{s}(V) \rightarrow \mathfrak{s}(W)$ stands for the corresponding projection.

Recall that $\mathcal{O}[X]$ denotes the $S(V)$ -orbit in $\mathfrak{s}(V)$ of $X \in \mathfrak{s}(V)$.

Lemma 4.7. Let $X \in \mathfrak{s}(V)$. Then

- a) $\text{rk } \phi(X) \leq \text{rk } X$;
- b) if $d_W > \text{rk } X$, the image $\phi(\mathcal{O}[X])$ contains an element of rank $\text{rk } X$.

Proof. Part a) follows immediately from the formula

$$\phi(X) = \text{pr}_W \circ (X|_W).$$

To prove part b) we consider a generic subspace $\tilde{W} \subset V$ of dimension d_W . Note that for $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$ the restriction of the fixed form on V to \tilde{W} is nondegenerate. Furthermore, $\text{rk } X|_{\tilde{W}} = \text{rk } X$, and $\tilde{W}^\perp \cap \text{Im } X = 0$ where $\text{Im } X \subset V$ is the image of $X \in \text{End } V$. Therefore $\text{rk}(\text{pr}_{\tilde{W}} \circ X|_{\tilde{W}}) = \text{rk } X$.

Since there exists $g \in S(V)$ with $g(W) = \tilde{W}$, we have

$$\text{rk } \phi(g(X)) = \text{rk } X.$$

\square

Lemma 4.8. Consider the projection $\varphi : \mathfrak{sp}(V)^* \rightarrow \mathfrak{sl}(V^{iso})^*$ dual to the embedding $\mathfrak{sl}(V^{iso}) \rightarrow \mathfrak{sp}(V)$, where V^{iso} is a maximal isotropic subspace of V . Let Z be an $\mathrm{Sp}(V)$ -invariant subset of $\mathfrak{sp}(V)$ such that $Z \subset \phi^{-1}(\mathfrak{sl}(V^{iso})^{\leq r})$. Then

$$Z \subset \mathfrak{sp}(V)^{\leq r}.$$

Proof. Let $X \in \mathfrak{sp}(V)$ be such that the $\mathrm{Sp}(V)$ -orbit of X lies in $\phi^{-1}(\mathfrak{sl}(V^{iso})^{\leq r})$. Arguments similar to the arguments in the proof of Lemma 4.7 b) show that $X \in \mathfrak{sp}(V)^{\leq r}$. \square

Lemma 4.9. Let $x \in \mathrm{Gr}(r; V)$ and let $V(x) \subset V$ be the corresponding subspace. Let W' be the $\mathrm{S}(W)$ -stable complement to W in V . Assume that

$$\dim(V(x) \cap W) = 0 \text{ and } \dim(V(x) \cap W') = r - d_W.$$

Then the stabilizer of $x \in \mathrm{Gr}(r; V)$ in $\mathrm{SL}(W)$ (and therefore also in $\mathrm{SO}(W)$, $\mathrm{Sp}(W)$) is trivial.

Proof. Since $\dim(V(x) \cap W') = r - d_W$, the projection of $V(x)$ to W is surjective. Thus $V(x)/(V(x) \cap W')$ can be considered as the graph of a linear operator from W to $W'/(W' \cap V(x))$. The condition $\dim(V(x) \cap W) = 0$ shows that this linear operator is invertible, i.e. can be represented by an invertible $d_W \times d_W$ matrix. The stabilizer of such a matrix under the action of $\mathrm{SL}(W)$ on $\mathrm{GL}(W)$ by left multiplication is trivial. \square

For a subset $S \subset \mathbb{F}$ we set $V_S^X := \bigoplus_{\lambda \in S} V_\lambda^X$. We set also $\hat{0} := \mathbb{F} \setminus \{0\}$; then $V = V_0^X \oplus V_{\hat{0}}^X$. Let $X_{nn} := X|_{V_0}$, $X_r := X|_{V_{\hat{0}}}$, and $X_r = X_s + X_{ns}$ be the Jordan decomposition as sum of commuting semisimple and nilpotent elements. The decomposition $V = V_0^X \oplus V_{\hat{0}}^X$ allows us to consider all four operators X_{nn}, X_r, X_s, X_{ns} as endomorphisms of V . Furthermore,

$$\mathrm{rk}(X_s + X_{ns}) = \mathrm{rk} X_s, \quad \mathrm{rk}(X_s + X_{nn}) = \mathrm{rk} X,$$

and $X_s + X_{nn} \in \overline{\mathcal{O}[X]}$.

Definition 4.10. We say that $X \in \mathrm{End} V$ is *rank-reduced* whenever

$$X_{nn}^2 = 0 \text{ and } X_{ns} = 0.$$

Lemma 4.11. If $2\mathrm{rk} X \leq d_V$ for some $X \in \mathfrak{s}(V)$, then there exists a (unique up to conjugation) rank-reduced element $X' \in \mathfrak{s}(V)$ such that $X' \in \overline{\mathcal{O}[X]}$ and $\mathrm{rk} X = \mathrm{rk} X'$.

Proof. The condition $(X'_{nn})^2 = 0$ means that X'_{nn} is nilpotent and the sizes of its Jordan blocks are at most 2×2 . Moreover, the number of non-zero Jordan blocks of X'_{nn} equals $\mathrm{rk} X'_{nn}$. In particular, all nilpotent rank-reduced endomorphisms in $\mathfrak{s}(V)$ of fixed rank are conjugate (possibly, by outer automorphisms of $\mathfrak{s}(V)$). Since $2\mathrm{rk} X_{nn} \leq d_V$, there exists a nilpotent rank-reduced endomorphism X_{rr} of rank equal to $\mathrm{rk} X$ and such that $X_{rr} \in \overline{\mathcal{O}[X_{nn}]}$ [CM]. Then a rank-reduced endomorphism $X' \in \mathfrak{s}(V)$ such that $X'_s = X_s$ and X'_{nn} is conjugate to X_{rr} , is as desired. By construction, X' is unique up to conjugation. \square

Lemma 4.12. Assume $d_V > 3d_W$. If for some $X \in \mathfrak{s}(V)$ the image $\phi(\mathcal{O}[X])$ is not dense in $\mathfrak{s}(W)$, then there exists a unique $\lambda \in \mathbb{F}$ such that

$$\mathrm{rk}(X - \lambda \mathrm{Id}_V) < d_W.$$

Proof. Set $d_V(\mu) := \dim V_\mu^X$ and $d_V(S) := \sum_{\mu \in S} d_V(\mu)$ for any subset $S \subset \mathbb{F}$. We have an $\mathrm{S}(V)$ -equivariant morphism

$$\mathcal{O}[X] \rightarrow \text{Gr}(d_V(S); V), \quad X \mapsto \bigoplus_{\mu \in S} V_\mu^X.$$

As $\phi(\mathcal{O}[X])$ is not dense in $\mathfrak{s}(W)$, the inequality $\text{m}_{\text{S}(W)}(\mathcal{O}[X]) < \dim \text{S}(W)$ holds. Therefore, for any $S \subset \mathbb{F}$, there exists k such that

$$\text{m}_{\text{S}(W)}(\text{Gr}^k(d_V(S); V)) < \dim \text{S}(W). \quad (2)$$

We now show that, for any $S \subset \mathbb{F}$, either $d_V(S) < d_W$ or $d_V(S) > d_V - d_W$. Assume to the contrary that $d_W \leq d_V(S) \leq d_V - d_W$. Let $x \in \text{Gr}^k(d_V(S); V)$ be a generic point and $V(x) \subset V$ be the corresponding subspace. Then

$$\dim(V(x) \cap W') = \dim V(x) - d_W,$$

where W' is the $\text{S}(W)$ -stable complement to W in V , and Lemma 4.9 implies that the stabilizer of x in $\text{S}(W)$ is trivial. This contradicts (2).

We claim next that there exists $\lambda \in \mathbb{F}$ such that $d_V(\lambda) > d_V - d_W$. Indeed, assuming that $d_V(\lambda) < d_W$ for all $\lambda \in \mathbb{F}$, we see that the inequality $3d_W < d_V$, together with the alternative $d_V(S) < d_W$ or $d_V(S) > d_V - d_W$, implies $d_V(S) < d_W$ for any finite set S . This is again a contradiction.

Fix now λ with $d_V(\lambda) > d_V - d_W$. It is easy to see that $\text{rk}(X - \lambda \text{Id}_V) = d_V(\mathbb{C} \setminus \lambda) + \text{rk}(X - \lambda \text{Id}_V)|_{V_\lambda^X}$. Furthermore, we have a decomposition

$$X - \lambda \text{Id}_V = (X - \lambda \text{Id}_V)_s + (X - \lambda \text{Id}_V)_{ns} + (X - \lambda \text{Id}_V)_{nn}.$$

Let $r' \in \mathbb{Z}_{\geq 0}$ for $\mathfrak{s} = \mathfrak{sl}$ and $r' \in 2\mathbb{Z}_{\geq 0}$ for $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$. If $r' < \text{rk}((X - \lambda \text{Id}_V)_{nn}|_{V_\lambda^X})$, there exists $X_{r'}^\lambda \in \overline{\mathcal{O}[(X - \lambda \text{Id}_V)_{nn}|_{V_\lambda^X}]} \subset \mathfrak{s}(V_\lambda^X)$ with $\text{rk} X_{r'}^\lambda = r'$. Hence, if $\text{rk}((X - \lambda \text{Id}_V)_s) \leq r' \leq \text{rk}(X - \lambda \text{Id}_V)$, there exists

$$X_{r'} \in \overline{\mathcal{O}[X]} \subset \mathfrak{s}(V)$$

with $X_{r'} - \lambda \text{Id} = (X - \lambda \text{Id})_s + X_{r' - \text{rk}((X - \lambda \text{Id})_s)}^\lambda$, and thus $\text{rk}(X_{r'} - \lambda \text{Id}_V) = r'$.

As, for any d with $d_W \leq d \leq d_V - d_W$ the equality $\text{m}_{\text{S}(W)}(\text{Gr}^k(d; V)) = \dim \text{S}(W)$ holds for some k , for any r' we have either $\dim \text{Im}(X_{r'} - \lambda \text{Id}_V) < d_W$ or $\dim \text{Im}(X_{r'} - \lambda \text{Id}_V) > d_V - d_W$. Therefore, either $\text{rk}(X - \lambda \text{Id}_V) < d_W$ or $\text{rk}(X - \lambda \text{Id}_V)_s > d_V - d_W > d_W$. On the other hand, $\text{rk}(X - \lambda \text{Id}_V)_s < d_W$. This implies

$$\text{rk}(X - \lambda \text{Id}_V) < d_W.$$

□

Lemma 4.13. Assume $d_V \geq 2d_W$, $d_W > 2r$, and let $X \in \mathfrak{s}(V)^{\leq r}$. If

$$\text{rk}(X - \lambda \text{Id}_V) = r$$

for some $\lambda \in \mathbb{F}$, then $\phi(\mathcal{O}[X])$ is dense in $\mathfrak{s}(W)^{\leq r}$.

We consider separately the cases $\mathfrak{s} = \mathfrak{sl}$ and $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$.

Proof of Lemma 4.13 for $\mathfrak{s} = \mathfrak{sl}$. By Lemma 4.11 we can assume that $X - \lambda \text{Id}_V$ is rank-reduced. Therefore the linear operator $X - \lambda \text{Id}_V : V \rightarrow V$ is conjugate to a direct sum

$$\bigoplus_{1 \leq i \leq \text{rk} X_s} A(t_i) \quad \bigoplus_{1 \leq j \leq \text{rk} X_{nn}} B_j \bigoplus C,$$

where C is the zero operator, and $A(t), B_j$ are operators of the following forms:

$$A(t) := \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \quad B_j = B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We denote by $a_1[i], a_2[i]$ and $b_1[j], b_2[j]$ the elements of the bases of the 2-dimensional subspaces of V corresponding respectively to $A(t_i)$ and B_j , and by $c[k]$ the elements of the basis of the subspace corresponding to C . We can further assume that

$$W := \text{span}\{a_1[i], b_1[j], c[k] \mid 1 \leq i \leq \text{rk} X_s, 1 \leq j \leq \text{rk} X_{nn}, 1 \leq k \leq d_W - r\}$$

and that the $\text{SL}(W)$ -invariant complement W' of W is the span of the remaining elements of the basis $\{a_1[i], a_2[i], b_1[j], b_2[j], c_k\}_{i,j,k}$ of V . Then

$$\phi\left(\bigoplus_i \mathcal{O}[A(t_i)] \oplus \bigoplus_j \mathcal{O}[B_j] \oplus C\right) \subset \mathfrak{sl}\left(\bigoplus_{1 \leq i \leq \text{rk} X_s} \mathfrak{gl}\langle a_1[i], a_2[i] \rangle \oplus \bigoplus_{1 \leq j \leq \text{rk} X_{nn}} \mathfrak{sl}\langle b_1[j], b_2[j] \rangle\right).$$

Moreover, $\phi\left(\bigoplus_i \mathcal{O}[A(t_i)] \oplus \bigoplus_j \mathcal{O}[B_j] \oplus C\right)$ consists of all diagonal matrices of the form

$$\text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0), \quad \lambda_1 + \dots + \lambda_r = 0.$$

This follows from the obvious statement that the matrices in $\mathcal{O}[A(t)]$ for $t \neq 0$, and in $\mathcal{O}[B_j]$, can have arbitrary values in their left top corner. Therefore $\phi(\mathcal{O}[X])$ contains all semisimple elements of rank $r = \text{rk} X$. As such elements are dense in $\mathfrak{sl}(W)^{\leq r}$, we conclude that $\phi(\mathcal{O}[X])$ is dense in $\mathfrak{sl}(W)^{\leq r}$. \square

In the rest of this section $\mathfrak{s} = \mathfrak{so}$, \mathfrak{sp} . To prove Lemma 4.13 in this case, we need some preliminary lemmas.

Lemma 4.14. Let $X \in \mathfrak{s}(V)$. Assume that $2\text{rk}(X - \lambda \text{Id}_V) < d_V$ for some $\lambda \in \mathbb{F}$. Then $\lambda = 0$.

Proof. Obviously, $\text{rk}(X - \lambda \text{Id}_V) + d_V(\lambda) \geq d_V$ (recall that $d_V(\lambda) = \dim V_\lambda^X$). Therefore $2\text{rk}(X - \lambda \text{Id}_V) < d_V$ implies $2d_V(\lambda) > d_V$.

However, for \mathfrak{so} , \mathfrak{sp} , we have $d_V(\lambda) = d_V(-\lambda)$. Therefore $\lambda \neq 0$ would mean that $\lambda \neq -\lambda$ and $\dim V_\lambda^X + \dim V_{-\lambda}^X > d_V$, which is false. Hence $\lambda = 0$. \square

Fix $r \in \mathbb{Z}_{\geq 0}$. If r is even, we denote by $\mathfrak{s}(V)_{reg}^r$ the set of semisimple elements of $\mathfrak{s}(V)$ which have rank r . If r is odd, let $\mathfrak{sp}(V)_{reg}^r$ be the set of elements X of $\mathfrak{sp}(V)$ such that

$$\text{rk} X_s = r - 1, X_{ns} = 0, \text{rk} X_{nn} = 1.$$

In particular, X_{nn} is nilpotent and has a single non-zero Jordan block of size 2×2 .

Lemma 4.15. Let d_V be even. Then $\mathfrak{sp}(V)_{reg}^{d_V-1}$ is dense in the set of elements of $\mathfrak{sp}(V)$ of rank less or equal to $d_V - 1$.

Proof. We denote by Z the set of elements of $\mathfrak{sp}(V)$ of rank at most $d_V - 1$. By definition Z coincides with the set of elements of $\mathfrak{sp}(V)$ whose determinant is 0. The determinant function on $\text{End}(V)$ restricted to $\mathfrak{sp}(V)$ is irreducible, and thus Z is a closed irreducible subvariety of $\mathfrak{sp}(V)$.

The set of elements of $\mathfrak{sp}(V)$ which have at least $d_V - 1$ distinct eigenvalues is an open subvariety of $\mathfrak{sp}(V)$, and is non-empty as it contains $\mathfrak{sp}(V)_{reg}^{d_V-1}$. Therefore Z contains a dense open subvariety \tilde{Z} which consists of elements with at least $d_V - 1$ distinct eigenvalues.

Let $X \in \tilde{Z}$. The multiplicity of the eigenvalue 0 of X equals 2 and the multiplicity of any non-zero eigenvalue of X equals 1. Thus X is either semisimple or belongs to $\mathfrak{sp}(V)_{reg}^{\leq d_V-1}$. In both cases we clearly have

$$X \in \overline{\mathfrak{sp}(V)_{reg}^{\leq d_V-1}}.$$

Therefore $\mathfrak{sp}(V)_{\text{reg}}^{\leq d_V - 1}$ is dense in \tilde{Z} . \square

In connection with the following two lemmas, note that $\text{rk } X$ is even for any $X \in \mathfrak{so}(V)$. Recall also that on $\mathfrak{s}(V)$ there is the following partial order: X_1 is lower than X_2 if $\mathcal{O}[X_1] \subset \overline{\mathcal{O}[X_2]}$. This order can be described explicitly in terms of Young diagrams (or equivalently, in terms of the Jordan normal forms of X_1, X_2), see [CM].

Lemma 4.16. Let d_V be even and r be odd such that $1 \leq r \leq \frac{d_V}{2}$. Then

$$\mathfrak{sp}(V)^{\leq r} = \overline{\mathfrak{sp}(V)_{\text{reg}}^r}.$$

Proof. By definition $\mathfrak{sp}(V)_{\text{reg}}^r \subset \mathfrak{sp}(V)^{\leq r}$. Therefore it is sufficient to prove that $\mathfrak{sp}(V)^{\leq r} \subset \overline{\mathfrak{sp}(V)_{\text{reg}}^r}$. Fix $X \in \mathfrak{sp}(V)^{\leq r}$ and recall that $\text{rk } X \leq r$ by Lemma 4.14. There exists an X -stable orthogonal decomposition $V_s \oplus V_n = V$ such that $X|_{V_n}$ is nilpotent and $X|_{V_s}$ is given (in an appropriate basis) by a matrix of the form

$$A := \begin{pmatrix} F & 0 \\ 0 & -F^t \end{pmatrix}.$$

Set $r_s := \text{rk } X|_{V_s}$, $r_n := \text{rk } X|_{V_n}$. Then r_s and $d_{V_s} := \dim V_s$ are even, and $r_s + r_n = r$.

To prove that $X \in \overline{\mathfrak{sp}(V)_{\text{reg}}^r}$, it suffices to prove that $X|_{V_s} \in \overline{\mathfrak{sp}(V)_{\text{reg}}^{r_s}}$ and $X|_{V_n} \in \overline{\mathfrak{sp}(V_n)_{\text{reg}}^{r_n}}$. First of all we check that $X|_{V_s} \in \overline{\mathfrak{sp}(V_s)_{\text{reg}}^{r_s}}$. The matrices of type A form a subalgebra isomorphic to \mathfrak{gl}_m for $m = \frac{d_{V_s}}{2}$. As $X|_{V_s} \in \mathfrak{gl}_m^{\leq \frac{r_s}{2}}$, we have $X|_{V_s} \in \overline{(\mathfrak{gl}_m)_{\text{reg}}^{\frac{r_s}{2}}}$. Therefore $X|_{V_s} \in \overline{\mathfrak{sp}(V_s)_{\text{reg}}^{r_s}}$.

Set $X' := X|_{V_n}$. It remains to show that $X' \in \overline{\mathfrak{sp}(V_n)_{\text{reg}}^{r_n}}$ (note that $\text{rk } X' = r_n$).

Let $X_{\text{max}} \in \mathfrak{sp}(V_n)$ be a nilpotent element of rank r_n with a single non-zero Jordan block of size $(r_n + 1) \times (r_n + 1)$. Such an element of $\mathfrak{sp}(V_n)$ exists by [CM], and moreover, $X' \in \overline{\mathcal{O}[X_{\text{max}}]}$.

It suffices to prove that $X_{\text{max}} \in \overline{\mathfrak{sp}(V_n)_{\text{reg}}^{r_n}}$. Using the explicit description of the coadjoint orbits of a symplectic group given in [CM] one can check that there exists an X_{max} -stable orthogonal decomposition $V'_n \oplus V''_n = V$ such that

$$X_{\text{max}}|_{V''_n} = 0, \quad d_{V'_n} = \text{rk } X_{\text{max}} + 1.$$

By Lemma 4.15, $X_{\text{max}}|_{V'_n} \in \overline{\mathfrak{sp}(V'_n)_{\text{reg}}^{r_n}}$, and thus $X_{\text{max}} \in \overline{\mathfrak{sp}(V_n)_{\text{reg}}^{r_n}}$. \square

Lemma 4.17. Let r be an even integer such that $2 \leq r < \frac{d_V}{2}$. Then

$$\mathfrak{s}(V)^{\leq r} = \overline{\mathfrak{s}(V)_{\text{reg}}^r}.$$

Proof. By definition, $\mathfrak{s}(V)_{\text{reg}}^r \subset \mathfrak{s}(V)^{\leq r}$. Therefore, taking into account Lemma 4.14, it suffices to prove that any element X of rank at most r lies in $\overline{\mathfrak{s}(V)_{\text{reg}}^r}$. For $\mathfrak{s} = \mathfrak{so}$ the arguments are the same as in the proof of Lemma 4.16. This applies also to the case $\mathfrak{s} = \mathfrak{sp}$ if $\text{rk } X$ is odd. Thus in the remainder of the proof we can assume that $\mathfrak{s} = \mathfrak{sp}$ and $\text{rk } X$ is even.

Let $X_{\text{max}} \in \mathfrak{sp}(V_n)$ be a nilpotent element of rank r_n with two non-zero Jordan blocks of respective sizes $r_n \times r_n$ and 2×2 . Such an element of $\mathfrak{sp}(V_n)$ exists by [CM], and moreover, $X' \in \overline{\mathcal{O}[X_{\text{max}}]}$.

It suffices to prove that $X_{\text{max}} \in \overline{\mathfrak{sp}(V)_{\text{reg}}^r}$. There exists an X_{max} -stable orthogonal decomposition $V'_n \oplus V''_n \oplus V'''_n = V$ such that

$$d_{V_n''} = 2, d_{V_n'} = \text{rk } X_{max}, \text{ and } X|_{V_n'''} = 0.$$

Since $X|_{V_n'} \in \overline{\mathfrak{sp}(V_n')_{reg}^{d_{V_n'}-1}}$, $X|_{V_n' \oplus V_n''}$ lies in the closure in $\mathfrak{sp}(V_n' \oplus V_n'')$ of the set of elements $Y \in \mathfrak{sp}(V_n' \oplus V_n'')^{\leq r}$ such that

$$\text{rk } Y_{ss} = \text{rk } X_{max} - 2, \text{ rk } Y_{nn} = 2.$$

As $Y_{nn} \in \overline{\mathfrak{sp}(\mathbb{F}^4)_{reg}^2}$, we have $X_{max} \in \overline{\mathfrak{sp}(V)_{reg}^r}$. \square

Proof of Lemma 4.13 for $\mathfrak{s} = \mathfrak{so}, \mathfrak{sp}$. Lemma 4.14 implies that $\lambda = 0$. Therefore $\text{rk } X = r$. By Lemma 4.11 we can assume that X is rank-reduced. Note that the standard \mathfrak{gl}_2 -subalgebra of $\mathfrak{s}_4 \cong \mathfrak{so}_4, \mathfrak{sp}_4$ is given by the matrices

$$\begin{pmatrix} F & 0 \\ 0 & -F^t \end{pmatrix}, \quad F \in \mathfrak{gl}_2$$

in an appropriate basis.

Assume that $\text{rk } X$ is even. Then X is conjugate (via $S(V)$) to a direct sum

$$\bigoplus_{1 \leq i \leq \text{rk } X_s} A(t_i) \bigoplus_{1 \leq j \leq \text{rk } X_{nn}} B_j \bigoplus C$$

where C is the zero-operator, and $A(t), B_j$ are of the form

$$A(t) := \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_j = B := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Let $a_1[i], a_2[i], a_3[i], a_4[i]$ and $b_1[j], b_2[j], b_3[j], b_4[j]$ denote the elements of the basis of the 4-dimensional subspaces of V corresponding to $A(t_i)$ and B_j respectively, and let $c[k]$ be the elements of the basis of the subspace corresponding to C . The restriction of the fixed form on V to

$$E := \text{span}\{a_1[i], a_3[i], b_1[j], b_3[j], c[k] \mid 1 \leq i \leq \text{rk } X_s, 1 \leq j \leq \text{rk } X_{nn}, 1 \leq k \leq d_W - 2r\}$$

is nondegenerate and therefore we can assume that $W = E$. The orthogonal complement W^\perp to W is the span of the remaining elements in the above basis of V . Then

$$\phi\left(\bigoplus_i \mathcal{O}[A(t_i)] \bigoplus_j \mathcal{O}[B_j] \bigoplus C\right) \subset \bigoplus_{1 \leq i \leq \text{rk } X_s} \mathfrak{gl}_2 \bigoplus_{1 \leq j \leq \text{rk } X_{nn}} \mathfrak{gl}_2 \subset \mathfrak{s}(V),$$

and we claim that $\phi\left(\bigoplus_i \mathcal{O}[A(t_i)] \bigoplus_j \mathcal{O}[B_j] \bigoplus C\right)$ contains all diagonal matrices of the form

$$\text{diag}(\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \dots, \lambda_{\text{rk } X}, -\lambda_{\text{rk } X}, 0, \dots, 0).$$

This follows from the obvious statement that matrices in $\mathcal{O}[A(t)]$ for $t \neq 0$ and in $\mathcal{O}[B]$ can have arbitrary values in their left top 2×2 -corner for $\mathfrak{s} = \mathfrak{sp}$ and arbitrary antisymmetric values for $\mathfrak{s} = \mathfrak{so}$.

Therefore $\phi(\mathcal{O}[X])$ contains all semisimple elements of rank $r = \text{rk } X$. By Lemma 4.17, such elements are dense in $\mathfrak{s}(W)^{\leq r}$ for r even, therefore $\phi(\mathcal{O}[X])$ is dense in $\mathfrak{s}(W)^{\leq r}$.

It remains to consider the case when $\text{rk } X$ is odd. Here $\mathfrak{s} = \mathfrak{sp}$. Arguments similar to those above show that $\phi(\mathcal{O}[X])$ contains all elements of $\mathfrak{sp}(W)$ which have the form

$$\text{diag}(\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \dots, \lambda_{\frac{r-1}{2}}, -\lambda_{\frac{r-1}{2}}, 0, \dots, 0) \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in an appropriate basis. Then $\phi(\mathcal{O}[X])$ is dense in $\mathfrak{sp}(W)^{\leq r}$ by Lemma 4.16. \square

5. PROOF OF THEOREMS 3.1 AND 3.3

In the rest of the paper we assume that \mathfrak{g}_∞ is a locally simple Lie algebra which may be finitary (i.e. isomorphic to $\mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$), diagonal, or non-diagonal. Let $\mathfrak{g}_\infty = \varinjlim \mathfrak{g}_n$ for a fixed sequence of embeddings (1). In the special case of $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ we assume in addition that the simple Lie algebras \mathfrak{g}_n satisfy the assumptions from section 3. Let G_n be the adjoint group of \mathfrak{g}_n . Recall that, if $J \subset \mathbf{S}(\mathfrak{g})$ is a Poisson ideal of locally infinite codimension, then the intersections $J_n := J \cap \mathbf{S}(\mathfrak{g}_n)$ determine proper G_n -stable closed subvarieties $\text{Var}(J_n) \subset \mathfrak{g}_n^*$ which form an inverse system

$$\dots \rightarrow \text{Var}(J_n) \xrightarrow{p_n} \text{Var}(J_{n-1}) \rightarrow \dots$$

under the natural projections $\mathfrak{g}_n^* \rightarrow \mathfrak{g}_{n-1}^*$. Moreover, $\text{Var}(J_{n-1}) = \overline{\text{pr}_n \text{Var}(J_n)}$ where the closure is taken in \mathfrak{g}_{n-1}^* .

Let \mathfrak{k}_1 be a simple finite-dimensional Lie algebra and $\mathfrak{k}_2 \subset \mathfrak{k}_1$ be a simple finite-dimensional subalgebra. Fix invariant non-degenerate symmetric forms f_1 on \mathfrak{k}_1 and f_2 on \mathfrak{k}_2 so that in each case the square length of a long coroot equals 2. Then the restriction of f_1 to \mathfrak{k}_2 is proportional to f_2 with a coefficient which we denote $I_{\mathfrak{k}_2}^{\mathfrak{k}_1}$. This coefficient, known as the *Dynkin index*, is multiplicative: if $\mathfrak{k}_3 \subset \mathfrak{k}_2 \subset \mathfrak{k}_1$ is a chain of inclusions, then

$$I_{\mathfrak{k}_3}^{\mathfrak{k}_1} = I_{\mathfrak{k}_3}^{\mathfrak{k}_2} I_{\mathfrak{k}_2}^{\mathfrak{k}_1}.$$

The Dynkin index is always a positive integer [Dy]. Moreover, if $\mathfrak{k}_2 \subset \mathfrak{k}_1$ are classical simple Lie algebras of the same type and of rank at least 5, then $I_{\mathfrak{k}_2}^{\mathfrak{k}_1} = 1$ if and only if the natural module of \mathfrak{k}_1 decomposes over \mathfrak{k}_2 as a natural plus a trivial module [DP, Proposition 2.3].

Proof of Theorem 3.1. Let J be a non-zero Poisson ideal of locally infinite codimension in $\mathbf{S}(\mathfrak{g}_\infty)$. Without loss of generality we may assume that J is a radical ideal, as the radical of a Poisson of locally infinite codimension ideal in $\mathbf{S}(\mathfrak{g}_\infty)$ is again Poisson and of locally infinite codimension.

Fix n so that $J_n = J \cap \mathbf{S}(\mathfrak{g}_n)$ is non-zero and of infinite codimension in $\mathbf{S}(\mathfrak{g}_n)$. The image of any G_{n+m} -orbit in $\text{Var}(J_{n+m})$ under the morphism $\text{Var}(J_{n+m}) \rightarrow \mathfrak{g}_n^*$ is not dense in \mathfrak{g}_n^* since it lies in the proper closed subvariety $\text{Var}(J_n) \subset \mathfrak{g}_n^*$. Therefore Proposition 4.2 implies that $\dim(\mathfrak{g}_n \cdot V_{n+m})$ is bounded as a function on m . Hence the number of non-trivial simple \mathfrak{g}_n -constituents in V_{n+m} and their dimensions are simultaneously bounded. As a consequence, $I_{\mathfrak{g}_n}^{\mathfrak{g}_{n+m}}$ is bounded as a function of $m > 0$. Therefore there exists $N > 0$ such that $I_{\mathfrak{g}_n}^{\mathfrak{g}_{n+N}} = 1$ for all $n > N$. Now [DP, Corollary 2.4] implies $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty$ or \mathfrak{sp}_∞ . \square

Proof of Theorem 3.3. If J is of locally finite codimension in $\mathbf{S}(\mathfrak{g}_\infty)$, then

$$\text{Var}(J) = 0 = \mathfrak{s}^{\leq 0}.$$

Therefore, in the rest of the proof we can assume that J is non-zero and of locally infinite codimension in $\mathbf{S}(\mathfrak{g}_\infty)$.

Fix n so that $J_n := J \cap \mathbf{S}(\mathfrak{g}_n)$ is non-zero and of infinite codimension in $\mathbf{S}(\mathfrak{g}_n)$. Since for any non-zero $X \in \text{Var}(J_{m+n})$ the image in \mathfrak{g}_n^* of the G_{m+n} -orbit $\mathcal{O}[X] \subset \mathfrak{g}_{m+n}^*$ is not dense in \mathfrak{g}_n^* , Lemma 4.12 implies that $\text{Var}(J_{m+n}) \subset \mathfrak{g}_{m+n}^{\leq r'_n}$ for some r'_n which depends on n only. Let r_n be the minimal such r'_n . By Lemma 4.12, $r_n < n$.

The inequality $r_n < n$ allows us to apply Lemma 4.13 when $m > n$. It implies that the image in $\mathfrak{g}_n^{\leq r_n}$ of $\mathcal{O}[X] \subset \mathfrak{g}_{m+n}^*$ is dense in $\mathfrak{g}_n^{\leq r_n}$ for any $X \in \text{Var}(J_{m+n})$ with $\text{rk } X = r_n$. Furthermore, $r_n \leq r_{n+m}$ by definition. On the other hand, Lemma 4.7 implies $r_{m+n} \geq r_n$, and therefore $r_n = r_{m+n}$. Set $r := r_n$. Then $J = J^{\leq r}$. \square

6. SOME COROLLARIES

Corollary 3.2 implies that if \mathfrak{g}_∞ is a locally simple Lie algebra which is not finitary, then any ideal in $U(\mathfrak{g}_\infty)$ is of locally finite codimension. Moreover, a result of A. Zhilinskii [Zh2] states that $U(\mathfrak{g}_\infty)$ admits an ideal of locally finite codimension which is not the augmentation ideal if and only if \mathfrak{g}_∞ is diagonal. These two facts yield the following corollary.

Corollary 6.1 (Baranov's Conjecture). Let \mathfrak{g}_∞ be any locally simple Lie algebra. Then the augmentation ideal is the only non-zero ideal in $U(\mathfrak{g}_\infty)$ if and only if \mathfrak{g}_∞ is not diagonal.

Furthermore, Theorems 3.1, 3.3 imply the following.

Corollary 6.2. Let \mathfrak{g}_∞ be any locally simple Lie algebra and $I \subset U(\mathfrak{g}_\infty)$ be an ideal. Then

- a) $\text{Var}(I) \neq 0$ implies that $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$;
- b) if $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ and $I \subset U(\mathfrak{g}_\infty)$ is an ideal of locally infinite codimension, then $\text{Var}(I) = \mathfrak{s}_\infty^{\leq r}$ for some $r \in \mathbb{Z}_{\geq 1}$.

A. Zhilinskii [Zh3] (see also [Zh1]) has given a description of all ideals of locally finite codimension in $U(\mathfrak{g}_\infty)$ for an arbitrary locally simple Lie algebra \mathfrak{g}_∞ . Therefore the problem of describing all ideals in $U(\mathfrak{g}_\infty)$ gets reduced to the problem of describing all ideals of locally infinite codimension in $U(\mathfrak{g}_\infty)$ for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$. In the following two sections we will show in particular that all pro-varieties $\mathfrak{s}_\infty^{\leq r}$ arise as associated ‘‘varieties’’ of ideals respectively of $U(\mathfrak{sl}_\infty), U(\mathfrak{so}_\infty), U(\mathfrak{sp}_\infty)$.

7. COHERENT LOCAL SYSTEMS OF MODULES AND A CLASSIFICATION OF PRIME INTEGRABLE IDEALS OF $U(\mathfrak{g}_\infty)$

In this section we review some published and unpublished results of A. Zhilinskii and draw corollaries.

Definition 7.1. a) A \mathfrak{g}_∞ -module M is *integrable*, if for any finitely generated subalgebra $U' \subset U(\mathfrak{g}_\infty)$ and any $m \in M$ we have $\dim(U' \cdot m) < \infty$.

b) An ideal $I \subset U(\mathfrak{g}_\infty)$ is *integrable*, if I is the annihilator of an integrable \mathfrak{g}_∞ -module.

Integrable ideals in $U(\mathfrak{g}_\infty)$ are described as annihilators of coherent local systems of finite-dimensional \mathfrak{g}_n -modules as introduced by A. Zhilinskii in [Zh2]. We discuss this topic below.

7.1. Integrable ideals and coherent local systems.

Definition 7.2. A *coherent local system of modules* (further shortened as *c.l.s.*) for $\mathfrak{g}_\infty = \varinjlim \mathfrak{g}_n$ is a collection of sets

$$\{Q_n\}_{n \in \mathbb{Z}_{\geq 1}} \subset \prod_{n \in \mathbb{Z}_{\geq 1}} \text{Irr } \mathfrak{g}_n$$

such that $Q_m = \langle Q_n \rangle_m$ for any $n > m$, where $\langle Q_n \rangle_m$ denotes the set of all simple \mathfrak{g}_m -constituents of the \mathfrak{g}_n -modules from Q_n .

A. Zhilinskii [Zh1], [Zh2], [Zh3] has classified c.l.s. for any locally simple Lie algebra \mathfrak{g}_∞ . In subsection 7.5 below we show how this classification leads to a description of integrable ideals of $U(\mathfrak{g}_\infty)$ for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

A c.l.s. Q is *irreducible* if $Q \neq Q' \cup Q''$ with $Q' \not\subset Q''$ and $Q'' \not\subset Q'$. Let Q be any c.l.s. and let $\mathcal{S}(Q)$ be the set of irreducible c.l.s. which are contained in Q . Then $\mathcal{S}(Q)$ has a finite subset of elements $Q(1), \dots, Q(r)$ which are maximal by inclusion, and $Q = \cup_r Q(r)$ [Zh1]; we call $Q(r)$ the *irreducible components* of Q . This makes apparent the analogy between c.l.s. and algebraic varieties.

Any integrable \mathfrak{g}_∞ -module M determines a c.l.s. $Q := \{Q_n\}_{n \in \mathbb{Z}_{\geq 1}}$, where

$$Q_n := \{z \in \text{Irr } \mathfrak{g}_n \mid \text{Hom}_{\mathfrak{g}_n}(z, M) \neq 0\}.$$

We denote this relation by $Q \leftarrow M$. We also recall that an integrable \mathfrak{g}_∞ -module M is *locally simple* if $M = \varinjlim M_n$ for a chain $\dots \subset M_n \subset M_{n+1} \subset \dots$ of simple finite-dimensional \mathfrak{g}_n -submodules M_n of M .

On the other hand, if Q is a c.l.s., then $\cap_{z \in Q_m} \text{Ann}_{U(\mathfrak{g}_m)} z \subset \cap_{z \in Q_n} \text{Ann}_{U(\mathfrak{g}_n)} z$ for any $n > m$. Therefore $\cup_m (\cap_{z \in Q_m} \text{Ann}_{U(\mathfrak{g}_m)} z)$ is an ideal of $U(\mathfrak{g})$; we denote this ideal by $I(Q)$, and say that $I(Q)$ is the *annihilator* of Q .

Proposition 7.3 ([Zh1, Lemma 1.1.2]). If Q is an irreducible c.l.s., then $I(Q)$ is the annihilator of some locally simple integrable \mathfrak{g}_∞ -module. In particular, $I(Q)$ is primitive and hence prime.

Corollary 7.4. The ideal $I(Q)$ is integrable for any c.l.s. Q .

Proof. Let $Q = \cup_r Q(r)$, $Q(r)$ being the irreducible components of Q . By Proposition 7.3, each irreducible c.l.s. $Q(r)$ is the annihilator of a simple integrable \mathfrak{g}_∞ -module $M(r)$. Then $I(Q)$ is the annihilator of the integrable \mathfrak{g}_∞ -module $\oplus_r M(r)$. \square

Fix n . The set $\text{Irr } \mathfrak{g}_n$ is parametrized by the lattice Λ_n of integral dominant weights of \mathfrak{g}_n . Let z_1, z_2 be isomorphism classes of simple \mathfrak{g}_n -modules with respective highest weights λ_1, λ_2 . We denote by $z_1 z_2$ the isomorphism class of a simple module with highest weight $\lambda_1 + \lambda_2$. If $S_1, S_2 \subset \text{Irr } \mathfrak{g}_n$ we set

$$S_1 S_2 := \{z \in \text{Irr } \mathfrak{g}_n \mid z = z_1 z_2 \text{ for some } z_1 \in S_1 \text{ and } z_2 \in S_2\}.$$

If Q', Q'' are c.l.s., we denote by $Q'Q''$ the smallest c.l.s. such that $(Q')_i (Q'')_i \subset (Q'Q'')_i$. By definition, $Q'Q''$ is the *product* of Q' and Q'' . The role of the identity for this product is played by the trivial c.l.s., i.e. the c.l.s. consisting of the isomorphism class of the trivial 1-dimensional module at all levels.

If $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{sp}_\infty$, then by [Zh1]

$$(Q')_n (Q'')_n = (Q'Q'')_n.$$

7.2. Zhilinskii's classification of c.l.s.. In this subsection we reproduce A. Zhilinskii's classification of irreducible c.l.s. for $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

In the rest of the paper $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty$ or \mathfrak{sp}_∞ , and $\mathfrak{g}_n = \mathfrak{s}(V_n)$ is a sequence (1) of finite-dimensional simple Lie algebras satisfying the assumptions of section 3. We set $V_\infty := \varinjlim V_n$ and $(V_\infty)_* := \varinjlim V_n^*$.

The following irreducible c.l.s. are by definition the *basic c.l.s.*:

$$\begin{aligned} &\text{for } \mathfrak{g}_\infty = \mathfrak{sl}_\infty : \mathcal{E} \leftarrow \Lambda V_\infty, \quad \mathcal{L}_p \leftarrow \Lambda^p V_\infty, \quad \mathcal{L}_p^\infty \leftarrow \mathbf{S}(V_\infty \otimes \mathbb{F}^p), \\ &\quad \mathcal{R}_q \leftarrow \Lambda^q (V_\infty)_*, \quad \mathcal{R}_q^\infty \leftarrow \mathbf{S}((V_\infty)_* \otimes \mathbb{F}^q), \quad \mathcal{E}^\infty \text{ (all modules);} \\ &\text{for } \mathfrak{g}_\infty = \mathfrak{sp}_\infty : \mathcal{E} \leftarrow \Lambda V_\infty, \quad \mathcal{L}_p \leftarrow \Lambda^p V_\infty, \quad \mathcal{L}_p^\infty \leftarrow \mathbf{S}(V_\infty \otimes \mathbb{F}^p), \\ &\quad \mathcal{E}^\infty \text{ (all modules);} \\ &\text{for } \mathfrak{g}_\infty = \mathfrak{so}_\infty : \mathcal{E} \leftarrow \Lambda V_\infty, \quad \mathcal{L}_p \leftarrow \Lambda^p V_\infty, \quad \mathcal{L}_p^\infty \leftarrow \mathbf{S}(V_\infty \otimes \mathbb{F}^p), \\ &\quad \mathcal{R} \text{ (spinor modules), } \mathcal{E}^\infty \text{ (all modules),} \end{aligned}$$

where $p, q \in \mathbb{Z}_{\geq 1}$.

Proposition 7.5 (Unique factorization property [Zh1]). Any irreducible c.l.s. can be expressed uniquely as a product as follows:

$$\begin{aligned} &(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m (\mathcal{R}_w^\infty \mathcal{R}_{w+1}^{z_{w+1}} \mathcal{R}_{w+2}^{z_{w+2}} \dots \mathcal{R}_t^{z_t}) && \text{for } \mathfrak{g}_\infty = \mathfrak{sl}_\infty, \quad (3) \\ &(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m \text{ or } (\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m \mathcal{R} && \text{for } \mathfrak{g}_\infty = \mathfrak{so}_\infty, \quad (4) \\ &(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m && \text{for } \mathfrak{g}_\infty = \mathfrak{sp}_\infty, \quad (5) \end{aligned}$$

where

$$\begin{aligned} &m, n, v, w \in \mathbb{Z}_{\geq 0}, \quad v \leq n, \quad w \leq t \\ &x_i, z_j \in \mathbb{Z}_{\geq 0} \text{ for } v+1 \leq i \leq n \text{ and } w+1 \leq j \leq t. \end{aligned}$$

Here, for $v=0$, \mathcal{L}_v^∞ is assumed to be the trivial c.l.s., and, similarly, \mathcal{R}_w^∞ is assumed to be the trivial c.l.s. for $w=0$.

We say that an irreducible c.l.s. for \mathfrak{so}_∞ is of *integer type* if its expression (4) does not contain \mathcal{R} , and of *semiinteger type* otherwise. Furthermore, we define a c.l.s. Q to be of *finite type* if the set Q_n is finite for all $n \geq 1$. It is easy to see that the ideal $I(Q)$ is of locally finite-codimension in $U(\mathfrak{g}_\infty)$ if and only if Q is of finite type.

If $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{sp}_\infty$, the irreducible c.l.s. of finite type form a free lattice (by means of product) generated by $\mathcal{E}, \mathcal{L}_p, \mathcal{R}_q$ for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ and by $\mathcal{E}, \mathcal{L}_p$ for $\mathfrak{g}_\infty = \mathfrak{sp}_\infty$. For $\mathfrak{g} = \mathfrak{so}_\infty$ the set of irreducible c.l.s. of finite type equals the union $\mathcal{N} \sqcup \mathcal{NR}$, where \mathcal{N} is a free lattice generated by $\mathcal{E}, \mathcal{L}_p$, and $\mathcal{NR} := \{N\mathcal{R} \mid N \in \mathcal{N}\}$ [Zh1].

7.3. Partial order by inclusion on irreducible c.l.s.. To an irreducible c.l.s. Q for \mathfrak{sl}_∞ in the form (3) A. Zhilinskii assigns the following two non-increasing sequences of elements of $\mathbb{Z}_{\geq 0} \cup \{+\infty\}$:

$$\{l_i := m + \sum_{j \geq i} x_j\} \text{ and } \{r_i := m + \sum_{j \geq i} z_j\},$$

where it is assumed $l_1 = l_2 = \dots = l_v := +\infty =: r_1 = \dots = r_w$. Note that

$$\lim_{i \rightarrow \infty} l_i = \lim_{i \rightarrow \infty} r_i = m.$$

Similarly, to an irreducible c.l.s. for \mathfrak{sp}_∞ or \mathfrak{so}_∞ in the form (4) or (5) A. Zhilinskii assigns the non-increasing sequence

$$\{l_i := m + \sum_{j \geq i} x_j\},$$

where $l_1 = \dots = l_v := +\infty$. Again, $\lim_{i \rightarrow \infty} l_i = m$.

Zhilinskii [Zh1] establishes the following inclusion criterion. An irreducible c.l.s. Q for \mathfrak{sl}_∞ contains an irreducible c.l.s. Q' if and only if, for some $a, b \in \mathbb{Z}_{\geq 0}$, we have

$$a + b = m - m', l_i \geq l'_i + a, r_i \geq r'_i + b$$

for the corresponding sequences $\{l_i\}, \{r_i\}$ and $\{l'_i\}, \{r'_i\}$. If Q and Q' are irreducible c.l.s. for \mathfrak{sp}_∞ , then $Q' \subset Q$ if and only if

$$l_i \geq l'_i.$$

Finally, if Q and Q' are irreducible c.l.s. for \mathfrak{so}_∞ , then $Q' \subset Q$ if and only if Q and Q' have the same integer or semiinteger type, and

$$l_i \geq l'_i.$$

Corollary 7.6. If $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{sp}_\infty$, the only minimal (with respect to inclusion) c.l.s. is the trivial c.l.s.. If $\mathfrak{g}_\infty = \mathfrak{so}_\infty$, there are two minimal c.l.s.: the trivial one and \mathcal{R} .

Proof. A minimal c.l.s. is irreducible. Thus our claim follows from the inclusion criterion of A. Zhilinskii. \square

7.4. Tensor product and ideals. Fix i . If $S_1, S_2 \subset \text{Irr } \mathfrak{g}_n$ we set

$$S_1 \otimes S_2 := \{z \in \text{Irr } \mathfrak{g}_n \mid \text{Hom}(z, z_1 \otimes z_2) \neq 0 \text{ for some } z_1 \in S_1 \text{ and } z_2 \in S_2\}.$$

Furthermore, it is clear that the tensor product of two c.l.s. is a well defined c.l.s.. One can check that

$$(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m (\mathcal{R}_w^\infty \mathcal{R}_{w+1}^{z_{w+1}} \mathcal{R}_{w+2}^{z_{w+2}} \dots \mathcal{R}_t^{z_t}) = (\mathcal{L}_1^\infty)^{\otimes v} \otimes (\mathcal{R}_1^\infty)^{\otimes w} \otimes ((\mathcal{L}_1^{x_{v+1}} \mathcal{L}_2^{x_{v+2}} \dots \mathcal{L}_{n-v}^{x_n}) \mathcal{E}^m (\mathcal{R}_1^{z_{w+1}} \mathcal{R}_2^{z_{w+2}} \dots \mathcal{R}_{t-w}^{z_t}))$$

for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$, and

$$(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m = (\mathcal{L}_1^\infty)^{\otimes v} \otimes ((\mathcal{L}_1^{x_{v+1}} \mathcal{L}_2^{x_{v+2}} \dots \mathcal{L}_{n-v}^{x_n}) \mathcal{E}^m),$$

$$(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} \dots \mathcal{L}_n^{x_n}) \mathcal{E}^m \mathcal{R} = (\mathcal{L}_1^\infty)^{\otimes v} \otimes ((\mathcal{L}_1^{x_{v+1}} \mathcal{L}_2^{x_{v+2}} \dots \mathcal{L}_{n-v}^{x_n}) \mathcal{E}^m \mathcal{R}) \quad (6)$$

for $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, where $(\mathcal{L}_1^\infty)^{\otimes v} := \mathcal{L}_1^\infty \otimes \mathcal{L}_1^\infty \otimes \dots \otimes \mathcal{L}_1^\infty$ (v times). Note that formula (6) applies to the \mathfrak{so}_∞ -case only.

The above formulas yield a different parametrization of the irreducible c.l.s.: for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ the irreducible c.l.s. are parametrized by triples (v, w, Q_f) , where $v, w \in \mathbb{Z}_{\geq 0}$ and Q_f is an irreducible c.l.s. of finite type, and for $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ the irreducible c.l.s. are parametrized by pairs (v, Q_f) , where $v \in \mathbb{Z}_{\geq 0}$ and Q_f is an irreducible c.l.s. of finite type.

For $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ we set

$$\text{cls}(v, w, Q_f) := (\mathcal{L}_1^\infty)^{\otimes v} \otimes (\mathcal{R}_1^\infty)^{\otimes w} \otimes Q_f$$

and

$$I(v, w, Q_f) := I(\text{cls}(v, w, Q_f)).$$

However, it is easy to check that the annihilators of \mathcal{L}_1^∞ and \mathcal{R}_1^∞ in $U(\mathfrak{sl}_\infty)$ coincide. Therefore

$$I(v, w, Q_f) = I(v + w, 0, Q_f).$$

In what follows we call an irreducible c.l.s. for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ of the form $\text{cls}(v, 0, Q_f)$ a *left irreducible* c.l.s. An arbitrary *left* c.l.s. is defined as a finite union of left irreducible c.l.s. We denote the left irreducible c.l.s. for $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ by $\text{cls}(v, Q_f)$. We also set $I(v, Q_f) := I(v, 0, Q_f)$.

Note that for $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ an arbitrary irreducible c.l.s. corresponds to a pair (v, Q_f) . Therefore the notations $\text{cls}(v, Q_f)$ and $I(v, Q_f)$ make sense in all three cases: $\mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

7.5. Classification of integrable ideals in $U(\mathfrak{g}_\infty)$. Recall that $\mathfrak{g}_n = \mathfrak{sl}_{n+1}$ for $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$. In this case, the space of weights of \mathfrak{g}_n is identified with the set of $(n+1)$ -tuples $(\lambda_1, \dots, \lambda_{n+1})$, $\lambda_i \in \mathbb{F}$, up to the equivalence relation

$$(\lambda_1, \dots, \lambda_{n+1}) \sim (\lambda_1 + k, \dots, \lambda_{n+1} + k)$$

for $k \in \mathbb{F}$. The lattice of integral dominant weights is identified with the set of weights such that $\lambda_{i+1} - \lambda_i \in \mathbb{Z}_{\leq 0}$ for $1 \leq i \leq n$.

For $\mathfrak{g}_\infty \cong \mathfrak{sp}_\infty$, the space of weights of $\mathfrak{g}_n \cong \mathfrak{sp}_{2n}$ is identified with the set of n -tuples $(\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \mathbb{F}$. The lattice of integral dominant weights is identified with the set of weights such that $\lambda_i \in \mathbb{Z}_{\geq 0}$ and $\lambda_{i+1} \leq \lambda_i$ for $1 \leq i \leq n$.

For $\mathfrak{g}_\infty \cong \mathfrak{so}_\infty$, we assume that $\mathfrak{g}_n \cong \mathfrak{so}_{2n}$. The space of weights of $\mathfrak{g}_n = \mathfrak{so}_{2n}$ is identified with the set of n -tuples $(\lambda_1, \dots, \lambda_n)$, $\lambda_i \in \mathbb{F}$. The lattice of integral dominant weights is identified with the set of weights such that $\lambda_i \in \frac{1}{2}\mathbb{Z}$, $\lambda_{i+1} \leq \lambda_i$ for $1 \leq i \leq n-1$, and $\lambda_{n-1} \geq |\lambda_n|$.

Let \mathcal{W}_n denote the Weyl group of \mathfrak{g}_n , and let ρ_n be the half-sum of positive roots of \mathfrak{g}_n . The set of radical ideals of $Z_{U(\mathfrak{g}_n)}$ is identified with the set of Zariski-closed \mathcal{W}_n -stable subsets of the space of weights of \mathfrak{g}_n .

Let Q be a c.l.s.. For any $n \in \mathbb{Z}_{\geq 1}$, we denote by \overline{Q}_n the Zariski-closure of the set of highest weights of the isomorphism classes of simple \mathfrak{g}_n -modules from Q_n in the space of weights of \mathfrak{g}_n .

Lemma 7.7. a) For any $v \in \mathbb{Z}_{\geq 0}$ and any c.l.s. Q_f of finite type, the following conditions are equivalent:

- (i) $(v, Q_f) = (v', Q'_f)$,
- (ii) $\mathcal{W}_n(\rho_n + \overline{\text{cls}(v, Q_f)_n}) = \mathcal{W}_n(\rho_n + \overline{\text{cls}(v', Q'_f)_n})$ for all $n \in \mathbb{Z}_{\geq 1}$.

b) Let $z \in \text{Irr } \mathfrak{g}_n$ and let λ_z be the highest weight of a representative of z . For $\mathfrak{g}_\infty \cong \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, we have

$$\lambda_z + \rho_n \in \mathcal{W}_n(\rho_n + \overline{\text{cls}(v, Q_f)_n})$$

if and only if $z \in \text{cls}(v, Q_f)_n$. For $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$, we have

$$\lambda_z + \rho_n \in \mathcal{W}_n(\rho_n + \overline{\text{cls}(v, Q_f)_n})$$

if and only if there exist $v', v'' \in \mathbb{Z}_{\geq 0}$ such that $v' + v'' = v$ and $z \in \text{cls}(v', v'', Q_f)_n$.

c) Let $v \in \mathbb{Z}_{\geq 0}$ and Q_f be a c.l.s. of finite type for \mathfrak{g}_∞ . Then

$$Q(I(v, Q_f)) = \text{cls}(v, Q_f)$$

for $\mathfrak{g}_\infty \cong \mathfrak{so}_\infty, \mathfrak{sp}_\infty$. If $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$, we have

$$Q(I(v, Q_f)) = \cup_{v'+v''=v} \text{cls}(v', v'', Q_f).$$

Proof. Part a) is implied by the following explicit description of the closure $\overline{\text{cls}(v, Q_f)_{n+v}}$: if Q_f is an irreducible c.l.s. of finite type, then $(\lambda_1, \dots, \lambda_{n+v}) \in \text{cls}(v, Q_f)_{n+v}$ if and only if the simple \mathfrak{g}_n -module with highest weight $(\lambda_{v+1}, \dots, \lambda_{n+v})$ lies in $(Q_f)_n$.

Part b) is a straightforward corollary of part a).

We proceed to c). Fix v, Q_f , and n . Let $z \in Q(I(v, Q_f))_n$ and λ_z be the highest weight of z . By definition, $I(v, Q_f) \cap U(\mathfrak{g}_n) \subset \text{Ann}_{U(\mathfrak{g}_n)} z$. Therefore

$$I(v, Q_f) \cap Z(U(\mathfrak{g}_n)) \subset (\text{Ann}_{U(\mathfrak{g}_n)} z) \cap Z(U(\mathfrak{g}_n)).$$

The latter inclusion is equivalent to the condition

$$\lambda_z + \rho_n \in \mathcal{W}_n(\rho_n + \overline{\text{cls}(v, Q_f)_n}).$$

By b) we have $z \in \text{cls}(v, Q_f)_n$ for $\mathfrak{g}_\infty \cong \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, and $z \in \cup_{v'+v''=v} \text{cls}(v', v'', Q_f)_n$ for $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$.

Hence, $Q(I(v, Q_f)) = \text{cls}(v, Q_f)$ for $\mathfrak{g}_\infty \cong \mathfrak{so}_\infty, \mathfrak{sp}_\infty$.

Assume that $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$. Then $I(v', v'', Q_f) = I(v' + v'', Q_f)$, and thus

$$\text{cls}(v', v'', Q_f) \subset Q(I(v' + v'', Q_f)).$$

Therefore $Q(I(v, Q_f)) = \cup_{v'+v''=v} \text{cls}(v', v'', Q_f)$. \square

For any ideal $I \subset U(\mathfrak{g}_\infty)$, set

$$Q(I)_n := \{z \in \text{Irr } \mathfrak{g}_n \mid I \cap U(\mathfrak{g}_n) \subset \text{Ann}_{U(\mathfrak{g}_n)} z\}$$

and note that $Q(I)$ is a well-defined c.l.s.. If I is integrable, then I is the annihilator of $Q(I)$.

Proposition 7.8. Assume $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$. An integrable ideal of $U(\mathfrak{g}_\infty)$ is prime if and only if it is primitive. ²

Proof. Let I be a prime integrable ideal of $U(\mathfrak{g}_\infty)$. Then I is the annihilator of $Q(I)$. Let Q_1, \dots, Q_s be the irreducible components of $Q(I)$. We have

$$I = I(Q(I)) \subset \cap_{i \leq s} I(Q_s)$$

and

$$I(Q_1)I(Q_2)\dots I(Q_s) \subset I(Q(I)) = I.$$

Therefore I coincides with $I(Q_i)$ for some irreducible c.l.s. Q_i . Consequently, the ideal $I(Q_i)$ is primitive by Proposition 7.3. On the other hand, any primitive ideal is prime. \square

For $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ we denote by $Q_l(I)$ the union of all left irreducible components of $Q(I)$. For $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ we put $Q_l(I) = Q(I)$.

Theorem 7.9. a) The maps

$$\begin{aligned} I &\mapsto Q_l(I), \\ Q &\mapsto I(Q) \end{aligned}$$

are mutually inverse bijections between the set of prime integrable ideals $I \subset U(\mathfrak{g}_\infty)$ and the set of irreducible left c.l.s..

b) If $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, the maps in a) extend to mutually inverse anti-isomorphisms

$$I \mapsto Q(I), \tag{7}$$

$$Q \mapsto I(Q) \tag{8}$$

between the lattice of integrable ideals in $U(\mathfrak{g}_\infty)$ and the lattice of c.l.s. for \mathfrak{g}_∞ ³.

c) If $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$, any integrable ideal of $U(\mathfrak{sl}_\infty)$ equals $I(Q)$ for some left c.l.s. Q for \mathfrak{g}_∞ .

²The analogous statement is false for \mathfrak{g}_n . For instance, $I(\mathcal{L}_1) \cap U(\mathfrak{g}_n)$, $n \geq 2$, is an integrable prime ideal of $U(\mathfrak{g}_n)$ which is not primitive.

³Contrary to our conventions from section 2, here we consider $U(\mathfrak{g}_\infty)$ as an integrable ideal and $\{0\}$ as a c.l.s..

Proof. Let I be a prime integrable ideal of $U(\mathfrak{g}_\infty)$. By the proof of Proposition 7.8, I coincides with $I(Q_i)$ for some irreducible c.l.s. Q_i . Hence, according to subsection 7.4, $I = I(v, Q_f)$ for some $v \in \mathbb{Z}_{\geq 0}$ and some c.l.s. of finite type Q_f .

To prove a) it remains to check that the annihilators of the c.l.s. corresponding to distinct pairs (v, Q_f) and (v', Q'_f) are distinct. For $\mathfrak{g}_\infty = \mathfrak{sl}_\infty$ we have to prove that $I(v, 0, Q_f) = I(v', 0, Q'_f)$ if and only if $v = v'$ and $Q_f = Q'_f$. For $\mathfrak{g}_\infty = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, we have to check that $I(v, Q_f) = I(v', Q'_f)$ if and only if $v = v'$ and $Q_f = Q'_f$. Both statements follow from Lemma 7.7. Hence, a) is proved.

To prove b) we assume that $\mathfrak{g}_\infty \cong \mathfrak{so}_\infty, \mathfrak{sp}_\infty$. Let Q be any c.l.s.. Then $Q = \cup_{i \leq s} Q_i$ for some irreducible c.l.s. Q_i , and $I(Q) = \cap_{i \leq s} I(Q_i)$. On the other hand, $Q_I(\cap_{i \leq s} I(Q_i)) = Q(\cap_{i \leq s} I(Q_i))$ by definition, and

$$Q(\cap_{i \leq s} I(Q_i)) = \cup_{i \leq s} (Q(I(Q_i))) = \cup_{i \leq s} Q_i = Q$$

by a). Therefore the maps (7) and (8) are mutually inverse. In addition, it is now obvious that both maps are anti-homomorphisms of lattices. This proves b).

To prove c) we assume that $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$. Let I be an integrable ideal of $U(\mathfrak{sl}_\infty)$. Then $I = \cap_{i \leq s} I_i$, where I_i are prime integrable ideals of $U(\mathfrak{sl}_\infty)$. For any $i \leq s$, $I_i = I(v_i, (Q_f)_i)$ for some $(v_i, (Q_f)_i)$ as in a). In particular, I_i is the annihilator of some irreducible left local system $Q_i := (v_i, 0, (Q_f)_i)$. Then I is the annihilator of $\cup_{i \leq s} Q_i$. This proves c). \square

Note that for $\mathfrak{g}_\infty \cong \mathfrak{sl}_\infty$ the annihilators of \mathcal{L}_2^∞ and $(\mathcal{L}_2^\infty) \cup (\mathcal{L}_1^\infty \mathcal{R}_1)$ coincide. This shows in particular that the one-to-one correspondence between left irreducible c.l.s. for \mathfrak{g}_∞ and prime integrable ideals of $U(\mathfrak{g}_\infty)$ can not be extended to an anti-isomorphism between the corresponding lattices.

Nevertheless, Theorem 7.9 c) provides a certain description of general integrable ideals of $U(\mathfrak{sl}_\infty)$: it yields a surjection from the set of c.l.s. of \mathfrak{sl}_∞ to the set of integrable ideals of $U(\mathfrak{sl}_\infty)$. Two c.l.s. Q', Q'' determine the same integrable ideal $I(Q') = I(Q'')$ if and only if $Q(I(Q')) = Q(I(Q''))$. For any irreducible c.l.s. Q , the c.l.s. $Q(I(Q))$ is described by Lemma 7.7 c). If $Q = \cup_{i \leq s} Q_i$ is a reducible c.l.s. (Q_1, \dots, Q_s being irreducible c.l.s.), then $Q(I(Q)) = \cup_{i \leq s} Q(I(Q_i))$. This allows in principle to check when $Q(I(Q')) = Q(I(Q''))$.

Furthermore, let's point out that Theorem 7.9, together with A. Zhilinskii's result [Zh1], [Zh3] that any coherent local system is the union of finitely many coherent local systems, implies that the lattice of integrable ideals of $U(\mathfrak{g}_\infty)$ satisfies the ascending chain condition. This has already been stated in [Zh1] and [Zh3]. We would like to think of this result as of "relative Nötherianity" of the algebra $U(\mathfrak{g}_\infty)$. We don't know whether $U(\mathfrak{g}_\infty)$ is two-sided Nötherian.

Theorem 7.9 and Corollary 7.6 imply immediately that if $\mathfrak{g}_\infty = \mathfrak{sl}_\infty, \mathfrak{sp}_\infty$, the augmentation ideal is the unique integrable maximal ideal of $U(\mathfrak{g}_\infty)$. For $\mathfrak{g}_\infty = \mathfrak{so}_\infty$ there are two integrable maximal ideals: the augmentation ideal and the "spinor ideal" $I(\mathcal{R})$.

Corollary 7.10. The algebras $U(\mathfrak{sl}_\infty), U(\mathfrak{so}_\infty)$ and $U(\mathfrak{sp}_\infty)$ are pairwise nonisomorphic.

Proof. The algebras $U(\mathfrak{sl}_\infty), U(\mathfrak{sp}_\infty)$ have each a unique integrable maximal ideal, while the algebra $U(\mathfrak{so}_\infty)$ has two integrable maximal ideals. Hence $U(\mathfrak{sl}_\infty) \not\cong U(\mathfrak{so}_\infty)$ and $U(\mathfrak{sp}_\infty) \not\cong U(\mathfrak{so}_\infty)$.

Consider now prime submaximal integrable ideals in $U(\mathfrak{sl}_\infty)$ and $U(\mathfrak{sp}_\infty)$, i.e. integrable prime ideals which are properly contained only in integrable maximal ideals. Using the inclusion criterion of irreducible c.l.s. from subsection 7.3, one checks immediately (using Theorem 7.9 a)) that $U(\mathfrak{sl}_\infty)$ has two such submaximal ideals, namely $I(\mathcal{L}_1)$ and $I(\mathcal{R}_1)$, while $U(\mathfrak{sp}_\infty)$ has a single such ideal $I(\mathcal{L}_1)$. Hence $U(\mathfrak{sl}_\infty) \not\cong U(\mathfrak{sp}_\infty)$. \square

Finally, note that

$$\text{Var}(I(v, Q_f)) = \begin{cases} \mathfrak{sl}_\infty^{\leq v} & \text{for } \mathfrak{g}_\infty = \mathfrak{sl}_\infty, \\ \mathfrak{so}_\infty^{\leq 2v} & \text{for } \mathfrak{g}_\infty = \mathfrak{so}_\infty, \\ \mathfrak{sp}_\infty^{\leq 2v} & \text{for } \mathfrak{g}_\infty = \mathfrak{sp}_\infty. \end{cases} \quad (9)$$

Since $\mathfrak{so}_\infty^{\leq 2v+1} = \mathfrak{so}_\infty^{\leq 2v}$, all possible associated “varieties” of ideals in $U(\mathfrak{sl}_\infty)$ and $U(\mathfrak{so}_\infty)$ appear in the right-hand side of (9). The only possible associated “varieties” which do not appear in the right-hand side of (9) are $\mathfrak{sp}_\infty^{\leq 2v+1}$ for $v \in \mathbb{Z}_{\geq 0}$, and these pro-varieties are not associated “varieties” of integrable ideals of $U(\mathfrak{sp}_\infty)$.

8. SOME NON-INTEGRABLE IDEALS OF $U(\mathfrak{sp}_\infty)$

We will now provide non-integrable ideals of $U(\mathfrak{sp}_\infty)$ whose associated “varieties” are the pro-varieties $\mathfrak{sp}_\infty^{\leq 2v+1}$ for $z \in \mathbb{Z}_{\geq 0}$. We start with a lemma and a corollary.

Let \mathfrak{g} be a semisimple Lie algebra. Consider the coproduct

$$\Delta : U(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g}), \quad x \mapsto x \otimes 1 + 1 \otimes x \text{ for } x \in \mathfrak{g}.$$

In what follows we will denote the “diagonal”, “left” and “right” copies of \mathfrak{g} respectively by \mathfrak{g}_Δ , \mathfrak{g}_l , \mathfrak{g}_r , i.e. we have

$$\Delta : U(\mathfrak{g}_\Delta) \subset U(\mathfrak{g}_l) \otimes U(\mathfrak{g}_r).$$

Lemma 8.1. Let I_l, I_r be ideals of $U(\mathfrak{g})$. Then

$$\text{Var}(I_l) + \text{Var}(I_r) \subset \text{Var}(U(\mathfrak{g}_\Delta)/(U(\mathfrak{g}_\Delta) \cap (I_l \otimes 1 + 1 \otimes I_r))),$$

where $\text{Var}(I_l) + \text{Var}(I_r)$ is the pointwise sum of the varieties $\text{Var}(I_l)$ and $\text{Var}(I_r)$ inside \mathfrak{g}^* .

Proof. We have

$$\text{gr}(U(\mathfrak{g}_\Delta) \cap (I_l \otimes U(\mathfrak{g}_r) + U(\mathfrak{g}_l) \otimes I_r)) \subset \text{gr}(I_l \otimes U(\mathfrak{g}_r) + U(\mathfrak{g}_l) \otimes I_r) = \text{gr } I_l \otimes S^*(\mathfrak{g}_r) + S^*(\mathfrak{g}_l) \otimes \text{gr } I_r.$$

Therefore

$$\text{Var}(\text{gr}(I_l)) + \text{Var}(\text{gr}(I_r)) \subset \text{Var}(\text{gr}(U(\mathfrak{g}_\Delta) \cap (I_l \otimes U(\mathfrak{g}_r) + U(\mathfrak{g}_l) \otimes I_r))).$$

\square

Corollary 8.2. Let M_l, M_r be \mathfrak{g} -modules with respective annihilators I_l, I_r . Let I_Δ be the annihilator of $M_l \otimes M_r$. Then

$$\text{Var}(I_l) + \text{Var}(I_r) \subset \text{Var}(I_\Delta).$$

Proof. We first consider $M_l \otimes M_r$ as $U(\mathfrak{g}_l) \otimes U(\mathfrak{g}_r)$ -module. Then the structure of \mathfrak{g} -module on $M_l \otimes M_r$ comes from the homomorphism Δ . Therefore

$$\text{Ann}_{U(\mathfrak{g})}(M_l \otimes M_r) = U(\mathfrak{g}_\Delta) \cap (I_l \otimes 1 + 1 \otimes I_r).$$

Hence $\text{Var}(I_l) + \text{Var}(I_r) \subset \text{Var}(I_\Delta)$ by Lemma 8.1. \square

We now provide a maximal ideal $I \subset U(\mathfrak{sp}_\infty)$ such that $\text{Var}(I) = \mathfrak{sp}(V)^{\leq 1}$.

Consider the Weyl algebra \mathbb{W}_∞ of infinitely many variables, i.e. the associative algebra generated by $x_1, \partial_{x_1}, x_2, \partial_{x_2}, \dots, x_n, \partial_{x_n}, \dots$ with relations

$$\begin{aligned} [x_i, x_j] &= 0, & [\partial_{x_i}, \partial_{x_j}] &= 0 \text{ for all } i, j; \\ [\partial_{x_i}, x_i] &= 1 \text{ for } 1 \leq i \leq n-1; & [\partial_{x_i}, x_j] &= 0 \text{ for all } i \neq j. \end{aligned}$$

The subspace spanned by $1, x_i x_j, x_i \partial_{x_j}, \partial_{x_j} \partial_{x_i}$ is a Lie subalgebra of the Lie algebra associated with \mathbb{W}_∞ . This subalgebra $\tilde{\mathfrak{s}}$ has a 1-dimensional center generated by 1, and its derived subalgebra $[\tilde{\mathfrak{s}}, \tilde{\mathfrak{s}}]$ is isomorphic to \mathfrak{sp}_∞ . Therefore we have a homomorphism $U(\mathfrak{sp}_\infty) \rightarrow \mathbb{W}_\infty$; the kernel of this homomorphism is an ideal $I_{\mathbb{W}}$. For any n , $I_{\mathbb{W}} \cap U(\mathfrak{sp}_{2n})$ is a Joseph ideal [Jo].

It is well known that the ideal $I_{\mathbb{W}} \cap U(\mathfrak{g}_n)$ is maximal (thus primitive and prime) in $U(\mathfrak{sp}_{2n})$ and not integrable. Hence $I_{\mathbb{W}}$ is a maximal ideal in $U(\mathfrak{sp}_\infty)$ which is not integrable. The zero-set $\text{Var}(I_{\mathbb{W}})$ coincides with $\mathfrak{sp}_\infty^{\leq 1}$ (note that any element of rank 1 in \mathfrak{sp}_{2n} is nilpotent).

More generally, we have the following.

Proposition 8.3. Let $V_{\mathbb{W}}$ denote the algebra $\mathbb{F}[x_1, x_2, \dots]$ considered as an \mathfrak{sp}_∞ -module. Then for any $v \in \mathbb{Z}_{\geq 0}$ we have

$$\text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}})) = \mathfrak{sp}_\infty^{\leq 2v+1}.$$

Proof. By Corollary 8.2, we have

$$\begin{aligned} \overline{\text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}(V_\infty \otimes \mathbb{F}^v)))} + \overline{\text{Var}(I_{\mathbb{W}})} &= \overline{\mathfrak{sp}_\infty^{\leq 2v} + \mathfrak{sp}_\infty^{\leq 1}} \\ &= \overline{\mathfrak{sp}_\infty^{\leq 2v+1}} \subset \overline{\text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}}))}. \end{aligned}$$

For any n there is a natural embedding $\mathfrak{sl}(V_n) \rightarrow \mathfrak{sp}(V_n \oplus V_n^*)$, and these embeddings define an embedding $\mathfrak{sl}_\infty \rightarrow \mathfrak{sp}_\infty$. Furthermore,

$$\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}}) \cap U(\mathfrak{sl}_\infty) = I(2v+1, \mathbb{F})$$

where \mathbb{F} stands for the trivial c.l.s.. Therefore the image of

$$\text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}}))$$

under the projection $\mathfrak{sp}_\infty^* \rightarrow \mathfrak{sl}_\infty^*$ lies in $\mathfrak{sl}_\infty^{\leq 2v+1}$. Hence, by Lemma 4.8,

$$\text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}})) \subset \mathfrak{sp}_\infty^{\leq 2v+1}.$$

This shows that $\text{Var}(\text{Ann}_{U(\mathfrak{sp}_\infty)}(\mathbf{S}(V_\infty \otimes \mathbb{F}^v) \otimes V_{\mathbb{W}})) = \mathfrak{sp}_\infty^{\leq 2v+1}$. \square

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