Loewy Lengths of Indecomposable Injective Objects in Certain Universal Tensor Categories

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Abstract. We study socle filtrations of indecomposable injective objects in certain universal tensor categories obtained as categories of representations of the classical locally finite Lie algebras $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$ introduced in **[13]**, and the $\aleph_t$-dimensional Mackey Lie algebra $\mathfrak{gl}^M$ introduced in **[2]**. It was known that these injective objects have exhaustive and finite length socle filtrations, however no general formula for their Loewy length was available. The indecomposable injectives over $\mathfrak{gl}_\infty$ and $\mathfrak{sl}_\infty$ are parameterised by Young diagrams λ and μ. We prove that these objects have Loewy length $|\lambda \cap \mu| + 1$. The indecomposable objects over $\mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$ are parameterised by one Young diagram λ. We show that in both cases the Loewy length is $|\gamma| + 1$ where γ is maximal such that $2\gamma \subset \lambda^\top$ in case of $\mathfrak{sp}_\infty$ and $2\gamma \subset \lambda$ in case of $\mathfrak{so}_\infty$. The indecomposable injective objects, denoted $\tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu}$, in the category $\overline{\mathbb{T}}_{\aleph_t}$ are parameterised by $t+3$ Young diagrams $\lambda_t,\dots,\lambda_0,\mu,\nu$. We prove that these objects have Loewy length $1 + (t+1)|\mu| + \sum_{u=0}^{t}(t-u)|\lambda_u|$. In addition, we also show that whenever all diagrams for all objects that appear in the socle filtration of $\tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu}$ are conjugated, we obtain the socle filtration of $\tilde{V}_{\lambda_t^\top,\dots,\lambda_0^\top,\mu^\top,\nu^\top}$.

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CHAPTER 1

Introduction

This work studies universal monoidal categories obtained as subcategories of representations of certain infinite dimensional Lie algebras. The Lie algebras themselves are generalisations of the classical Lie algebras. The first generalisation is to the "limiting" Lie algebra obtained by taking the union of the classical Lie algebras under the obvious inclusions. The next generalisation is obtained by studying the so-called Mackey Lie algebra, i.e. the Lie algebra of endormorphisms of a diagonalisable paring of two vector spaces of dimension $\aleph_t$ for the same but arbitrary t.

Let $\mathfrak{g}$ be one such Lie algebra. Our main focus here are subcategories $\mathbb{T}_{\mathfrak{g}}$ of $\mathfrak{g}$-**mod**. The chosen $\aleph_t$-dimensional vector spaces have a $\mathfrak{g}$-module structure and are called the natural and the conatural representations of $\mathfrak{g}$. We require $\mathbb{T}_{\mathfrak{g}}$ to contain these representations and be closed with respect to tensor products, finite direct sums, submodules, and quotients. It turns out that these requirements are sufficient to make $\mathbb{T}_{\mathfrak{g}}$ universal among all $\mathbf{K}$-linear symmetric monoidal categories generated by two objects a and b where a has a finite filtration and there is pairing $a \otimes b \to e$ to the monoidal unit e. A review of results about the Lie algebras involved and the category $\mathbb{T}_{\mathfrak{g}}$ is available in [**2**], [**12**] and [**13**]. A proof of universality is given in [**2**]. We summarise some of these results along with the required mathematical prerequisites in sections 5 and 4 of Chapter 2.

In this general setup, we restrict to representations of of $\mathfrak{g}$ that can be studied as indecomposable injective objects in $\mathbb{T}_{\mathfrak{g}}$. Since an arbitrary object of this type need not be semi-simple, it is therefore worthwhile to study its "largest semi-simple part" called the socle. Successively computing the socles, we get the socle filtration of an indecomposable injective object. A explicit description of the simple objects that appear in various "layers" of the socle filtration is given in [**13**] and [**2**]. We summarise these facts in Chapter 3. This filtration is an important invariant and whenever it terminates, the object is said to have finite Loewy length. Previously it was known that the indecomposable injective objects in $\mathbb{T}_{\mathfrak{g}}$ have finite Lowey length. However the exact Loewy length and, in particular, the dependence of this length on the indices was not known. The goal of the present work was to tackle precisely this problem and determine a formula for the Loewy length in terms of the indices.

Results from [**13**] and [**2**] describe a parameterisation of the indecomposable injectives in terms of partitions of non-negative integers. The number of such partitions required depends on the dimension of $\mathfrak{g}$. The multiplicities of the simple objects are given in terms of Littlewood-Richardson coefficients that encode some combinatorial data related to partitions. Hence to determine the Loewy length, we need to determine the lowest level at which all simple objects appear with zero multiplicity.

We were able to use this combinatorial description of the multiplicities to first write a computer program that computed socle filtrations for many objects. We then studied the generated computations and were able to conjecture a formula for the Loewy length. Proofs of these conjectures appear as theorems 4.10, 4.11 and 4.16. In our proof, in order to show that a special class of Littlewood-Richardson coefficients is non-zero, we introduce the "Sorted Filling Algorithm" in Definition 4.1. The relationship between our combinatorial algorithm and Littlewood - Richardson coefficients is proven in Lemma 4.8. Furthermore, in Theorem 5.1 we show that the socle filtrations have a symmetry with respect to conjugation of partitions. In particular, this fact provides combinatorial evidence for the existence of a functor of auto-equivalence of the category $\mathbb{T}_{\mathfrak{g}}$. We state this as Conjecture 5.2. The details of our results are available in chapters 4 and 5. A description of the computer program and a selection of generated computations is given in appendices A and B respectively.

CHAPTER 2

Preliminaries

In this chapter we discuss the basic mathematical objects and list some of their properties that are relevant for our work. Note that in the rest of the document, we call the discussion of objects in $\mathfrak{gl}_\infty$**-mod**, $\mathfrak{sl}_\infty$**-mod**, $\mathfrak{sp}_\infty$**-mod** and $\mathfrak{so}_\infty$**-mod** the "locally-finite case" and that of objects in $\overline{\mathbb{T}}_{\aleph_t}$ the "$\aleph_t$-case".

1. Indecomposable injective objects and tensor categories

All definitions stated here closely follow the ones given in [**8**], [**7**], [**3**] and [**14**].

1.1. Injective objects. Let $\mathcal{C}$ be a category. A morphism $m : a \to b$ is a ***monomorphism*** in $\mathcal{C}$ whenever

$$m \circ f = m \circ f' \implies f = f'$$

for every f and f'. Similarly a morphism $h : a \to b$ is an ***epimorphism*** if

$$g \circ h = g' \circ h \implies g = g'$$

for every g and g'. An object p is ***projective*** if every morphism $h : p \to c$ factors through every epimorphism $g : b \to c$. An object q is ***injective*** if every morphism $h : c \to q$ factors through every monomorphism $g : c \to d$. Equivalently, the diagrams

$$\begin{tikzcd} & p \arrow[dl, dashed, "\exists h'"'] \arrow[d, "h"] \\ b \arrow[r, "g"] & c \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} c \arrow[r, "g"] \arrow[d, "h"'] & d \arrow[dl, dashed, "\exists h'"] \\ q & \end{tikzcd}$$

commute when p and q are, respectively, projective and injective.

1.2. Indecomposable objects. The ***product*** of two objects a, b in a category is an object $a \times b$ (or $a \prod b$) together with morphism $p : a \times b \to a$ and $q : a \times b \to b$ called ***projections*** such that morphisms $f : c \to a$ and $g : c \to b$ factor through a unique $h : c \to a \times b$. The ***coproduct***, dual to a product, is an object $a + b$ (or $a \coprod b$) together with morphisms $i : a \to a + b$ and $j : b \to a + b$ called ***injections*** such that morphisms $f : a \to c$ and $g : b \to c$ factor through a unique $h : a + b \to c$. For products and coproducts the diagrams

$$\begin{tikzcd} & c \arrow[dl, "f"'] \arrow[d, dashed, "\exists! h"] \arrow[dr, "g"] & \\ a & a \times b \arrow[l, "p"] \arrow[r, "q"'] & b \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a \arrow[r, "i"] \arrow[dr, "f"'] & a + b \arrow[d, dashed, "\exists! h"] & b \arrow[l, "j"'] \arrow[dl, "g"] \\ & c & \end{tikzcd}$$

commute respectively. An object t is ***terminal*** if for every object a there is exactly one morphism $a \to t$. An object s is ***initial*** if for for every object a there is exactly one morphism $s \to a$. A terminal object which is also initial is called a ***null object*** (or ***zero object***). An object x is called ***indecomposable*** if every isomorphism $x \cong x_1 + x_2$ implies that either x_1 or x_2 is a zero object.

1.3. Tensor categories. Let $\mathcal{B}$ be a category and let $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be a bifunctor. The category $\mathcal{B}$ is ***monoidal*** if there are natural isomorphisms α, λ, ϱ and an object e in $\mathcal{B}$ such that for all objects a, b, c and d in $\mathcal{B}$, the following conditions hold

(1) associativity of $\otimes$ is given by the natural isomorphism α and hence

$$\alpha = \alpha_{a,b,c} : a \otimes (b \otimes c) \cong (a \otimes b) \otimes c,$$

(2) the pentagonal diagram

$$\begin{tikzcd} & (a \otimes b) \otimes (c \otimes d) \arrow[dr, "\alpha"] & \\ a \otimes (b \otimes (c \otimes d)) \arrow[ur, "\alpha"] \arrow[d, "1\otimes\alpha"'] & & ((a \otimes b) \otimes c) \otimes d \\ a \otimes ((b \otimes c) \otimes d) \arrow[rr, "\alpha"'] & & (a \otimes (b \otimes c)) \otimes d \arrow[u, "\alpha\otimes 1"'] \end{tikzcd}$$

commutes,

(3) the object e is the left and the right unit of $\otimes$ with the respeective isomorphisms given by λ and ϱ and

$$\lambda_a : e \otimes a \cong a, \quad \varrho_a : a \otimes e \cong a,$$

(4) the triangular diagram

$$\begin{tikzcd} & a \otimes c & \\ a \otimes (e \otimes c) \arrow[ur, "1\otimes\lambda"] \arrow[rr, "\alpha"'] & & (a \otimes e) \otimes c \arrow[ul, "\varrho\otimes 1"'] \end{tikzcd}$$

commutes.

A ***tensor category*** (or a ***symmetric monoidal category***) is a monoidal category together with a natural isomorphism

$$\beta = \beta_{a,b} : a \otimes b \to b \otimes a$$

such that for all objects a, b and c

(1) the hexagonal diagram

$$\begin{tikzcd} & (a \otimes b) \otimes c \arrow[r, "\beta"] & c \otimes (a \otimes b) \arrow[dr, "\alpha"] & \\ a \otimes (b \otimes c) \arrow[ur, "\alpha"] \arrow[dr, "1\otimes\beta"'] & & & (c \otimes a) \otimes b \\ & a \otimes (c \otimes b) \arrow[r, "\alpha"'] & (a \otimes c) \otimes b \arrow[ur, "\beta\otimes 1"'] & \end{tikzcd}$$

commutes,

(2) the natural isomorphism β satisfies

$$\beta_{b,a} \circ \beta_{a,b} = 1_{a\otimes b}.$$

2. Littlewood-Richardson coefficients

This section introduces Young diagrams, skew diagrams and their tableaux, Littlewood-Richardson coefficients, and the Schur functor. The discussion on Young diagrams and skew tableaux is based on [**4**] and [**20**], the results on Littlewood-Richardson coefficients and related combinatorics summarise some results from [**5**], [**15**] and [**18**]. The description of the Schur functor and the Young symmetrizer is based on the discussion in [**5**].

2.1. Skew diagrams and tableaux.

A ***partition*** λ of a positive integer n, denoted $\lambda \vdash n$, is a tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integers such that

$$n = \lambda_1 + \cdots + \lambda_k \quad \text{and} \quad \lambda_1 \geq \cdots \geq \lambda_k.$$

The ***Young diagram*** associated to this partition, also denoted by λ, is a collection of boxes (or cells) indexed by the set of positive integers

$$\{(i, j) : 1 \leq i \leq k, 1 \leq j \leq \lambda_j\}.$$

A Young diagram is usually represented by drawing a box at the coordinates $(-i, j)$ for each index (i, j). The ***conjugate diagram*** to λ is the Young diagram $\lambda^\top$ obtained by transposing λ. The number of boxes in the k-th row of $\lambda^\top$ is equal to the number of boxes in the k-th column of λ. We also use the convention that $\lambda_l = 0$ for all $l > k$ and say that the empty set (with the diagram $\varnothing$) is a partition of 0. If μ is another partition whose diagram is contained in the diagram of λ, that is $\mu \subset \lambda$, then the ***skew diagram*** λ/μ consists of boxes of λ not present in μ. Since a Young diagram λ can be thought of as the skew diagram $\lambda/\varnothing$, we only focus on the skew diagrams in the rest of the discussion. Note that the cardinalities $|\lambda|$ and $|\lambda/\mu|$ indicate the number of boxes in λ and λ/μ respectively. In particular, $|\lambda/\mu| = |\lambda| - |\mu|$.

A ***filling*** of a skew diagram is a map $T : \lambda/\mu \to \mathbb{N}$ that assigns a non-negative integer to each cell in the diagram. The value of T on each cell is called the ***entry*** assigned to that cell. A filling T is called a ***tableau*** (or a ***semistandard tableau***) on λ/μ if all entries

(1) are weakly increasing across each row, i.e. $T(i, j) \leq T(i + 1, j)$, and
(2) strictly increasing down each column, i.e. $T(i, j) < T(i, j + 1)$.

We will denote the set of all semi-standard tableaux on the shape λ/μ by $\mathrm{SST}(\lambda/\mu)$. A ***standard tableau*** on λ is a semistandard tableau with entries in the set

$$\{1, \dots, |\lambda|\} = [|\lambda|] = [n].$$

The ***content*** (or ***weight***, or ***type***) of a tableau T is a tuple $\eta = (\eta_1, \dots, \eta_r)$ such that T has η_i entries equal to i.

Example 2.1. Let $\lambda = (5, 4, 3, 2)$ and $\mu = (3, 3, 1)$. The following shows the skew diagram λ/μ and a tableau $T(\lambda/\mu)$ of weight $(2, 2, 2, 1)$:

$$\lambda/\mu = \begin{array}{ccccc} & & & \boxed{} & \boxed{} \\ & & & \boxed{} & \\ & \boxed{} & \boxed{} & & \\ \boxed{} & \boxed{} & & & \end{array}, \qquad T(\lambda/\mu) = \begin{array}{ccccc} & & & \boxed{1} & \boxed{1} \\ & & & \boxed{2} & \\ & \boxed{2} & \boxed{3} & & \\ \boxed{3} & \boxed{4} & & & \end{array}.$$

2.2. Littlewood-Richardson coefficients.

A ***word*** is a string of positive integers. The ***reverse*** of a word $x_1 \dots x_r$ is the word $x_r \dots x_1$. A ***lattice word*** (or a ***lattice permutation***) is a word every prefix of which contains as many positive integers i as integers $i + 1$. The word w is a ***reverse lattice word*** if its reverse is a lattice word. The ***word*** (or ***row word***) of a skew tableau T, denoted by $w(T)$ or $w_{\mathrm{row}}(T)$, is defined as the string obtained by writing down the entries in all rows of T

from left to right starting at the bottom row. The skew tableau T is a ***Littlewood-Richardson skew tableau*** if $w_{\text{row}}(T)$ is a reverse lattice word. If ν is a Young diagram then the total number of Littlewood-Richardson skew tableaux of shape λ/μ with content ν is denoted by $\mathrm{N}^{\lambda}_{\mu,\nu}$ and is called the ***Littlewood-Richardson coefficient***.

EXAMPLE 2.2. Let T be the skew tableau in Example 2.1. Then $w_{\text{row}}(T) = 3423211$. One can check that the word $w_{\text{row}}(T)$ is a reverse lattice word and thus T is a Littlewood-Richardson tableau of shape λ/μ. Moreover this is the only Littlewood-Richardson tableau with content $\nu = (2, 2, 2, 1)$. Hence $\mathrm{N}^{\lambda}_{\mu,\nu} = 1$.

In the sequel we will be interested in checking when products of Littlewood-Richardson coefficients are non-zero. In general it is not easy to determine necessary and sufficient conditions on the diagrams such that the corresponding Littlewood-Richardson coefficients are non-zero. However, as shown by the following result from [**4**, p. 66], this is possible in cases when some relationship between the diagrams is known.

LEMMA 2.3. *Let μ and ν be Young diagrams, and let $\lambda_i = \nu_i + \mu_i$ for all i. Then $\mathrm{N}^{\lambda}_{\mu,\nu} = 1$.*

Moreover, several necessary conditions and symmetries between tableaux are available, for instance, in [**4**] and [**20**]. If $\mathrm{N}^{\lambda}_{\mu,\nu}$ is non-zero, then the number of boxes in λ/μ is equal to the total number of entries $|\nu|$. Hence one can conclude

LEMMA 2.4. *If $\mathrm{N}^{\lambda}_{\mu,\nu}$ is non-zero then $|\lambda| = |\mu| + |\nu|$.*

An equivalent definition states that the Littlewood Richardson coefficients are given by the relation

$$s_\mu s_\nu = \sum_{\lambda} \mathrm{N}^{\lambda}_{\mu,\nu} s_\lambda$$

between Schur symmetric polynomials s_λ, s_μ and s_ν. This allows us to also conclude

LEMMA 2.5. $\mathrm{N}^{\lambda}_{\mu,\nu} = \mathrm{N}^{\lambda}_{\nu,\mu}$.

Moreover, as proved in [**6**] we also have symmetry of Littlewood-Richardson coefficients with respect to partition conjugation.

LEMMA 2.6. $\mathrm{N}^{\lambda}_{\mu,\nu} = \mathrm{N}^{\lambda^\top}_{\mu^\top,\nu^\top}$.

2.3. Schur functor. Young diagrams play a crucial role in representation theory in general and in the representations of the symmetric group in particular. If $\mathfrak{S}_d$ is the symmetric group on d objects, then an irreducible representation of $\mathfrak{S}_d$ can be obtained inside the group algebra $\mathbf{C}[\mathfrak{S}_d]$. These irreducible representations are parametrized by partitions of d. Say λ is one such partition and say $\{e_\sigma\}_{\sigma\in\mathfrak{S}_d}$ is a basis of $\mathbf{C}[\mathfrak{S}_d]$. Since $\mathfrak{S}_d$ acts on $[d]$ it also acts on the set of standard fillings on λ. The ***Young symmetrizer*** is defined as the element

$$c_\lambda = \left(\sum_{\sigma\in R} e_\sigma\right)\left(\sum_{\sigma\in C} \mathrm{sgn}(\sigma) e_\sigma\right)$$

in the group algebra where R [resp. C] denote the elements of $\mathfrak{S}_d$ that preserve each row [resp column] of the chosen filling of λ. The irreducible representations of $\mathfrak{S}_d$ are given by

$$H_\lambda = \mathbf{C}[\mathfrak{S}_d]c_\lambda.$$

If V is any vector space then the symmetric group $\mathfrak{S}_d$ has a natural action on the d-th tensor power $V^{\otimes d}$ by permuting the factors. For any partition, we can use the Young symmetrizer to define the ***Schur functor***

$$\mathbb{S}_\lambda : \mathbf{Vect}_{\mathbf{C}} \to \mathbf{Vect}_{\mathbf{C}}$$

given by the correspondence

$$V \rightsquigarrow \mathbb{S}_\lambda V = \mathrm{im}(c_\lambda : V^{\otimes d} \to V^{\otimes d}).$$

We will often denote $\mathbb{S}_\lambda V$ by V_λ. In this setup the Littlewood-Richardson coefficients $\mathrm{N}^\lambda_{\mu,\nu}$ appear as coefficients in the decomposition of the composition $\mathbb{S}_\mu \mathbb{S}_\nu$ as a sum of applications of $\mathbb{S}_\lambda$. Proofs for these facts, including a systematic treatment of the representation theory of $\mathfrak{S}_d$, is available in [**5**] and [**15**].

3. Semi-simplicity and socle filtrations

Recall that a module is ***semi-simple*** when it is a direct sum of a family of simple submodules. Moreover the following conditions on a module M are equivalent

(1) M is a sum of a family of simple sub-modules,
(2) M is the direct sum of a family of simple submodules, and
(3) Every submodule N of M is a direct summand and there is another sub-module N' such that $M = N \oplus N'$.

A proof of the above is available, for instance, in [**9**, XVII.2]. Since an arbitrary module need not be semi-simple, it is worthwhile to study its "largest semi-simple submodule". If M is a module, then the sum of all of its simple sub-modules, denoted $\mathrm{soc}(M)$, is called the ***socle*** of M. Equivalently, the socle is the maximal semisimple submodule of M. Hence

$$\mathrm{soc}(\mathrm{soc}(M)) = \mathrm{soc}(M)$$

and whenever M is semisimple

$$\mathrm{soc}(M) = M.$$

The semisimplicity of the socle implies that we can write

$$\mathrm{soc}(M) = \sum_{i \in I} M_i = \bigoplus_{j \in J} M_j$$

where the M_i are the simple submodules of M and $J \subseteq I$. The ***socle filtration*** of a module M is defined inductively by

$$0 \subset \mathrm{soc}(M) = \mathrm{soc}^1(M) \subset \mathrm{soc}^2(M) = \pi_1^{-1}(M/\mathrm{soc}(M)) \subset \dots$$

where π_i are the canonical projections. For such a filtration we define the q-th ***socle layer*** by the quotient

$$\underline{\mathrm{soc}}^q(M) = \mathrm{soc}^q(M)/\mathrm{soc}^{q-1}(M).$$

The socle filtration of an object is ***exhaustive*** if M is the union of all of its socles. The socle filtration of an object M has finite length if for some q, the layer $\underline{\mathrm{soc}}^{q+1}(M)$ is zero. In this case, the smallest such q is called the ***Loewy length*** of M. If no such q exists, then the Loewy length is said to be infinite.

4. Classical locally finite Lie algebras

In this section we define the classical locally finite Lie algebras $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, $\mathfrak{so}_\infty$, and $\mathfrak{sp}_\infty$. We also some facts about their tensor representations and socles. The content in this section summarises some of the main results and definitions from [**13**].

We fix the base field $\mathbf{C}$ and two countable dimensional vector spaces V and V_*. Let $\langle\cdot,\cdot\rangle : V \otimes V_* \to \mathbf{C}$ be a non-degenerate pairing of the two vector spaces. A result of Mackey [**10**] implies that in this case one can always find countable dual bases $\{\xi_i : i \in \mathcal{I}\}$ and $\{\xi_i^* : i \in \mathcal{I}\}$ of V and V_* respectively such that the pairing is diagonal. That is,

$$\langle \xi_i, \xi_j^* \rangle = \delta_{i,j} \quad \text{for all} \quad i, j \in \mathcal{I}.$$

Hence the tensor product space $V \otimes V_*$ is spanned by the basis

$$\{E_{i,j} = \xi_i \otimes \xi_j^* : i, j \in \mathcal{I}\}.$$

4.1. The Lie algebras $\mathfrak{gl}_\infty$ and $\mathfrak{sl}_\infty$. The Lie algebra $\mathfrak{gl}_\infty$ is the tensor product vector space $V \otimes V_*$ together with the bracket

$$[u \otimes u^*, v \otimes v^*] = \langle v, u^* \rangle u \otimes v^* - \langle u, v^* \rangle v \otimes u^* \tag{2.1}$$

for $u, v \in V$ and $u^*, v^* \in V_*$. In the basis $\{E_{i,j}\}$ the bracket is given by the expected commutation relation

$$[E_{i,j}, E_{k,l}] = \delta_{j,k} E_{i,l} - \delta_{i,l} E_{k,j}.$$

The Lie subalgebra $\mathfrak{sl}_\infty$ of $\mathfrak{gl}_\infty$ is defined as the kernel of the map $\langle \cdot, \cdot \rangle$. A tensor representation of these Lie algebras is obtained on the vector space $V^{\otimes(p,q)} = V^{\otimes p} \otimes (V_*)^{\otimes q}$ with the action of $\mathfrak{gl}_\infty$ given by

$$\begin{aligned}(u \otimes u^*) \cdot (x \otimes x^*) &= (u \otimes u^*) \cdot (v_1 \otimes \cdots v_p \otimes v_1^* \otimes \cdots \otimes v_q^*) \\ &= \sum_{i=1}^{p} \langle v_i, u^* \rangle x_{[v_i \to u]} - \sum_{j=1}^{q} \langle u, v_j^* \rangle x^*_{[v_j^* \to u^*]}\end{aligned}$$

where the notation $x_{[v_i \to u]}$ means that we replace the i-th tensor factor v_i in x by u and similarly for $x^*_{[v_j^* \to u^*]}$.

The symmetric groups $\mathfrak{S}_p$ and $\mathfrak{S}_q$ act, respectively, on the tensor powers $V^{\otimes p}$ and $(V_*)^{\otimes q}$ by permuting factors and it is easy to see that this action commutes with the action of $\mathfrak{gl}_\infty$. Hence we can regard $V^{\otimes(p,q)}$ as a $(\mathfrak{gl}_\infty, \mathfrak{S}_p \times \mathfrak{S}_q)$-module. Moreover for every choice of indices i, j with $1 \leq i \leq p$ and $1 \leq j \leq q$, we can define the contraction $\Phi_{(i,j)} : V^{\otimes(p,q)} \to V^{\otimes(p-1,q-1)}$ of the factors v_i and v_j^* by

$$\Phi_{(i,j)}(v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^*) = \langle v_i, v_j^* \rangle v_1 \otimes \cdots \hat{v}_i \cdots \otimes v_p \otimes v_1^* \otimes \cdots \hat{v}_j^* \cdots \otimes v_q^*.$$

Considering all choices of (i, j) we define the submodule $V^{\{p,q\}}$ of $V^{\otimes(p,q)}$ by

$$V^{\{p,q\}} = \bigcap_{(i,j)} \ker(\Phi_{(i,j)} : V^{\otimes(p,q)} \to V^{\otimes(p-1,q-1)}).$$

We also set $V^{\{p,0\}} = V^{\otimes p}$ and $V^{\{0,q\}} = V^{\otimes q}$. For partitions λ and μ of p and q the $\mathfrak{gl}_\infty$-submodule $\Gamma_{\lambda;\mu}$ is defined by

$$\Gamma_{\lambda;\mu} = V^{\{p,q\}} \cap (\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V_*).$$

This module is actually irreducible as shown by the following theorem.

Theorem 2.7 (Theorem 2.1 in [**13**]). *For any p, q there is an isomorphism of $(\mathfrak{gl}_\infty, \mathfrak{S}_p \times \mathfrak{S}_q)$-modules*

$$V^{\{p,q\}} \cong \bigoplus_{\lambda \vdash p} \bigoplus_{\mu \vdash q} \Gamma_{\lambda;\mu} \otimes H_\lambda \otimes H_\mu$$

For all partitions λ and μ the $\mathfrak{gl}_\infty$-module $\Gamma_{\lambda;\mu}$ is irreducible. Furthermore, $\Gamma_{\lambda;\mu}$ is also irreducible when regarded by restriction as an $\mathfrak{sl}_\infty$-module.

Observe that in the special case $p = 0$ or $q = 0$, we get

$$V^{\otimes p} \cong \bigoplus_{\lambda \vdash p} \Gamma_{\lambda;\varnothing} \otimes H_\lambda, \qquad (V_*)^{\otimes q} \cong \bigoplus_{\mu \vdash q} \Gamma_{\varnothing;\mu} \otimes H_\mu.$$

4.2. The Lie algebra $\mathfrak{sp}_\infty$. Let V be a countable dimensional vector space as before and consider a non-degenerate anti-symmetric bilinear form $\Omega : V \otimes V \to \mathbf{C}$. Let $\mathfrak{gl}_\infty$ be the Lie algebra from the last subsection corresponding to $V_* = V$ and $\Omega = \langle \cdot, \cdot \rangle$. The Lie algebra $\mathfrak{sp}_\infty$ is the maximal subalgebra of $\mathfrak{g}l_\infty$ such that the form Ω is invariant. That is

$$\mathfrak{sp}_\infty = \{X \in \mathfrak{gl}_\infty : \Omega(Xu, v) + \Omega(u, Xc) = 0 \text{ for all } u, v \in V\}.$$

Picking a basis $\{\xi_i : i \in \mathcal{I}\}$ of V with indexing set $\mathcal{I} = \mathbb{Z} \setminus \{0\}$ such that

$$\Omega(\xi_i, \xi_j) = \operatorname{sign}(i)\delta_{i+j,0}.$$

In coordinates, the Lie algebra is spanned by the basis

$$\{\operatorname{sign}(j)E_{i,j} - \operatorname{sign}(i)E_{-j,-i} : i, j \in \mathcal{I}\}.$$

Since the dual basis $\{\xi_i^* : i \in \mathcal{I}\}$ is given by $\xi_i^* = \operatorname{sign}(i)$, we have $\mathfrak{sp}_\infty = S^2V$. The Lie bracket on $\mathfrak{sp}_\infty$ is induced by the Lie bracket (2.1) on $\mathfrak{gl}_\infty$. Moreover, as the $\mathfrak{gl}_\infty$ action on $V^{\otimes(p,q)}$ coincides with the $\mathfrak{gl}_\infty$ action on $V^{\otimes(p+q)}$ we can restrict our study to the tensor representation $V^{\otimes d}$.

Similar to the previous subsection, any pair of indices (i, j) satisfying $1 \leq i < j \leq d$ defines a contraction $\Phi_{\langle i,j\rangle} : V^{\otimes d} \to V^{\otimes(d-2)}$ by

$$\Phi_{\langle i,j\rangle}(v_1 \otimes \cdots \otimes v_d) = \Omega(v_i, v_j)v_1 \otimes \cdots \hat{v}_i \cdots \hat{v}_j \cdots \otimes v_d.$$

Now we set $V^{\langle 0\rangle} = \mathbf{C}$, $V^{\langle 1\rangle} = V$, and obtain

$$V^{\langle d\rangle} = \bigcap_{(i,j)} \ker(\Phi_{\langle i,j\rangle} : V^{\otimes d} \to V^{\otimes(d-2)})$$

for $d \geq 2$ where we look at all pairs of indices (i, j), Finally for any partition λ of d, we define the $\mathfrak{sp}_\infty$ module

$$\Gamma_{\langle\lambda\rangle} = V^{\langle d\rangle} \cap \mathbb{S}_\lambda V.$$

Analogous to the result earlier we have

THEOREM 2.8 (Theorem 3.1 in **[13]**). *For any non-negative integer d there is an isomorphism of $(\mathfrak{sp}_\infty, \mathfrak{S}_d)$-modules*

$$V^{\langle d\rangle} \cong \bigoplus_{\lambda \vdash d} \Gamma_{\langle\lambda\rangle} \otimes H_\lambda.$$

For every partition λ the $\mathfrak{sp}_\infty$-module $\Gamma_{\langle\lambda\rangle}$ is irreducible.

4.3. The Lie algebra $\mathfrak{so}_\infty$. Pick a countable dimensional vector space V and a non-degenerate symmetric bilinear form $Q : V \otimes V \to \mathbf{C}$. Realising $\mathfrak{gl}_\infty$ as the Lie algebra with $V_* = V$ and $Q = \langle \cdot, \cdot \rangle$, we obtain the Lie algebra

$$\mathfrak{so}_\infty = \{X \in \mathfrak{gl}_\infty : Q(Xu, v) + Q(u, Xv) = 0 \text{ for all } u, v \in V\}.$$

Picking basis $\{\xi_i : i \in \mathcal{I}\}$ indexed by $\mathcal{I} = \mathbb{Z} \setminus \{0\}$ such that $Q(\xi_i, \xi_j) = \delta_{i+j,0}$, $\mathfrak{so}_\infty$ is spanned by the basis

$$\{E_{i,j} - E_{-j,-i} : i, j \in \mathcal{I}\}.$$

Since the dual basis is $\{\xi_i^* = \xi_{-i} : i \in \mathcal{I}\}$, $\mathfrak{so}_\infty = \bigwedge^2 V$. As before, the bracket is obtained using (2.1).

Furthermore indices (i, j) satisfying $1 \leq i < j \leq d$ define a contraction $\Phi_{[i,j]} : V^{\otimes d} \to V^{\otimes(d-2)}$ by

$$\Phi_{[i,j]}(v_1 \otimes \cdots \otimes v_d) = Q(v_i, v_j)v_1 \otimes \cdots \hat{v}_i \cdots \hat{v}_j \cdots \otimes v_d.$$

We set $V^{[0]} = \mathbf{C}$, $V^{[1]} = V$ and

$$V^{[d]} = \bigcap_{(i,j)} \ker(\Phi_{[i,j]} : V^{\otimes d} \to V^{\otimes(d-2)})$$

for $d \geq 2$ with intersection over all pairs (i,j). Finally for any partition λ of d, we define the $\mathfrak{so}_\infty$ module

$$\Gamma_{[\lambda]} = V^{[d]} \cap \mathbb{S}_\lambda V.$$

The irreducibility of the $\Gamma_{[\lambda]}$ is proved in the following theorem.

THEOREM 2.9 (Theorem 4.1 in **[13]**). *For any non-negative integer d there is an isomorphism of $(\mathfrak{so}_\infty, \mathfrak{S}_d)$-modules*

$$V^{[d]} \cong \bigoplus_{\lambda \vdash d} \Gamma_{[\lambda]} \otimes H_\lambda.$$

Moreover for every partition λ, the module $\Gamma_{[\lambda]}$ is irreducible.

Alternatively, the Lie algebras described above can also be obtained by taking the union of the respective finite dimensional Lie algebras under the obvious inclusions.

5. The category $\overline{\mathbb{T}}_{\aleph_t}$

Recall that to define the classical locally finite Lie algebras, we first fixed countable dimensional vector spaces and then considered a suitable pairing between them. Following **[2]**, we describe here a generalisation of the same idea to higher dimensions. While **[2]** treats vector spaces whose dimensions are given by an arbitrary cardinal α, here we only restrict to the special case $\alpha = \aleph_t$. Moreover we fix the base field $\mathbf{K} = \mathbf{C}$.

Consider a fixed diagonalizable pairing $\mathbf{p} : V \otimes V_* \to \mathbf{C}$ of $\aleph_t$-dimensional vector spaces V and V_*. The pairing gives us the inclusion $V_* \subseteq V^*$ where $V^* = \hom(V, \mathbf{C})$ is the dual space to V. Picking bases for V_* and V that diagonalize $\mathbf{p}$, we think of V_* [resp. V] as containing $\aleph_t$-sized row vectors [resp. column vectors] with finitely many non-zero entries. Let $\beta \leq \aleph_{t+1}$ and denote the subspace of V^* consisting of row vectors with strictly fewer than β non-zero entries by V^*_β. By definition, $V^*_{\aleph_{t+1}} = V^*$ and $V^*_{\aleph_0} = V_*$ and we get the chain of inclusions

$$0 \subset V^*_{\aleph_0} \subset \cdots \subset V^*_{\aleph_t} \subset V^*. \tag{2.2}$$

The ***Mackey Lie algebra*** $\mathfrak{gl}^M = \mathfrak{gl}^M(V_*, V)$ associated to the pairing $\mathbf{p}$ is the Lie algebra of endomorphisms of $\mathbf{p}$. It is given by the formula

$$\mathfrak{gl}^M(V_*, V) = \{x \in \mathrm{End}(V_*) : x^*(V) \subseteq V\} = \{y \in \mathrm{End}(V) : y^*(V_*) \subseteq V_*\}$$

where x^* and y^* respectively denote the duals of x and y. If we think of V and V_* as α-sized column and row vectors then elements of $\mathfrak{gl}^M(V_*, V)$ are $\aleph_t \times \aleph_t$-matrices with finite rows and columns. Moreover, V and V_* are modules over $\mathfrak{gl}^M$ with the action of $g \in \mathfrak{gl}^M$ given by

$$\begin{aligned} g \cdot v &= gv && \text{for } v \in V, \\ g \cdot v_* &= -v_* g && \text{for } v_* \in V_*. \end{aligned}$$

Note that this action keeps the filtration in (2.2) invariant. Let $\mathfrak{gl}^M$-**mod** denote the category of modules over $\mathfrak{gl}^M$. The category $\mathbb{T}_{\aleph_t}$ is obtained as the smallest full monoidal subcategory of $\mathfrak{gl}^M$-**mod** which contains V and V_* and is closed under taking subquotients. The category $\overline{\mathbb{T}}_{\aleph_t}$ is the full subcategory of $\mathfrak{gl}^M$-**mod** whose objects are arbitrary direct sums of objects in $\mathbb{T}_{\aleph_1}$. In fact the category just constructed is universal among all $\mathbf{C}$-linear symmetric monoidal categories generated by two objects a and b such that the object a has a possible transfinite filtration, and there is a pairing $a \otimes b \to e$ to the monoidal unit e. A proof of this fact is available in **[2]**.

To describe the simple objects in $\overline{\mathbb{T}}_{\aleph_t}$, we first introduce $V_{\mu,\nu}$. Let μ and ν be Young diagrams then $(V_*)_\mu \subseteq (V_*)^{\otimes|\mu|}$ and $V_\nu \subseteq V^{|\nu|}$. Hence $(V_*)_\mu \otimes V_\nu \subseteq$

$(V_*)^{\otimes|\mu|} \otimes V^{\otimes|\nu|}$. Let us use the pairing $\mathbf{p}$ to contract one factor of V with another factor of V_* to obtain the composition

$$(V_*)_\mu \otimes V_\nu \subseteq (V_*)^{\otimes|\mu|} \otimes V^{|\nu|} \to (V_*)^{\otimes(|\mu|-1)} \otimes V^{\otimes(|\nu|-1)}.$$

We denote the space annihilated by all possible $|\mu| \cdot |\nu|$ applications of $\mathbf{p}$ in the composition above by $V_{\mu,\nu}$. The following proposition classifies the simple objects of $\mathbb{T}_{\aleph_t}$.

THEOREM 2.10 (Proposition 4.2 in [**2**]). *Given Young diagrams $\lambda_t, \ldots, \lambda_0, \mu, \nu$ the object*

$$V_{\lambda_t,\ldots,\lambda_0,\mu,\nu} = \bigotimes_{s=0}^{t} (V^*_{\aleph_{s+1}}/V^*_{\aleph_s})_{\lambda_s} \otimes V_{\mu,\nu}$$

is simple over $\mathfrak{gl}^M$, and objects obtained for distinct choices of Young diagrams are mutually non-isomorphic.

Studying the injective objects of this category is crucial to understanding its structure. Since the injective objects in $\overline{\mathbb{T}}_{\aleph_t}$ appear as arbitrary direct sums of indecomposable injective objects, it suffices to study the indecomposable injectives. These objects are classified by the following result:

THEOREM 2.11 (Corollary 4.25(b) in [**2**]). *The indecomposable injective objects in the category $\overline{\mathbb{T}}_{\aleph_t}$ are (up to isomorphism)*

$$\tilde{V}_{\lambda_t,\ldots,\lambda_0,\mu,\nu} = \bigotimes_{s=0}^{t} \left(V^*/V^*_{\aleph_s}\right)_{\lambda_s} \otimes (V^*)_\mu \otimes V_\nu \tag{2.3}$$

for arbitrary Young diagrams $\lambda_t, \ldots, \lambda_0, \mu, \nu$.

CHAPTER 3

Socle Filtrations

In both the locally finite case and the $\aleph_t$-case the indecomposable injective objects admit well-defined socle filtrations. The goal of this chapter is to summarise results about their socle-layers, review some existing results on the Loewy lengths and state the main question we investigate.

1. Socle filtrations: Locally finite case

The socle layers that appear in the socle filtrations of the indecomposable $\mathfrak{gl}_\infty$-, $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$- and $\mathfrak{so}_\infty$-modules $\Gamma_{\lambda;\mu}$, $\Gamma_{\langle\lambda\rangle}$ and $\Gamma_{[\lambda]}$ are given, respectively, by theorems 2.3, 3.3 and 4.3 in [**13**]. Let r be a non-negative integer and let λ and μ be Young diagrams. Then, for the $\mathfrak{gl}_\infty$-module $\Gamma_{\lambda;\varnothing} \otimes \Gamma_{\varnothing;\mu}$,

$$\underline{\mathrm{soc}}^{r+1}(\Gamma_{\lambda;\varnothing} \otimes \Gamma_{\varnothing;\mu}) \cong \bigoplus_{\lambda',\mu'} \left(\sum_{\gamma \vdash r} \mathrm{N}^{\lambda}_{\lambda',\gamma} \mathrm{N}^{\mu}_{\mu',\gamma} \right) \Gamma_{\lambda';\mu'}. \tag{3.1}$$

The equation above also holds for $\Gamma_{\lambda;\varnothing} \otimes \Gamma_{\varnothing;\mu}$ regarded as an $\mathfrak{sl}_\infty$-module. Say $d = |\lambda|$ and $r = 1, \ldots, \left[\frac{d}{2}\right]$, then the $\mathfrak{sp}_\infty$-module $\Gamma_{\lambda;\varnothing}$ is indecomposable and

$$\underline{\mathrm{soc}}^{r+1}(\Gamma_{\lambda;\varnothing}) \cong \bigoplus_{\mu} \left(\sum_{\gamma \vdash r} \mathrm{N}^{\lambda}_{\mu,(2\gamma)^\top} \right) \Gamma_{\langle\mu\rangle}. \tag{3.2}$$

Above $2\gamma = 2(\gamma_1, \ldots, \gamma_k) = (2\gamma_1, \ldots, 2\gamma_k)$. Similarly, $\mathfrak{so}_\infty$-module $\Gamma_{\lambda;\varnothing}$ is indecomposable and

$$\underline{\mathrm{soc}}^{r+1}(\Gamma_{\lambda;\varnothing}) \cong \bigoplus_{\mu} \left(\sum_{\gamma \vdash r} \mathrm{N}^{\lambda}_{\mu,2\gamma} \right) \Gamma_{[\mu]}. \tag{3.3}$$

2. Socle filtrations: $\aleph_t$-case

A series of results in [**2**] describe the socle filtration of indecomposable injective objects in $\overline{\mathbb{T}}_{\aleph_t}$. From [**2**, Corollary 4.25(b)], the indecomposable injective $\tilde{V}_{\lambda_t,\ldots,\lambda_0,\mu,\nu}$ has socle $V_{\lambda_t,\ldots,\lambda_0,\mu,\nu}$ for arbitrary Young diagrams $\lambda_t, \ldots, \lambda_0, \mu, \nu$. We describe the socle filtration in two stages. First we use the decomposition (2.3) of an indecomposable injective as a tensor product to define the socle layers for the individual tensorands (see lemmas 4.28 and 4.29 in [**2**]) and then describe a method (see Proposition 4.30 in [**2**]) to combine the layers of the tensorands.

Let $u \leq t$ and q be given. Say λ is an arbitrary Young diagram. If $\eta_{t-u}, \ldots, \eta_0$ are Young diagrams, then the multiplicity of the simple object $V_{\eta_{t-u},\ldots,\eta_0,\varnothing,\ldots,\varnothing}$ in the layer $\underline{\mathrm{soc}}^q((V^*/V^*_{\aleph_u})_\lambda)$ is given by

$$[\underline{\mathrm{soc}}^q((V^*/V^*_{\aleph_u})_\lambda) : V_{\eta_{t-u},\ldots,\eta_0,\varnothing,\ldots\varnothing}] = \sum \mathrm{N}^{\alpha_1}_{\eta_0,\eta_1} \mathrm{N}^{\alpha_2}_{\alpha_1,\eta_2} \cdots \mathrm{N}^{\lambda}_{\alpha_{t-u-1},\eta_{t-u}} \tag{3.4}$$

whenever the indices $\eta_{t-u}, \ldots, \eta_0$ satisfy

$$\sum_{x=0}^{t-u} |\eta_x| = |\lambda| \tag{3.5}$$

and

$$q = 1 + \sum_{x=0}^{t-u} x|\eta_x|. \tag{3.6}$$

Above in (3.4) the summation is over all repeated indices. The multiplicity is zero in all other cases.

Let Young diagrams μ and ν be given. If $\eta_t, \ldots, \eta_0, \xi, \zeta$ are Young diagrams that satisfy

$$q = 1 + |\nu| - |\zeta| + \sum_{x=0}^{t} (x+1)|\eta_x|, \tag{3.7}$$

then the multiplicity of the simple object $V_{\eta_t,\ldots,\eta_0,\xi,\zeta}$ in the q-th layer of $(V^*)_\mu \otimes V_\nu$ is given by

$$[\underline{\mathrm{soc}}^q((V^*)_\mu \otimes V_\nu) : V_{\eta_t,\ldots,\eta_0,\xi,\zeta}] = \sum \mathrm{N}^{\mu}_{\pi_t,\eta_t} \mathrm{N}^{\pi_t}_{\pi_{t-1},\eta_{t-1}} \ldots \mathrm{N}^{\pi_1}_{\pi_0,\eta_0} \mathrm{N}^{\pi_0}_{\xi,\delta} \mathrm{N}^{\nu}_{\zeta,\delta} \tag{3.8}$$

with summation over repeated indices. In all other cases, the multiplicity is zero.

We can now describe the socle layers of an arbitrary indecomposable injective object. Let $\tilde{V}_{\lambda_t,\ldots,\lambda_0,\mu,\nu}$ be an arbitrary indecomposable injective. According to Propostion 4.30 in **[2]**, we have an isomorphism

$$\underline{\mathrm{soc}}^q(\tilde{V}_{\lambda_t,\ldots,\lambda_0,\mu,\nu}) \cong \sum_{u_x,y} \bigotimes_{x=0}^{t} \underline{\mathrm{soc}}^{u_x}((V^*/V^*_{\aleph_x})_{\lambda_x}) \otimes \underline{\mathrm{soc}}^{y}((V^*)_\mu \otimes V_\nu) \tag{3.9}$$

with summation over all indices u_x and y satisfying

$$\sum_{x=0}^{t} (u_x - 1) + (y-1) = q - 1. \tag{3.10}$$

3. Problem statement

Note that even thought the socle filtration of an object might be exhaustive, the object need not have a finite Loewy length. That is, the object need not have a finite Loewy length. However, for the indecomposable injectives that we study it was known that the socle filtrations are exhaustive and have finite Loewy length. In the locally finite case it was previously known that the $\mathfrak{gl}_\infty$-module $V^{\otimes(p,q)}$ has Loewy length $\min(p,q)+1$ (Theorem 2.2 in **[13]**) and that the $\mathfrak{sp}_\infty$- and $\mathfrak{so}_\infty$-module $V^{\otimes d}$ has Loewy length $[\frac{d}{2}]+1$ (theorems 3.2 and 4.2 in **[13]**). In the $\aleph_t$-case it was known that the socle filtration of the indecomposable injectives is exhaustive and finite, however a general formula for the Loewy length was not available. A formula for the special case $t = 0$ was proved in **[17]**. Our goal is to generalise this result and determine a formula based on the indices that parameterise the indecomposable injective objects.

CHAPTER 4

Loewy Lengths of Indecomposable Injectives

In this chapter we state our main results for Lowey lengths for two classes of indecomposable injective modules. At first in Section 1 we introduce our results on intersections of Young diagrams and properties of related Littlewood-Richardson coefficients. Then we use these results in Section 2 to determine Lowey lengths of indecomposable injective modules over $\mathfrak{gl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$. Finally in Section 3 we find the Loewy lengths for indecomposable injective objects in the category $\overline{\mathbb{T}}_{\aleph_t}$.

1. Sorted filling of skew diagrams

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_l)$ be Young diagrams. The **intersection diagram** $\lambda \cap \mu$ is the Young diagram that has $\min(k, l)$ rows with

$$(\lambda \cap \mu)_i = \min(\lambda_i, \mu_i)$$

boxes in each row. Note that $\lambda \cap \mu$ is just the diagram obtained by the set-theoretic intersection of the boxes of λ with the boxes of μ when they are aligned at the top-left corner. Let λ/μ be a skew diagram with $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_k)$ (where some μ_i are possibly zero), then the **sort of the skew diagram** λ/μ is the Young diagram obtained by sorting the tuple $(\lambda_1 - \mu_1, \dots, \lambda_k - \mu_k)$ in descending order. We denote this diagram by $\mathrm{sort}(\lambda/\mu)$. Sorting the boxes of λ/μ can be used to define a bijection $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$ between the rows of λ/μ and rows of $\mathrm{sort}(\lambda/\mu)$ such that

$$\mathrm{sort}(\lambda/\mu)_i = \lambda_{\sigma(i)} - \mu_{\sigma(i)} = (\lambda/\mu)_{\sigma(i)}.$$

Moreover this bijection is unique when we require

$$\mathrm{sort}(\lambda/\mu)_i = \mathrm{sort}(\lambda/\mu)_{i+1} \text{ if and only if } \sigma(i) > \sigma(i+1).$$

When there are no ambiguities, we refer to rows of (λ/μ) by $\sigma(j)$ and those of $\mathrm{sort}(\lambda/\mu)$ by j. We now introduce a tableau $\mathrm{SF}(\lambda/\mu)$ on λ/μ with weight $\mathrm{sort}(\lambda/\mu)$.

Definition 4.1 (Sorted Filling Algorithm). The filling $\mathrm{SF}(\lambda/\mu)$ is defined by iterating on the rows of $\mathrm{sort}(\lambda/\mu)$. Fill all boxes of $\sigma(1)$ with 1s and set $j = 1$. If $j \leq k$ repeat the following steps.

- SF.I Increment j by one.
- SF.II Set $n_j = \mathrm{sort}(\lambda/\mu)_j$.
- SF.III Note that rows $\sigma(1), \dots, \sigma(j-1)$ have been filled already.
  - SF.1 If there are no pre-filled rows below $\sigma(j)$ then fill all boxes of $\sigma(j)$ with j
  - SF.2 Else let $\sigma(j_2) < \cdots < \sigma(j_k)$ be the pre-filled rows below $\sigma(j_1) = \sigma(j)$.
    - SF.a For all $l < k$ refill the rightmost n_j boxes of $\sigma(j_l)$ with the values from the last n_j boxes of $\sigma(j_{l+1})$ while maintaining the left-to-right order of the entries in the boxes.
    - SF.b Fill the rightmost n_j boxes of $\sigma(j_k)$ with j.

The algorithm clearly terminates when $j > n$. Since σ was chosen to be unique and since the filling of boxes in λ/μ only depends on σ, the resulting tableau $\mathrm{SF}(\lambda/\mu)$

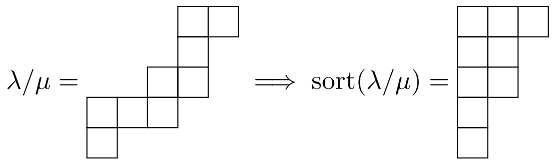
is well defined. Observe that at each step of the algorithm we obtain a filling on the diagram

$$(\lambda/\mu)^{(j)} = (\lambda_{\sigma(i_1)}, \dots, \lambda_{\sigma(i_j)})/(\mu_{\sigma(i_1)}, \dots, \mu_{\sigma(i_j)}) \tag{4.1}$$

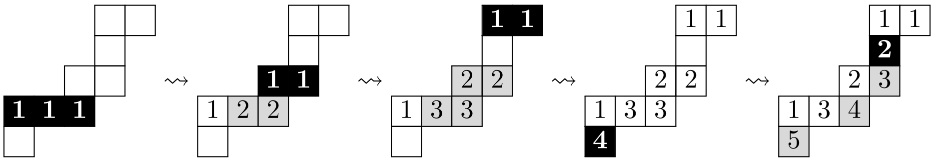
corresponding to the rows $\sigma(1), \dots, \sigma(j)$. Here the i_l are picked such that the relative order of rows in $(\lambda/\mu)^{(j)}$ is consistent with those in λ/μ. At each step, the partial filling obtained will be denoted by $\mathrm{SF}\big((\lambda/\mu)^{(j)}\big)$. Clearly $\mathrm{SF}\big((\lambda/\mu)^{(k)}\big) = \mathrm{SF}(\lambda/\mu)$.

The initialisation of the algorithm and step SF.1 ensure that that $\sigma(j)$ gets filled with js whenever $\sigma(j)$ is the bottom-most unfilled row at step j. When this is not the case, step SF.a just moves the rightmost n_j boxes from the pre-filled rows to fill the boxes in $\sigma(j)$. The example below illustrates this process.

EXAMPLE 4.2. Consider the Young diagrams $\lambda = (5, 4, 4, 3, 1)$ and $\mu = (3, 3, 2)$. Then



The sequence of diagrams below show $\mathrm{SF}\big((\lambda/\mu)^{(j)}\big)$ for various j. At step j the $\sigma(j)$ row has been shaded in black and the n_j boxes that move as a result of step SF.a are shaded in grey.



We now prove several properties of the filling $\mathrm{SF}(\lambda/\mu)$.

LEMMA 4.3. *The filling* $\mathrm{SF}(\lambda/\mu)$ *has weight* $\mathrm{sort}(\lambda/\mu)$.

PROOF. None of the steps decrease the total number of boxes filled with j. Since n_j boxes are filled with j at the start of the algorithm and at steps SF.1 and SF.b, the total number of js is exactly $n_j = \mathrm{sort}(\lambda/\mu)_j$. ☐

LEMMA 4.4. *For all steps j, the largest entry in* $\mathrm{SF}\big((\lambda/\mu)^{(j)}\big)$ *only appears in the bottom-most row of* $(\lambda/\mu)^{(j)}$.

PROOF. At the end of step j, the largest entry in $\mathrm{SF}\big((\lambda/\mu)^{(j)}\big)$ is j. If row $\sigma(j)$ was filled using step SF.1, then it is the bottom-most row and contains j-s at the end of step j. Otherwise, the j-s get added to row $\sigma(j_r)$ which was the bottom-most row. ☐

LEMMA 4.5. *Let x_i^s be the s-th entry from the right in row i of* $\mathrm{SF}(\lambda/\mu)$. *If both x_i^s and x_{i+1}^s are valid entries then $x_i^s < x_{i+1}^s$.*

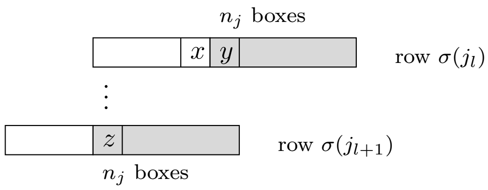
PROOF. Suffices to show that the statement holds for all $\mathrm{SF}\big((\lambda/\mu)^{(j)}\big)$. We proceed by induction. The statement vacuously holds for $j = 1$. Inductively assume that the statement holds for all $i < j$. If $s > n_j$ or if SF.1 was used to fill the row $\sigma(j)$ the claim follows from the inductive step. Now let $\tilde{x}^s_{\sigma(j_2)}, \dots, \tilde{x}^s_{\sigma(j_r)}$ be the entries before step SF.2 is executed. After step SF.2, as a result of the move, the entries are $x^s_{\sigma(j_1)} = \tilde{x}^s_{\sigma(j_2)}, \dots, x^s_{\sigma(j_r)} = j$ and the claim holds. ☐

LEMMA 4.6. $\mathrm{SF}(\lambda/\mu) \in \mathrm{SST}(\lambda/\mu)$.

Proof. It suffices to check that for all $j \leq n$, $\mathrm{SF}\big((\lambda/\mu)^{(j)}\big) \in \mathrm{SST}(\lambda/\mu)$. We proceed by induction. When $j = 1$, $\mathrm{SF}\big((\lambda/\mu)^{(1)}\big)$ has one row filled with 1-s hence the filling is semi-standard. Now suppose that for all $i < j$, $\mathrm{SF}\big((\lambda/\mu)^{(i)}\big) \in \mathrm{SST}(\lambda/\mu)$ also suppose that we are at the end of step j. Consider two cases:

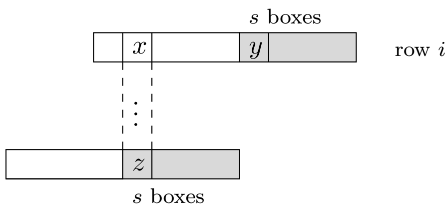
Case I: *Step SF.1 was used to fill row $\sigma(j)$.* The only entries in $\sigma(j)$ are j-s hence the entries weakly decrease along the row. From the inductive hypothesis, the entries weakly decrease along the rest of the rows. Now Lemma 1, the entries strictly increase down each column.

Case II: *Step SF.2 was used to fill row $\sigma(j)$.* From the inductive hypothesis, the rows above $\sigma(j)$ have weakly increasing entries. Moreover the n_j entries in row $\sigma(j)$ come from the last n_j boxes of $\sigma(j_2)$ and hence have weakly increasing entries. For rows below $\sigma(j)$, the situation is illustrated in the following diagram:



In this case $x \leq y$ from the inductive hypothesis and $y < z$ from Lemma 4.5. Hence the entries are weakly increasing in row $\sigma(j_l)$.

We now check the conditions on the columns. Since the right-most boxes of a row are also the right-most boxes of the corresponding row in λ, as a result of SF.2 entries in a box are not allowed to move to the left of their position before SF.2. It suffices to check columns with entries that are affected by the move. Suppose x is in row i and z is a box below x in the column containing x. Let z be the s-th box from the right. Let y be the s-th box from the right in row i. The situation is illustrated below:



Above, $x \leq y$ since the row-entries are weakly increasing and $y < z$ from Lemma 4.5. $\square$

We now show that the word of $\mathrm{SF}(\lambda/\mu)$ is a reverse lattice word. It is easy to see that any initial segment in the reverse word corresponds to a sequence of boxes starting at the top-rightmost box in $\mathrm{SF}(\lambda/\mu)$. Alternatively, any initial segment corresponds to the word of the tableau obtained by removing some boxes starting at the bottom-leftmost box. Hence it suffices to show that removing any number of boxes (in the order just described) results in a tableau with weakly increasing weight. We now use this strategy to show that $\mathrm{SF}(\lambda/\mu)$ is a reverse lattice word.

Lemma 4.7. *The word $w = w_{\mathrm{row}}(\mathrm{SF}(\lambda/\mu))$ is a reverse lattice word.*

Proof. Let $w = w_1 \dots w_N$. Proceeding by induction, when no boxes are removed, the tableau has weight $\mathrm{sort}(\lambda/\mu)$ by Lemma 4.3. Inductively assume that k boxes have been removed and the resulting weight is weakly decreasing. For the $(k+1)$-th removed box w_{N-k} and let p be the number of boxes to the right of w_{N-k}. We now consider two cases.

Case I: $p = 0$**.** From Lemma 4.5, the rightmost boxes of the rows above will have entries strictly smaller than w_{N-k}. Moreover the entries in each of the rows are weakly decreasing by Lemma 4.6. Therefore w_{N-k} is the largest entry in $w_1 \dots w_{N-k}$ and removing it keeps the weight weakly decreasing.

Case II: $p \neq 0$**.** Since the word $w_1 \dots w_{N-k}$ has weakly decreasing weight it suffices to check that the number of occurrences of w_{N-k} is strictly greater than the number of occurrences of $w_{N-k}+1$. Since there are p boxes to the right of w_{N-k}, the number of occurrences of w_{N-k} in $w_1 \dots w_{N-k}$ is at least $p+1$ (as $p+1$ entries of w_{N-k} must have been added by step SF.b). Moreover the number of occurrences of $w_{N-k}+1$ is at most p as only p entries could have moved to the last p boxes in this row by SF.a and therefore to the rest of the rows above. □

LEMMA 4.8 (Sorted Filling Lemma). *If λ, μ are diagrams with then there exists a Littlewood-Richardson tableau of shape λ/μ with content* $\operatorname{sort}(\lambda/\mu)$*. In particular, if $\nu = \operatorname{sort}(\lambda/\mu)$, then $\mathrm{N}^{\lambda}_{\mu,\nu} \neq 0$*

PROOF. The required Littlewood-Richardson tableau is given by $\mathrm{SF}(\lambda/\mu)$. The filling $\mathrm{SF}(\lambda/\mu)$ has weight $\operatorname{sort}(\lambda/\mu)$ by Lemma 4.3 and is a Littlewood-Richardson tableau as by lemmas 4.6 and 4.7. □

2. Loewy lengths of indecomposable injectives: Locally finite case

LEMMA 4.9. *If $\mathrm{N}^{\lambda}_{\mu,\nu}$ is non-zero then $\nu \subseteq \lambda$.*

PROOF. The symmetry $\mathrm{N}^{\lambda}_{\mu,\nu} = \mathrm{N}^{\lambda}_{\nu,\mu}$ of the Littlewood-Richardson coefficients implies that whenever Littlewood-Richardson tableaux of type ν on shape λ/μ exist, so do Littlewood-Richardson tableaux of type μ on shape λ/ν. In particular the skew diagram λ/ν is well defined and $\nu \subseteq \lambda$. □

THEOREM 4.10. *The Loewy length of the indecomposable $\mathfrak{gl}_\infty$-module $\Gamma_{\lambda;\varnothing} \otimes \Gamma_{\varnothing;\mu}$ is $|\lambda \cap \mu| + 1$.*

PROOF. Let $r = |\lambda \cap \mu|$. We first check that the $(r+1)$-th layer is non-zero. If $\alpha = \lambda \cap \mu$ then clearly $|\alpha| = r$. As $\alpha \subseteq \lambda$ and $\alpha \subseteq \mu$, set $\lambda' = \operatorname{sort}(\lambda/\alpha)$ and $\mu' = \operatorname{sort}(\mu/\alpha)$. From the Sorted Filling Lemma, the coefficients $\mathrm{N}^{\lambda}_{\alpha,\lambda'}$ and $\mathrm{N}^{\mu}_{\alpha,\mu'}$ are non-zero. Using the formula (3.1) the multiplicity of the object $\Gamma_{\lambda',\mu'}$ in the layer, we obtain

$$1 \leq \mathrm{N}^{\lambda}_{\alpha,\lambda'} \mathrm{N}^{\mu}_{\alpha,\mu'} \leq \sum_{|\gamma|=r} \mathrm{N}^{\lambda}_{\lambda',\gamma} \mathrm{N}^{\mu}_{\mu',\gamma} = [\underline{\mathrm{soc}}^{r+1}(\Gamma_{\lambda;\varnothing} \otimes \Gamma_{\varnothing;\mu}) : \Gamma_{\lambda';\mu'}].$$

This implies that the $(r+1)$-th layer is non-empty.

Now suppose the layer $r+2$ is non-empty. Then there exist Young diagrams $\tilde{\lambda}$, $\tilde{\mu}$ and γ with $|\gamma| = r+1$ such that the product $\mathrm{N}^{\lambda}_{\tilde{\lambda},\gamma} \mathrm{N}^{\mu}_{\tilde{\mu},\gamma}$ is non-zero. From Lemma 4.9, $\gamma \subseteq \lambda$ and $\gamma \subseteq \mu$. Hence for all indices i

$$\gamma_i \leq \min(\lambda_i, \mu_i) = (\lambda \cap \mu)_i \implies \gamma \subseteq (\lambda \cap \mu).$$

However the equations $|\gamma| = r+1$ and $|\lambda \cap \mu| = q$ lead to a contradiction. Therefore all layers after $r+1$ are empty and, as claimed, the Loewy length is $|\lambda \cap \mu| + 1$. □

Since the socle layers for indecomposable $\mathfrak{gl}_\infty$- and $\mathfrak{sl}_\infty$-modules coincide, we get the following corollary.

COROLLARY. *The Loewy length of the indecomposable $\mathfrak{sl}_\infty$-module $\Gamma_{\lambda;\varnothing} \otimes \Gamma_{\varnothing;\mu}$ is $|\lambda \cap \mu| + 1$.*

To treat the case of indecomposable $\mathfrak{sl}_\infty$- and $\mathfrak{so}_\infty$-modules we first introduce some terminology. A tuple $\gamma' = (\gamma'_1, \dots, \gamma'_k)$ of non-negative integers will be called even if all γ'_i are even. Hence $\gamma' = 2\gamma$ for some tuple γ. If λ is a Young diagram then consider the tuple γ with entries γ_i satisfying

$$\lambda_i = 2\gamma_i + \delta_i, \qquad 0 \leq \delta_i < 2.$$

Since $\lambda_i \geq \lambda_{i+1}$, $\gamma_i \geq \gamma_{i+1}$ and γ is a valid Young diagram. Since γ is unique by construction, we will call 2γ the ***maximal even sub-diagram*** of λ. Term "maximal" is justified as for any other even subdiagram $2\varepsilon \subseteq \lambda$, $\varepsilon \subseteq \gamma$.

THEOREM 4.11. *Let λ be a Young diagram and 2γ be the maximal even subdiagram of $\lambda^\top$. The Loewy length of the indecomposable $\mathfrak{sp}_\infty$-module $\Gamma_{\lambda;\varnothing}$ is $|\gamma| + 1$.*

PROOF. Pick γ as indicated and set $r = |\gamma|$. We first check that the layer $\underline{\mathrm{soc}}^{r+1}(\Gamma_{\lambda;\varnothing})$ is non-empty. Since $(2\gamma)^\top \subseteq \lambda$, let $\mu = \mathrm{sort}\big(\lambda/(2\gamma)^\top\big)$. As

$$r = |\gamma| \leq |2\gamma| \leq |\lambda|,$$

we use (3.2) to obtain

$$1 = \mathrm{N}^\lambda_{\mu,(2\gamma)^\top} \leq \sum_{|\eta|=r} \mathrm{N}^\lambda_{\mu,(2\eta)^\top} = [\underline{\mathrm{soc}}^{r+1}(\Gamma_{\lambda;\varnothing}) : \Gamma_{\langle\mu\rangle}].$$

Above the first equality is a consequence of the Sorted Filling Lemma and the inequality follows by picking the index $\eta = \gamma$.

Now suppose that the layer $\underline{\mathrm{soc}}^{r+2}(\Gamma_{\lambda;\varnothing})$ is non-zero. There are diagrams ν and δ satisfying $|\delta| = r + 1$ such that the Littlewood-Richardson coefficient $\mathrm{N}^\lambda_{\nu,(2\delta)^\top}$ is non-zero. From Lemma 4.9, the the skew diagram $\lambda/(2\delta)^\top$ is well defined and $2\delta \subseteq \lambda^\top$ is an even sub-diagram. As 2γ was the maximal even diagram,

$$r + 1 = |\delta| \leq |\gamma| = r$$

which is not possible.

Hence the $\mathfrak{sp}_\infty$-module $\Gamma_{\lambda;\varnothing}$ has Loewy length $|\gamma| + 1$. □

The proof above can be used, *mutatis mutandis*, to show the analogous statement for indecomposable $\mathfrak{so}_\infty$-modules.

THEOREM 4.12. *Let λ be a Young diagram and 2γ be the maximal even subdiagram of λ. The Loewy length of the indecomposable $\mathfrak{so}_\infty$-module $\Gamma_{\lambda;\varnothing}$ is $|\gamma| + 1$.*

3. Loewy lengths of indecomposable injectives: $\aleph_t$-case

We will first calculate Loewy lengths for the indecomposable injective objects $(V^*)_\mu \otimes V_\nu$ and $(V^*/V^*_{\aleph_u})_\lambda$. We then use formula (3.9) to obtain the Loewy length of $\tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu}$.

THEOREM 4.13. *The indecomposable injective object $(V^*)_\mu \otimes V_\nu$ has Loewy length $(t+1)|\mu| + 1$.*

PROOF. We first show that the layer $\underline{\mathrm{soc}}^q((V^*)_\mu \otimes V_\nu)$ for $q = (t+1)|\mu| + 1$ is non-zero. Consider the simple object $V_{\eta_t,\dots,\eta_0,\xi,\zeta}$ where $\eta_t = \mu$, $\zeta = \nu$, and $\eta_{t-1} = \cdots = \eta_0 = \xi = \varnothing$. Then the indices satisfy the required condition (3.7) as

$$\begin{aligned} 1 + |\nu| - |\zeta| + \sum_{x=0}^{t} (x+1)|\eta_x| &= 1 + |\nu| - |\nu| + \sum_{x=0}^{t-1} (x+1)|\varnothing| + (t+1)|\eta_t| \\ &= 1 + (t+1)|\mu| = q. \end{aligned}$$

Setting $\pi_t = \cdots = \pi_0 = \delta = \varnothing$, from (3.8) we see that the multiplicity of $V_{\eta_t,\ldots,\eta_0,\xi,\zeta}$ in $\underline{\mathrm{soc}}^q((V^*)_\mu \otimes V_\nu)$ is at least

$$\mathrm{N}^{\mu}_{\pi_t,\eta_t}\mathrm{N}^{\pi_t}_{\pi_{t-1},\eta_{t-1}}\cdots\mathrm{N}^{\pi_0}_{\xi,\delta}\mathrm{N}^{\nu}_{\zeta,\delta} = \mathrm{N}^{\mu}_{\varnothing,\mu}\mathrm{N}^{\varnothing}_{\varnothing,\varnothing}\ldots\mathrm{N}^{\varnothing}_{\varnothing,\varnothing}\mathrm{N}^{\nu}_{\nu,\varnothing} = 1$$

and hence the q-th layer is non-zero.

Now suppose the $(q+1)$-th layer is non-empty. Then there is a simple object $V_{\eta_t,\ldots,\eta_0,\xi,\zeta}$ that appears with non-zero multiplicity in $\underline{\mathrm{soc}}^{q+1}((V^*)_\mu \otimes V_\nu)$. Hence we can find indices $\pi_t,\ldots,\pi_0,\delta$ for which the product of Littlewood-Richardson coefficients

$$\mathrm{N}^{\mu}_{\pi_t,\eta_t}\mathrm{N}^{\pi_t}_{\pi_{t-1},\eta_{t-1}}\cdots\mathrm{N}^{\pi_0}_{\xi,\delta}\mathrm{N}^{\nu}_{\zeta,\delta}$$

is non-zero. In particular, each Littlewood-Richardson coefficient is non-zero and from Lemma 2.4 we obtain the set of equations

$$\begin{aligned} |\mu| &= |\pi_1| + |\eta_t|, \\ |\pi_t| &= |\pi_{t-1}| + |\eta_{t-1}|, \\ &\cdots \\ |\pi_0| &= |\xi| + |\delta|, \\ |\nu| &= |\zeta| + |\delta|. \end{aligned}$$

Using the convention $\pi_{t+1} = \mu$, for any x we can write

$$|\eta_x| = |\pi_{x+1}| - |\pi_x|.$$

This implies that

$$\begin{aligned} 1 + |\nu| - |\zeta| + \sum_{x=0}^{t}(x+1)|\eta_x| &= 1 + |\nu| - |\zeta| + \sum_{x=0}^{t}(x+1)(|\pi_{x+1}| - |\pi_x|) \\ &= 1 + |\delta| + \sum_{x=0}^{t}(x+1)|\pi_{x+1}| - \sum_{x=0}^{t}(x+1)|\pi_x| \\ &= 1 + |\delta| + (t+1)|\mu| - \sum_{x=0}^{t}|\pi_x|. \end{aligned}$$

Since the $(q+1)$-th layer was assumed to be non-zero, the expression above must satisfy the condition in (3.7), that is

$$\begin{aligned} q + 1 &= 1 + |\delta| + (t+1)|\mu| - \sum_{x=0}^{t}|\pi_x| \\ \implies (t+1)|\mu| + 2 &= 1 + |\delta| + (t+1)|\mu| - \sum_{x=0}^{t}|\pi_x| \\ \implies \sum_{x=0}^{t}|\pi_x| + 1 &= |\delta|. \end{aligned}$$

However since $|\delta| \leq |\pi_0|$, the above equality leads to a contradiction. Hence the $(q+1)$-th layer has to be zero and the Loewy length is $q = (1+t)|\mu| + 1$ as was claimed. □

THEOREM 4.14. *The indecomposable injective object $(V^*/V^*_{\aleph_u})_\lambda$ has Loewy length $1 + (t-u)|\lambda|$.*

PROOF. Let $s = t - u$ and $q = 1 + (t-u)|\lambda| = 1 + s|\lambda|$. We first show that the q-th layer is non-zero. Consider the simple object $V_{\eta_s,\dots,\eta_0,\varnothing,\dots,\varnothing} = V_{\lambda,\varnothing,\dots,\varnothing}$. Then as required by (3.6) and (3.5) the indices satisfy

$$1 + \sum_{x=0}^{s} x|\eta_x| = 1 + s|\eta_s| = 1 + s|\lambda| = q,$$

and

$$|\lambda| = \sum_{x=0}^{s} |\eta_x| = |\eta_s|$$

respectively. Setting $\alpha_1 = \cdots = \alpha_{s-1} = \varnothing$, we see from (3.4) that the multiplicity of $V_{\eta_s,\dots,\eta_0,\varnothing,\dots,\varnothing}$ is at least

$$\mathrm{N}^{\lambda}_{\alpha_{s-1},\eta_s} \cdots \mathrm{N}^{\alpha_2}_{\alpha_1,\eta_2} \mathrm{N}^{\alpha_1}_{\eta_0,\eta_1} = \mathrm{N}^{\lambda}_{\varnothing,\lambda} \cdots \mathrm{N}^{\varnothing}_{\varnothing,\varnothing} \mathrm{N}^{\varnothing}_{\varnothing,\varnothing} = 1.$$

Hence the q-th layer is non-zero.

Now suppose that the $(q+1)$-th layer contains some simple object $V_{\eta_s,\dots,\eta_0,\varnothing,\dots,\varnothing}$. Then from (3.4) there are indices $\alpha_{s-1},\dots,\alpha_1$ such that the product of Littlewood-Richardson coefficients

$$\mathrm{N}^{\lambda}_{\alpha_{s-1},\eta_s} \cdots \mathrm{N}^{\alpha_2}_{\alpha_1,\eta_2} \mathrm{N}^{\alpha_1}_{\eta_0,\eta_1}$$

is non-zero. In particular, each of the Littlewood-Richardson coefficients are non-zero and from Lemma 2.4 we get

$$\begin{aligned} |\lambda| &= |\alpha_{s-1}| + |\eta_s|, \\ &\cdots \\ |\alpha_2| &= |\alpha_1| + |\eta_2|, \\ |\alpha_1| &= |\eta_0| + |\eta_1|. \end{aligned}$$

Using the convention $\alpha_s = \lambda$ and $\alpha_0 = \eta_0$, for all $x > 0$ we can write

$$|\eta_x| = |\alpha_x| - |\alpha_{x-1}|.$$

Hence we now have

$$\begin{aligned} \sum_{x=0}^{s} x|\eta_x| &= \sum_{x=1}^{s} x(|\alpha_x| - |\alpha_{x-1}|) \\ &= s|\alpha_s| + \sum_{x=0}^{s-1} x|\alpha_x| - \sum_{x=0}^{s-1}(x+1)|\alpha_x| \\ &= s|\lambda| - \sum_{x=0}^{s-1} |\alpha_x|. \end{aligned}$$

Since the $(q+1)$-th layer was assumed to be non-zero from (3.6) we must additionally have

$$\begin{aligned} q + 1 &= 1 + \sum_{x=0}^{s} x|\eta_x| \\ \implies 1 + s|\lambda| &= s|\lambda| - \sum_{x=0}^{s-1} |\alpha_x| \\ \implies 1 + \sum_{x=0}^{s-1} |\alpha_x| &= 0. \end{aligned}$$

Since for all x, $|\alpha_x| \geq 0$ and the equality above leads to a contradiction. Hence the $(q+1)$-th layer is zero and the Loewy length is $q = 1 + s|\lambda| = 1 + (t-u)|\lambda|$. $\square$

We can now compute the Loewy length of an arbitrary indecomposable injective object. First observe that for any u and partition λ the only simple objects that appear in any socle layer of $(V^*/V^*_{\aleph_u})_\lambda$ have empty indices in the two right-most positions, that is these simple objects are of the form $V_{\eta_t,\dots,\eta_0,\varnothing,\varnothing}$.

LEMMA 4.15. *Let $V_{\eta_t,\dots,\eta_0,\varnothing,\varnothing}$ and $V_{\theta_t,\dots,\theta_0,\varnothing,\varnothing}$ be two simple objects. If for all s the inequality $0 \le s \le t$ and the equality $\kappa_s = \eta_s + \theta_s$ are satisfied, then the $V_{\kappa_t,\dots,\kappa_0,\varnothing,\varnothing}$ appears as a sub-module in $V_{\eta_t,\dots,\eta_0,\varnothing,\varnothing} \otimes V_{\theta_t,\dots,\theta_0,\varnothing,\varnothing}$.*

PROOF. Writing out the definition of the simple objects, we obtain

$$\begin{aligned} V_{\eta_t,\dots,\eta_0,\varnothing,\varnothing} \otimes V_{\theta_t,\dots,\theta_0,\varnothing,\varnothing} &= \left(\bigotimes_{s=0}^{t} (V^*_{\aleph_{s+1}}/V^*_{\aleph_s})_{\eta_s}\right) \otimes \left(\bigotimes_{s=0}^{t} (V^*_{\aleph_{s+1}}/V^*_{\aleph_s})_{\theta_s}\right) \\ &= \bigotimes_{s=0}^{t} \Big((V^*_{\aleph_{s+1}}/V^*_{\aleph_s})_{\eta_s} \otimes (V^*_{\aleph_{s+1}}/V^*_{\aleph_s})_{\theta_s} \Big) \\ &= \bigotimes_{s=0}^{t} \left(\sum_{\pi_s} \mathrm{N}^{\pi_s}_{\eta_s,\theta_s} (V^*_{\aleph_{s+1}}/V^*_{\aleph_s})_{\pi_s} \right) \end{aligned}$$

Above we use the Littlewood-Richardson rule to expand the tensor product at s. From Lemma 2.3 we know that for all s the coefficient $\pi_s = \kappa_s = \eta_s + \theta_s$ is certainly non-zero. Expanding the sums and then the tensors shows that the simple object $V_{\kappa_t,\dots,\kappa_0,\varnothing,\varnothing}$ appears as a sub-module in the tensor product. $\square$

THEOREM 4.16. *The indecomposable injective object $\tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu}$ has Loewy length*

$$1 + (t+1)|\mu| + \sum_{x=0}^{t} (t-x)|\lambda_x|.$$

PROOF. Recall that the isomorphism (3.9) describes the socle layer of $\tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu}$ in terms of layers of objects discussed in Theorem 4.13 and in Theorem 4.14. The previous lemma implies that a summand in (3.9) is non-zero only when all the layers appearing in the tensor product are non-zero. From Theorem 4.13 and Theorem 4.14 this is only possible when

$$u_x \le 1 + (t-x)|\lambda_x| \qquad \text{and} \qquad y \le 1 + (t+1)|\mu|.$$

Moreover since the indices u_x and y satisfy (3.10), we obtain the bound

$$q = y + \sum_{x=0}^{t} (u_x - 1) \le 1 + (t+1)|\mu| + \sum_{x=0}^{t} (t-x)|\lambda_x| = q_{\max}.$$

Hence for any $q' > q_{\max}$, the layer $\underline{\mathrm{soc}}^{q'}(\tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu})$ is necessarily zero.

Now consider the layer $\underline{\mathrm{soc}}^{q_{\max}}(\tilde{V}_{\lambda_t,\dots,\lambda_0,\mu,\nu})$. The indices $u_x = 1 + (t-x)|\lambda_x|$ and $y = 1 + (t+1)|\mu|$ satisfy (3.10) with $q = q_{\max}$. Moreover, from Theorem 4.13,

$$V_{\mu,\varnothing,\dots,\varnothing,\nu} \subseteq \underline{\mathrm{soc}}^{y}((V^*)_\mu \otimes V_\nu)$$

and from Theorem 4.14,

$$V_{\lambda_x,\varnothing,\dots,\varnothing} \subseteq \underline{\mathrm{soc}}^{u_x}((V^*/V^*_{\aleph_x})_{\lambda_x}).$$

From the previous lemma, the tensor product of these simple modules is non-zero. Hence the $q_{\max}$-th layer is non-zero and the Loewy length is

$$q_{\max} = 1 + (t+1)|\mu| + \sum_{x=0}^{t} (t-x)|\lambda_x|.$$

$\square$

CHAPTER 5

Symmetry of Socle Filtrations

We now consider the symmetry of the socle filtrations in $\overline{\mathbb{T}}_{\aleph_t}$ with respect to partition conjugation. The theorem below is a generalisation of Theorem 1 in [**17**]. The general proof technique used is also similar.

THEOREM 5.1. *Let $\lambda_t, \ldots, \lambda_0, \mu, \nu$ and $\eta_t, \ldots, \eta_0, \xi, \zeta$ be Young diagrams. Then for all q,*

$$[\underline{\mathrm{soc}}^q(\tilde{V}_{\lambda_t,\ldots,\lambda_0,\mu,\nu}) : V_{\eta_t,\ldots,\eta_0,\xi,\zeta}] = [\underline{\mathrm{soc}}^q(\tilde{V}_{\lambda_t^\top,\ldots,\lambda_0^\top,\mu^\top,\nu^\top}) : V_{\eta_t^\top,\ldots,\eta_0^\top,\xi^\top,\zeta^\top}].$$

PROOF. Let k be arbitrary and consider Young diagrams $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$, and $\gamma_1, \ldots, \gamma_k$. Using Lemma 2.6 we immediately get

$$\mathrm{N}^{\alpha_1}_{\beta_1,\gamma_1} \cdots \mathrm{N}^{\alpha_k}_{\beta_k,\gamma_k} = \mathrm{N}^{\alpha_1^\top}_{\beta_1^\top,\gamma_1^\top} \cdots \mathrm{N}^{\alpha_k^\top}_{\beta_k^\top,\gamma_k^\top}.$$

This fact along with formula (3.9) and possibly repeated applications of the Littlewood - Richardson rule in Lemma 4.15, allows us to reduce to the case of indecomposable injectives of type $(V^*)_\mu \otimes V_\nu$ and $(V^*/V^*_{\aleph_u})_{\lambda_u}$.

Let q be given. Since $|\alpha| = |\alpha^\top|$, whenever the indices $\eta_t, \ldots, \eta_0, \xi, \zeta$ satisfy the conditions (3.5), (3.6) and (3.7), so do the indices $\eta_t^\top, \ldots, \eta_0^\top, \xi^\top, \zeta^\top$. In case of $(V^*/V^*_{\aleph_u})_{\lambda_u}$, from formula (3.4) we obtain

$$\begin{aligned}
[\underline{\mathrm{soc}}^q((V^*/V^*_{\aleph_u})_{\lambda_u}) : V_{\eta_{t-u},\ldots,\eta_0,\varnothing,\ldots\varnothing}] &= \sum \mathrm{N}^{\alpha_1}_{\eta_0,\eta_1} \mathrm{N}^{\alpha_2}_{\alpha_1,\eta_2} \cdots \mathrm{N}^{\lambda_u}_{\alpha_{t-u-1},\eta_{t-u}} \\
&= \sum \mathrm{N}^{\alpha_1^\top}_{\eta_0^\top,\eta_1^\top} \mathrm{N}^{\alpha_2^\top}_{\alpha_1^\top,\eta_2^\top} \cdots \mathrm{N}^{\lambda_u^\top}_{\alpha_{t-u-1}^\top,\eta_{t-u}^\top} \\
&\leq \sum \mathrm{N}^{\alpha_1}_{\eta_0^\top,\eta_1^\top} \mathrm{N}^{\alpha_2}_{\alpha_1,\eta_2^\top} \cdots \mathrm{N}^{\lambda_u^\top}_{\alpha_{t-u-1},\eta_{t-u}^\top} \\
&= [\underline{\mathrm{soc}}^q((V^*/V^*_{\aleph_u})_{\lambda_u^\top}) : V_{\eta_{t-u}^\top,\ldots,\eta_0^\top,\varnothing,\ldots\varnothing}].
\end{aligned}$$

Similarly for $(V^*)_\mu \otimes V_\nu$, formula (3.8) gives us

$$\begin{aligned}
[\underline{\mathrm{soc}}^q((V^*)_\mu \otimes V_\nu) : V_{\eta_t,\ldots,\eta_0,\xi,\zeta}] &= \sum \mathrm{N}^{\mu}_{\pi_t,\eta_t} \mathrm{N}^{\pi_t}_{\pi_{t-1},\eta_{t-1}} \ldots \mathrm{N}^{\pi_1}_{\pi_0,\eta_0} \mathrm{N}^{\pi_0}_{\xi,\delta} \mathrm{N}^{\nu}_{\zeta,\delta} \\
&= \sum \mathrm{N}^{\mu^\top}_{\pi_t^\top,\eta_t^\top} \mathrm{N}^{\pi_t^\top}_{\pi_{t-1}^\top,\eta_{t-1}^\top} \ldots \mathrm{N}^{\pi_1^\top}_{\pi_0^\top,\eta_0^\top} \mathrm{N}^{\pi_0^\top}_{\xi^\top,\delta^\top} \mathrm{N}^{\nu^\top}_{\zeta^\top,\delta^\top} \\
&\leq \sum \mathrm{N}^{\mu^\top}_{\pi_t,\eta_t^\top} \mathrm{N}^{\pi_t}_{\pi_{t-1},\eta_{t-1}^\top} \ldots \mathrm{N}^{\pi_1}_{\pi_0,\eta_0^\top} \mathrm{N}^{\pi_0}_{\xi^\top,\delta} \mathrm{N}^{\nu^\top}_{\zeta^\top,\delta} \\
&= [\underline{\mathrm{soc}}^q((V^*)_{\mu^\top} \otimes V_{\nu^\top}) : V_{\eta_t^\top,\ldots,\eta_0^\top,\xi^\top,\zeta^\top}].
\end{aligned}$$

The equality of multiplicities follows by first noting that $(\alpha^\top)^\top = \alpha$ and repeating the process above starting with the objects $(V^*/V^*_{\aleph_u})_{\lambda_u^\top}$ and $(V^*)_{\mu^\top} \otimes V_{\nu^\top}$. $\square$

The above theorem allow us state a general version of Conjecture 2 from [**17**] in the setting of the category $\overline{\mathbb{T}}_{\aleph_t}$.

CONJECTURE 5.2. *There exists a functor of autoequivalence*

$$(\cdot)^\top : \overline{\mathbb{T}}_{\aleph_t} \to \overline{\mathbb{T}}_{\aleph_t}$$

such that

$$(\tilde{V}_{\lambda_t,\ldots,\lambda_0,\mu,\nu})^\top \cong \tilde{V}_{\lambda_t^\top,\ldots,\lambda_0^\top,\mu^\top,\nu^\top}.$$

As remarked in [**17**], we also note that the functor conjectured below resembles Serganova's functor of autoequivalence of the category $\mathbb{T}_{\mathfrak{gl}(\infty)}$ (Theorem 7 in [**16**]) and can perhaps be constructed by moving to the appropriate superalgebra.

APPENDIX A

Interface to the socle filtration calculator

We used the computer algebra system SageMath [**19**] to program a socle filtration calculator which we have made available at [**11**]. The program in particular relies on the library `lrcalc` [**1**] to compute values for various Littlewood-Richardson coefficients and an implementation of [**21**] to compute integer partitions. Both of these components were available in the combinatorics toolbox of SageMath.

To compute the socle filtrations, we first find the all possible indices for simple objects that can appear in a given layer q of the filtration. For each Young diagram η in the index, we first find possible values of $|\eta|$. A particularly useful trick here is to first use Lemma 2.4 and then the conditions relating q and $|\eta|$ to narrow down the choices for $|\eta|$. Once this is done, we generate all possible partitions and compute the multiplicity of simples in the q-th layer. The whole socle filtration is now just obtained by iterating on q until no simples appear in q-th layer.

The main interface to the program consists of the following sets of functions:

- The function `socle_filtration_aleph_t` in `socles_aleph_t.py` accepts a list of indices $\lambda_t, \ldots, \lambda_0, \mu, \nu$ and returns the socle filtration of the indecomposable injective object $\tilde{V}_{\lambda_t,\ldots,\lambda_0,\mu,\nu}$. Note that the category $\overline{\mathbb{T}}_{\aleph_t}$ is automatically determined by the number of indices.
- The functions `socle_filtration_gl_mod`, `socle_filtration_sp_mod` and `socle_filtration_so_mod` in `socles_two_diag.py` accept two, one and one indices and return the filtration of the corresponding $\mathfrak{gl}_\infty$-, $\mathfrak{sp}_\infty$- and $\mathfrak{so}_\infty$ modules respectively.

APPENDIX B

Socle computations

Here we present some computations that were produced by our program [**11**]. Since the total number of simple objects that appear in the socle filtration grows very large for even small values of t, we only show a small selection of explicit computations.

EXAMPLE B.1. We construct the socle filtration of the indecomposable injective object

$$\tilde{V}_{\varnothing,(1,1),(1),(1,1),\varnothing} = \tilde{V}_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing} \otimes \tilde{V}_{\varnothing,\varnothing,(1),\varnothing,\varnothing} \otimes \tilde{V}_{\varnothing,\varnothing,\varnothing,(1,1),\varnothing}$$

in the category $\overline{\mathbb{T}}_{\aleph_2}$. First note that we have

$V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing}$
$V_{(1),(1),\varnothing,\varnothing,\varnothing}$
$V_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing}$

,

$V_{(1),\varnothing,\varnothing,\varnothing,\varnothing}$
$V_{\varnothing,(1),\varnothing,\varnothing,\varnothing}$
$V_{\varnothing,\varnothing,(1),\varnothing,\varnothing}$

,

$V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing}$
$V_{(1),(1),\varnothing,\varnothing,\varnothing}$
$V_{(1),\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing}$
$V_{(1),\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,(1),(1),\varnothing,\varnothing}$
$V_{\varnothing,(1),\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,(1,1),\varnothing,\varnothing}$
$V_{\varnothing,\varnothing,(1),(1),\varnothing}$
$V_{\varnothing,\varnothing,\varnothing,(1,1),\varnothing}$

.

Combining these using the formula 3.9 we obtain the full socle filtration

$V_{(1,1,1,1,1),\varnothing,\varnothing,\varnothing,\varnothing} \oplus 2\,V_{(2,1,1,1),\varnothing,\varnothing,\varnothing,\varnothing} \oplus 2\,V_{(2,2,1),\varnothing,\varnothing,\varnothing,\varnothing}$ $\oplus V_{(3,1,1),\varnothing,\varnothing,\varnothing,\varnothing} \oplus V_{(3,2),\varnothing,\varnothing,\varnothing,\varnothing}$
$3\,V_{(1,1,1,1),(1),\varnothing,\varnothing,\varnothing} \oplus 5\,V_{(2,1,1),(1),\varnothing,\varnothing,\varnothing} \oplus 3\,V_{(2,2),(1),\varnothing,\varnothing,\varnothing} \oplus 2\,V_{(3,1),(1),\varnothing,\varnothing,\varnothing}$
$2\,V_{(1,1,1,1),\varnothing,(1),\varnothing,\varnothing} \oplus 5\,V_{(1,1,1),(1,1),\varnothing,\varnothing,\varnothing} \oplus 3\,V_{(1,1,1),(2),\varnothing,\varnothing,\varnothing} \oplus 3\,V_{(2,1,1),\varnothing,(1),\varnothing,\varnothing}$ $\oplus 6\,V_{(2,1),(1,1),\varnothing,\varnothing,\varnothing} \oplus 4\,V_{(2,1),(2),\varnothing,\varnothing,\varnothing} \oplus 2\,V_{(2,2),\varnothing,(1),\varnothing,\varnothing} \oplus V_{(3,1),\varnothing,(1),\varnothing,\varnothing}$ $\oplus V_{(3),(1,1),\varnothing,\varnothing,\varnothing} \oplus V_{(3),(2),\varnothing,\varnothing,\varnothing}$
$V_{(1,1,1,1),\varnothing,\varnothing,(1),\varnothing} \oplus 5\,V_{(1,1,1),(1),(1),\varnothing,\varnothing} \oplus 5\,V_{(1,1),(1,1,1),\varnothing,\varnothing,\varnothing} \oplus 6\,V_{(1,1),(2,1),\varnothing,\varnothing,\varnothing}$ $\oplus V_{(1,1),(3),\varnothing,\varnothing,\varnothing} \oplus 2\,V_{(2,1,1),\varnothing,\varnothing,(1),\varnothing} \oplus 6\,V_{(2,1),(1),(1),\varnothing,\varnothing} \oplus V_{(2,2),\varnothing,\varnothing,(1),\varnothing}$ $\oplus 3\,V_{(2),(1,1,1),\varnothing,\varnothing,\varnothing} \oplus 4\,V_{(2),(2,1),\varnothing,\varnothing,\varnothing} \oplus V_{(2),(3),\varnothing,\varnothing,\varnothing}$ $\oplus V_{(3,1),\varnothing,\varnothing,(1),\varnothing} \oplus V_{(3),(1),(1),\varnothing,\varnothing}$
$3\,V_{(1,1,1),(1),\varnothing,(1),\varnothing} \oplus 2\,V_{(1,1,1),\varnothing,(1,1),\varnothing,\varnothing} \oplus V_{(1,1,1),\varnothing,(2),\varnothing,\varnothing} \oplus 7\,V_{(1,1),(1,1),(1),\varnothing,\varnothing}$ $\oplus 4\,V_{(1,1),(2),(1),\varnothing,\varnothing} \oplus 3\,V_{(1),(1,1,1,1),\varnothing,\varnothing,\varnothing} \oplus 5\,V_{(1),(2,1,1),\varnothing,\varnothing,\varnothing} \oplus 3\,V_{(1),(2,2),\varnothing,\varnothing,\varnothing}$ $\oplus 2\,V_{(1),(3,1),\varnothing,\varnothing,\varnothing} \oplus 4\,V_{(2,1),(1),\varnothing,(1),\varnothing} \oplus 2\,V_{(2,1),\varnothing,(1,1),\varnothing,\varnothing} \oplus V_{(2,1),\varnothing,(2),\varnothing,\varnothing}$ $\oplus 4\,V_{(2),(1,1),(1),\varnothing,\varnothing} \oplus 3\,V_{(2),(2),(1),\varnothing,\varnothing} \oplus V_{(3),(1),\varnothing,(1),\varnothing}$
$2\,V_{(1,1,1),\varnothing,(1),(1),\varnothing} \oplus 4\,V_{(1,1),(1,1),\varnothing,(1),\varnothing} \oplus 4\,V_{(1,1),(1),(1,1),\varnothing,\varnothing} \oplus 2\,V_{(1,1),(1),(2),\varnothing,\varnothing}$ $\oplus 3\,V_{(1,1),(2),\varnothing,(1),\varnothing} \oplus 5\,V_{(1),(1,1,1),(1),\varnothing,\varnothing} \oplus 6\,V_{(1),(2,1),(1),\varnothing,\varnothing} \oplus V_{(1),(3),(1),\varnothing,\varnothing}$ $\oplus 2\,V_{(2,1),\varnothing,(1),(1),\varnothing} \oplus 3\,V_{(2),(1,1),\varnothing,(1),\varnothing} \oplus 2\,V_{(2),(1),(1,1),\varnothing,\varnothing} \oplus V_{(2),(1),(2),\varnothing,\varnothing}$ $\oplus 2\,V_{(2),(2),\varnothing,(1),\varnothing} \oplus V_{\varnothing,(1,1,1,1,1),\varnothing,\varnothing,\varnothing} \oplus 2\,V_{\varnothing,(2,1,1,1),\varnothing,\varnothing,\varnothing} \oplus 2\,V_{\varnothing,(2,2,1),\varnothing,\varnothing,\varnothing}$ $\oplus V_{\varnothing,(3,1,1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(3,2),\varnothing,\varnothing,\varnothing}$
$V_{(1,1,1),\varnothing,\varnothing,(1,1),\varnothing} \oplus 4\,V_{(1,1),(1),(1),(1),\varnothing} \oplus V_{(1,1),\varnothing,(1,1,1),\varnothing,\varnothing} \oplus V_{(1,1),\varnothing,(2,1),\varnothing,\varnothing}$ $\oplus 3\,V_{(1),(1,1,1),\varnothing,(1),\varnothing} \oplus 4\,V_{(1),(1,1),(1,1),\varnothing,\varnothing} \oplus 2\,V_{(1),(1,1),(2),\varnothing,\varnothing} \oplus 4\,V_{(1),(2,1),\varnothing,(1),\varnothing}$ $\oplus 2\,V_{(1),(2),(1,1),\varnothing,\varnothing} \oplus V_{(1),(2),(2),\varnothing,\varnothing} \oplus V_{(1),(3),\varnothing,(1),\varnothing} \oplus V_{(2,1),\varnothing,\varnothing,(1,1),\varnothing}$ $\oplus 2\,V_{(2),(1),(1),(1),\varnothing} \oplus 2\,V_{\varnothing,(1,1,1,1),(1),\varnothing,\varnothing} \oplus 3\,V_{\varnothing,(2,1,1),(1),\varnothing,\varnothing}$ $\oplus 2\,V_{\varnothing,(2,2),(1),\varnothing,\varnothing} \oplus V_{\varnothing,(3,1),(1),\varnothing,\varnothing}$
$2\,V_{(1,1),(1),\varnothing,(1,1),\varnothing} \oplus V_{(1,1),\varnothing,(1,1),(1),\varnothing} \oplus V_{(1,1),\varnothing,(2),(1),\varnothing} \oplus 4\,V_{(1),(1,1),(1),(1),\varnothing}$ $\oplus V_{(1),(1),(1,1,1),\varnothing,\varnothing} \oplus V_{(1),(1),(2,1),\varnothing,\varnothing} \oplus 2\,V_{(1),(2),(1),(1),\varnothing} \oplus V_{(2),(1),\varnothing,(1,1),\varnothing}$ $\oplus V_{\varnothing,(1,1,1,1),\varnothing,(1),\varnothing} \oplus 2\,V_{\varnothing,(1,1,1),(1,1),\varnothing,\varnothing} \oplus V_{\varnothing,(1,1,1),(2),\varnothing,\varnothing} \oplus 2\,V_{\varnothing,(2,1,1),\varnothing,(1),\varnothing}$ $\oplus 2\,V_{\varnothing,(2,1),(1,1),\varnothing,\varnothing} \oplus V_{\varnothing,(2,1),(2),\varnothing,\varnothing} \oplus V_{\varnothing,(2,2),\varnothing,(1),\varnothing} \oplus V_{\varnothing,(3,1),\varnothing,(1),\varnothing}$
$V_{(1,1),\varnothing,(1),(1,1),\varnothing} \oplus 2\,V_{(1),(1,1),\varnothing,(1,1),\varnothing} \oplus V_{(1),(1),(1,1),(1),\varnothing} \oplus V_{(1),(1),(2),(1),\varnothing}$ $\oplus V_{(1),(2),\varnothing,(1,1),\varnothing} \oplus 2\,V_{\varnothing,(1,1,1),(1),(1),\varnothing} \oplus V_{\varnothing,(1,1),(1,1,1),\varnothing,\varnothing}$ $\oplus V_{\varnothing,(1,1),(2,1),\varnothing,\varnothing} \oplus 2\,V_{\varnothing,(2,1),(1),(1),\varnothing}$
$V_{(1),(1),(1),(1,1),\varnothing} \oplus V_{\varnothing,(1,1,1),\varnothing,(1,1),\varnothing} \oplus V_{\varnothing,(1,1),(1,1),(1),\varnothing}$ $\oplus V_{\varnothing,(1,1),(2),(1),\varnothing} \oplus V_{\varnothing,(2,1),\varnothing,(1,1),\varnothing}$
$V_{\varnothing,(1,1),(1),(1,1),\varnothing}$

.

EXAMPLE B.2. Let $\mu = (1,1)$ and $\nu = \varnothing$ we will now consider the socle filtration of the object $(V^*)_\mu \otimes V_\nu$ in $\overline{\mathbb{T}}_{\aleph_t}$ for increasing values of t.

$$\begin{array}{|c|}\hline V_{(1,1),\varnothing,\varnothing} \\ \hline V_{(1),(1),\varnothing} \\ \hline V_{\varnothing,(1,1),\varnothing} \\ \hline \end{array} \quad \begin{array}{|c|}\hline V_{(1,1),\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),(1),\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,(1),\varnothing} \oplus V_{\varnothing,(1,1),\varnothing,\varnothing} \\ \hline V_{\varnothing,(1),(1),\varnothing} \\ \hline V_{\varnothing,\varnothing,(1,1),\varnothing} \\ \hline \end{array} \quad \begin{array}{|c|}\hline V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),(1),\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,(1),(1),\varnothing,\varnothing} \\ \hline V_{\varnothing,(1),\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,(1,1),\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,(1),(1),\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,(1,1),\varnothing} \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),(1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,(1),(1),\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,(1),\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,(1),\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,(1),(1),\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,(1),\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,(1,1),\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,(1),(1),\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,\varnothing,(1,1),\varnothing} \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),(1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,(1),\varnothing,\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1),(1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,(1),\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,(1),\varnothing,\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,(1),(1),\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,(1),\varnothing,\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,(1),\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,(1),\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,(1),(1),\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,(1),\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,\varnothing,(1,1),\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,\varnothing,(1),(1),\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,(1,1),\varnothing} \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),(1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,(1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,(1),\varnothing,\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1),(1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1),\varnothing,(1),\varnothing,\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,\varnothing,\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,(1),\varnothing,\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,(1),(1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,(1),\varnothing,\varnothing,\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,(1),\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,(1),\varnothing,\varnothing,\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,(1),\varnothing,\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,(1),(1),\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,(1),\varnothing,\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,(1),\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,(1),\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,\varnothing,(1),(1),\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,\varnothing,(1),\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,(1,1),\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,(1),(1),\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,(1,1),\varnothing} \\ \hline \end{array}$$

EXAMPLE B.3. Let $t = 4$. We will now consider the case when all but one diagram are empty. Let us pick $(1,1)$ to be the non-empty diagram.

$$\begin{array}{|c|}\hline V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline \end{array}, \quad \begin{array}{|c|}\hline V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),(1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline \end{array}, \quad \begin{array}{|c|}\hline V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),(1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,(1),\varnothing,\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,(1),(1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline \end{array},$$

$$\begin{array}{|c|}\hline V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),(1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,(1),\varnothing,\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1),(1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,(1),\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,(1),(1),\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing} \\ \hline \end{array},$$

$$\begin{array}{|c|}\hline V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),(1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,(1),\varnothing,\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1),(1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{(1),\varnothing,\varnothing,\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,(1),\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,(1),\varnothing,\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,(1),(1),\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,(1),\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,(1),(1),\varnothing,\varnothing} \\ \hline V_{\varnothing,\varnothing,\varnothing,\varnothing,(1,1),\varnothing,\varnothing} \\ \hline \end{array},$$

$$\begin{array}{|c|}
\hline
V_{(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\
\hline
V_{(1),(1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\
\hline
V_{(1),\varnothing,(1),\varnothing,\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing,\varnothing} \\
\hline
V_{(1),\varnothing,\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,(1),(1),\varnothing,\varnothing,\varnothing,\varnothing} \\
\hline
V_{(1),\varnothing,\varnothing,\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,(1),\varnothing,(1),\varnothing,\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing,\varnothing} \\
\hline
V_{(1),\varnothing,\varnothing,\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,(1),\varnothing,\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,(1),(1),\varnothing,\varnothing,\varnothing} \\
\hline
V_{\varnothing,(1),\varnothing,\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,(1),\varnothing,(1),\varnothing,\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,(1,1),\varnothing,\varnothing,\varnothing} \\
\hline
V_{\varnothing,\varnothing,(1),\varnothing,\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,(1),(1),\varnothing,\varnothing} \\
\hline
V_{\varnothing,\varnothing,\varnothing,(1),\varnothing,(1),\varnothing} \oplus V_{\varnothing,\varnothing,\varnothing,\varnothing,(1,1),\varnothing,\varnothing} \\
\hline
V_{\varnothing,\varnothing,\varnothing,\varnothing,(1),(1),\varnothing} \\
\hline
V_{\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,(1,1),\varnothing} \\
\hline
\end{array},$$

$$\boxed{V_{\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,\varnothing,(1,1)}}.$$

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