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DECOMPOSITION OF COHOMOLOGY OF VECTOR BUNDLES ON HOMOGENEOUS IND-SPACES

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Abstract

Let G be a locally semisimple ind-group, P be a parabolic subgroup, and E be a finite-dimensional P-module. We show that, under a certain condition on E, the nonzero cohomologies of the homogeneous vector bundle $\mathcal{O}_{G/P}(E^*)$ on G/P induced by the dual P-module E^* decompose as direct sums of cohomologies of bundles of the form $\mathcal{O}_{G/P}(R)$ for (some) simple constituents R of E^*. In the finite-dimensional case, this result is a consequence of the Bott–Borel–Weil theorem and Weyl's semisimplicity theorem. In the infinite-dimensional setting we consider, there is no relevant semisimplicity theorem. Instead, our results are based on the injectivity of the cohomologies of the bundles $\mathcal{O}_{G/P}(R)$.

Key words: ind-groups, cohomology of vector bundles, Bott–Borel–Weil theory, injective modules

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1. Introduction. The Bott–Borel–Weil theorem is a basic result which connects algebraic geometry with representation theory. More precisely, if G is a connected complex reductive linear algebraic group and P is a parabolic subgroup, the Bott–Borel–Weil theorem states that a homogeneous bundle on G/P, induced by a simple P-module, has at most one nonzero cohomology group. Moreover, this group is a simple highest weight G-module. Using Weyl's semisimplicity

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theorem, one can further show that, for any finite-dimensional P-module, the corresponding nonzero cohomologies are semisimple G-modules. More specifically, these cohomologies split as direct sum of cohomologies of homogeneous bundles induced by (some) simple constituents of the inducing P-module.

Analogues of the Bott–Borel–Weil theorem have been proved in various contexts. In particular, in [2,3], the case of locally reductive ind-groups G has been studied. For a locally reductive ind-group G, a homogeneous bundle on G/P induced by a simple finite-dimensional P-module still has at most one nonzero cohomology group. However, the difference with the finite-dimensional case is that this cohomology group is dual to a simple G-module, and hence is not irreducible being uncountable dimensional. In addition, the cohomology group may or may not be an integrable $\mathfrak{g} = \mathrm{Lie}(G)$-module.

In this article we consider locally semisimple ind-groups G and their homogeneous ind-spaces G/P for parabolic ind-groups $P \hookrightarrow G$. To any finite-dimensional P-module E we attach the homogeneous vector bundle $\mathcal{O}_{G/P}(E^*)$ induced by the dual P-module E^*. We impose the condition that all non-zero cohomology groups $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ are integrable $\mathfrak{g}$-modules. Then, our main result (Theorem 4.5) is that, despite the absence of a relevant analogue of Weyl's semisimplicity theorem, any such nonzero cohomology group $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ is isomorphic to a direct sum of cohomologies of homogeneous bundles on G/P induced by (some) simple constituents of E^*. The proof uses the fact that any nonzero cohomology group of a homogeneous bundle on G/P induced by a simple constituent of E^* is injective in the category of integrable $\mathfrak{g}$-modules.

2. Preliminaries. In this section we summarize the definitions and properties from the field of ind-groups, which we need throughout the article. More detailed expositions on the subject can be found in [2,3].

A *locally algebraic ind-group* over $\mathbb{C}$, which we briefly refer to as an *ind-group*, is a set G which is the inductive limit of embeddings of connected algebraic groups, i.e. $G = \varinjlim G_n$, where

$$G_1 \subset G_2 \subset \cdots G_n \subset \cdots \tag{1}$$

An ind-group G is *locally (semi)simple* if the filtration (1) can be chosen so that each G_n is a (semi)simple linear algebraic group. A first important class of locally simple ind-groups is that of diagonal locally simple ind-groups. For its definition we need to recall the notion of a diagonal embedding. An embedding of classical simple finite-dimensional groups $G' \subset G$ is called *diagonal* if the induced injection of the Lie algebras $\mathfrak{g}' \subset \mathfrak{g}$ has the following property: the natural representation of $\mathfrak{g}$ decomposes over $\mathfrak{g}'$ as a direct sum of copies of the natural representation of $\mathfrak{g}'$, of its dual, and of the trivial $\mathfrak{g}'$-representation. A locally simple ind-group G is called *diagonal* if the filtration (1) can be chosen so that each embedding $G_n \subset G_{n+1}$ is a diagonal embedding of classical simple groups. First examples of diagonal ind-groups are the *finitary simple ind-groups*

$\mathrm{SL}(\infty) = \varinjlim \mathrm{SL}(n)$, $\mathrm{SO}(\infty) = \varinjlim \mathrm{SO}(n)$, and $\mathrm{Sp}(\infty) = \varinjlim \mathrm{Sp}(2n)$, where the inclusions $\mathrm{SL}(n) \subset \mathrm{SL}(n+1)$ and $\mathrm{SO}(n) \subset \mathrm{SO}(n+1)$ are given by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the inclusion $\mathrm{Sp}(2n) \subset \mathrm{Sp}(2n+2)$ is given by $A \mapsto \left(\begin{smallmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

Let $G = \varinjlim G_n$ be an ind-group. A *Cartan (resp., Borel) subgroup* of G is an ind-subgroup H (resp., B) of G such that for a well-chosen filtration (1), the group $H_n = G_n \cap H$ (resp., $B_n = G_n \cap B$) is a Cartan (resp., Borel) subgroup of G_n for each n. A *parabolic subgroup* P of G is an ind-subgroup P which contains a Borel subgroup B. A *G-module* is a vector space V equipped with compatible structures of G_n-modules for all n.

An ind-group is an example of the more general notion of an ind-variety (see [3,6]). Briefly, an *ind-variety* $X = \varinjlim X_n$ is determined by a sequence of closed embeddings of algebraic varieties

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$$

An ind-variety X is automatically a topological space, and one defines the structure sheaf $\mathcal{O}_X$ of X as the projective limit sheaf of the structure sheaves $\mathcal{O}_{X_n}$ of X_n. More generally, a sheaf $\mathcal{F}$ on X is the limit of a projective system of sheaves $\mathcal{F}_n$ on X_n.

Let $G = \varinjlim G_n$ be a locally semisimple ind-group and $P = \varinjlim P_n$ be a parabolic subgroup. Let E be a finite-dimensional P-module. The inductive limit $G/P = \varinjlim G_n/P_n$ has a natural structure of an ind-variety. The sheaf $\mathcal{O}_{G/P}(E^*)$ is defined as the projective limit $\mathcal{O}_{G/P}(E^*) = \varprojlim \mathcal{O}_{G_n/P_n}(E_n^*)$, where $E_n = E_{|P_n}$ and $\mathcal{O}_{G_n/P_n}(E_n^*)$ is the sheaf of regular local sections of the homogeneous vector bundle over G_n/P_n induced by the module E_n^*. Using the Mittag-Leffler principle, it is shown in [3] that the cohomology group $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ is canonically isomorphic to the projective limit $\varprojlim H^q(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*))$ for each q.

By definition (see [3]), the Lie algebra of an ind-group $G = \varinjlim G_n$ is the inductive limit Lie algebra $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ for the filtration

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n \subset \cdots ,$$

where $\mathfrak{g}_n$ is the Lie algebra of G_n and the inclusions $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ are the differentials of the group embeddings $G_n \subset G_{n+1}$. A Lie algebra $\mathfrak{g}$ which is isomorphic to a direct limit of finite-dimensional Lie algebras is called *locally finite*. Therefore, the Lie algebra of an ind-group is locally finite. A locally finite Lie algebra $\mathfrak{g}$ which is isomorphic to a direct limit of (semi)simple finite-dimensional Lie algebras, is called *locally (semi)simple*. First examples of locally simple Lie algebras are the *simple finitary* Lie algebras $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, and $\mathrm{sp}(\infty)$, which are the Lie algebras of the ind-groups $\mathrm{SL}(\infty)$, $\mathrm{SO}(\infty)$, and $\mathrm{Sp}(\infty)$.

A module M over $\mathfrak{g}$ is called *integrable* if $\dim \operatorname{span}\{m, g \cdot m, g^2 \cdot m, \cdots\} < \infty$ for any $m \in M$ and $g \in \mathfrak{g}$. Below we use extensively the properties of integrable

$\mathfrak{g}$-modules established in [8]. By $\mathrm{Int}_{\mathfrak{g}}$ we denote the category of integrable $\mathfrak{g}$-modules.

3. On integrable $\mathfrak{g}$-modules. In this section $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ denotes a locally semisimple Lie algebra.

Lemma 3.1. *Let $M = \varinjlim M_n$ be an integrable $\mathfrak{g}$-module obtained as the inductive limit of finite-dimensional $\mathfrak{g}_n$-modules M_n. If the length of M_n is bounded by a number N not depending on n, the length of the $\mathfrak{g}$-module M is also bounded by N.*

Proof. Assume, on the contrary, that M possesses a chain of submodules of the form

$$0 \subsetneq U^1 \subsetneq U^2 \subsetneq \cdots \subsetneq U^N \subsetneq U^{N+1} = M.$$

We choose elements $x_1 \in U^1 \setminus \{0\}$, $x_2 \in U^2 \setminus U^1$, $\ldots, x_{N+1} \in U^{N+1} \setminus U^N$, and choose n large enough so that $x_1, x_2, \ldots, x_{N+1} \in M_n$. Then we consider the filtration of M_n

$$0 \subset (U^1 \cap M_n) \subset (U^2 \cap M_n) \subset \cdots \subset (U^N \cap M_n) \subset M_n$$

and note that $U^i \cap M_n \neq U^{i+1} \cap M_n$ for $i = 1, \ldots, N$. This contradicts the assumption that the length of M_n is bounded by N. □

We recall the following results from [8] which are very useful for our considerations.

Lemma 3.2 ([8], Lemma 4.1, Proposition 3.2). *Let M be an integrable $\mathfrak{g}$-module.*

- (a) *The $\mathfrak{g}$-module $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ is an integrable $\mathfrak{g}$-module if and only if for any n, M considered as a $\mathfrak{g}_n$-module has finitely many isotypic $\mathfrak{g}_n$-components.*
- (b) *If the $\mathfrak{g}$-module M^* is integrable, then M^* is an injective object in the category $\mathrm{Int}_{\mathfrak{g}}$.*

Corollary 3.3. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$ be a module of finite length N with simple constituents M_i, $i = 1, \ldots N$, and such that $M^* \in \mathrm{Int}_{\mathfrak{g}}$. Then there is an isomorphism of $\mathfrak{g}$-modules*

$$M^* \cong \bigoplus_i M_i^*.$$

Proof. The case $N = 1$ is clear and the general case follows from a simple induction argument using Lemma 3.2 (b) and the fact that $\mathrm{Int}_{\mathfrak{g}}$ is closed with respect to taking submodules and quotients. □

In the setting of Section 2, let B_n be a Borel subgroup of G_n. A weight μ of $B = \varinjlim B_n$ is the (projective) limit of a projective system of integral G_n-weights $\{\mu_n\}$. By definition, μ is *B-dominant* if all μ_n are B_n-dominant. Let

μ be a B-dominant weight. By $V_B(\mathbb{C}_\mu)$ we denote the inductive limit of finite-dimensional simple G_n-modules $V_{B_n}(\mathbb{C}_{\mu_n})$ with B_n-highest weight μ_n, where the highest weight space $\mathbb{C}_{\mu_n}$ of $V_{B_n}(\mathbb{C}_{\mu_n})$ is mapped to $\mathbb{C}_{\mu_{n+1}}$. Assuming that G_n is classical simple, following BOURBAKI [1], we write the weights of G_n as linear combinations of standard functions $\varepsilon_n^1, \ldots, \varepsilon_n^{r_n+1}$ if G_n is of type A, and $\varepsilon_n^1, \ldots, \varepsilon_n^{r_n}$ otherwise, where $r_n = \mathrm{rk}\mathfrak{g}_n$.

Proposition 3.4. *Let G be one of the finitary simple ind-groups and let $\mu = \varprojlim \mu_n$ be a dominant weight of G.*

(i) If $G \cong \mathrm{SL}(\infty)$, then $V_B(\mathbb{C}_\mu)^*$ is an integrable $\mathfrak{g}$-module if and only if there exists an integer m_0 such that for any n if $\mu_n = \sum_i a_n^i \varepsilon_n^i$ then $a_n^1 - a_n^{r_n+1} \leq m_0$.

(ii) If $G \cong \mathrm{SO}(\infty)$ or $\mathrm{Sp}(\infty)$, then $V_B(\mathbb{C}_\mu)^*$ is an integrable $\mathfrak{g}$-module if and only if there exists an integer m_0 such that for any n if $\mu_n = \sum_i a_n^i \varepsilon_n^i$, then $a_n^1 \leq m_0$.

Proof. According to Lemma 3.2 (a), the $\mathfrak{g}$-module $V_B(\mathbb{C}_\mu)^*$ is integrable if and only if, when considered as a $\mathfrak{g}_n$-module, it has finitely many isotypic components. Using the standard branching rules for embeddings of classical groups (see e.g. [9]) one can show that this latter condition is equivalent to the explicit conditions of (i) and (ii). □

Notice that the simple tensor sl(∞)-, so(∞)-, sp(∞)-modules discussed in [8] are a special case of the modules discussed in Proposition 3.4.

More generally, if G is a diagonal nonfinitary locally simple ind-group, it turns out that the condition of $\mathfrak{g}$-integrability of the module $V_B(\mathbb{C}_\mu)^*$ is equivalent to the condition that $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(V_B(\mathbb{C}_\mu)) \neq 0$, see ([7], Proposition 4.5) and the references therein. Here we present an explicit sufficient condition for the integrability of $V_B(\mathbb{C}_\mu)^*$ for all diagonal locally simple ind-groups.

Proposition 3.5. *Let G be a diagonal locally simple ind-group and let $\mu = \varprojlim \mu_n$ be a dominant weight of G. Assume that there exists an integer m_0 with the property that for any n, the weight μ_n has an expression $\mu_n = \sum_i a_n^i \varepsilon_n^i$ such that $\sum_i |a_n^i| \leq m_0$. Then $V_B(\mathbb{C}_\mu)^*$ is integrable as a $\mathfrak{g}$-module.*

Proof. Since G is diagonal, for large enough n and for all k the embeddings $\mathfrak{g}_n \subset \mathfrak{g}_{n+k}$ are diagonal. Therefore, we can decompose $V_{B_{n+k}}(\mathbb{C}_{\mu_{n+k}})$ over $\mathfrak{g}_n$ using branching rules for diagonal embeddings of classical Lie algebras, see [4,5]. These branching rules show that for the highest weight ν of any simple $\mathfrak{g}_n$-module U which enters the decomposition of $V_{B_{n+k}}(\mathbb{C}_{\mu_{n+k}})$ over $\mathfrak{g}_n$, there exists an expression $\nu = \sum_i b_i \varepsilon_n^i$ which satisfies $\sum_i |b_i| \leq \sum_i |a_{n+k}^i| \leq m_0$. This holds for any k. Thus there are only finitely many isotypic $\mathfrak{g}_n$-components in the decomposition of $V_{B_{n+k}}(\mathbb{C}_{\mu_{n+k}})$ and their number stabilizes for large k. By Lemma 3.2 (a), $V_B(\mathbb{C}_\mu)^*$ is an integrable $\mathfrak{g}$-module. □

4. Decomposition of cohomology. In this section we fix $G = \varinjlim G_n$ to be a locally semisimple ind-group and $P = \varinjlim P_n$ a parabolic subgroup. By $\mathfrak{g}$ we denote the Lie algebra of G.

Lemma 4.1. *Let E be a finite-dimensional P-module. Then*

$$H^q(G/P, \mathcal{O}_{G/P}(E^*))$$

is the dual module of an integrable $\mathfrak{g}$-module of finite length.

Proof. By Theorem 10.3 in [3] we have

$$H^q(G/P, \mathcal{O}_{G/P}(E^*)) = \varprojlim H^q(G_n/P_n, \mathcal{O}_{G_n/P_n}(E_n^*)),$$

where $E_n := E_{|P_n}$. The classical Bott–Borel–Weil theorem implies that

$$H^q(G_n/P_n, \mathcal{O}_{G/P}(E_n^*)) \cong M_n^*,$$

where M_n is some finite-dimensional G_n-module. In other words,

$$H^q(G/P, \mathcal{O}_{G/P}(E^*)) = \varprojlim M_n^*,$$

hence $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ is the dual module to the module $M = \varinjlim M_n$. Obviously, $M \in \mathrm{Int}_{\mathfrak{g}}$. Since the length of each M_n is bounded by the length of E, by Lemma 3.1 the same holds for the length of M. □

Lemma 4.2. *For any finite-dimensional P-module E, the integrability of the $\mathfrak{g}$-module $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ implies the injectivity of this module in the category* $\mathrm{Int}_{\mathfrak{g}}$.

Proof. The statement follows directly from Lemma 4.1 and Lemma 3.2 (b). □

Let E be a finite-dimensional irreducible P-module. Then Theorem 11.1 (ii) from [3] states that $H^q(G/P, \mathcal{O}_{G/P}(E^*)) \neq 0$ for at most one integer $q \geq 0$. Moreover, if q is such an integer, then $H^q(G/P, \mathcal{O}_{G/P}(E^*)) = V^*$ for some irreducible G-module V. We call the module E *strongly finite* if V^* is an integrable $\mathfrak{g}$-module.

Furthermore, we call an arbitrary finite-dimensional P-module E *strongly finite* if all its simple constituents are strongly finite.

We now describe several classes of strongly finite modules. Let $G = \varinjlim G_n$ be a diagonal locally simple ind-group, let $H = \varinjlim H_n$ be a Cartan subgroup and $B = \varinjlim B_n$ be a Borel subgroup containing H.

Proposition 4.3. *Let E be a finite-dimensional B-module satisfying the following condition: if $\lambda = \varprojlim \lambda_n$ is a weight of a simple constituent of E, then there exists an integer m_0, such that for any n the weight λ_n has an expression $\lambda_n = \sum_i a_n^i \varepsilon_n^i$ with $\sum_i |a_n^i| \leq m_0$. Then E is a strongly finite B-module.*

Proof. Let $L = \mathbb{C}_\lambda$ denote the one-dimensional B-module of weight $\lambda = \varprojlim \lambda_n$, such that λ satisfies the condition of the proposition. In [2] an analogue

W_B of the Weyl group for diagonal locally simple ind-groups is defined and it is proved that (see Theorem 4.27 in [2])

$$H^q(G/B, \mathcal{O}_{G/B}(L^*)) = V_B(\mathbb{C}_{w\cdot\lambda})^*,$$

where $w \in W_B$ and $w \cdot \lambda$ is a dominant weight defined as the (projective) limit of a projective system of G_n-weights $\{w(n)(\lambda_n + \rho_n) - \rho_n\}$. Here $w(n)$ is an element of the Weyl group of G_n, determined uniquely by w, and ρ_n denotes as usual the half-sum of the positive roots of $\mathfrak{g}_n$. The definitions of w and of $w(n)$ can be found in [2]. It is important for our considerations that given w, there exists an integer N such that for each n, the weight $w(n)\rho_n - \rho_n$ is a linear combination of at most N of the ε_n^i's. Furthermore, it is proved in [2] that $\{w(n)\rho_n - \rho_n\}$ is a projective system of weights of G_n. It follows that there exists an integer m, such that for all n the weight $w(n)\rho_n - \rho_n$ has an expression $w(n)\rho_n - \rho_n = \sum_i b_n^i \varepsilon_n^i$ with $\sum_i |b_n^i| \leq m$. This shows that if $w(n)(\lambda_n + \rho_n) - \rho_n = \sum_i (k_n^i + b_n^i)\varepsilon_n^i$, then $\sum_i |k_n^i + b_n^i| \leq m_0 + m$. Therefore, by Proposition 3.5, $V_B(\mathbb{C}_{w\cdot\lambda})^*$ is an integrable $\mathfrak{g}$-module. □

When G is a finitary simple ind-group the following stronger result holds.

Proposition 4.4. *Let G be one of the finitary simple ind-groups and let E be a finite-dimensional B-module. Then E is strongly finite if and only if the following holds:*

- *If $G \cong \mathrm{SL}(\infty)$ and $\lambda = \varprojlim \lambda_n$ is the weight of a simple constituent of E, then there exists an integer m_0 such that for any n, if $\lambda_n = \sum_i a_n^i \varepsilon_n^i$, then $a_n^{\max} - a_n^{\min} \leq m_0$. Here $a_n^{\max} = \max_i\{a_n^i\}$ and $a_n^{\min} = \min_i\{a_n^i\}$.*

- *If $G \cong \mathrm{SO}(\infty)$ or $\mathrm{Sp}(\infty)$ and $\lambda = \varprojlim \lambda_n$ is the weight of a simple constituent of E, then there exists an integer m_0 such that for any n, if $\lambda_n = \sum_i a_n^i \varepsilon_n^i$, then $\max_i\{|a_n^i|\} \leq m_0$.*

The proof follows the same ideas as in Proposition 4.3, and uses also Proposition 3.4.

We now return to the general setting of this section and state the main result of this note.

Theorem 4.5. *Let E be a strongly finite P-module. Then*

(i) $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ is an integrable $\mathfrak{g}$-module for any q, and $H^q(G/P, \mathcal{O}_{G/P}(E^*)) \neq 0$ for at most N values of q, where N denotes the length of E.

(ii) $H^q(G/P, \mathcal{O}_{G/P}(E^*)) \neq 0$ implies that as a $\mathfrak{g}$-module $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ is isomorphic to a direct sum of cohomologies of homogeneous bundles of the form $\mathcal{O}_{G/P}(R)$ for simple constituents R of E^*.

Proof. We use induction on N. When E is irreducible, Theorem 11.1 from [3] tells us that $\mathcal{O}_{G/P}(E^*)$ has at most one non-vanishing cohomology group. The integrability of the nonzero cohomology group follows from the assumption that E is strongly finite. This proves (i) and (ii) for an irreducible P-module E. Next, assume that (i) and (ii) hold for stongly finite modules E' of length $N-1$. Then we consider our module E of length N together with a short exact sequence

$$0 \to E'' \to E \to E' \to 0$$

such that E' is of length $N-1$ and E'' is irreducible. If $H^q(G/P, \mathcal{O}_{G/P}(E''^*)) = 0$ for all q, both (i) and (ii) follow trivially from the induction assumption. Suppose that $H^q(G/P, \mathcal{O}_{G/P}(E''^*)) \neq 0$. Then the short exact sequence

$$0 \to \mathcal{O}_{G/P}(E'^*) \to \mathcal{O}_{G/P}(E^*) \to \mathcal{O}_{G/P}(E''^*) \to 0$$

yields an exact sequence

$$\begin{aligned} &0 \to H^q(G/P, \mathcal{O}_{G/P}(E'^*)) \to H^q(G/P, \mathcal{O}_{G/P}(E^*)) \to \\ &H^q(G/P, \mathcal{O}_{G/P}(E''^*)) \xrightarrow{g} H^{q+1}(G/P, \mathcal{O}_{G/P}(E'^*)) \to \\ &H^{q+1}(G/P, \mathcal{O}_{G/P}(E^*)) \to 0. \end{aligned} \tag{2}$$

Since $\mathrm{Int}_{\mathfrak{g}}$ is an abelian category and E is strongly finite, it follows that $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ and $H^{q+1}(G/P, \mathcal{O}_{G/P}(E^*))$ are integrable $\mathfrak{g}$-modules. Moreover, for $p \neq q, q+1$ we have $H^p(G/P, \mathcal{O}_{G/P}(E^*)) \cong H^p(G/P, \mathcal{O}_{G/P}(E'^*))$, and this proves (i).

To prove (ii) we use Lemma 4.6 below and consider two cases for the connecting homomorphism g in the exact sequence (2).

Case 1. $\ker g = H^q(G/P, \mathcal{O}_{G/P}(E''^*))$. Then the integrability of $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ proven in (i), and Lemmas 4.1, 4.2 imply that

$$H^q(G/P, \mathcal{O}_{G/P}(E^*)) \cong H^q(G/P, \mathcal{O}_{G/P}(E'^*)) \oplus H^q(G/P, \mathcal{O}_{G/P}(E''^*)).$$

Furthermore, $H^{q+1}(G/P, \mathcal{O}_{G/P}(E'^*)) \cong H^{q+1}(G/P, \mathcal{O}_{G/P}(E^*))$. Therefore, by induction (ii) holds for both $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ and $H^{q+1}(G/P, \mathcal{O}_{G/P}(E^*))$.

Case 2. $\ker g = 0$. Then $\mathrm{im} g$ is a direct summand of $H^{q+1}(G/P, \mathcal{O}_{G/P}(E'^*))$ by Lemma 4.2, i.e.

$$H^{q+1}(G/P, \mathcal{O}_{G/P}(E'^*)) \cong H^q(G/P, \mathcal{O}_{G/P}(E''^*)) \oplus H^{q+1}(G/P, \mathcal{O}_{G/P}(E^*)).$$

Furthermore, $H^q(G/P, \mathcal{O}_{G/P}(E'^*)) \cong H^q(G/P, \mathcal{O}_{G/P}(E^*))$.

Since by induction both $H^q(G/P, \mathcal{O}_{G/P}(E'^*))$ and $H^{q+1}(G/P, \mathcal{O}_{G/P}(E'^*))$ are isomorphic to direct sums of cohomologies of bundles $\mathcal{O}_{G/P}(R')$ for simple constituents R' of $(E')^*$, and hence also of E^*, the same follows for $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ and $H^{q+1}(G/P, \mathcal{O}_{G/P}(E^*))$. □

Lemma 4.6. *Consider the connecting homomorphism*

$$g : H^q(G/P, \mathcal{O}_{G/P}(E''^*)) \to H^{q+1}(G/P, \mathcal{O}_{G/P}(E'^*)),$$

where E'' *is an irreducible* P*-module. Then* $\ker g = H^q(G/P, \mathcal{O}_{G/P}(E''^*))$ *or* $\ker g = 0$.

Proof. We have the following commuting diagram of homomorphisms

$$\begin{array}{ccc} \dots \to H^q(G_{n+1}/P_{n+1}, \mathcal{O}_{G_{n+1}/P_{n+1}}(E''^*_{n+1})) \to & H^q(G_n/P_n, \mathcal{O}_{G_n/P_n}(E''^*_n)) \to \dots \\ \downarrow g_{n+1} & \downarrow g_n \\ \dots \to H^{q+1}(G_{n+1}/P_{n+1}, \mathcal{O}_{G_{n+1}/P_{n+1}}(E'^*_{n+1})) \to & H^{q+1}(G_n/P_n, \mathcal{O}_{G_n/P_n}(E'^*_n)) \to \dots \end{array}$$

where

$$H^q(G/P, \mathcal{O}_{G/P}(E''^*)) = \varprojlim H^q(G_n/P_n, \mathcal{O}_{G_n/P_n}(E''^*_n)),$$

$$H^{q+1}(G/P, \mathcal{O}_{G/P}(E'^*)) = \varprojlim H^{q+1}(G_n/P_n, \mathcal{O}_{G_n/P_n}(E'^*_n)),$$

and $g = \varprojlim g_n$.

For each n it holds that $H^q(G_n/P_n, \mathcal{O}_{G_n/P_n}(E''^*_n))$ is an irreducible G_n-module. Hence for g_n we have $\ker g_n = H^q(G_n/P_n, \mathcal{O}_{G_n/P_n}(E''^*_n))$ or $\ker g_n = 0$. By the commutativity of the above diagram the statement of the lemma follows. □

Theorem 4.5 together with Lemma 4.2 imply

Corollary 4.7. *If* E *is strongly finite, then all nonzero cohomology groups* $H^q(G/P, \mathcal{O}_{G/P}(E^*))$ *are injective objects of* $\mathrm{Int}_{\mathfrak{g}}$.

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