Borel subalgebras of $gl(\infty)$ *

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Abstract

Our main object of study are Borel subalgebras of the Lie algebra $gl(\infty)$ of finitary infinite matrices. By definition, a Borel subalgebra of $gl(\infty)$ is a maximal locally solvable subalgebra. We give an explicit description of Borel subalgebras as stabilizers of certain chains of subspaces in the natural representation of $gl(\infty)$. More precisely, we claim that each Borel subalgebra of $gl(\infty)$ is the stabilizer of a unique maximal closed generalized flag in the natural representation. We also discuss the relationship between Borel subalgebras and toral subalgebras of $gl(\infty)$. The paper is a self–contained statement of results and examples. Proofs will appear elsewhere.

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Introduction

In this talk we announce our recent general description of all maximal locally solvable subalgebras of the Lie algebra $gl(\infty)$ or, equivalently, of its maximal simple subalgebra $sl(\infty)$. In fact, our main result applies to any Lie algebra associated with a linear system, see Section 1 below. This result is part of our ongoing study of the structure of locally finite Lie algebras and in particular of the classical simple locally finite Lie algebras $sl(\infty)$, $o(\infty)$ and $sp(\infty)$.

Sections 1, 2 and 3 are of preliminary nature. In Section 1 we review some basic properties of linear systems, i.e. of pairs of vectors spaces in duality, see also [M], and discuss the propertues of the Lie algebras associated with linear systems. Section 2 recalls the definition and main properties of generalized flags. Generalized flags, see [DP2], are certain chains of subspaces in an infinite dimensional vector space which generalize the notion of a flag in a finite dimensional vector space. Section 3 summarizes results on maximal toral subalgebras of $gl(\infty)$ following [NP]. The main result, Theorem 1, is stated in Section 4. If U denotes the natural representation of $gl(\infty)$, the theorem claims that each Borel, i.e. maximal locally solvable subalgebra of $gl(\infty)$, is the stabilizer of a unique generalized flag in U which is closed

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with respect to a natural closure operation. We give examples and explain the relations to existing more specific results. In Section 5 we state results about the relation between Borel subalgebras and toral subalgebras of $gl(\infty)$. In particular we describe all Borel subalgebras which contain a given maximal toral subalgebra of a certain type, Theorem 2. We also discuss the toral subalgebras contained in a given Borel subalgebra. In particular we construct a somewhat unexpected example of a Borel subalgebra of $gl(\infty)$ which contains no nonzero toral subalgeras. This example also solves an open problem posed in [NP] as it is an example of a selfnormalizing locally nilpotent subalgebra of $gl(\infty)$ whose adjoint representation is not locally finite.

The present talk contains no proofs. The proofs of all new results announced here will appear in a complete paper to follow.

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Conventions The base field is $\mathbb{C}$. $\mathbb{N} = \{1, 2, \dots\}$. All vector spaces and Lie algebras are assumed to be defined over $\mathbb{C}$. The countable ordinal is denoted as usual by $\aleph_0$. A Lie algebra is *locally finite* (respectively, *locally nilpotent* or *locally solvable*) if every finite set of elements generates a finite dimensional (respectively, nilpotent or solvable) subalgebra. A module M over a Lie algebra $\mathfrak{k}$ is *locally finite* if every vector $m \in M$ is contained in a finite dimensional $\mathfrak{k}$–submodule of M. The superscript * denotes dual space.

1 Linear systems and the Lie algebra $gl(\infty)$

Let U and V be a pair of vector spaces equipped with a fixed bilinear form

$$\langle \circ, \circ \rangle : U \times V \to \mathbb{C}. \tag{1}$$

G. Mackey calls such a pair a *linear system.* In what follows we will always assume that the bilinear form $\langle \circ, \circ \rangle$ is non–degenerate. If (U, V) is a linear system, the vector space $U \otimes V$ is naturally endowed with the structure of an associative algebra over $\mathbb{C}$ such that

$$(u_1 \otimes v_1)(u_2 \otimes v_2) = \langle u_2, v_1 \rangle u_1 \otimes v_2, \tag{2}$$

where $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Furthermore, U is a left $U \otimes V$–module such that $(u_1 \otimes v_1) \cdot u_2 = \langle u_2, v_1 \rangle u_1$, and V is a right $U \otimes V$–module such that $v_1 \cdot (u_2 \otimes v_2) = \langle u_2, v_1 \rangle v_1$. Note also that (1) induces inclusions $U \subset V^*$ and $V \subset U^*$.

If both the dimensions of U and V are finite or countable (in that case they are necessarily equal), G. Mackey has shown, [M], Ch. III, Lemma in Sec. 5, that U and V always admit bases $\{u_\alpha\}$ and $\{v_\alpha\}$ with the property $\langle u_\alpha, v_\beta\rangle = \delta_{\alpha,\beta}$, where $\delta_{\alpha,\beta}$ stands for Kronecker's symbol. An immediate corollary of Mackey's result is that if $\dim U$ and $\dim V$ are countable dimensional or finite, the associative algebra $U \otimes V$ depends up to isomorphism only on $\dim U$. If $\dim U = \dim V = n$, $U \otimes V$ is isomorphic to $\mathrm{End}\, U$, and if $\dim U = \dim V = \aleph_0$, $U \otimes V$ is isomorphic to the algebra Mat^f_∞ of infinite matrices with finitely many nonzero entities. If either $\dim U$ or $\dim V$ is uncountable, the isomorphism class of the associative algebra $U \otimes V$ is not determined by $\dim U$ and $\dim V$ only. An example of a linear system with different dimensions of U and V is the pair $(U, V = U^*)$, where U is a countable dimensional vector space and the bilinear form $\langle \circ, \circ \rangle$ is the canonical pairing $U \times U^* \to \mathbb{C}$.

For the rest of this talk we fix a linear system (U, V). We denote by $\mathfrak{g}$ the Lie algebra corresponding to the associative algebra $U \otimes V$, i.e. $\mathfrak{g} = U \otimes V$ with Lie bracket induced by the product (2). Each of the spaces U and V is a $\mathfrak{g}$–module. When both U and V are countable dimensional, $\mathfrak{g}$ is isomorphic to $gl(\infty)$, the Lie algebra of infinite matrices with finitely many nonzero entries.

We also fix the following notation. For any subspace $W \subset U$ we set $W^\perp := \{v \in V \mid \langle w, v\rangle = 0, \text{ for every } w \in W\}$. By definition, $W^\perp$ is a subspace of V, and $W \subset (W^\perp)^\perp \subset U$. Following Mackey, [M], we call the correspondence

$$W \mapsto \overline{W} := (W^\perp)^\perp$$

closure, and call W *closed* if $W = \overline{W}$.

2 Generalized flags

Any Borel subalgebra of $gl(n)$ is the stabilizer of a unique maximal flag of subspaces in the natural (n–dimensional) representation. Our main result, Theorem 1 below, is an analog of this statement for $\mathfrak{g}$. In the present section we introduce a class of chains of subspaces which we call generalized flags and which appear in Theorem 1.

Let X be a vector space. A *chain* of subspaces in X is a set $\mathcal{C}$ of subspaces in X linearly ordered by inclusion. A *generalized flag* in X, [DP2], is a chain of subspaces $\mathcal{F}$ in X satisfying the following properties:

(i) each space $F \in \mathcal{F}$ has an immediate predecessor or an immediate successor;

(ii) for every $0 \neq x \in X$ there exists a pair $F', F'' \in \mathcal{F}$, such that $x \in F'' \backslash F'$ and F'' is the immediate successor of F'.

Condition (i) implies that $\mathcal{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$, where F'_α is the immediate predecessor of F''_α, and A is an index set which is linearly ordered as follows: $\alpha \prec \beta$ if and only if F'_α is a proper subspace of F'_β. If a subspace $F \in \mathcal{F}$ has both an immediate successor and an immediate predecessor, then $F = F'_\alpha = F''_\beta$ for some $\alpha, \beta \in A$. (In the latter case, β is the immediate predecessor of α in A.) A generalized flag $\mathcal{F}$ such that the corresponding index set A of $\mathcal{F}$ is isomorphic as an ordered set to a subset of $\mathbb{Z}$, is called a *flag* in X. For the rest of the talk the superscripts $'$ and $''$ will be used to denote two subspaces in a generalized flag, such that the subspace with superscript $''$ is the immediate successor of the subspace with superscript $'$.

Example 1.

a) Any chain of subspaces in X of one of the following forms:

(i)	$0 \subset F_1 \subset F_2 \subset \ldots \subset F_n \subset X$;	
(ii)	$0 \subset F_1 \subset F_2 \subset \ldots,$	such that $\cup_{i \in \mathbb{N}} F_i = X$;
(iii)	$\ldots \subset F_{-2} \subset F_{-1} \subset X,$	such that $\cap_{i \in \mathbb{N}} F_{-i} = 0$;
(iv)	$\ldots \subset F_{-2} \subset F_{-1} \subset F_0 \subset F_1 \subset \ldots,$	such that $\cap_{i \in \mathbb{Z}} F_i = 0$ and $\cup_{i \in \mathbb{Z}} F_i = V$

is a flag in X. Furthermore, any flag in X is of one of the above types.

b) An infinite chain of subspaces in X of the form

$$0 \subset F_1 \subset F_2 \subset \ldots \subset F_{-2} \subset F_{-1} \subset X,$$

such that $\cup_{i \in A_+} F_i = \cap_{j \in A_-} F_{-j}$, is a generalized flag but not a flag. Here A_+ and A_- are nonempty subsets of $\mathbb{N}$ not both of which are finite. A simple case which we will consider below is the case when $A_+ = \mathbb{N}$ and $A_- = \{1\}$.

c) Let $\dim X = \aleph_0$ and let $\{x_q\}_{q \in \mathbb{Q}}$ be a basis of X enumerated by $\mathbb{Q}$. For each $q \in \mathbb{Q}$, set $F'_q = \mathrm{span}\{x_s \,|\, s < q\}$ and $F''_q = \mathrm{span}\{x_s \,|\, s \leq q\}$. Then the chain of subspaces $\mathcal{F} = \{F'_q, F''_q\}_{q \in Q}$ is a generalized flag with $A = \mathbb{Q}$.

A generalized flag $\mathcal{F}$ is *maximal* if it is not properly contained in another generalized flag in X. Clearly, $\mathcal{F} = \{F'_\alpha, F''_\alpha\}_{\alpha \in A}$ is maximal if and only if $\dim F''_\alpha / F'_\alpha = 1$ for every $\alpha \in A$. In particular, the generalized flag $\mathcal{F}$ from Example 1, c) is a maximal generalized flag. However, it is not a maximal chain of subspaces in X for the chain $\mathcal{C} = \{F'_q, F''_q, F_\iota\}_{q \in \mathbb{Q}, \iota \in \mathbb{R} \setminus \mathbb{Q}}$, where $F_\iota := \mathrm{span}\{e_s \,|\, s < \iota\}$, properly contains $\mathcal{F}$. In fact, one can check that $\mathcal{C}$ is the unique maximal chain in X containing the maximal generalized flag $\mathcal{F}$, see also [DP2]. More generally, any chain $\mathcal{C}$ of subspaces in X determines a unique generalized flag. Indeed, if $\mathcal{C} = \{C_\kappa\}$ is a chain and $x \in X$ is a nonzero vector, put $F''_x(\mathcal{C}) := \cup_{F \in \mathcal{C}, x \notin F} F$ and $F'_x(\mathcal{C}) := \cap_{F \in \mathcal{C}, x \in F} F$. Then $\mathcal{F} := \{F'_x(\mathcal{C}), F''_x(\mathcal{C})\}_{0 \neq x \in X}$ is a generalized flag in X which we denote by $\mathit{fl}(\mathcal{C})$. (See [DP2] for more details on the relation between $\mathcal{C}$ and $\mathit{fl}(\mathcal{C})$.)

If $\mathcal{F}$ is a generalized flag in X and $\{x_\beta\}_{\beta \in B}$ is a basis of X, we say that $\mathcal{F}$ and $\{x_\beta\}$ are compatible if there exists an order preserving injection $\phi : A \to B$ such that $F'_\alpha = \mathrm{span}\{x_\beta \,|\, \beta \prec \phi(\alpha)\}$ and $F''_\alpha = \mathrm{span}\{x_\beta \,|\, \phi(\alpha) \not\prec \beta\}$. For instance, the generalized flag in Example 3, c) is compatible with the basis $\{x_q\}_{q \in \mathbb{Q}}$.

Proposition 3 in [DP2] claims that if $\dim X \leq \aleph_0$, then every generalized flag in X admits a compatible basis. There are generalized flags in X with $\dim X > \aleph_0$ which do not admit compatible bases.

Consider now generalized flags in the space U of a linear system (U, V). In all considerations below U may be replaced by V. To every chain $\mathcal{C} = \{C_\kappa\}$ in V we can assign the chain $\mathcal{C} := \{C_\kappa^\perp\}$ in V, and by iteration, the chain $(\mathcal{C}^\perp)^\perp$ in U. It is not true that $\mathcal{C}$ is a subchain of $(\mathcal{C}^\perp)^\perp$, as for instance we may have $\mathcal{C}^\perp = \{0\}$ and, consequently, $(\mathcal{C}^\perp)^\perp = \{X\}$, while $\mathcal{C}$ has infinitely many spaces. If $\mathcal{F}$ is a generalized flag, then $\mathcal{F}^\perp$ and $(\mathcal{F}^\perp)^\perp$ are not necessarily generalized flags but are in general well defined chains in V and U respectively. Therefore we can define $\overline{\mathcal{F}}$ as *fl*$((\mathcal{F}^\perp)^\perp)$. We call $\mathcal{F}$ *closed* if $\overline{\mathcal{F}} = \mathcal{F}$, and *strongly closed* if $(\mathcal{F}^\perp)^\perp = \mathcal{F}$. Clearly every strongly closed generalized flag in U is a closed generalized flag. The converse is not true. Here is an explicit characterization of closed and strongly closed generalized flags in U.

Proposition 1 (i) *$\mathcal{F}$ is strongly closed if and only if $\overline{F} = F$ for every $F \in \mathcal{F}$.*

(ii) *$\mathcal{F}$ is closed if and only if $\overline{F''_\alpha} = F''_\alpha$ and $\overline{F'_\alpha}$ equals either F'_α or F''_α for every $\alpha \in A$.*

Example 2.

a) Let U be a countable dimensional space with basis $\{u_\alpha\}_{\alpha \in A}$, $V = \operatorname{span}\{u_\alpha^*\} \subset U^*$, where $u_\alpha^*(u_\beta) = \delta_{\alpha,\beta}$, and the bilinear form $U \times V \to \mathbb{C}$ be the restriction of the canonical pairing $U \times U^* \to \mathbb{C}$ to $U \times V$. Any generalized flag $\mathcal{F}$ which is compatible with the basis $\{u_\alpha\}$ is automatically strongly closed.

b) Let (U, V) be as in a) and let $\{a_\alpha\}$ denote the coordinates of a vector $u \in U$ with respect to the basis $\{u_\alpha\}$. Identify A with $\mathbb{N} \times \mathbb{N}$ and let U_j for $j \in \mathbb{N}$ be the subspace of U given by the system of j equations

$$\sum_{k,l \in \mathbb{N}} a_{k,l} = 0, \quad \sum_{k,l \in \mathbb{N}, k \geq 2} a_{k,l} = 0, \quad \ldots \quad , \sum_{k,l \in \mathbb{N}, k \geq j} a_{k,l} = 0.$$

Then the chain $\mathcal{F}$

$$\ldots \subset U_2 \subset U_1 \subset U$$

is a (maximal) flag for which $\mathcal{F}^\perp = \{0\}$ and $(\mathcal{F}^\perp)^\perp = \{U\}$, i.e. $\mathcal{F}$ is not closed.

c) Let now $U = \operatorname{span}\{u_i, \tilde{u}\}_{i \in \mathbb{N}}$, $V = \operatorname{span}\{v_i\}_{i \in \mathbb{N}}$, and $\langle u_i, v_j \rangle = \delta_{i,j}$, $\langle \tilde{u}, v_i \rangle = 1$. Then the chain $\mathcal{F}$

$$0 \subset U_1 \subset U_2 \subset \ldots \subset U' \subset U,$$

where $U_j = \operatorname{span}\{u_i\}_{i \leq j}$, $U' = \cup_{j \in \mathbb{N}} U_j$, is a maximal generalized flag in U. We have $\overline{U_j} = U_j$ for every j and $\overline{U'} = U$. Hence $\mathcal{F}$ is closed but not strongly closed.

3 Maximal toral subalgebras of $\mathfrak{g}$

In this section we review some results from [NP] which are relevant to our topic.

We call an element $g \in \mathfrak{g}$ *semisimple* (respectively, *nilpotent*) if it is semisimple (respectively, nilpotent) as a linear operator on the vector space U. A subalgebra $\mathfrak{t} \subset \mathfrak{g}$ is *toral* if all its elements are semisimple. Similarly, to the classical case of a semisimple finite dimensional Lie algebra, any toral subalgebra of $\mathfrak{g}$ is necessarily abelian, [NP], Lemma 1.3.

A *dual system* of one dimensional subspaces in the linear system (U, V) is a pair of sets of one dimensional subspaces U^α, V^α, α running over some index set A, such that $\langle U^\alpha, V^\beta \rangle = 0$ if and only if $\alpha \neq \beta$. There is the following correspondence between maximal dual systems of one dimensional subspaces (i.e. dual systems which are not proper subsets of any dual system), and maximal toral subalgebras of $\mathfrak{g}$. If $U^\alpha \subset U$, $V^\alpha \subset V$ is a maximal dual system, we set

$$\mathfrak{t} := \oplus_{\alpha \in A} U^\alpha \otimes V^\alpha.$$

Conversely, if $\mathfrak{t}$ is a maximal toral subalgebra, we define the families of one dimensional subspaces in U and V as eigenspaces of $\mathfrak{t}$ with nonzero eigenvalues in U and V respectively. The following proposition is a reformulation of [NP], Proposition 3.7.

Proposition 2 *The above correspondence is a well–defined bijection between the set of maximal toral subalgebras of $\mathfrak{g}$ and the set of maximal dual systems of one dimensional subspaces in the linear system (U, V).*

A maximal toral subalgebra $\mathfrak{t} \subset \mathfrak{g}$ determines also the following subspaces of U and V:

$$U^0_\mathfrak{t} := \{u \in U \,|\, t \cdot u = 0 \text{ for every } t \in \mathfrak{t}\},$$

$$V^0_\mathfrak{t} := \{v \in V \,|\, t \cdot v = 0 \text{ for every } t \in \mathfrak{t}\}.$$

It is shown in [NP] that $\langle U^0_\mathfrak{t}, V^0_\mathfrak{t} \rangle = 0$. We call a maximal toral subalgebra $\mathfrak{t}$ *splitting* if $U = \oplus_{\alpha \in A} U^\alpha$ and $V = \oplus_{\alpha \in A} V^\alpha$.

Example 3.

a) Let U and V be as in Example 2, a). The subalgebra $\mathfrak{t} = \oplus_{\alpha \in A} (\mathbb{C} u_\alpha) \otimes (\mathbb{C} u^*_\alpha)$ is a splitting maximal toral subalgebra of $\mathfrak{g}$, and any splitting maximal toral subalgebra of $\mathfrak{g}$ is of this form for some basis $\{u'_\alpha\}_{\alpha \in A}$ such that $\text{span}\{(u'_\alpha)^*\} = V$.

b) Let U and V be as in Example 2, c). The subalgebra $\mathfrak{t} = \oplus_{i \in \mathbb{N}} (\mathbb{C} u_i) \otimes (\mathbb{C} u^*_i)$ is a maximal toral subalgebra of $\mathfrak{g}$ which is not splitting.

c) Let U and V be as in Example 2, a) with $A = \mathbb{N}$. Consider the following maximal dual system

$$U^k = \mathbb{C}(u_k - u_2), \quad V^k = \mathbb{C}(u^*_k - u^*_1), \quad \text{for } k \geq 3.$$

The corresponding maximal toral subalgebra

$$\mathfrak{t} = \oplus_{k \geq 3} U^k \otimes V^k$$

is not splitting, and furthermore, both $U^0_\mathfrak{t} = \mathbb{C} u_2$ and $V^0_\mathfrak{t} = \mathbb{C} u^*_1$ are nonzero.

In [NP] a Cartan subalgebra of $\mathfrak{g}$ is defined as a self–normalizing locally nilpotent subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ for which the adjoint module of $\mathfrak{h}$ is locally finite. It is shown ([NP], Theorem 4.1 and Proposition 3.8) that any such subalgebra of $\mathfrak{g}$ is the centralizer $C(\mathfrak{t})$ of a unique maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Moreover, $C(\mathfrak{t}) = \mathfrak{t} \oplus (U_{\mathfrak{t}}^0 \otimes V_{\mathfrak{t}}^0)$. Imposing the additional condition of locally finite action in the above definition is in contrast with the definitions of a toral subalgebra or a Borel subalgebra of $\mathfrak{g}$. Indeed, the latter are very straightforward extensions of the definitions in the finite dimensional case. Therefore the problem whether locally finite action is a redundant condition is quite natural and was posed in [NP]. We show in Section 4 that this condition is in fact essential by constructing an example of a self–normalizing locally nilpotent subalgebra of $\mathfrak{g}$ whose adjoint representation is not locally finite, see Example 4 below.

4 Borel subalgebras of $\mathfrak{g}$

We are now ready to announce and discuss the main result of the talk.

We define a *Borel subalgebra of* $\mathfrak{g}$ as a maximal locally solvable subalgebra of $\mathfrak{g}$. For the finite dimensional Lie algebra $gl(n)$, every Borel (i.e. maximal solvable) subalgebra is the stabilizer of a unique maximal flag in the natural representation of $gl(n)$. The following theorem is a far reaching generalization of this result.

Theorem 1 *Every Borel subalgebra* $\mathfrak{b}$ *of* $\mathfrak{g}$ *is the stabilizer of a unique maximal closed generalized flag* $\mathcal{F}_{\mathfrak{b}}$ *in* U*, and the map*

$$\mathfrak{b} \mapsto \mathcal{F}_{\mathfrak{b}}$$

is a bijection between the set of Borel subalgebras in $\mathfrak{g}$ *and the set of maximal closed generalized flags in* U*. The inverse map is*

$$\mathcal{F} \mapsto \mathrm{St}_{\mathcal{F}},$$

where $\mathrm{St}_{\mathcal{F}}$ *denotes the stabilizer of* $\mathcal{F}$*.*

In fact, both maps in Theorem 1 are very explicit. Firstly, $\mathcal{F}_{\mathfrak{b}} = fl(\{\overline{\mathfrak{b} \cdot u}\}_{u \in U})$, where $\{\mathfrak{b} \cdot u\}_{u \in U}$ is the chain of cyclic $\mathfrak{b}$–submodules of U, and, secondly, $\mathrm{St}_{\mathcal{F}} = \sum_\alpha F''_\alpha \otimes (F'_\alpha)^\perp$ for any generalized flag $\mathcal{F} = \{F'_\alpha, F''_\alpha\}$ in U. Furthermore, the maximal closed generalized flags in U have a simple description: $\mathcal{F}$ is a maximal closed generalized flag in U if and only if it is closed and $\dim F''_\alpha / F'_\alpha = 1$ whenever $\overline{F'_\alpha} = F'_\alpha$, cf. Proposition 1.

Next we describe the nilradical of a Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ with $\mathcal{F}_{\mathfrak{b}} = \{F'_\alpha, F''_\alpha\}$, and let $\mathfrak{n}_{\mathfrak{b}}$ denote the subspace of nilpotent elements in $\mathfrak{b}$.

Proposition 3 (i) $\mathfrak{n}_{\mathfrak{b}}$ *is an ideal of* $\mathfrak{b}$*. Moreover,* $\mathfrak{n}_{\mathfrak{b}} = \sum_\alpha F''_\alpha \otimes (F''_\alpha)^\perp = [\mathfrak{b}, \mathfrak{b}]$*, and* $\mathfrak{b}$ *is the normalizer of* $\mathfrak{n}_{\mathfrak{b}}$ *in* $\mathfrak{g}$*.*

(ii) *There exists a toral subalgebra* $\mathfrak{l}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{b} = \mathfrak{l} \oplus \mathfrak{n}_{\mathfrak{b}}$*, and* $[\mathfrak{l}, \mathfrak{b}] \subset \mathfrak{n}_{\mathfrak{b}}$*.*

Unlike the case of $gl(n)$, the toral subalgebra $\mathfrak{l}$ need not be a maximal toral subalgebra of $\mathfrak{g}$. For more details on the relation between Borel subalgebras and toral subalgebras of $\mathfrak{g}$ in the case when $\mathfrak{g} \cong gl(\infty)$ see Section 5.

In the rest of this section we use Theorem 1 to provide examples of Borel subalgebras of $\mathfrak{g}$ by describing explicitly their corresponding generalized flags $\mathcal{F}_{\mathfrak{b}}$.

The simplest maximal closed generalized flags in U are the maximal strongly closed generalized flags in U. Note that $\mathcal{F}$ is a maximal strongly closed generalized flag in U if and only if $\mathcal{F}$ is a strongly closed generalized flag which is also a maximal generalized flag. Let $\{u_\alpha\}_{\alpha \in A}$ be a basis of U such that for every $\alpha \in A$ there is an element $u^*_\alpha \in V$ with $\langle u_\beta, u^*_\alpha \rangle = \delta_{\alpha,\beta}$ for every $\beta \in A$. Then every maximal generalized flag in U compatible with $\{u_\alpha\}$ is a maximal strongly closed generalized flag in U. Conversely, if $\mathcal{F}$ is a maximal strongly closed generalized flag in U, then $\mathcal{F}$ admits a compatible basis $\{u_\alpha\}$ with the above property.

A simple example of a maximal closed generalized flag in U which is not strongly closed is the generalized flag $\mathcal{F}$ from Example 2, c).

Our next example is an example of a Borel subalgebra $\mathfrak{b}$ of $gl(\infty)$ for which $\mathfrak{n}_{\mathfrak{b}} = \mathfrak{b}$. As every Borel subalgebra is self–normalizing, $\mathfrak{b}$ is an example of a self–normalizing locally nilpotent subalgebra of $\mathfrak{l}(\infty)$ whose adjoint module is not locally finite. The latter follows directly from the explicit description of $\mathfrak{b}$.

Example 4. Let $U = \mathrm{span}\{\tilde{u}_q\}_{q \in \mathbb{Q}}$, $V = \mathrm{span}\{u^*_q\}_{q \in \mathbb{Q}}$, and where

$$\langle \tilde{u}_q, u^*_s \rangle = \begin{cases} 1 & \text{if} \quad q > s \\ 0 & \text{if} \quad q \leq s. \end{cases}$$

Then $\langle \circ, \circ \rangle$ is non–degenerate and $\dim U = \dim V = \aleph_0$, hence $\mathfrak{g} \cong gl(\infty)$. Similarly to Example 1, c), set $F'_q = \mathrm{span}\{\tilde{u}_s \,|\, s < q\}$ and $F''_q = \mathrm{span}\{\tilde{u}_s \,|\, s \leq q\}$. Then $\mathcal{F} = \{F'_q, F''_q\}$ is a maximal closed generalized flag in U for which $\overline{F'_q} = F''_q$ for every $q \in \mathbb{Q}$. Thus, for $\mathfrak{b} = \mathrm{St}_{\mathcal{F}}$, $\mathfrak{n}_{\mathfrak{b}} = \mathfrak{b}$ by Proposition 3 (i). Moreover, $\mathfrak{b}$ contains no nonzero semisimple elements, and hence no nontrivial toral subalgebras.

5 The case when $\mathfrak{g} = gl(\infty)$

In this section we restrict ourselves to the case when $\mathfrak{g} \cong gl(\infty)$, i.e. $\dim U = \dim V = \aleph_0$, and study the relationship between maximal toral subalgebras and Borel subalgebras of $\mathfrak{g}$. As the examples at the end of the previous section show, Theorem 1 is a powerful tool for constructing Borel subalgebras of $gl(\infty)$. However, not all relevant information about a Borel subalgebra $\mathfrak{b}$ can be easily read off the generalized flag $\mathcal{F}_{\mathfrak{b}}$. In fact, it is very useful to look at the $\mathfrak{b}$–stable maximal closed generalized flags in both spaces U and V of $\mathfrak{g}$. The consideration of both representations U and V leads naturally to connections between Borel subalgebras and toral subalgebras of $\mathfrak{g}$.

In the case of $gl(n)$, every Borel subalgebra contains a maximal toral subalgebra, and in fact, infinitely many maximal toral subalgebras. As Example 4 shows, it is no longer true in the case of $gl(\infty)$. On the other hand, every toral subalgebra of $\mathfrak{g}$ is abelian, thus solvable, and hence it is contained in a Borel subalgebra, and, in fact, in infinitely many Borel subalgebras. The best understood Borel subalgebras are those containing a splitting maximal toral subalgebra of $\mathfrak{g}$, and we discuss them first.

Define a *splitting Borel subalgebra* $\mathfrak{b}$ of $\mathfrak{g}$ as a Borel subalgebra $\mathfrak{b}$ containing a splitting maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. If $\mathfrak{b}$ is splitting, all $\mathfrak{b}$–stable subaspaces in U are of the form $\text{span}\{U^\alpha\}_{\alpha\in B}$ for varying subsets B of the set of indices A of the maximal dual system $\{U^\alpha, V^\alpha\}_{\alpha\in A}$ corresponding to $\mathfrak{t}$. This follows from the fact that all $\mathfrak{t}$–invariant subspaces have that form. The following proposition characterizes the splitting Borel subalgebras of $\mathfrak{g}$.

Proposition 4 *Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. The following statements are equivalent.*

(i) $\mathfrak{b}$ *is splitting.*

(ii) *The $\mathfrak{b}$–stable maximal closed generalized flags in both U and V are strongly closed.*

(iii) *There exists a direct system of subalgebras $\mathfrak{g}_n \subset \mathfrak{g}$, such that $\mathfrak{g}_n \cong gl(n)$ and $\varinjlim \mathfrak{g}_n = \mathfrak{g}$, and for which the intersection $\mathfrak{b}\cap\mathfrak{g}_n$ is a Borel subalgebra of $\mathfrak{g}_n$ for every n.*

Informally, Proposition 4 shows that if one thinks of $gl(\infty)$ as the direct limit of $gl(n)$, one is naturally led to consider splitting Borel subalgebras only. Proposition 4 implies also that the splitting Borel subalgebras of $\mathfrak{g}$ containing a fixed maximal toral subalgebra $\mathfrak{t}$ are in a bijective correspondence with maximal generalized flags in U compatible with a fixed basis $\{u_\alpha\}$ of U such that $u_\alpha \in U^\alpha$ for every $\alpha \in A$. In other words, the splitting Borel subalgebras containing $\mathfrak{t}$ are in a bajective correspondence with permutations of the index set A. This result is well known, and has appeared in particular in [DP1], [N] and [LN].

Here are examples of splitting Borel subalgebras of $\mathfrak{g}$. We assume that the maximal toral subalgebra $\mathfrak{t}$ and its corresponding dual system $\{U^\alpha, V^\alpha\}$ are fixed.

Example 5. Here U and V are as in Example 2, a).

a) If $A = \mathbb{N}$, set $U_i = \text{span}\{u_j\}_{j\le i}$ and $U_{-i} = \text{span}\{u_j\}_{j\ge i}$. Then the generalized flags

$$0 \subset U_1 \subset U_2 \subset \dots$$

and

$$\dots \subset U_{-2} \subset U_{-1} \subset U$$

are maximal, they are compatible with the basis $\{u_i\}$, and their respective stabilizers are splitting Borel subalgebras of $\mathfrak{g}$.

If $A = \mathbb{Z}$, we set $U_i = \text{span}\{u_j\}_{j\le i}$ and the generalized flag

$$\dots \subset U_{-1} \subset U_0 \subset U_1 \subset \dots$$

is also maximal and compatible with the basis $\{u_j\}$ and its stabilizer is a splitting Borel subalgebra of $\mathfrak{g}$.

All generalized flags above are flags and the corresponding Borel subalgebras play a special role among all splitting Borel subalgebras as each of them admits a basis of simple roots. We do not discuss roots in this talk and refer the interested reader to [DP1].

b) Let $A = \mathbb{Q}$. Set, as in Example 1, c), $U'_q = \mathrm{span}\{u_s \,|\, s < q\}$, $U''_q = \mathrm{span}\{u_s \,|\, s \leq q\}$. Then the generalized flag $\mathcal{F} = \{U'_q, U''_q\}_{q\in\mathbb{Q}}$ is maximal and compatible with the basis $\{u_q\}$. It's stabilizer is a Borel subalgebra of $\mathfrak{g}$, and this Borel subalgebra does not stabilize any maximal flag in U.

Here are some comments to Example 5. First of all, both parts a) and b) show that a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ does not necessarily have a one dimensional (or even a finite dimensional) $\mathfrak{b}$–stable subspace in U. On the other hand, a splitting Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ always stabilizes a unique maximal chain in U. This is the unique maximal chian $\mathcal{C}_\mathfrak{b}$ such that $\mathit{fl}(\mathcal{C}_\mathfrak{b}) = \mathcal{F}_\mathfrak{b}$. In Example 5, a) $\mathcal{C}_\mathfrak{b}$ is obtained from $\mathcal{F}_\mathfrak{b}$ by adding either U or 0, while in Example 5, b) $\mathcal{C}_\mathfrak{b}$ equals the unique maximal chain containing $\mathcal{F}_\mathfrak{b}$ introduced in Section 2:

$$\mathcal{C}_\mathfrak{b} = \mathcal{F}_\mathfrak{b} \cup \{U_\iota\}_{\iota\in\mathbb{R}\setminus\mathbb{Q}},$$

where $U_\iota = \mathrm{span}\{u_s \,|\, s < \iota\}$ for $\iota \in \mathbb{R}\setminus\mathbb{Q}$.

As Example 4 shows, not every Borel subalgebra of $\mathfrak{g}$ is splitting. A simpler example of a non–splitting Borel subalgebra of $\mathfrak{g}$ is the stabilizer $\mathfrak{b}$ of the maximal closed generalized flag $\mathcal{F}$ from Example 2, c). Note however that eventhough $\mathfrak{b}$ is not a splitting Borel subalgebra of $\mathfrak{g}$, it is isomophic to a splitting Borel subalgebra of $\mathfrak{g}$, e.g. to the subalgebra corresponding to the second flag in Example 5, a). This phenomenon is related to the fact that $\mathfrak{b}$ contains the maximal toral subalgebra $\mathfrak{t}$ from Example 3, b) which is not splitting in $\mathfrak{g}$ but is splitting in a subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ with $\mathfrak{g}' \cong gl(\infty)$.

We complete this section by describing all Borel subalgebras of $\mathfrak{g}$ which contain a fixed self–normalizing maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Since $\mathfrak{t}$ is self–normalizing, $U^0_\mathfrak{t} = 0$ or $V^0_\mathfrak{t} = 0$. Without restriction of generality we assume that $U^0_\mathfrak{t} = 0$. For each eigenspace U^α of $\mathfrak{t}$ fix a nonzero vector $u_\alpha \in U^\alpha$. Complete the set $\{u_\alpha\}_{\alpha\in A}$ to a basis $\{u_\alpha\}_{\alpha\in A} \cup \{\tilde{u}_\beta\}_{\beta\in B}$ of U. Consider an index set C with an order $\prec$ such that the relation "$\gamma_1{\sim}\gamma_2$ if and only if neither $\gamma_1 \prec \gamma_2$ nor $\gamma_2 \prec \gamma_1$" is an equivalence relation on C. Suppose, furthermore, that a surjection $\pi : A \cup B \to C$ is given and satisfies the following properties:

(i) the restriction of π on A is injective and $\pi^{-1}(\pi(A)) = A$;

(ii) for every $\beta \in B$, $\langle \tilde{u}_\beta, u^*_\alpha \rangle = 0$ if $\pi(\beta) \prec \pi(\alpha)$;

(iii) for every $\beta \in B$ and every $\alpha \in A$ with $\pi(\alpha) \prec \pi(\beta)$ there exists $\alpha' \in A$ such that $\pi(\alpha) \prec \pi(\alpha') \prec \pi(\beta)$ and $\langle \tilde{u}_\beta, u^*_{\alpha'} \rangle \neq 0$.

For every $\gamma \in C$, set $F'_\gamma := \mathrm{span}\{u_\alpha, \tilde{u}_\beta \,|\, \pi(\alpha) \prec \gamma, \pi(\beta) \prec \gamma\}$ and $F''_\gamma := \mathrm{span}\{u_\alpha, \tilde{u}_\beta \,|\, \pi(\gamma) \not\prec \alpha, \pi(\gamma) \not\prec \beta\}$. Finally, set $\mathcal{F}^\pi := \{F'_\gamma, F''_\gamma\}_{\gamma\in C}$. (In fact, $F'_{\gamma_1} = F'_{\gamma_2}$ and $F''_{\gamma_1} = F''_{\gamma_2}$ if γ_1 and

γ_2 are equivalent with respect to the equivalence relation above, so $\mathcal{F}^\pi$ is indexed by the the quotient $C/\sim$ of C with respect to this equivalence relation.)

Theorem 2 *$\mathcal{F}^\pi$ is a maximal closed generalized flag in U such that $\mathrm{St}_{\mathcal{F}}$ contains $\mathfrak{t}$. Conversely, if $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$, then $\mathcal{F}_{\mathfrak{b}}$ equals $\mathcal{F}^\pi$ for some π as above.*

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