ON CATEGORIES OF ADMISSIBLE $\big(\mathfrak{g}, \mathrm{sl}(2)\big)$-MODULES

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Abstract.
Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra and $\mathfrak{k}$ be any sl(2)-subalgebra of $\mathfrak{g}$. In this paper we prove an earlier conjecture by Penkov and Zuckerman claiming that the first derived Zuckerman functor provides an equivalence between a truncation of a thick parabolic category $\mathcal{O}$ for $\mathfrak{g}$ and a truncation of the category of admissible $(\mathfrak{g}, \mathfrak{k})-$modules. This latter truncated category consists of admissible $(\mathfrak{g}, \mathfrak{k})-$modules with sufficiently large minimal $\mathfrak{k}$-type. We construct an explicit functor inverse to the Zuckerman functor in this setting. As a corollary we obtain an estimate for the global injective dimension of the inductive completion of the truncated category of admissible $(\mathfrak{g}, \mathfrak{k})-$modules.

Key words: generalized Harish-Chandra module, Zuckerman functor, thick category $\mathcal{O}$.

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1. Introduction

Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra and $\mathfrak{k} \subseteq \mathfrak{g}$ be a reductive in $\mathfrak{g}$ subalgebra. An *admissible* $(\mathfrak{g}, \mathfrak{k})$-*module* is a $\mathfrak{g}$-module on which $\mathfrak{k}$ acts semisimply, locally finitely, and with finite multiplicities. The study of the category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules is a main objective of the theory of generalized Harish-Chandra modules, see [PZ1].

In the case of a general reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$, a central result of the existing theory of generalized Harish-Chandra modules is the classification of simple admissible $(\mathfrak{g}, \mathfrak{k})$-modules with generic minimal $\mathfrak{k}$-type [PZ1]. Other notable

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results for a general $\mathfrak{k}$ are established in [PSZ], [PS], [PZ2], and [PZ4].

There are three special cases for $\mathfrak{k}$ in which more detailed information on admissible $(\mathfrak{g}, \mathfrak{k})$-modules is available. First of all, this is the case when $\mathfrak{k}$ is a symmetric subalgebra of $\mathfrak{g}$, i.e., $\mathfrak{k}$ coincides with the fixed points of an involution on $\mathfrak{g}$. This case, the theory of Harish-Chandra modules, is in the origin of the studies of generalized Harish-Chandra modules. There is an extensive literature on Harish-Chandra modules, see for instance [V], [KV], and references therein. (In particular, some remarks on the history of Harish-Chandra modules can be found in [KV].) Another case which has drawn considerable attention is the case when $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$, see for instance [BL], [BBL], [F], [Fe], [M], [GS1], [GS2], and references therein. In both these cases, a classification of simple admissible $(\mathfrak{g}, \mathfrak{k})$-modules is available and there has been progress in the study of the category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules.

A third natural choice for $\mathfrak{k}$ is to let $\mathfrak{k}$ be isomorphic to sl(2). This case "interpolates" between the above two cases and is a natural experimentation ground when aiming at the case of a general $\mathfrak{k}$. For $\mathfrak{k} \simeq \mathrm{sl}(2)$, there is no classification of simple admissible $(\mathfrak{g}, \mathfrak{k})$-modules for a general $\mathfrak{g}$ and an arbitrary sl(2)-subalgebra $\mathfrak{k} \subset \mathfrak{g}$; however, for $\mathfrak{k} \simeq \mathrm{sl}(2)$ the partial classification of [PZ1] can be carried out under much less severe restrictions on the minimal $\mathfrak{k}$-type: the details are explained in [PZ3] and [PZ4]. Since the $\mathfrak{k}$-types are parametrized here simply by nonnegative integers, one can talk about a truncated category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules: it consists of finite-length admissible modules whose minimal $\mathfrak{k}$-type is larger than or equal to a bound Λ depending on the pair $(\mathfrak{g}, \mathfrak{k})$. The simple objects of this truncated category have been classified in [PZ3] (see also [PZ4]).

The purpose of this paper is to describe the above truncated category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules for $\mathfrak{k} = \mathrm{sl}(2)$ by proving that it is equivalent to an explicit full subcategory of a thick parabolic category $\mathcal{O}$ for $\mathfrak{g}$. In fact, the objects of the truncated category of $(\mathfrak{g}, \mathfrak{k})$-modules are constructed by simply applying the Zuckerman (first derived) functor Γ^1 to a subcategory of a thick parabolic category $\mathcal{O}$. It was conjectured in [PZ3] that the functor Γ^1 yields an equivalence of these categories, and here we prove this conjecture. We construct a left adjoint to Γ^1 defined on all finitely generated admissible $(\mathfrak{g}, \mathfrak{k})$-modules, and then show that, when restricted to the truncated category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules, this functor is an inverse to the appropriately restricted functor Γ^1.

The history of $\big(\mathfrak{g}, \mathrm{sl}(2)\big)$-modules goes back to the 1940's: a classical example here is the Lorentz pair $\big(\mathrm{sl}(2) \oplus \mathrm{sl}(2), \text{diagonal } \mathrm{sl}(2)\big)$ studied by Harish-Chandra [HC], Gelfand-Minlos-Shapiro [GMS], and others. Explaining how exactly the theorem proved in this paper fits in the 70-year history of the topic is a task so complex that we do not really attempt to tackle it. Nonetheless, we would like to mention that in this subject many equivalences of categories have been established; some relate algebraic categories of $\mathfrak{g}$-modules to geometric categories of sheaves, others relate algebraic categories of $\mathfrak{g}$-modules to other algebraic categories of $\mathfrak{g}$-modules. The equivalence we establish is clearly of the second kind and could be seen as an analogue of Bernstein-Gelfand's equivalence of a certain subcategory of Harish-Chandra bimodules (or $(\mathfrak{g} \oplus \mathfrak{g}, \text{diagonal } \mathfrak{g})$-modules) with category $\mathcal{O}$. An extension of the geometric techniques introduced by Beilinson and Bernstein from the theory of Harish-Chandra modules to generalized Harish-Chandra modules is not straightforward (see some results in this direction in [PSZ], [PS], and [Pe]), and fitting the main result of the present paper into a geometric contex is an open problem. We show, however, that the algebraic methods from the 1970's (where, in addition to the third author's contribution, we would like to mention the important contributions by Enright-Varadarajan and Enright), together with the more recent ideas of [PZ1], [PZ3], and [PZ4] (which are building up on Vogan's work), are well suited to yield concrete results about the structure of categories of generalized Harish-Chandra modules.

The paper is structured as follows. We state the main result in Section 3. In particular, we introduce the functor B_1 which will then be shown to be inverse to the functor Γ^1. In Section 4 we present some results which deal mostly with the structure of a semi-thick parabolic category $\mathcal{O}$ we work with. Section 5 contains the proof of the adjointness of Γ^1 and B_1. The proof of the fact that Γ^1 and B_1 are mutually inverse equivalences of categories is carried out in steps throughout Sections 6,7 and 8. In Section 9 we show that for some blocks of the semi-thick parabolic category $\mathcal{O}$, the truncation condition is vacuous, which then implies a stronger equivalence of categories for certain central characters. Finally, in Section 10 we provide an application of our equivalence of categories by proving an estimate for the global dimension of the truncated category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules via a correponding estimate for the truncated semi-thick parabolic category $\mathcal{O}$.

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2. Notations and Conventions

The ground field is $\mathbb{C}$. The superscript * indicates dual spaces. By $\mathfrak{g}$ we will denote a fixed finite-dimensional semisimple Lie algebra. We fix also an sl(2)-subalgebra $\mathfrak{k} \subseteq \mathfrak{g}$. By $\mathfrak{k}^\perp$ we denote the orthogonal (with respect to the Killing form) complement of $\mathfrak{k}$ in $\mathfrak{g}$. The classification of all possible subalgebras $\mathfrak{k}$ up to conjugacy is equivalent to describing all nilpotent orbits in $\mathfrak{g}$, and goes back to Malcev and Dynkin (see [D] and the references therein). By a $\mathfrak{k}$-*type* we mean a simple finite-dimensional sl(2)-module. A simple finite-dimensional sl(2)-module with highest weight $\mu \in \mathbb{Z}_{\geq 0}$ is denoted by $V_{\mathfrak{k}}(\mu)$. By $\mathrm{Soc}\, M$ (respectively, $\mathrm{Top}\, M$), we denote the socle (respectively, the top) of a $\mathfrak{g}$-module M of finite length. $\mathrm{Soc}\, M$ is the maximal semisimple submodule of M, and $\mathrm{Top}\, M$ is the maximal semisimple quotient of M. By $[A : B]$ we denote the multiplicity as a subquotient of a simple module B in a module A. $\mathrm{Res}_{\mathfrak{q}}$ stands for the restriction of a module M to a subalgebra $\mathfrak{q}$, and $M^{\oplus t}$ stands for the direct sum of t copies of M. The sign $\supsetplus$ denotes semidirect sum of Lie algebras (the round side of the sign points to the ideal).

3. Statement of Main Result

The main result of this paper states that certain categories of $\mathfrak{g}$-modules are equivalent via explicit mutually inverse functors. In this section we define these categories and functors.

Recall that $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra and $\mathfrak{k}$ is an arbitrary sl(2)-subalgebra of $\mathfrak{g}$. Fix a standard basis $\left\{e, f, h = [e, f]\right\}$ of $\mathfrak{k}$ and note that h is a semisimple element of $\mathfrak{g}$. Let $\mathfrak{t} = \mathbb{C}h$ be the toral subalgebra of $\mathfrak{g}$ spanned by h. For any $\alpha \in \mathfrak{t}^*$ let $\mathfrak{g}^\alpha$ denote the subspace of $\mathfrak{g}$ of weight α:

$$\mathfrak{g}^\alpha = \{g \in \mathfrak{g} \,|\, [t, g] = \alpha(t) g \;\; \forall t \in \mathfrak{t}\}.$$

Observe that if $\mathfrak{g}^\alpha \neq 0$ for some $\alpha \in \mathfrak{t}^*$ then $\alpha(h) \in \mathbb{Z}$.

Define the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ by setting

$$\mathfrak{p} := C(\mathfrak{t}) \supsetplus \left(\bigoplus_{\substack{\alpha \in \mathfrak{t}^* \\ \alpha(h) > 0}} \mathfrak{g}^\alpha \right),$$

where $C(\mathfrak{t})$ is the centralizer of h in $\mathfrak{g}$. By $\bar{\mathfrak{p}}$ we denote the opposite parabolic subalgebra

$$\bar{\mathfrak{p}} = C(\mathfrak{t}) \supsetplus \left(\bigoplus_{\substack{\alpha \in \mathfrak{t}^* \\ \alpha(h) < 0}} \mathfrak{g}^\alpha \right).$$

We also set

$$\mathfrak{n} := \bigoplus_{\substack{\alpha \in \mathfrak{t}^* \\ \alpha(h) > 0}} \mathfrak{g}^\alpha.$$

Let $\mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t}}$ be the category of finite-length $\mathfrak{g}$-modules which are $\bar{\mathfrak{p}}$-locally finite, $\mathfrak{t}$-semisimple, and $\mathfrak{t}$-integral (i.e., h acts with integer eigenvalues). Informally, $\mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t}}$ is a "semi-thick" ("thick in all directions except the $\mathfrak{t}$-direction") parabolic category $\mathcal{O}$. By $\mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n}$ for $n \in \mathbb{Z}_{\geq 0}$, we denote the n-truncated category $\mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t}}$, i.e., the full subcategory of $\mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t}}$ consisting of objects all $\mathfrak{t}$-weights μ of which satisfy $\mu(h) \geq n$. We also assign an integer Λ to the pair $(\mathfrak{g}, \mathfrak{k})$: we set $\Lambda = \frac{1}{2}(\lambda_1 + \lambda_2)$, where λ_1 (respectively, λ_2) is the maximum (resp., submaximum) weight of $\mathfrak{t}$ in $\mathfrak{g}/\mathfrak{k}$. Here and below, we identify $\mathfrak{t}$-weights with integers via the correspondence $\mu \rightsquigarrow \mu(h)$.

Denote by $\mathcal{C}_{\mathfrak{k}}$ the category of admissible $(\mathfrak{g}, \mathfrak{k})$-modules of finite length, i.e., the category of $\mathfrak{g}$-modules M of finite length on which $\mathfrak{k}$ acts locally finitely and such that $\dim \mathrm{Hom}_{\mathfrak{k}}(L, M) < \infty$ for any $\mathfrak{k}$-type L. By $\mathcal{C}_{\mathfrak{k},n}$ for $n \in \mathbb{Z}_{\geq 0}$, we denote the full subcategory of $C_{\mathfrak{k}}$ consisting of $\mathfrak{g}$-modules M such that $\mathrm{Hom}_{\mathfrak{k}}(L, M) \neq 0$ implies $\dim L > n$.

We now describe two functors: $\Gamma_{\mathfrak{k},\mathfrak{t}}$ and $\mathrm{B}^{\mathfrak{k},\mathfrak{t}}$. The functor $\Gamma_{\mathfrak{k},\mathfrak{t}}$ is the functor of $\mathfrak{k}$-finite vectors in a $(\mathfrak{g}, \mathfrak{t})$-module. That is, if M is a $(\mathfrak{g}, \mathfrak{t})$-module then

$$\Gamma_{\mathfrak{k},\mathfrak{t}} M := \Big\{ m \in M \,\Big|\, \dim U(\mathfrak{k}) \cdot m < \infty \Big\},$$

and $\Gamma_{\mathfrak{k},\mathfrak{t}} M$ is a $\mathfrak{g}$-submodule of M. It is well known (and easy to see) that $\Gamma_{\mathfrak{k},\mathfrak{t}}$ is a left-exact functor. In what follows we set $\Gamma := \Gamma_{\mathfrak{k},\mathfrak{t}}$ and denote the right derived functors $R^i\Gamma_{\mathfrak{k},\mathfrak{t}}$ by Γ^i. The functor Γ^i is known as the *i-th Zuckerman functor*. By definition, Γ^i is a functor from $(\mathfrak{g}, \mathfrak{t})$-mod to $(\mathfrak{g}, \mathfrak{k})$-mod. It is proved in [PZ3] that the restriction of Γ^i to $C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ is a well-defined functor from $C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ to $C_{\mathfrak{k},n}$. We denote this functor also by Γ^i.

Next, we define a functor

$$\mathrm{B}^{\mathfrak{k},\mathfrak{t}} : (\mathfrak{g}, \mathfrak{t})^{\mathrm{fg}}\text{-mod} \rightsquigarrow C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}\,,$$

where $(\mathfrak{g}, \mathfrak{t})^{\mathrm{fg}}$-mod stands for the category of finitely generated $(\mathfrak{g}, \mathfrak{t})$-modules. For this we need to fix some further notation.

Throughout the rest of the paper, $\theta : Z_{U(\mathfrak{g})} \to \mathbb{C}$ denotes a fixed central character. If M is a $\mathfrak{g}$-module, then M^θ stands for the vectors in M on which $z - \theta(z)$ acts locally nilpotently for any $z \in Z_{U(\mathfrak{g})}$. By ℓ we denote a variable positive integer. We also fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $h \in \mathfrak{h}$. Then $\mathfrak{h}$ is also a Cartan subalgebra of the reductive subalgebra $C(\mathfrak{t})$ of $\mathfrak{g}$.

Let $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ be the subcategory of $C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ consisting of modules M with $M = M^\theta$ and such that $\mathfrak{h}$ acts via Jordan blocks of size at most ℓ. We note that $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ is a finite-length category which has an injective cogenerator $I^{\theta,\ell}_{n+2}$. This fact is proved in Lemma 6 below. We set

$$\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell} X := X \Bigg/ \left(\bigcap_{\varphi \in \mathrm{Hom}_{\mathfrak{g}}\left(X, I^{\theta,\ell}_{n+2}\right)} \ker \varphi \right)$$

for $X \in (\mathfrak{g}, \mathfrak{t})^{\mathrm{fg}}$-mod. Lemma 14 below claims that $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell} X \in C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$, which shows that $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell} X$ is the "largest quotient" of X lying in $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$.

Next, we notice that there is a canonical surjective homomorphism

$$\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell} X \twoheadrightarrow \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell-1} X\,,$$

i.e., that $\left\{ \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell} X \right\}$ is an inverse system of $\bar{\mathfrak{p}}$-locally finite $(\mathfrak{g}, \mathfrak{t})$-modules. We set

$$\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta} X := \varprojlim \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell} X\,.$$

It is easy to see that $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}$ is a right-exact functor from $(\mathfrak{g},\mathfrak{t})^{\mathrm{fg}}$-mod to $\mathfrak{g}$-mod, and we denote by $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}_{j}$ its left derived functors, that is $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}_{j} X = L_j \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta} X$ for $X \in (\mathfrak{g},\mathfrak{t})^{\mathrm{fg}}$-mod.

Let $\mathcal{C}^{\theta}_{\mathfrak{t},n}$ and $\mathcal{C}^{\theta}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ be the respective subcategories of $\mathcal{C}_{\mathfrak{k},n}$ and $\mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ consisting of $\mathfrak{g}$-modules M with $M = M^{\theta}$. Corollary 18 below states that in fact $\left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}_{j}$ is a well-defined functor from $\mathcal{C}^{\theta}_{\mathfrak{t},n}$ to $\mathcal{C}^{\theta}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$. As $\mathfrak{k}$ and $\mathfrak{t}$ are fixed, in what follows we set $\mathrm{B}^{\theta,\ell} := \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta,\ell}$, $\mathrm{B}^{\theta} := \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}$, and $\mathrm{B}^{\theta}_{j} := \left(\mathrm{B}^{\mathfrak{k},\mathfrak{t}}\right)^{\theta}_{j}$. By the same letters we also denote the restrictions of these functors to the category $\mathcal{C}^{\theta}_{\mathfrak{t},n}$.

The main result of this paper is the following:

Theorem 1. *For $X \in \mathcal{C}_{\mathfrak{k},n}$, let $\mathrm{B}_j X := \bigoplus_{\theta} \mathrm{B}^{\theta}_{j} X$. Then, for any $n \geq \Lambda$, the functors*

$$\Gamma^1 : \mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2} \rightsquigarrow \mathcal{C}_{\mathfrak{k},n}$$

and

$$\mathrm{B}_1 : \mathcal{C}_{\mathfrak{k},n} \rightsquigarrow \mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$$

are mutually inverse equivalences of categories.

Remark 1. Let us note that for the Lorentz pair $\left(\mathrm{sl}(2)\oplus\mathrm{sl}(2), \text{diagonal}\,\mathrm{sl}(2)\right)$ a description of the category $\mathcal{C}_{\mathfrak{k}}$ was given by I. Gelfand and V. Ponomarev in [GP] already in 1967.

4. Preparatory Results

We start with two general results.

Proposition 2. *Let $\mathfrak{n}_{\mathfrak{k}} := \mathfrak{n} \cap \mathfrak{k}$ ($\dim \mathfrak{n}_{\mathfrak{k}} = 1$).*

a) For any $(\mathfrak{g},\mathfrak{k})$-module X there exists a singly graded spectral sequence converging to $H_i(\mathfrak{n}, X)$ such that its E^1-term has the form

$$E^1_i = H_0(\mathfrak{n}_{\mathfrak{k}}, X) \otimes \Lambda^i(\mathfrak{n} \cap \mathfrak{k}^{\perp}) \oplus H_1(\mathfrak{n}_{\mathfrak{k}}, X) \otimes \Lambda^{i-1}(\mathfrak{n} \cap \mathfrak{k}^{\perp}). \tag{1}$$

b) If $X \in \mathcal{C}_{\mathfrak{k},n}$ for $n \geq 0$, then the $\mathfrak{n}$-homology $H_{\bullet}(\mathfrak{n}, X)$ is finite dimensional.

Proof. a) The statement follows from Proposition 3.1 in [PZ3] and the formula for the singly graded E^1-term right after Proposition 3.1. Formula (1) is a direct cosequence of the above formula in [PZ3] if one takes into account that in our case $\mathfrak{k}$ is isomorphic to sl(2). In fact, the statement holds more generally for any $\mathfrak{g}$-module X but the assumption that X is a $(\mathfrak{g},\mathfrak{k})$-module is sufficient for us.

b) This follows from the more general statement of Proposition 3.5 in [PZ3].

□

Let $M = \oplus_{p \in \mathbb{C}} M_p$ be an admissible $(\mathfrak{g}, \mathfrak{t})$-module where M_p is the $\mathfrak{t}$-weight space in M of weight p: by definition, $hm = pm$ for $m \in M_p$. Set $M^*_{\mathfrak{t}} := \oplus_{p \in \mathbb{C}} M^*_p$. Then $M^*_{\mathfrak{t}}$ is a well-defined admissible $(\mathfrak{g}, \mathfrak{t})$-module. Similarly let $X = \oplus_{\mu \in \mathbb{Z}_{\geq 0}} \tilde{V}_{\mathfrak{k}}(\mu)$ be an admissible $(\mathfrak{g}, \mathfrak{k})$-module. Here $\tilde{V}_{\mathfrak{k}}(\mu)$ stands for the $V_{\mathfrak{k}}(\mu)$-isotypic component in X. Then $X^*_{\mathfrak{k}} := \oplus_{\mu \in \mathbb{Z}_{\geq 0}} \tilde{V}_{\mathfrak{k}}(\mu)^*$ is a well-defined admissible $(\mathfrak{g}, \mathfrak{k})$-module. Moreover, $(\bullet)^*_{\mathfrak{t}}$ and $(\bullet)^*_{\mathfrak{k}}$ are well-defined contravariant functors (in fact antiequivalences) on the respective categories of admissible $(\mathfrak{g}, \mathfrak{t})$-modules and $(\mathfrak{g}, \mathfrak{k})$-modules.

In what follows we will use the composition of the functors $(\bullet)^*_{\mathfrak{t}}$ and $(\bullet)^*_{\mathfrak{k}}$ with the twist by the Cartan involution of $\mathfrak{g}$ which acts as $-\mathrm{id}$ on $\mathfrak{h}$. The so obtained new functors are denoted respectively by $(\cdot)^\vee_{\mathfrak{t}}$ and $(\cdot)^\vee_{\mathfrak{k}}$. These functors preserve the respective $\mathfrak{t}$- and $\mathfrak{k}$-characters of the modules.

A duality theorem proved in [EW] implies the following:

Proposition 3. *For any admissible* $(\mathfrak{g}, \mathfrak{t})$*-module* M *there is a natural isomorphism of admissible* $(\mathfrak{g}, \mathfrak{k})$*-modules*

$$(\Gamma^i M)^\vee_{\mathfrak{k}} \simeq \Gamma^{2-i}(M^\vee_{\mathfrak{t}}).$$

In the remainder of the paper E stands for a finite-dimensional simple $C(\mathfrak{t})$-module on which h acts via a natural number $|E|$. Often, we consider E as a $\bar{\mathfrak{p}}$-module by setting $\bar{\mathfrak{n}} \cdot E := 0$. In this case we set also

$$M(E) := U(\mathfrak{g}) \underset{U(\bar{\mathfrak{p}})}{\otimes} E$$

and let $L(E)$ be the unique simple quotient of $M(E)$. Then $L(E)^\vee \simeq L(E)$, and $M(E)^\vee$ is an indecomposable object of $C_{\bar{\mathfrak{p}},\mathfrak{t}}$ with $\mathrm{Soc}\, M(E)^\vee = L(E)^\vee \simeq L(E)$.

The following proposition is a summary of preliminary results concerning the specific categories we study in this paper.

Proposition 4. *Let* $n \geq 0$*. Then*

a) $\Gamma^1 : C_{\bar{\mathfrak{p}},\mathfrak{t},n+2} \rightsquigarrow C_{\mathfrak{k},n}$ *is a faithful exact functor;*

b) under the assumption that $n \geq \Lambda$*, the functor from a) maps a simple object to a simple object and induces a bijection on the isomorphism classes of simple objects in* $C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ *and in* $C_{\mathfrak{k},n}$*, respectively. Moreover, the simple* $(\mathfrak{g}, \mathfrak{k})$*-module* $\Gamma^1 L(E)$ *has minimal* $\mathfrak{k}$*-type* $|E| - 2$.

c) under the assumption that $n \geq \Lambda$*,* $\Gamma^1 L(E)$ *and* $\mathrm{Top}\, \Gamma^1 M(E)$ *are isomorphic simple* $(\mathfrak{g}, \mathfrak{k})$*-modules with minimal* $\mathfrak{k}$*-type* $|E| - 2$*, and the isotypic components of the minimal* $\mathfrak{k}$*-types of* $\Gamma^1 M(E)$ *and* $\Gamma^1 L(E)$ *are isomorphic.*

Proof. Parts a) and b) follow directly from the results of [PZ3], see Proposition 7.8 and Corollary 6.4. Part c) is a consequence of the above mentioned results and the fact that the functor Γ^1 commutes with $(\bullet)^\vee$ according to Proposition 3. □

Remark 2. Since Γ^1 preserves central characters, Proposition 4, b) implies in particular the existence of a bijection between the isomorphism classes of simple objects in $C^\theta_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ and $C^\theta_{\mathfrak{k},n}$ for $n \geq \Lambda$. Without the condition $n \geq \Lambda$, no such bijection exists in general. For instance, if $(\mathfrak{g}, \mathfrak{k})$ is the Lorentz pair and θ is the central character of a finite-dimensional $\mathfrak{g}$-module of the form $V \boxtimes V$ for a simple finite-dimensional sl(2)-module V, then $C^{\theta}_{\bar{\mathfrak{p}},\mathfrak{t},2}$ has 3 pairwise nonisomorphic simple objects while $C^{\theta}_{\mathfrak{t},0}$ has two nonisomorphic simple objects.

The rest of the section is devoted to results on $\bar{\mathfrak{p}}$-locally finite modules.

By ß we denote the derived subalgebra of the reductive Lie algebra $C(\mathfrak{t})$, and by $\mathfrak{c}$ the center of $C(\mathfrak{t})$. Then $\mathfrak{c} \subset \mathfrak{h}$. Let $\mathcal{F}_{C(\mathfrak{t}),\mathfrak{t}}$ be the category of locally finite $C(\mathfrak{t})$-modules semisimple over $\mathfrak{t}$ with integral h-eigenvalues. By $\mathcal{F}^{\ell}_{C(\mathfrak{t}),\mathfrak{t}}$ we denote the subcategory of $\mathcal{F}_{C(\mathfrak{t}),\mathfrak{t}}$ consisiting of modules on which $\mathfrak{c}$ acts via Jordan blocks of size less than or equal to ℓ. Clearly

$$\mathcal{F}_{C(\mathfrak{t}),\mathfrak{t}} = \varinjlim \mathcal{F}^{\ell}_{C(\mathfrak{t}),\mathfrak{t}}.$$

Note that $(\bullet)^{\vee}$ is also a well-defined functor on the category $\mathcal{F}^{\ell}_{C(\mathfrak{t}),\mathfrak{t}}$ (but not on $\mathcal{F}_{C(\mathfrak{t}),\mathfrak{t}}$).

Lemma 5. *Let S be a simple finite-dimensional ß-module and $\lambda \in \mathfrak{c}^*$ be a $\mathfrak{t}$-integral weight. Define E as $S \otimes \mathbb{C}_{\lambda}$ where $\mathbb{C}_{\lambda}$ is a one-dimensional $\mathfrak{c}$-module with weight λ. Let I^{ℓ}_{λ} denote the ideal in $S(\mathfrak{c})$ generated by $h - \lambda(h)$ and $(z - \lambda(z))^{\ell}$ for all $z \in \mathfrak{c}$. Then*

a) Every simple object in $\mathcal{F}_{C(\mathfrak{t}),\mathfrak{t}}$ is isomorphic to E for some choice of S and λ. Furthermore, $E^{\vee} \simeq E$.

b) $E^{\ell} := E \otimes (S(\mathfrak{c})/I^{\ell}_{\lambda})$ is a projective cover of E and $\left(E^{\ell}\right)^{\vee}$ is an injective hull of E in $\mathcal{F}^{\ell}_{C(\mathfrak{t}),\mathfrak{t}}$.

c) $\bar{E} := \varinjlim \left(E^{\ell}\right)^{\vee}$ is an injective hull of E in $\mathcal{F}_{C(\mathfrak{t}),\mathfrak{t}}$.

Proof. (a) is obvious. To show (b), note that S is projective in the category of locally finite ß-modules, and that E^{ℓ} is the maximal quotient of the induced module $U(C(\mathfrak{t})) \otimes_{U(ß)} S$ lying in in $\mathcal{F}^{\ell}_{C(\mathfrak{t}),\mathfrak{t}}$. Then (c) is clearly a corollary of (b). □

Recall that, for any two Lie algebras $\mathfrak{a}' \subset \mathfrak{a}$, the functor

$$\mathrm{pro}^{\mathfrak{a}}_{\mathfrak{a}'} : \mathfrak{a}'\text{-mod} \rightsquigarrow \mathfrak{a}\text{-mod}$$

is defined as

$$\mathrm{pro}^{\mathfrak{a}}_{\mathfrak{a}'} K' = \mathrm{Hom}_{U(\mathfrak{a}')}\left(U(\mathfrak{a}), K'\right)$$

for an $\mathfrak{a}'$-module K'. In addition, if $\mathfrak{a}''$ is an abelian Lie subalgebra of $\mathfrak{a}$, we have the functor

$$\Gamma_{\mathfrak{a}''} : \mathfrak{a}\text{-mod} \rightsquigarrow \mathfrak{a}\text{-mod}$$

of $\mathfrak{a}''$-weight vectors defined as

$$\Gamma_{\mathfrak{a}''} K = \bigoplus_{\alpha \in (\mathfrak{a}'')^*} K^{\alpha},$$

where $K \in \mathfrak{a}$-mod and

$$K^{\alpha} = \left\{k \in K \,|\, a''k = \alpha(a'')\, k \text{ for all } a'' \in \mathfrak{a}''\right\}.$$

The following generalizes basic results in [BGG].

Lemma 6. *For any $\ell > 0$ and any $n \geq 0$, the abelian category $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ has a unique, up to isomorphism, minimal injective cogenerator $I^{\theta,\ell}_{n+2}$. Moreover, the* $\mathfrak{t}$*-weight spaces of $I^{\theta,\ell}_{n+2}$ are finite dimensional.*

Proof. We denote by $\mathcal{F}^{\ell}_{\mathfrak{p},\mathfrak{t}}$ the category of $\mathfrak{p}$-modules whose restrictions to $C(\mathfrak{t})$ belong to $\mathcal{F}^{\ell}_{C(\mathfrak{t}),t}$, and by $\mathcal{F}^{\ell}_{\mathfrak{p},\mathfrak{t},n+2}$ the subcategory consisting of modules whose $\mathfrak{t}$-weights are bounded from below by $n+2$. Let E be as in Lemma 5. Endow E with a $\mathfrak{p}$-module structure by letting $\mathfrak{n}$ act trivially on E, and consider the $\mathfrak{p}$-module

$$\Gamma_{\mathfrak{t}}\mathrm{pro}^{\mathfrak{p}}_{C(\mathfrak{t})}\left(\left(E^{\ell}\right)^{\vee}\right) = \Gamma_{\mathfrak{t}}\,\mathrm{Hom}_{U(C(\mathfrak{t}))}\left(U(\mathfrak{p}), \left(E^{\ell}\right)^{\vee}\right). \tag{2}$$

Recall that $\mathrm{pro}^{\mathfrak{p}}_{C(\mathfrak{t})}(\cdot)$ preserves injectivity and that the functor of $\mathfrak{t}$-weight vectors $\Gamma_{\mathfrak{t}}$ is right adjoint to the inclusion of the category of $\mathfrak{p}$-modules semisimple over $\mathfrak{t}$ into the category of all $\mathfrak{p}$-modules. Therefore $\Gamma_{\mathfrak{t}}$ also preserves injectivity, and the $\mathfrak{p}$-module (2) is injective in $\mathcal{F}^{\ell}_{\mathfrak{p},\mathfrak{t}}$. It is straightforward to verify that this module is an injective hull of E in $\mathcal{F}^{\ell}_{\mathfrak{p},\mathfrak{t}}$. Consequently, the truncated submodule $\left(\Gamma_{\mathfrak{t}}\mathrm{pro}^{\mathfrak{p}}_{C(\mathfrak{t})}\left(\left(E^{\ell}\right)^{\vee}\right)\right)_{\geq n+2}$ of (2), spanned by all $\mathfrak{t}$-weight spaces with weights greater or equal than $n+2$, is an injective hull of E in $\mathcal{F}^{\ell}_{\mathfrak{p},\mathfrak{t},n+2}$.

Set

$$J^{\ell}(E) := \left[\Gamma_{\mathfrak{t}}\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{p}}\left(\left(\Gamma_{\mathfrak{t}}\mathrm{pro}^{\mathfrak{p}}_{C(\mathfrak{t})}\left(\left(E^{\ell}\right)^{\vee}\right)\right)_{\geq n+2}\right)\right]^{\theta}.$$

Then, by a similar argument, $J^{\ell}(E)$ is injective in $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ and we have an embedding of $\mathfrak{g}$-modules $L(E) \hookrightarrow J^{\ell}(E)$ induced by the embedding of $\mathfrak{p}$-modules $E \hookrightarrow \left(\Gamma_{\mathfrak{t}}\mathrm{pro}^{\mathfrak{p}}_{C(\mathfrak{t})}\left(\left(E^{\ell}\right)^{\vee}\right)\right)_{\geq n+2}$. The $\mathfrak{p}$-module $\left(\Gamma_{\mathfrak{t}}\mathrm{pro}^{\mathfrak{p}}_{C(\mathfrak{t})}\left(\left(E^{\ell}\right)^{\vee}\right)\right)_{\geq n+2}$ is finite dimensional. Moreover, it is easy to check that the $\mathfrak{g}$-module $J^{\ell}(E)$ has finite-dimensional $\mathfrak{t}$-weight spaces.

Note that, up to isomorphisms, $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ has finitely many simple objects $L(E_1)$, $\dots$, $L(E_r)$. Each of them has a unique, up to isomorphism, injective hull $I^{\ell}\left(E_j\right)$ which is a submodule of $J^{\ell}(E_j)$. Then $I^{\theta,\ell}_{n+2}$ is the direct sum $\bigoplus_{j=1}^{r} I^{\ell}\left(E_j\right)$. □

Corollary 7. *Let $A^{\theta,\ell}_{n+2} := \mathrm{End}_{\mathfrak{g}}\, I^{\theta,\ell}_{n+2}$. Then $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ is equivalent to the category of finite-dimensional $A^{\theta,\ell}_{n+2}$-modules.*

Let $C^{\theta,\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ be the category of inductive limits of objects from $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$.

Corollary 8. *For any n, the category $C^{\theta,\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ has a unique, up to isomorphism, minimal injective cogenerator I^{θ}_{n+2}. Moreover, $I^{\theta}_{n+2} = \varinjlim I^{\theta,\ell}_{n+2}$. In particular, the category $C^{\theta,\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ has enough injectives.*

In fact, if I is any injective object in $C^{\theta,\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$, then I is a direct limit, $\varinjlim I^{\ell}$, for injective objects $I^{\ell} \in C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$.

Recall the definition of $\bar{E}$ from Lemma 5, and let

$$\overline{W}(E) := \Gamma_{\mathfrak{t}}\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{p}}(\bar{E}).$$

Lemma 9. *Let $\mathcal{F}_{\bar{\mathfrak{p}},\mathrm{t}}$ be the category of locally finite $\bar{\mathfrak{p}}$-modules such that their restrictions to $C(\mathrm{t})$ lie in $\mathcal{F}_{C(\mathrm{t}),t}$. Then $\mathrm{Res}_{\bar{\mathfrak{p}}}\,\overline{W}(E)$ is an injective hull of E in $\mathcal{F}_{\bar{\mathfrak{p}},\mathrm{t}}$. Moreover, $\mathrm{Res}_{C(\mathrm{t})}\,\overline{W}(E)$ is isomorphic to a finite direct sum $\bigoplus_\alpha \bar{F}_\alpha$ for some finite-dimensional irreducible $C(\mathrm{t})$-modules F_α.*

Proof. By the Poincare–Birkhoff–Witt Theorem we have an isomorphism

$$\mathrm{Res}_{\bar{\mathfrak{p}}}\,\overline{W}(E) \simeq \Gamma_{\mathrm{t}}\,\mathrm{Hom}_{C(\mathrm{t})}(U(\bar{\mathfrak{p}}), \bar{E}) = \Gamma_{\mathrm{t}}\mathrm{pro}^{\bar{\mathfrak{p}}}_{C(\mathrm{t})}\bar{E}.$$

Since $\bar{E}$ is an injective module in $\mathcal{F}_{C(\mathrm{t}),t}$, $\Gamma_{\mathrm{t}}\mathrm{pro}^{\bar{\mathfrak{p}}}_{C(\mathrm{t})}\bar{E}$ is an injective module in $\mathcal{F}_{\bar{\mathfrak{p}},\mathrm{t}}$. Hence $\mathrm{Res}_{\bar{\mathfrak{p}}}\,\overline{W}(E)$ is an injective module in $\mathcal{F}_{\bar{\mathfrak{p}},\mathrm{t}}$.

Let $S(E)$ be the socle of $\mathrm{Res}_{\bar{\mathfrak{p}}}\,\overline{W}(E)$ as a module over $C(\mathrm{t})$. Since $\mathrm{Res}_{C(\mathrm{t})}\,\overline{W}(E)$ is locally $C(\mathrm{t})$-finite, it is an essential extension of $S(E)$, and therefore $\mathrm{Res}_{C(\mathrm{t})}\,\overline{W}(E)$ is by definition an injective hull of $S(E)$.

Let $S(E) = \bigoplus_\alpha F_\alpha$ for some finite-dimensional irreducible $C(\mathrm{t})$-modules F_α. We now prove that $\mathrm{Res}_{C(\mathrm{t})}\,\overline{W}(E) \simeq \bigoplus_\alpha \bar{F}_\alpha$. For this it is enough to show that the $C(\mathrm{t})$-module $T(E) := \bigoplus_\alpha \bar{F}_\alpha$ is injective as $\mathrm{Soc}\,T(E) = S(E)$, $T(E)$ is an essential extension of $S(E)$, and any two injective hulls of $S(E)$ are isomorphic. Finally, the injectivity of $T(E)$ follows directly from the left Noetherian property of $U\big(C(\mathrm{t})\big)$ since any direct sum of injective modules over a left Noetherian algebra is injective. □

We define an object $M \in C^{\mathrm{ind}}_{\bar{\mathfrak{p}},\mathrm{t}}$ to admit a *parabolic co-Verma filtration* if there exists a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$$

whose successive quotients M_{i+1}/M_i are isomorphic to $\overline{W}(E_i)$ for simple $C(\mathrm{t})$-modules $E_1, \dots, E_t$. In what follows we say simply "co-Verma filtration".

Lemma 10. *Let M be an object of $C^{\mathrm{ind}}_{\bar{\mathfrak{p}},\mathrm{t}}$. Then M admits a co-Verma filtration if and only if $\mathrm{Res}_{\bar{\mathfrak{p}}}\,M$ is injective in $\mathcal{F}_{\bar{\mathfrak{p}},\mathrm{t}}$ with socle of finite length.*

Proof. If M admits a co-Verma filtration, then $\mathrm{Res}_{\bar{\mathfrak{p}}}\,M$ is a direct sum of modules of the form $\overline{W}(F)$, and by Lemma 9 $\mathrm{Res}_{\bar{\mathfrak{p}}}\,M$ is injective in $\mathcal{F}_{\bar{\mathfrak{p}},\mathrm{t}}$ with $\bar{\mathfrak{p}}$-socle of finite length.

To prove the opposite assertion, choose a simple $\bar{\mathfrak{p}}$-submodule $E \subset \mathrm{Res}_{\bar{\mathfrak{p}}}\,M$ with minimal $|E|$. The existence of E follows from the fact that the socle of $\mathrm{Res}_{\bar{\mathfrak{p}}}\,M$ has finite length. Let k be the multiplicity of E in $\mathrm{Soc}\,\mathrm{Res}_{\bar{\mathfrak{p}}}\,M$. Then we have a surjective morphism $\varphi : \mathrm{Res}_{\mathfrak{p}}\,M \to \bar{E}^{\oplus k}$ of $\mathfrak{p}$-modules ($\varphi|_{\bar{E}^{\otimes k}}$ being the identity map) which induces a morphism $\tilde{\varphi} : M \to \overline{W}(E)^{\oplus k}$ of $\mathfrak{g}$-modules by Frobenius reciprocity. Since $\mathrm{Res}_{\bar{\mathfrak{p}}}\,M$ is injective in $\mathcal{F}_{\bar{\mathfrak{p}},\mathrm{t}}$, and since $\mathrm{Res}_{\bar{\mathfrak{p}}}$ is $\bar{\mathfrak{p}}$-locally finite, we have that $\mathrm{Res}_{\bar{\mathfrak{p}}}\,M$ is an injective hull of its socle, i.e.,

$$\mathrm{Res}_{\bar{\mathfrak{p}}}\,M \simeq \mathrm{Res}_{\bar{\mathfrak{p}}}\left(\overline{W}(E)^{\oplus k} \oplus \bigoplus_{|F|>|E|} \overline{W}(F)\right)$$

by Lemma 9. Moreover, $\mathrm{Hom}_{\bar{\mathfrak{p}}}\left(\mathrm{Res}_{\bar{\mathfrak{p}}}\,\overline{W}(F), \mathrm{Res}_{\bar{\mathfrak{p}}}\,\overline{W}(E)\right) = 0$ if $|F| > |E|$. Therefore $\tilde{\varphi}\left(\mathrm{Res}_{\bar{\mathfrak{p}}}\left(\bigoplus_{|F|>|E|}\overline{W}(F)\right)\right) = 0$, and

$$\tilde{\varphi}\big|_{\mathrm{Res}_{\bar{\mathfrak{p}}}\,\overline{W}(E)^{\oplus k}} : \mathrm{Res}_{\bar{\mathfrak{p}}}\,\overline{W}(E)^{\oplus k} \to \mathrm{Res}_{\bar{\mathfrak{p}}}\,\overline{W}(E)^{\oplus k}$$

is an isomorphism of $\bar{\mathfrak{p}}$-modules since it is induced by the identity map $\varphi|_{\bar{E}^{\oplus k}} : \bar{E}^{\oplus k} \to \bar{E}^{\oplus k}$.

Set $Q := \ker\tilde{\varphi}$. Then $\mathrm{Res}_{\bar{\mathfrak{p}}}\,Q$ is isomorphic to $\mathrm{Res}_{\bar{\mathfrak{p}}}\left(\bigoplus_{|F|>|E|}\overline{W}(F)\right)$, and hence Q satisfies all conditions of the lemma. So we can finish the proof by induction on the length of the socle of $\mathrm{Res}_{\bar{\mathfrak{p}}}\,M$. □

Corollary 11. *Let $R = M \oplus N$ for some $M, N \in \mathcal{C}^{\theta,\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t}}$. Suppose that R admits a co-Verma filtration. Then M and N also admit co-Verma filtrations.*

Proof. A direct summand of an injective module is injective, so the statement follows from Lemma 10. □

Lemma 12. *Let $I(E)$ be an injective hull of $L(E)$ in $\mathcal{C}^{\theta,\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$. Then $I(E)/\overline{W}(E)$ admits a co-Verma filtration with successive quotients isomorphic to $\overline{W}(D)$ for $|D| < |E|$.*

Proof. Let $J(E) := \left[\Gamma_{\mathfrak{t}}\mathrm{pro}^{\mathfrak{g}}_{\mathfrak{p}}\left(\left(\Gamma_{\mathfrak{t}}\mathrm{pro}^{\mathfrak{p}}_{C(\mathfrak{t})}(\bar{E})\right)_{\geq n+2}\right)\right]^{\theta}$. The $\mathfrak{p}$-module $\Gamma_{\mathfrak{t}}\left(\left(\mathrm{pro}^{\mathfrak{p}}_{C(\mathfrak{t})}(\bar{E})\right)_{\geq n+2}\right)$ has a finite filtration with successive quotients $\bar{D}$ such that $|D| \leq |E|$. Moreover, the quotient $\left(\Gamma_{\mathfrak{t}}\left(\left(\mathrm{pro}^{\mathfrak{p}}_{C(\mathfrak{t})}(\bar{E})\right)_{\geq n+2}\right)\right)/\bar{E}$ has a filtration with successive quotients $\bar{D}$ for $|D| < |E|$. Therefore $J(E)/\overline{W}(E)$ admits a co-Verma filtration with successive quotients $\overline{W}(D)$ such that $|D| < |E|$.

Similarly as in the proof of Lemma 6, $I(E)$ is a direct summand of $J(E)$. Therefore, by Lemma 11, $I(E)$ has a filtration as desired. □

Corollary 13. *I^{θ}_{n+2} admits a co-Verma filtration.*

5. Adjointness of B_1 and Γ^1

In this section, n is an arbitrary nonnegative integer.

Lemma 14. *For any $\ell \in \mathbb{Z}_{>0}$, $\mathrm{B}^{\theta,\ell}$ is a right-exact functor from $(\mathfrak{g},\mathfrak{t})^{\mathrm{fg}}$-mod into $\mathcal{C}^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ (in particular, $\mathrm{B}^{\theta,\ell}X$ has finite length for $X \in (\mathfrak{g},\mathfrak{t})^{\mathrm{fg}}$-mod).*

Proof. Fix $X \in (\mathfrak{g},\mathfrak{t})^{\mathrm{fg}}$-mod. Then $\mathrm{Hom}_{\mathfrak{g}}\left(X, I^{\theta,\ell}_{n+2}\right)$ is finite dimensional. This follows from the fact that the $\mathfrak{t}$-weight spaces of $I^{\theta,\ell}$ are finite dimensional. As a consequence, $\mathrm{B}^{\theta,\ell}X$ is isomorphic to a submodule of a finite direct sum of copies of $I^{\theta,\ell}_{n+2}$. Since $I^{\theta,\ell}_{n+2}$ has finite length, $\mathrm{B}^{\theta,\ell}X$ also has finite length, and is an object of $\mathcal{C}^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$.

The fact that $\mathrm{B}^{\theta,\ell}$ is right-exact follows from the observation that $\mathrm{B}^{\theta,\ell}$ is left adjoint to the inclusion functor $\mathcal{C}^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2} \rightsquigarrow (\mathfrak{g},\mathfrak{t})^{\mathrm{fg}}$-mod, i.e.,

$$\mathrm{Hom}_{\mathcal{C}^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}}\left(\mathrm{B}^{\theta,\ell}X, M\right) \simeq \mathrm{Hom}_{\mathfrak{g},\mathfrak{t}}(X, M)$$

for any $X \in (\mathfrak{g}, \mathfrak{t})^{\mathrm{fg}}$-mod and $M \in C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$. Indeed, a left adjoint to a left-exact functor is right-exact. □

Next we need to recall the Mittag-Leffler principle. Let $K_\bullet$ be a complex which is the limit of an inverse system of morphisms of complexes $K^{l+1}_\bullet \to K^l_\bullet$ for $l \in \mathbb{Z}_{\geq 0}$. Then for each l and j the images of $H_j(K^{l+l'}_\bullet)$ in $H_j(K^l_\bullet)$ for $l' \geq 1$ form a descending chain of subspaces. The Mittag-Leffler principle asserts that if, for a fixed j, the filtration in $H_{j-1}(K^l_\bullet)$ stabilizes for every l, then $H_j(K_\bullet)$ is isomorphic to the inverse limit of $H_j(K^l_\bullet)$, see for example [W], Theorem 3.5.8.

Lemma 15. *For any* $X \in (\mathfrak{g}, \mathfrak{t})^{\mathrm{fg}}$-mod *and* $j \in \mathbb{Z}_{\geq 0}$, *we have*

$$\mathrm{B}^\theta_j X \simeq \varprojlim \mathrm{B}^{\theta,\ell}_j X\,,$$

where $\mathrm{B}^{\theta,\ell}_j$ *is the j-th left derived functor of* $\mathrm{B}^{\theta,\ell}$.

Proof. Let $P_\bullet$ be a projective resolution of X in the category $(\mathfrak{g}, \mathfrak{t})^{\mathrm{fg}}$-mod. By definition, $\mathrm{B}^\theta_\bullet X$ is the homology of the complex $\mathrm{B}^\theta P_\bullet$, and $\mathrm{B}^{\theta,\ell}_\bullet X$ is the homology of the complex $\mathrm{B}^{\theta,\ell} P_\bullet$. Moreover,

$$\mathrm{B}^\theta P_\bullet = \varprojlim \mathrm{B}^{\theta,\ell} P_\bullet\,.$$

By Lemma 14, for every ℓ, j and q, $\mathrm{B}^{\theta,\ell}_j X_q$ has finite length as a $\mathfrak{g}$-module. So, Lemma 15 follows from the Mittag-Leffler Principle. □

If $\left\{A^\ell\right\}$ is an inverse system of objects from $C^{\theta,\ell}_{\bar{\mathfrak{p}},\mathfrak{t}}$ for variable $\ell \to \infty$, we set (by analogy with the continuous dual of an inverse limit of topological spaces)

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathfrak{g}}\left(\varprojlim A^\ell, M\right) := \varinjlim \mathrm{Hom}_{\mathfrak{g}}\left(A^\ell, M\right)$$

for any $\mathfrak{g}$-module M.

Proposition 16. *Let I be injective in* $C^{\theta,\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$. *Then, for any finitely generated* $(\mathfrak{g}, \mathfrak{t})$-*module X and for any* $j \in \mathbb{Z}_{\geq 0}$,

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathfrak{g}}\left(\mathrm{B}^\theta_j X, I\right) \simeq \mathrm{Ext}^j_{\mathfrak{g},\mathfrak{t}}\left(X, I\right)\,.$$

Proof. Let $P_\bullet$ be as in the proof of Lemma 15. Then

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathfrak{g}}\left(\mathrm{B}^\theta_j X, I\right) = \mathrm{Hom}^{\mathrm{cont}}_{\mathfrak{g}}\left(H_j(\mathrm{B}^\theta P_\bullet), I\right) = \mathrm{Hom}^{\mathrm{cont}}_{\mathfrak{g}}\left(H_j\left(\varprojlim \mathrm{B}^{\theta,\ell} P_\bullet\right), I\right)\,.$$

Since

$$H_j\left(\varprojlim \mathrm{B}^{\theta,\ell} P_\bullet\right) \simeq \varprojlim H_j\left(\mathrm{B}^{\theta,\ell} P_\bullet\right)$$

by the Mittag-Leffler Principle, we have

$$\mathrm{Hom}_{\mathfrak{g}}^{\mathrm{cont}}\left(H_j\left(\varprojlim \mathrm{B}^{\theta,\ell}P_\bullet\right), I\right) \simeq \mathrm{Hom}_{\mathfrak{g}}^{\mathrm{cont}}\left(\varprojlim H_j\left(\mathrm{B}^{\theta,\ell}P_\bullet\right), I\right) = \varinjlim \mathrm{Hom}_{\mathfrak{g}}\left(H_j\left(\mathrm{B}^{\theta,\ell}P_\bullet\right), I\right).$$

Next, the injectivity of I in $\mathcal{C}^{\theta,\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ implies

$$\mathrm{Hom}_{\mathfrak{g}}\left(H_j\left(\mathrm{B}^{\theta,\ell}P_\bullet\right), I\right) \simeq H_j\Big(\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}^{\theta,\ell}P_\bullet, I\right)\Big).$$

Consequently,

$$\varinjlim \mathrm{Hom}_{\mathfrak{g}}\left(H_j\left(\mathrm{B}^{\theta,\ell}P_\bullet\right), I\right) \simeq \varinjlim H_j\Big(\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}^{\theta,\ell}P_\bullet, I\right)\Big),$$

and since homology commutes with direct limits,

$$\varinjlim H_j\Big(\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}^{\theta,\ell}P_\bullet, I\right)\Big) \simeq H_j\Big(\varinjlim \mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}^{\theta,\ell}P_\bullet, I\right)\Big). \tag{3}$$

Recalling that $I = \varinjlim I^\ell$, we notice that

$$\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}^{\theta,\ell}P_\bullet, I\right) = \mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}^{\theta,\ell}P_\bullet, I^\ell\right) = \mathrm{Hom}_{\mathfrak{g}}\left(P_\bullet, I^\ell\right). \tag{4}$$

Furthermore, since $P_\bullet$ is finitely generated,

$$\varinjlim \mathrm{Hom}_{\mathfrak{g}}\left(P_\bullet, I^\ell\right) \simeq \mathrm{Hom}_{\mathfrak{g}}\left(P_\bullet, I\right). \tag{5}$$

Therefore, (3), (4), and (5) yield

$$\varinjlim H_j\Big(\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}^{\theta,\ell}P_\bullet, I\right)\Big) \simeq H_j\Big(\mathrm{Hom}_{\mathfrak{g}}\left(P_\bullet, I\right)\Big).$$

Since $H_j\Big(\mathrm{Hom}_{\mathfrak{g}}\left(P_\bullet, I\right)\Big) = \mathrm{Ext}^j_{\mathfrak{g},\mathfrak{t}}\left(X, I\right)$, we obtain

$$\mathrm{Hom}_{\mathfrak{g}}^{\mathrm{cont}}\left(\mathrm{B}^\theta_j X, I\right) \simeq \mathrm{Ext}^j_{\mathfrak{g},\mathfrak{t}}\left(X, I\right)$$

as desired. □

Recall that I^θ_{n+2} is an injective cogenerator of the category $\mathcal{C}^{\theta,\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$.

Proposition 17. *For any $X \in \mathcal{C}_{\mathfrak{k},n}$ and any $j \geq 0$, $\mathrm{Ext}^j_{\mathfrak{g},\mathfrak{t}}\left(X, I^\theta_{n+2}\right)$ is finite dimensional.*

Proof. By Lemma 12, it suffices to show that $\dim \mathrm{Ext}^j_{\mathfrak{g},\mathfrak{t}}\left(X, \overline{W}(E)\right) < \infty$ for any E with $\overline{W}(E) \in \mathcal{C}^{\theta,\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$. Shapiro's Lemma yields

$$\mathrm{Ext}^j_{\mathfrak{g},\mathfrak{t}}\left(X, \overline{W}(E)\right) = \mathrm{Ext}^j_{\mathfrak{g},\mathfrak{t}}\left(X, \Gamma_t \mathrm{pro}^{\mathfrak{g}}_{\mathfrak{p}} \bar{E}\right) \simeq \mathrm{Ext}^j_{\mathfrak{p},\mathfrak{t}}\left(X, \bar{E}\right).$$

Since by the injectivity of $\bar{E}$ as a $C(\mathfrak{t})$-module we have $\mathrm{Ext}^j_{\mathfrak{p},\mathfrak{t}}\left(X, \bar{E}\right) \simeq \mathrm{Hom}_{C(\mathfrak{t})}\left(H_j(\mathfrak{n}, X), \bar{E}\right)$, we conclude that

$$\mathrm{Ext}^j_{\mathfrak{g},\mathfrak{t}}\left(X, \overline{W}(E)\right) \simeq \mathrm{Hom}_{C(\mathfrak{t})}\left(H_j(\mathfrak{n}, X), \bar{E}\right). \tag{6}$$

Now the statement follows from the finite-dimensionality of $H_j(\mathfrak{n}, X)$, see Proposition 2, b). □

Corollary 18. *For $X \in \mathcal{C}_{\mathfrak{k},n}$ and any $j \geq 0$, we have $\mathrm{B}_j X \in \mathcal{C}_{\bar{\mathfrak{p}},t,n+2}$.*

Proof. By Lemma 14, $\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_j^{\theta,\ell} X, I_{n+2}^{\theta}\right)$ is finite dimensional for any ℓ. By Propositions 16 and 17, $\mathrm{Hom}_{\mathfrak{g}}^{\mathrm{cont}}\left(\mathrm{B}_j^{\theta} X, I_{n+2}^{\theta}\right) = \varinjlim \mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_j^{\theta,\ell} X, I_{n+2}^{\theta}\right)$ is finite dimensional. By the definition of the direct limit functor, we have, for sufficiently large s,

$$\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_j^{\theta,s} X, I_{n+2}^{\theta}\right) \simeq \mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_j^{\theta,s+1} X, I_{n+2}^{\theta}\right)$$

under the $\mathfrak{g}$-module map $\alpha^s : \mathrm{B}_j^{\theta,s+1} X \to \mathrm{B}_j^{\theta,s} X$. Since I_{n+2}^{θ} is an injective cogenerator for the category $\mathcal{C}_{\bar{\mathfrak{p}},\mathrm{t},n+2}^{\theta,\mathrm{ind}}$, we conclude that the map α^s is an isomorphism for sufficiently large s.

By Lemma 15 and the definition of an inverse limit of functors, we conclude that $\mathrm{B}_j^{\theta} X \simeq \mathrm{B}_j^{\theta,s} X$ for large enough s. Since $\mathrm{B}_j^{\theta'} X \neq 0$ for only finitely many θ', $\mathrm{B}_j X$ has finite length, or equivalently $\mathrm{B}_j X \in \mathcal{C}_{\bar{\mathfrak{p}},\mathrm{t},n+2}$. □

Corollary 19. *For any $X \in \mathcal{C}_{\mathfrak{k},n}$ and any injective object $I \in \mathcal{C}_{\bar{\mathfrak{p}},\mathrm{t},n+2}^{\theta,\mathrm{ind}}$,*

$$\mathrm{Hom}_{\mathfrak{g}}^{\mathrm{cont}}\left(\mathrm{B}_j^{\theta} X, I\right) = \mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_j^{\theta} X, I\right).$$

Proof. This follows from the isomorphism $\mathrm{B}_j^{\theta} X \simeq \mathrm{B}_j^{\theta,s} X$ for large enough s. □

Proposition 20. *For any $X \in \mathcal{C}_{\mathfrak{k},n}$ and $M \in \mathcal{C}_{\bar{\mathfrak{p}},\mathrm{t},n+2}$,*

$$\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_1 X, M\right) \simeq \mathrm{Hom}_{\mathfrak{g}}\left(X, \Gamma^1 M\right), \tag{7}$$

so $\mathrm{B}_1 : \mathcal{C}_{\mathfrak{k},n} \rightsquigarrow \mathcal{C}_{\bar{\mathfrak{p}},\mathrm{t},n+2}$ and $\Gamma^1 : \mathcal{C}_{\bar{\mathfrak{p}},\mathrm{t},n+2} \rightsquigarrow \mathcal{C}_{\mathfrak{k},n}$ are adjoint functors.

Proof. It clearly suffices to prove (7) for $X \in \mathcal{C}_{\mathfrak{k},n}^{\theta}$ and $M \in \mathcal{C}_{\bar{\mathfrak{p}},\mathrm{t},n+2}^{\theta}$. By Proposition 7.9 in [PZ3],

$$\mathrm{Hom}_{\mathfrak{g}}\left(X, \Gamma^1 I\right) \simeq \mathrm{Ext}_{\mathfrak{g},\mathfrak{t}}^1(X, I) \tag{8}$$

for any injective object $I \in \mathcal{C}_{\bar{\mathfrak{p}},\mathrm{t},n+2}^{\theta,\mathrm{ind}}$. By Proposition 16 and Corollary 19,

$$\mathrm{Ext}_{\mathfrak{g},\mathfrak{t}}^1(X, I) \simeq \mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_1 X, I\right). \tag{9}$$

Consider a part of an injective resolution of M in $\mathcal{C}_{\bar{\mathfrak{p}},\mathrm{t}}^{\theta,\mathrm{ind}}$, $0 \to M \to I_0 \to I_1$. Since Γ^1 is an exact functor (Proposition 4), the following sequence is also exact: $0 \to \Gamma^1 M \to \Gamma^1 I_0 \to \Gamma^1 I_1$. Next, applying $\mathrm{Hom}_{\mathfrak{g}}(X, \bullet)$, we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{g}}\left(X, \Gamma^1 M\right) \to \mathrm{Hom}_{\mathfrak{g}}\left(X, \Gamma^1 I_0\right) \to \mathrm{Hom}_{\mathfrak{g}}\left(X, \Gamma^1 I_1\right).$$

By (8) and (9), we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathfrak{g}}\left(X, \Gamma^1 M\right) \arrow[r] & \mathrm{Hom}_{\mathfrak{g}}\left(X, \Gamma^1 I_0\right) \arrow[r, "\varphi"] \arrow[d, "\simeq"'] & \mathrm{Hom}_{\mathfrak{g}}\left(X, \Gamma^1 I_1\right) \arrow[d, "\simeq"] \\
& & \mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_1 X, I_0\right) \arrow[r, "\psi"] & \mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_1 X, I_1\right)
\end{tikzcd}$$

which is commutative as the identifications

$$\mathrm{Hom}_{\mathfrak{g}}\left(X, \Gamma^1 I\right) \simeq \mathrm{Ext}^1_{\mathfrak{g},\mathfrak{t}}(X, I) = \mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_1 X, I\right)$$

are functorial in I. Since $\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_1 X, \bullet\right)$ is left-exact, we conclude that

$$\ker \psi \simeq \mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_1 X, M\right) \ .$$

Finally, $\ker \varphi \simeq \ker \psi$, and we are done. □

Corollary 21. $\mathrm{B}_1 : C_{\mathfrak{k},n} \rightsquigarrow C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ *is a right-exact functor.*

Proof. This is a direct consequence of the fact that B_1 is a left adjoint functor. □

6. B_1 is a bijection on isomorphism classes of simple modules in $C_{\mathfrak{k},n}$ and $C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$.

As stated in Proposition 4, for $n \geq \Lambda$ the functor Γ^1 induces a bijection between the sets of isomorphism classes of simple objects in the categories $C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ and $C_{\mathfrak{k},n}$. The main result of this section is:

Proposition 22. *For $n \geq \Lambda$, the functor B_1 induces a bijection of sets of isomorphism classes of simple objects of $C_{\mathfrak{k},n}$ and $C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$, inverse to the bijection induced by Γ^1.*

In the rest of the paper we assume that $n \geq \Lambda$. Recall that E is a finite-dimensional simple $\bar{\mathfrak{p}}$-module (in particular, $\bar{\mathfrak{n}} \cdot E = 0$) on which h acts via a natural number $|E|$. We set $X(E) := \Gamma^1 L(E)$ for $L(E) \in C^{\theta}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$. Then $X(E)$ is a simple object of $C_{\mathfrak{k},n}$, and all simple objects in $C_{\mathfrak{k},n}$ are of this form for appropriate simple $C(\mathfrak{t})$-modules.

Lemma 23. *Let $X \in C^{\theta}_{\mathfrak{k},n}$ have the property that the isotypic component of the minimal $\mathfrak{k}$-type of X is isomorphic to the isotypic component of the minimal $\mathfrak{k}$-type of $X(E) \in C^{\theta}_{\mathfrak{k},n}$. Then*

$$\left[\mathrm{B}^{\theta}_1 X : L(F)\right] = \begin{cases} 0 & \textit{for } |F| < |E|\,, \\ \leq 1 & \textit{for } |F| = |E|\,. \end{cases}$$

Proof. Observe first that the $\mathfrak{t}$-weights of $H_0\left(\mathfrak{n}_{\mathfrak{k}}, X\right)$ are less than or equal to $-\left(|E|-2\right)$. Therefore the $\mathfrak{t}$-weights of $H_0\left(\mathfrak{n}_{\mathfrak{k}}, X\right) \otimes \left(\mathfrak{n}_{\mathfrak{k}} \cap \mathfrak{k}^{\perp}\right)$ are less than or equal to $-|E|+2+\lambda_1$. This shows that $\left(H_0\left(\mathfrak{n}_{\mathfrak{k}}, X\right) \otimes \left(\mathfrak{n}_{\mathfrak{k}} \cap \mathfrak{k}^{\perp}\right)\right)_p = 0$ for $n+2 \leq p < |E|$. Indeed, the inequalities $n+2 \leq p < |E|$ and $p \leq -|E|+2+\lambda_1$ yield $|E| \leq \frac{\lambda_1}{2} - \frac{\lambda_2}{2} \leq \frac{\lambda_1}{2}$, which contradicts our assumption that $|E| \geq 2 + \frac{\lambda_1+\lambda_2}{2} \geq 2 + \frac{\lambda_1}{2}$.

Next, note that Kostant's Theorem applied to $\mathfrak{k}$ gives

$$H_1\left(\mathfrak{n}_{\mathfrak{k}}, X\right)_p = \begin{cases} 0 & \text{for } p < |E|\,, \\ \mathbb{C} & \text{for } p = |E|\,. \end{cases}$$

Therefore the spectral sequence (1) of Proposition 2 implies

$$(E^1_1)_p = \begin{cases} 0 \text{ for } n+2 \leq p < |E| \\ \mathbb{C} \text{ for } p = |E| \end{cases} , \ (E^1_0)_{|E|} = (E^1_2)_{|E|} = 0\,.$$

Hence,

$$H_1(\mathfrak{n}, X)_p = \begin{cases} 0 & \text{for } n+2 \leq p < |E|\,, \\ \mathbb{C} & \text{for } p = |E|\,. \end{cases}$$

Furthermore, for any D

$$\dim \operatorname{Ext}^1_{\mathfrak{g},\mathfrak{t}}\left(X, \overline{W}(D)\right) = \dim \operatorname{Hom}_{C(\mathfrak{t})}\left(H_1(\mathfrak{n}, X), \bar{D}\right) \leq \dim H_1(\mathfrak{n}, X)_{|D|}$$

by (6). This yields

$$\dim \operatorname{Ext}^1_{\mathfrak{g},\mathfrak{t}}\left(X, \overline{W}(D)\right) = \begin{cases} 0 & \text{for } |D| < |E|\,, \\ \leq 1 & \text{for } |D| = |E|\,. \end{cases}$$

Consequently, since the injective hull $I(F)$ of $L(F)$ in $C^{\theta,\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ admits a co-Verma filtration with successive quotients isomorphic to $\overline{W}(D)$ for $|D| \leq |F|$, and $\overline{W}(E)$ enters $I(E)$ with multiplicity 1, we have

$$\dim \operatorname{Ext}^1_{\mathfrak{g},\mathfrak{t}}\left(X, I(F)\right) = \begin{cases} 0 & \text{for } |F| < |E|\,, \\ \leq 1 & \text{for } |F| = |E|\,. \end{cases}$$

Finally,

$$\dim \operatorname{Ext}^1_{\mathfrak{g},\mathfrak{t}}\left(X, I(F)\right) = \dim \operatorname{Hom}_{\mathfrak{g}}\left(\mathrm{B}^{\theta}_1 X, I(F)\right) = \left[\mathrm{B}^{\theta}_1 X : L(F)\right]\,,$$

and the lemma is proved. □

Corollary 24. *Set $Y(E) := \Gamma^1 M(E)$ under the assumption that $M(E) \in C^{\theta}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$. Then*

$$[\mathrm{B}_1 Y(E) : L(F)] = [\mathrm{B}_1 X(E) : L(F)] = \begin{cases} 0 & \textit{for } |F| < |E|\,, \\ 1 & \textit{for } F \simeq E\,. \end{cases}$$

Proof. For $|F| < |E|$, the statement follows directly from Lemma 23 as the isotypic components of the minimal $\mathfrak{k}$-types of $Y(E)$ and $X(E)$ are isomorphic by Proposition 4, c). If $F \simeq E$, then

$$\operatorname{Hom}_{\mathfrak{g}}\left(\mathrm{B}_1 X(E), L(E)\right) \simeq \operatorname{Hom}_{\mathfrak{g}}\left(X(E), X(E)\right),$$

so the identity homomorphism $X(E) \to X(E)$ provides a nonzero homomorphism $\mathrm{B}_1 X(E) \to L(E)$. Since Γ^1 is exact and B_1 is right-exact, this homomorphism is in fact a composition of surjections $\mathrm{B}_1 Y(E) \to \mathrm{B}_1 X(E) \to L(E)$, in particular, $[\mathrm{B}_1 Y(E) : L(E)] \geq 1$ and $[\mathrm{B}_1 X(E) : L(E)] \geq 1$. On the other hand, $[\mathrm{B}_1 Y(E) : L(E)] \leq 1$ by Lemma 23; hence,

$$[\mathrm{B}_1 Y(E) : L(E)] = [\mathrm{B}_1 X(E) : L(E)] = 1\,.$$

□

Corollary 25. $\mathrm{B}_1 Y(E) \simeq M(E)$.

Proof. By the adjointness of B_1 and Γ^1, we have a canonical nonzero homomorphism

$$\varphi : \mathrm{B}_1 Y(E) \to M(E)$$

induced by the identity homomorphism $Y(E) \to Y(E)$. Note that $\operatorname{Top} Y(E)$ is isomorphic to $X(E)$ by Proposition 4, c). Next (again by the adjointness of B_1 and Γ^1),

$$\operatorname{Hom}_{\mathfrak{g}}(\mathrm{B}_1 Y(E), L(F)) \simeq \operatorname{Hom}_{\mathfrak{g}}\left(Y(E), X(F)\right).$$

Since $\operatorname{Hom}_{\mathfrak{g}}\left(Y(E), X(F)\right) \neq 0$ only for $F \simeq E$ by Corollary 24, we see that

$$\operatorname{Top} \mathrm{B}_1 Y(E) \simeq L(E).$$

As $\varphi \neq 0$, φ induces an isomorphism

$$\operatorname{Top} \mathrm{B}_1 Y(E) \simeq L(E) = \operatorname{Top} M(E). \tag{10}$$

Consequently, φ is surjective.

Let $N = \ker \varphi$. The isomorphism (10) shows that the exact sequence

$$0 \to N \to \mathrm{B}_1 Y(E) \to M(E) \to 0$$

does not split. Therefore, the assumption $N \neq 0$ leads to the conclusion that $\operatorname{Ext}^1_{\mathfrak{g}}(M(E), L(F)) \neq 0$ for some simple subquotient $L(F)$ of N. However, $\operatorname{Ext}^1_{\mathfrak{g}}(M(E), L(F))$ implies $|F| < |E|$, while $[\mathrm{B}_1 Y(E) : L(F)] = 0$ for $|F| < |E|$ by Corollary 24. This contradiction shows that $N = 0$, i.e., that φ is an isomorphism. □

Corollary 26. $\operatorname{Top} \Gamma^1 \mathrm{B}_1 X(E) \simeq X(E)$.

Proof. B_1 is right-exact, hence the surjective homomorphism $Y(E) \to X(E)$ yields a surjective homomorphism $M(E) \simeq \mathrm{B}_1 Y(E) \to \mathrm{B}_1 X(E)$. By applying Γ^1 we obtain a surjective homomorphism $Y(E) \to \Gamma^1 \mathrm{B}_1 X(E)$, hence $\operatorname{Top}(\Gamma^1 \mathrm{B}_1 X(E)) \simeq \operatorname{Top} Y(E) \simeq X(E)$. □

Corollary 27. $\mathrm{B}_1 X(E) \simeq L(E)$.

Proof. By Corollary 24, $\mathrm{B}_1 X(E) \neq 0$. The adjointness of B_1 and Γ^1 yields an isomorphism

$$\operatorname{Hom}_{\mathfrak{g}}\left(X(E), \Gamma^1 \mathrm{B}_1 X(E)\right) \simeq \operatorname{Hom}_{\mathfrak{g}}(\mathrm{B}_1 X(E), \mathrm{B}_1 X(E)).$$

Hence, there is a nonzero (and therefore injective) homomorphism

$$\alpha : X(E) \to \Gamma^1 \mathrm{B}_1 X(E)$$

corresponding to the identity homomorphism $\mathrm{B}_1 X(E) \to \mathrm{B}_1 X(E)$.

Once again, Corollary 24 implies $[\mathrm{B}_1 X(E) : L(E)] = 1$. Since Γ^1 is exact and is a bijection on isomorphism classes of simple modules, we have

$$\left[\Gamma^1 \mathrm{B}_1 X(E) : X(E)\right] = 1. \tag{11}$$

By Corollary 26, there is a surjective homomorphism

$$\beta : \Gamma^1 \mathrm{B}_1 X(E) \to X(E)\,.$$

Equation (11) now implies that $\beta\alpha \neq 0$ and $\alpha\beta \neq 0$. Thus, $X(E)$ is a direct summand of $\Gamma^1\mathrm{B}_1X(E)$. But Corollary 26 shows that $\Gamma^1\mathrm{B}_1X(E)$ is indecomposable. We conclude that

$$\Gamma^1\mathrm{B}_1X(E) \simeq X(E) = \Gamma^1 L(E)\,.$$

As before, Γ^1 is exact and is a bijection on isomorphism classes of simple modules. So, $\mathrm{B}_1X(E)$ is a simple $\mathfrak{g}$-module, and $\mathrm{B}_1X(E) \simeq L(E)$. □

7. Exactness of B_1

The goal of this section is to prove the following:

Proposition 28. $\mathrm{B}_1 : \mathcal{C}_{\mathfrak{k},n} \rightsquigarrow \mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ *is an exact functor.*

Our main effort will go into proving the following lemma:

Lemma 29. $\mathrm{B}_2Y(E) = 0$ *for* $X(E) \in \mathcal{C}_{\mathfrak{k},n}$.

Note that it suffices to show that $H_2\Big(\mathfrak{n}, Y(E)\Big)_{|F|} = 0$ for all $X(E), X(F) \in \mathcal{C}_{\mathfrak{k},n}$. Indeed, the implication

$$\left(H_2\Big(\mathfrak{n}, Y(E)\Big)_{|F|} = 0 \text{ for all } X(E), X(F) \in \mathcal{C}_{\mathfrak{k},n}\right) \Rightarrow (\mathrm{B}_2Y(E) = 0 \text{ for all } X(E) \in \mathcal{C}_{\mathfrak{k},n})$$

follows from the following three facts:

(1) $\mathrm{Ext}^2_{\mathfrak{g},\mathfrak{t}}\Big(Y(E), I(F)\Big) \simeq \mathrm{Hom}_{\mathfrak{g}}\,(\mathrm{B}_2Y(E), I(F))$;

(2) $I(F)/\overline{W}(F)$ has a co-Verma filtration with factors isomorphic to $\overline{W}(F')$ for $|F'| < |F|$ (Lemma 12);

(3) $\dim \mathrm{Ext}^2_{\mathfrak{g},\mathfrak{t}}\Big(Y(E), \overline{W}(F)\Big) = \dim \mathrm{Hom}_{C(\mathfrak{t})}\Big(H_2\Big(\mathfrak{n}, Y(E)\Big), \bar{F}\Big) \leq \dim H_2\Big(\mathfrak{n}, Y(E)\Big)_{|F|}$, see (6).

To prove that $H_2\Big(\mathfrak{n}, Y(E)\Big)_{|F|} = 0$ for all $X(E), X(F) \in \mathcal{C}_{\mathfrak{k},n}$, we give another construction of the functor $\Gamma^1 : \mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2} \to \mathcal{C}_{\mathfrak{k},n}$. Denote by $U_e(\mathfrak{g})$ the enveloping algebra $U(\mathfrak{g})$ localized by the multiplicative set $\{e^n\}_{n\in\mathbb{Z}_{\geq 1}}$. The localized algebra $U_e(\mathfrak{k})$ is a subalgebra of $U_e(\mathfrak{g})$. For any $\mathfrak{g}$-module (resp., $\mathfrak{k}$-module) M, set

$$\mathcal{D}^{\mathfrak{g}}_e(M) := U_e(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M, \quad \mathcal{D}^{\mathfrak{k}}_e(M) := U_e(\mathfrak{k}) \otimes_{U(\mathfrak{k})} M\,.$$

Lemma 30. *If M is a $\mathfrak{g}$-module on which e acts injectively, we have an isomorphism of $\mathfrak{k}$-modules*

$$\mathrm{Res}_{\mathfrak{k}}\mathcal{D}^{\mathfrak{g}}_e(M) \simeq \mathcal{D}^{\mathfrak{k}}_e(M).$$

Proof. There is an embedding $\psi : M \hookrightarrow D^{\mathfrak{g}}_e(M)$. By Frobenius reciprocity, ψ induces a morphism $\tilde{\psi} : D^{\mathfrak{k}}_e(M) \to D^{\mathfrak{g}}_e(M)$. As $U_e(\mathfrak{g}) = U_e(\mathfrak{k})S(\mathfrak{k}^{\perp})$, the morphism $\tilde{\psi}$ is surjective. Let us show that $\tilde{\psi}$ is also injective. Since e acts injectively on M, we see that $\mathcal{D}^{\mathfrak{g}}_e(M)$ is an essential extension of M. Therefore, the fact that $\ker\tilde{\psi} \cap M = 0$ suffices to conclude that $\tilde{\psi}$ is injective. □

Suppose that a $\mathfrak{g}$-module M is free over $\mathbb{C}[e]$ and locally finite over $\mathbb{C}[f]$. Then we have an embedding

$$M \hookrightarrow \Gamma_{\mathbb{C}f}\mathcal{D}_e^{\mathfrak{g}}(M)$$

where $\Gamma_{\mathbb{C}f}$ is the functor of $\mathbb{C}f$-finite vectors. Set

$$\mathcal{E}M := (\Gamma_{\mathbb{C}f}\mathcal{D}_e^{\mathfrak{g}}(M))/M,$$

cf. [E]. Since $M \in \mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ satisfies the above assumptions, we have constructed a new functor

$$\mathcal{E} : \mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2} \rightsquigarrow \mathcal{C}_{\mathfrak{k},n}.$$

Lemma 31. *If $M \in \mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$, $n \geq 0$, then for some $\gamma(\mu) \in \mathbb{Z}_{\geq 0}$*

$$\mathrm{Res}_{\mathfrak{k}} M \simeq \bigoplus_{\mu \geq n+2} M_{\mathfrak{k}}(\mu)^{\oplus\gamma(\mu)}$$

and

$$\mathrm{Res}_{\mathfrak{k}}(\mathcal{E}M) \simeq \bigoplus_{\mu \geq n+2} V_{\mathfrak{k}}(\mu-2)^{\oplus\gamma(\mu)},$$

where $M_{\mathfrak{k}}(\mu) := U(\mathfrak{k}) \otimes_{U(\mathfrak{k}\cap\bar{\mathfrak{p}})} \mathbb{C}_\mu$ for an integral $\mathfrak{t}$*-weight μ.*

Proof. Any $\mathfrak{t}$-weight of M is not less than $n+2$. Therefore M is free over $\mathbb{C}e$, which implies that $\mathrm{Res}_{\mathfrak{k}} M$ has a filtration with quotients isomorphic to Verma modules $M_{\mathfrak{k}}(\mu)$ for $\mu \geq n+2$. Recall that if μ' and μ'' are positive then the central characters of $M_{\mathfrak{k}}(\mu')$ and $M_{\mathfrak{k}}(\mu'')$ coincide only if $\mu' = \mu''$, and there is no non-trivial $\mathfrak{t}$-semisimple self-extension of a Verma module of $\mathfrak{k}$. This implies $\mathrm{Ext}^1_{\mathfrak{k},\mathfrak{t}}(M_{\mathfrak{k}}(\mu'), M_{\mathfrak{k}}(\mu'')) = 0$ for positive μ' and μ'', therefore $\mathrm{Res}_{\mathfrak{k}} M$ is isomorphic to a direct sum of Verma modules. The first assertion follows.

Let us prove the second assertion. Recall that $\mathcal{D}_e^{\mathfrak{k}}(M_{\mathfrak{k}}(\mu)) = U_e(\mathfrak{k}) \otimes_{U(\mathfrak{k})} M_{\mathfrak{k}}(\mu)$. By Lemma 30 it suffices to check that for any $\mu \geq 2$

$$\Gamma_{\mathbb{C}f}\mathcal{D}_e^{\mathfrak{k}}(M_{\mathfrak{k}}(\mu))/M_{\mathfrak{k}}(\mu) \simeq V_{\mathfrak{k}}(\mu-2). \tag{12}$$

First, we show that $\mathcal{D}_e^{\mathfrak{k}}(M_{\mathfrak{k}}(\mu))$ is an indecomposable multiplicity-free weight sl(2)-module with socle filtration

$$M_{\mathfrak{k}}(\mu) \subset M_{\mathfrak{k}}(2-\mu) \subset \mathcal{D}_e^{\mathfrak{k}}(M_{\mathfrak{k}}(\mu)).$$

Indeed, let $v \in M_{\mathfrak{k}}(\mu)$ be a nonzero vector annihilated by f (v is unique up to proportionality). Then $\{e^k v \,|\, k \in \mathbb{Z}_{\geq 0}\}$ is a basis of h-eigenvectors in $M_{\mathfrak{k}}(\mu)$. In the localized algebra $U_e(\mathfrak{k})$ we have the relations

$$[h, e^{-1}] = -2e^{-1}, \quad [f, e^{-1}] = e^{-1}he^{-1} = e^{-2}(h-2), \quad [f, e^{-k}] = e^{-k-1}k(h-(k+1)).$$

Therefore $\mathcal{D}_e^{\mathfrak{k}}(M_{\mathfrak{k}}(\mu))$ has a basis $\{e^k v \,|\, k \in \mathbb{Z}\}$ and the action of h and f is defined by the above relations. In particular, $w = e^{1-\mu}v$ is annihilated by f and generates a submodule $M_{\mathfrak{k}}(2-\mu) \subset \mathcal{D}_e^{\mathfrak{k}}(M_{\mathfrak{k}}(\mu))$. The quotient $M_{\mathfrak{k}}(2-\mu)/M_{\mathfrak{k}}(\mu)$ is isomorphic to the finite-dimensional module $V_{\mathfrak{k}}(\mu-2)$, and the quotient $\mathcal{D}_e^{\mathfrak{k}}(M_{\mathfrak{k}}(\mu))/M_{\mathfrak{k}}(2-\mu)$ is isomorphic to the Verma module with respect to $\mathfrak{k} \cap \mathfrak{p}$ with highest weight $-\mu$. Since the latter quotient is free over f, (12) follows. $\square$

Corollary 32. *If $M \in C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$, $n \geq 0$, then*

$$\mathrm{Res}_{\mathfrak{k}}(\mathcal{E}M) \simeq \mathrm{Res}_{\mathfrak{k}}\left(\Gamma^1 M\right).$$

Corollary 33. *The functor $\mathcal{E} : C_{\bar{\mathfrak{p}},\mathfrak{t},n+2} \rightsquigarrow C_{\mathfrak{t},n}$ is exact.*

Proof. The exactness of Γ^1, together with Corollary 32, shows that the functor $\mathrm{Res}_{\mathfrak{k}} \circ \mathcal{E}$ is exact. Therefore $\mathcal{E}$ is also exact. □

Proposition 34. *The functors $\mathcal{E} : C_{\bar{\mathfrak{p}},\mathfrak{t},n+2} \rightsquigarrow C_{\mathfrak{t},n}$ and $\Gamma^1 : C_{\bar{\mathfrak{p}},\mathfrak{t},n+2} \rightsquigarrow C_{\mathfrak{t},n}$ are isomorphic.*

Proof. Let us start with the construction of a morphism of functors $\varphi : \mathcal{E} \rightsquigarrow \Gamma^1$. Let $M \in C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$. Then the exact sequence

$$0 \to M \to \Gamma_{\mathbb{C}f}\mathcal{D}^e_{\mathfrak{g}}(M) \xrightarrow{\pi_M} \mathcal{E}M \to 0$$

does not split over $\mathfrak{k}$, and therefore does not split over $\mathfrak{g}$. Set

$$R^i(M) := \Gamma_{\mathfrak{t}} \operatorname{Hom}_{\mathbb{C}}\left(U(\mathfrak{g}) \otimes_{U(\mathfrak{t})} \Lambda^i(\mathfrak{g}/\mathfrak{t}), M\right)$$

and let

$$0 \to M \xrightarrow{\partial_0} R^0(M) \xrightarrow{\partial_1} R^1(M) \xrightarrow{\partial_2} R^2(M) \xrightarrow{\partial_3} \dots$$

be the Koszul resolution of M as introduced in Lemma 2.2 of [Z].

The complex $R^\bullet(M)$ is functorial with respect to M and yields an injective resolution of M in the category of $(\mathfrak{g}, \mathfrak{t})$-modules. Hence, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\mathrm{id}_M"'] & \Gamma_{\mathbb{C}f}\mathcal{D}^e_{\mathfrak{g}}(M) \arrow[r] \arrow[d, "\eta_M"'] & \mathcal{E}M \arrow[r] \arrow[d, "\varphi_M"'] & 0 \arrow[d] \\
0 \arrow[r] & M \arrow[r, "\partial_0"] & R^0(M) \arrow[r, "\partial_1"] & R^1(M) \arrow[r, "\partial_2"] & R^2(M)
\end{tikzcd}$$

for some morphisms η_M and φ_M, unique up to homotopy. We recall from [PZ3] that $\Gamma M = 0$. By construction, $\Gamma\left(\Gamma_{\mathbb{C}f}\mathcal{D}^e_{\mathfrak{g}}(M)\right) = 0$ and $\Gamma\mathcal{E}M = \mathcal{E}M$. By applying Γ to the above diagram, we obtain a new commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathcal{E}M \arrow[r] \arrow[d, "\Gamma\varphi_M"'] & 0 \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \Gamma R^0(M) \arrow[r, "\Gamma\partial_1"] & \Gamma R^1(M) \arrow[r, "\Gamma\partial_2"] & \Gamma R^2(M).
\end{tikzcd}$$

The morphism $\Gamma\varphi_M$ induces a unique morphism $\psi_M : \mathcal{E}M \to \Gamma^1 M$, by the definition of Γ^1. Since our diagram is functorial in M, we obtain a morphism of functors $\psi : \mathcal{E} \rightsquigarrow \Gamma^1$.

It remains to show that ψ_M is an isomorphism for all $M \in \mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$. Since both functors $\mathcal{E}$ and Γ^1 are exact, it is sufficient to check this for simple M as the general case follows by an easy induction on the length of M.

Suppose that M is simple and ψ_M is nonzero. Then we have a surjective morphism $\psi_M : \mathcal{E}M \to \Gamma^1 M$, since $\Gamma^1 M$ is also simple. But then, by Corollary 32, ψ_M is an isomorphism. Now, suppose $\psi_M = 0$. Recall that $\psi_M : \mathcal{E}M \to \Gamma^1 M = \ker \Gamma\partial_2 / \operatorname{im}\Gamma\partial_1$. Therefore the equality $\psi_M = 0$ defines a non-zero morphism $\mathcal{E}M \to \operatorname{im}\Gamma\partial_1$, or equivalently a nonzero morphism $\kappa_M : \mathcal{E}M \to R^0(M)$ such that $\partial_1\kappa_M = \varphi_M$. Moreover, $\operatorname{im}\kappa_M \simeq \operatorname{im}\eta_M$. Because $\Gamma_{\mathbb{C}f}\mathcal{D}^e_{\mathfrak{g}}(M)$ is an essential extension of M, η_M is an injection, and hence $\Gamma \operatorname{im}\eta_M = \eta_M\Gamma\left(\Gamma_{\mathbb{C}f}\mathcal{D}^e_{\mathfrak{g}}(M)\right) = 0$. On the other hand,

$$\Gamma \operatorname{im}\kappa_M = \operatorname{im}\Gamma\kappa_M = \operatorname{im}\kappa_M \neq 0\,,$$

a contradiction. Hence $\psi_M \neq 0$, and the proposition is proved. □

Corollary 35. *There is an isomorphism of* $\mathfrak{g}$*-modules*

$$Y(E) \simeq \mathcal{E}M(E).$$

We are now ready to give a proof of Lemma 29.

Proof of Lemma 29. Set

$$D(E) := \mathcal{D}^{\mathfrak{g}}_e M(E), \quad C(E) := \Gamma_{\mathbb{C}f} D(E)\,.$$

From the explicit form of $D(E)$ as a $\mathfrak{k}$-module it is easy to verify that

$$H_0\left(\mathfrak{n}_{\mathfrak{k}}, D(E)\right) = H_1\left(\mathfrak{n}_{\mathfrak{k}}, D(E)\right) = 0\,.$$

By the spectral sequence of Proposition 2 this implies

$$H_i\Big(\mathfrak{n}, D(E)\Big) = 0 \text{ for all } i\,.$$

The exact sequence

$$0 \to C(E) \to D(E) \to F(E) \to 0\,,$$

where $F(E) := D(E)/C(E)$, yields $H_2\Big(\mathfrak{n}, C(E)\Big)_{|F|} = H_3\Big(\mathfrak{n}, F(E)\Big)_{|F|}$. It is easy to check that $H_0\left(\mathfrak{n}_{\mathfrak{k}}, F(E)\right) = 0$, hence the input into $H_3\Big(\mathfrak{n}, F(E)\Big)$ in the spectral sequence (1) comes from

$$H_1\left(\mathfrak{n}_{\mathfrak{k}}, F(E)\right) \otimes \Lambda^2\left(\mathfrak{n} \cap \mathfrak{k}^{\perp}\right)\,. \tag{13}$$

The maximum possible $\mathfrak{t}$-weight of $H_1\left(\mathfrak{n}_{\mathfrak{k}}, F(E)\right)$ is $2 - |E|$, hence the maximum possible $\mathfrak{t}$-weight of (13) is $2 - |E| + \lambda_1 + \lambda_2$. However, for any F such that $X(F) \in \mathcal{C}_{\mathfrak{t},n}$, we have $2 - |E| + \lambda_1 + \lambda_2 < |F|$ as $|E|, |F| \geq \frac{\lambda_1+\lambda_2}{2} + 2$. We obtain $H_2\Big(\mathfrak{n}, C(E)\Big)_{|F|} = 0$.

Next, we note that Corollary 35 shows the existence of an exact sequence

$$0 \to M(E) \to C(E) \to Y(E) \to 0$$

as $Y(E) \simeq \mathcal{E}M(E)$. Since $M(E)$ is free as an $\mathfrak{n}$-module, $H_1\Big(\mathfrak{n}, M(E)\Big) = 0$. Together with $H_2\Big(\mathfrak{n}, C(E)\Big)_{|F|} = 0$, this yields

$$H_2\Big(\mathfrak{n}, Y(E)\Big)_{|F|} \simeq H_1\Big(\mathfrak{n}, M(E)\Big)_{|F|} = 0$$

as $M(E)$ is free as an $\mathfrak{n}$-module. The proof of Lemma 29 is complete. □

To prove Proposition 28, it now suffices to establish the following.

Lemma 36. $\mathrm{B}_2X(E)=0$ *for any* $X(E)\in\mathcal{C}_{\mathfrak{k},n}$.

Proof. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{h}\subset\mathfrak{b}\subset\mathfrak{p}$ and $e\in\mathfrak{b}$. We will prove the statement by induction on the Bruhat height on the $\mathfrak{b}$-lowest weight of $L(E)$.

If λ is $\mathfrak{b}$-dominant (i.e. $\bar{\mathfrak{b}}$−antidominant), then $Y(E)=X(E)$ and we are done. For an arbitrary λ, we consider the exact sequences

$$0\to N(E)\xrightarrow{\nu}M(E)\xrightarrow{\mu}L(E)\to 0 \tag{14}$$

and

$$0\to\Gamma^1N(E)\to Y(E)\to X(E)\to 0\,. \tag{15}$$

The long exact sequence corresponding to (15) is the top row of the following commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & \mathrm{B}_2Y(E) \arrow[r] \arrow[d, equal] & \mathrm{B}_2X(E) \arrow[r] \arrow[d, equal] & \mathrm{B}_1\Gamma^1N(E) \arrow[r] \arrow[d, "\wr"] & \mathrm{B}_1\Gamma^1M(E) \arrow[r] \arrow[d, "\wr"] & \mathrm{B}_1X(E) \arrow[r] \arrow[d, "\wr"] & 0 \\
\ldots \arrow[r] & 0 \arrow[r] & \mathrm{B}_2X(E) \arrow[r] & N(E) \arrow[r, "\nu"] & M(E) \arrow[r, "\mu"] & L(E) \arrow[r] & 0\,.
\end{tikzcd}$$

The vertical isomorphisms are explained as follows:

$$\mathrm{B}_2Y(E)=0 \text{ by Lemma 29}\,,$$

$$\mathrm{B}_1\Gamma^1N(E)\simeq N(E) \text{ by the induction assumption}\,,$$
$$\mathrm{B}_1\Gamma^1M(E)\simeq M(E) \text{ by Corollary 25}\,,$$

and

$$\mathrm{B}_1(X)\simeq L(E) \text{ by Proposition 22}\,.$$

The exactness of the bottom row of the diagram yields $\mathrm{B}_2X(E)=0$, and Lemma 36 is proved. Proposition 28 now follows. □

8. End of Proof of Theorem 1

The results of Sections 5-7 imply that, under the assumption $n\geq\Lambda$, the functors

$$\Gamma^1:\mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}\rightsquigarrow\mathcal{C}_{\mathfrak{k},n}$$

and

$$\mathrm{B}_1:\mathcal{C}_{\mathfrak{k},n}\rightsquigarrow\mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$$

are exact functors between finite-length abelian categories which induce mutually inverse bijections on isomorphism classes of simple objects.

The isomorphisms

$$\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_1\Gamma^1M,M\right)\simeq\mathrm{Hom}_{\mathfrak{g}}\left(\Gamma^1M,\Gamma^1M\right)$$

and

$$\mathrm{Hom}_{\mathfrak{g}}\left(\mathrm{B}_1 X, \mathrm{B}_1 X\right) \simeq \mathrm{Hom}_{\mathfrak{g}}\left(X, \Gamma^1 \mathrm{B}_1 X\right),$$

for $X \in \mathcal{C}_{\mathfrak{k},n}$ and $M \in \mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$, induce morphisms of functors

$$\Delta : B_1 \circ \Gamma^1 \rightsquigarrow \mathrm{id}_{\mathcal{C}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}}$$

and

$$\nabla : \Gamma^1 \circ B_1 \rightsquigarrow \mathrm{id}_{\mathcal{C}_{\mathfrak{t},n}}.$$

As in the proof of Proposition 34, it suffices to show that Δ and ∇ are isomorphisms on simple objects as all functors involved are exact functors on finite-length abelian categories. Finally, for simple objects the claim follows from Proposition 22.

9. Discussion and examples

It is interesting to see when the functor B_1 establishes an equivalence of the category $\mathcal{C}^{\theta}_{\mathfrak{k},\Lambda}$ with the entire category $\mathcal{C}^{\theta}_{\bar{\mathfrak{p}},\mathfrak{t}}$. This is equivalent to the question: for which central characters θ does the equality $\mathcal{C}^{\theta}_{\bar{\mathfrak{p}},\mathfrak{t},\Lambda+2} = \mathcal{C}^{\theta}_{\bar{\mathfrak{p}},\mathfrak{t}}$ hold?

Consider in more detail the case when $\mathfrak{g}$ is simple and $\mathfrak{k}$ is a principal sl(2)-subalgebra of $\mathfrak{g}$. Here h is a regular element of $\mathfrak{g}$ and $\mathfrak{p} = \mathfrak{b}$ is a Borel subalgebra. Let the simple roots of $\mathfrak{b}$ be $\alpha_1, \dots, \alpha_r \in \mathfrak{h}^*$, and β be the highest root. Then $\beta = m_1\alpha_1 + \cdots + m_r\alpha_r$ for some positive integers $m_1, \dots, m_r$. Moreover, $\Lambda = \beta(h) - 1$. We would like to find central characters θ such that $\mathcal{C}^{\theta}_{\bar{\mathfrak{b}},\mathfrak{t},\Lambda+2} = \mathcal{C}^{\theta}_{\bar{\mathfrak{b}},\mathfrak{t}}$. For a weight $\nu \in \mathfrak{h}^*$ denote by θ_ν the central character of the Verma module $M(\nu) = U(\mathfrak{g}) \otimes_{U(\bar{\mathfrak{b}})} \mathbb{C}_\nu$. The equality $\theta_\nu = \theta_\eta$ holds if and only if $\nu - \rho$ and $\eta - \rho$ belong to the same orbit of the Weyl group, where ρ is the half-sum of roots of $\mathfrak{b}$. Consider the set $\mathbf{C}$ of all weights $\lambda \in \mathfrak{h}^*$ such that $\mathrm{Re}(\lambda, \alpha) \leq 0$ for all positive roots α. The orbit $W(\nu - \rho)$ contains at least one weight $\gamma - \rho$ lying in $\mathbf{C}$. Moreover, for any other $\eta = w(\gamma - \rho) + \rho$ on the Weyl group orbit $\mathrm{Re}(\gamma, \alpha) \leq \mathrm{Re}(\eta, \alpha)$ for all positive roots α. Hence $\mathrm{Re}\,\gamma(h) \leq \mathrm{Re}\,\eta(h)$. Thus, it suffices to find γ such that $\gamma - \rho \in \mathbf{C}$ and $\mathrm{Re}\,\gamma(h) \geq \beta(h) + 1$.

Let $h_1, \dots, h_r$ denote the simple coroots. Then $h = n_1h_1 + \cdots + n_rh_r$ for some positive integers $n_1, \dots, n_r$. We set $\gamma_i := \gamma(h_i)$. Since $\rho(h_i) = 1$ for all $i = 1, \dots, r$, the condition that $\gamma - \rho \in \mathbf{C}$ can be written in the form

$$\mathrm{Re}\,\gamma_i \leq 1 \text{ for all } i = 1, \dots r. \tag{16}$$

The equality $\alpha_i(h) = 2$ shows that $\beta(h) = 2\sum_{i=1}^r m_i$. Hence, the condition $\gamma(h) \geq \beta(h) + 1$ is equivalent to

$$\mathrm{Re}\sum_{i=1}^r n_i\gamma_i \geq 1 + 2\sum_{i=1}^r m_i. \tag{17}$$

Let $\Sigma(\mathfrak{g})$ denote the set of weights satisfying conditions (16) and (17). Clearly $\Sigma(\mathfrak{g})$ is not empty as soon as

$$\sum_{i=1}^r n_i \geq 1 + 2\sum_{i=1}^r m_i.$$

The latter inequality can be rewritten as

$$\rho(h) \geq 1 + \beta(h). \tag{18}$$

For example, let $\mathfrak{g} = \mathrm{sl}(r+1)$. Then $m_1 = \cdots = m_r = 1$, hence $\beta(h) = 1 + 2r$ and $\rho(h) = \frac{r(r+1)(r+2)}{6}$. Therefore (18) holds for $r \geq 3$.

Proposition 37. *Let $\mathfrak{g}$ be a simple Lie algebra not isomorphic to* $\mathrm{sl}(2)$ *or* $\mathrm{sl}(3)$*. Then $\Sigma(\mathfrak{g})$ is not empty. If in addition $\mathfrak{g}$ is not isomorphic to* $\mathrm{sp}(4)$*, then $\Sigma(\mathfrak{g})$ is infinite.*

Proof. $\Sigma(\mathfrak{g})$ is infinite as soon as the inequality (17) is strict. We can further rewrite (18) as

$$\frac{1}{2} \sum_{\alpha \in \Delta^+ \setminus \beta} \alpha(h) \geq 1 + \frac{1}{2}\beta(h). \tag{19}$$

If $\beta(h) \geq 8$, the inequality (19) is strict as in this case the sum of positive non-highest $\mathfrak{t}$-weights in the $\mathfrak{k}$-submodule generated by the highest root vector is greater than the highest $\mathfrak{t}$-weight. Therefore, the statement holds for all $\mathfrak{g}$ of rank greater than 2 and for $\mathfrak{g} = G_2$. For $\mathfrak{g} = B_2$ we have $\rho(h) = 7$, $\beta(h) = 6$, and hence $\Sigma(\mathfrak{g})$ consists of one element: $\Sigma(\mathfrak{g}) = \{\rho\}$. For $\mathfrak{g} = A_2$ we have $\Sigma(\mathfrak{g}) = \emptyset$. □

Note that the set of integral weights lying in $\Sigma(\mathfrak{g})$ is always finite since $\Sigma(\mathfrak{g})$ is compact. Moreover, the cardinality of this finite set grows with rank.

Using translation functors we can strengthen Theorem 1 for certain central characters. Let us call a central character θ $\mathfrak{k}$-*adapted* if $C^{\theta}_{\bar{\mathfrak{b}},\mathfrak{t},\Lambda+2} = C^{\theta}_{\bar{\mathfrak{b}},\mathfrak{t}}$. A central character $\tilde{\theta}$ is *weakly* $\mathfrak{k}$-*adapted* if there exists a $\mathfrak{k}$-adapted character θ and a translation functor T establishing an equivalence between the categories of $\mathfrak{g}$-modules admitting respective generalized central characters θ and $\tilde{\theta}$. Recall that, if $\tilde{\theta} = \theta_\eta$ for some η such that $\eta - \rho \in \mathbf{C}$ and $\theta = \theta_\gamma$ for some $\gamma \in \Sigma(\mathfrak{g})$, then $\gamma - \eta$ must be integral and the stabilizers of $\gamma - \rho$ and $\eta - \rho$ in the Weyl group of $\mathfrak{g}$ must be the same [BG].

Corollary 38. *Assume that $\tilde{\theta}$ is weakly $\mathfrak{k}$-adapted. Then*

(a) $\Gamma^1 L$ is simple for any simple module $L \in C^{\tilde{\theta}}_{\bar{\mathfrak{b}},\mathfrak{t}}$.

(b) Let $\Gamma^1 C^{\tilde{\theta}}_{\bar{\mathfrak{b}},\mathfrak{t}}$ denote the full subcategory of $C^{\tilde{\theta}}_{\mathfrak{k},\Lambda}$ consisiting of modules whose simple constituents are of the form $\Gamma^1 L$ for simple modules $L \in C^{\tilde{\theta}}_{\bar{\mathfrak{b}},\mathfrak{t}}$. Then the functor $\mathrm{B}_1 : \Gamma^1 C^{\tilde{\theta}}_{\bar{\mathfrak{b}},\mathfrak{t}} \rightsquigarrow C^{\theta}_{\bar{\mathfrak{b}},\mathfrak{t}}$ is an equivalence of categories, and is inverse to Γ^1.

Proof. Both assertions follow from the following commutative diagram of functors

$$\begin{tikzcd}
C^{\tilde{\theta}}_{\bar{\mathfrak{b}},\mathfrak{t}} \arrow[r, squiggly, shift left, "\Gamma^1"] \arrow[d, squiggly, shift right, "T_1"'] & \Gamma^1 C^{\tilde{\theta}}_{\bar{\mathfrak{b}},\mathfrak{t}} \arrow[l, squiggly, shift left, "\mathrm{B}_1"] \arrow[d, squiggly, shift right, "T_1"'] \\
C^{\theta}_{\bar{\mathfrak{b}},\mathfrak{t}} \arrow[r, squiggly, shift left, "\Gamma^1"] \arrow[u, squiggly, shift right, "T_2"'] & C^{\theta}_{\mathfrak{k},\Lambda} \arrow[l, squiggly, shift left, "\mathrm{B}_1"] \arrow[u, squiggly, shift right, "T_2"']
\end{tikzcd}.$$

where T_1, T_2 are appropriate translation functors. The commutativity of the diagram is a consequence of Theorem 1 and of the fact that the Zuckerman functor commutes with translation functors. This latter fact is essentially a reformulation of Proposition 2.6 and Corollary 2.8 in [Z]. $\square$

10. An application

In this section $\mathfrak{k}$ is an arbitrary sl(2)-subalgebra of $\mathfrak{g}$ and $n \geq \Lambda$. By $C^{\mathrm{ind}}_{\mathfrak{k},n}$ (respectively, $C^{\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$) we denote the category of inductive limits of objects from $C_{\mathfrak{k},n}$ (respectively, $C^{\ell}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$). Theorem 1 implies the following:

Corollary 39. *The functors Γ^1 and B_1 induce mutually inverse equivalences of the categories $C^{\mathrm{ind}}_{\mathfrak{k},n}$ and $C^{\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$.*

Recall that if an abelian category C has enough injectives, then the global dimension $\mathrm{gdim}\, C$ can be defined as

$$\mathrm{gdim}\, C = \sup_{M,N \in C} \left\{ i \in \mathbb{Z}_{\geq 0} \,\middle|\, \mathrm{Ext}^i_{\mathbb{C}}(M,N) \neq 0 \right\}.$$

Corollary 8 implies that $C^{\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ (and consequently also $C^{\mathrm{ind}}_{\mathfrak{k},n}$ by Theorem 1) has enough injectives. The goal of this section is to prove the following proposition.

Proposition 40. *We have*

$$\mathrm{gdim}\, C^{\mathrm{ind}}_{\mathfrak{k},n} = \mathrm{gdim}\, C^{\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2} \leq 2 \dim \mathfrak{n} + \dim \mathfrak{c} - 1$$

($\mathfrak{n}$ and $\mathfrak{c}$ are subalgebras of $\mathfrak{g}$ depending on the pair $\mathfrak{g}, \mathfrak{k}$ only).

Lemma 41. *For every simple $C(\mathfrak{t})$-module E such that $|E| \geq n+2$, the module $\overline{W}(E)$ has an injective resolution in $C^{\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ of length not greater than $\dim \mathfrak{n}$. Hence $\mathrm{Ext}^i_{C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}}(M, \overline{W}(E)) = 0$ for any $M \in C^{\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ and any $i > \dim \mathfrak{n}$.*

Proof. Consider the category of locally finite $\mathfrak{p}$-modules which are semisimple over $\mathfrak{t}$ and whose $\mathfrak{t}$-weights are at least $n+2$. Then $\bar{E}$ is an object of this category and has an injective resolution in it with terms

$$Z^i(E) := \left(\Gamma_{\mathfrak{t}} \operatorname{Hom}_{\mathbb{C}}(U(\mathfrak{p}) \otimes_{C(\mathfrak{t})} \Lambda^i(\mathfrak{p}/C(\mathfrak{t})), \bar{E}) \right)_{\geq n+2}.$$

Furthermore, $\Gamma_{\mathfrak{t}} \mathrm{pro}^{\mathfrak{g}}_{\mathfrak{p}} Z^i(E)$ provides an injective resolution for $\overline{W}(E)$ in $C^{\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ of length at most $\dim \mathfrak{n}$. $\square$

Lemma 42. *For every simple $C(\mathfrak{t})$-module E, let $W(E) := M(E)^{\vee} = \Gamma_{\mathfrak{t}} \mathrm{pro}^{\mathfrak{g}}_{\mathfrak{p}}(E)$. Then there exists an acyclic complex*

$$0 \to W(E) \to S^0 \to \cdots \to S^{\dim \mathfrak{c} - 1} \to 0$$

such that all S^i admit co-Verma filtrations.

Proof. Let
$$Q^i(E) = \mathrm{Hom}_{\mathbb{C}}\left(S^\bullet(\mathfrak{c}/\mathfrak{t}) \otimes \Lambda^i(\mathfrak{c}/\mathfrak{t}), E\right).$$

Consider the exact complex of $C(\mathfrak{t})$-modules $0 \to E \to Q^0(E) \to Q^1(E) \to \dots$ with usual Koszul differentials and set $S^i := \Gamma_{\mathfrak{t}}\mathrm{pro}_{\mathfrak{p}}^{\mathfrak{g}} T^i(E)$. ◻

Corollary 43. $\mathrm{Ext}^i_{C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}}(M, W(E)) = 0$ *for any* $M \in C^{\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$ *and* $i > \dim \mathfrak{n} + \dim \mathfrak{c} - 1$.

Proof. We note that, by Lemma 41, $\mathrm{Ext}^i_{\bar{\mathfrak{p}},\mathfrak{t},n+2}(M,N) = 0$ for $i > \dim \mathfrak{n}$ and N admitting a co-Verma filtration. In particular, $\mathrm{Ext}^i_{C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}}(M, S^j) = 0$ for $i > \dim \mathfrak{n}$ and $j = 0, \dots, \dim \mathfrak{c} - 1$. Hence the statement. ◻

Lemma 44. *For every E, $L(E)$ has a right resolution of length not greater than* $\dim \mathfrak{n}$ *by modules which admit finite filtrations with succesive quotients isomorphic to $W(F)$.*

Proof. This is a standard fact about parabolic category $\mathcal{O}$. Indeed let $V^i := \Gamma_{\mathfrak{t}}\mathrm{pro}_{\mathfrak{p}}^{\mathfrak{g}} \Lambda^i(\mathfrak{g}/\mathfrak{p})^*$. Then $V^i \simeq S^\bullet((\mathfrak{g}/\mathfrak{p})^*) \otimes \Lambda^i(\mathfrak{g}/\mathfrak{p})^*$. Consider the Koszul complex
$$0 \to V^0 \to V^1 \to \cdots \to V^{\dim \mathfrak{n}} \to 0$$
as the complex of polynomial differential forms on the open orbit of $\bar{P}$ on G/P, where G, P and $\bar{P}$ are appropriate connected algebraic groups with respective Lie algebras $\mathfrak{g}$, $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. It gives a resolution of the trivial module by modules which admit finite filtrations with succesive quotients isomorphic to $W(F)$. To obtain a similar resolution for $L(E)$, we tensor the above resolution with $L(E)$ and project to the subcategory of modules with the central character of $L(E)$. ◻

Corollary 45. $\mathrm{Ext}^i_{C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}}(M, L(E)) = 0$ *for any* $M, L(E) \in C^{\mathrm{ind}}_{\bar{\mathfrak{p}},\mathfrak{t},n+2}$*, and* $i > 2\dim \mathfrak{n} + \dim \mathfrak{c} - 1$.

Proposition 40 follows from the last corollary since $\mathrm{Ext}^i_{\bar{\mathfrak{p}},\mathfrak{t},n+2}(M,N) \neq 0$ implies $\mathrm{Ext}^i_{C_{\bar{\mathfrak{p}},\mathfrak{t},n+2}}(M, N') \neq 0$ for some submodule $N' \subset N$ of finite length.

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