To my parents

Classically Semisimple Locally Finite Lie Superalgebras

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Abstract

We introduce the class of classically semisimple locally finite Lie superalgebras over an algebraically closed field K of characteristic 0 and classify all Lie superalgebras in this class. By definition, a countably dimensional locally finite Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ over K is classically semisimple if it is semisimple (i.e. its locally solvable radical equals zero) and in addition it admits a generalized root decomposition, such that the root spaces generate $\mathfrak{g}$, and $\mathfrak{g}_0$ is a root-reductive Lie algebra in the sense of [DP] and [PS]. We prove that any classically semisimple Lie superalgebra is isomorphic to a direct sum of copies of the infinite dimensional Lie superalgebras $\mathfrak{sl}(\infty|n)$, $\mathfrak{sl}(\infty|\infty)$, $\mathfrak{osp}(m|\infty)$, $\mathfrak{osp}(\infty|2k)$, $\mathfrak{osp}(\infty|\infty)$, $\mathfrak{sp}(\infty)$, $\mathfrak{sq}(\infty)$ and of copies of simple finite dimensional Lie superalgebras with reductive even part. In particular, the above listed infinite dimensional Lie superalgebras are all (up to isomorphism) countably dimensional simple locally finite Lie superalgebras which admit a generalized root decomposition with root-reductive even part. Finally, we describe all generalized root decompositions of any classically semisimple locally finite Lie superalgebra.

Key words (2000 MSC): Primary 17B65, Secondary 17A70.

Introduction. Recently the class of root-reductive locally finite Lie algebras has been studied from several different points of view. According to

*Work supported in part by an NSF GIG Grant, by the Max-Planck-Institute for Mathematics, Bonn, and by MSRI.

[DP], a root-reductive Lie algebra is defined as the union of at most countably many finite-dimensional reductive Lie algebras $\mathfrak{k}_n$ via root embeddings, i.e. via embeddings $i_n : \mathfrak{k}_n \hookrightarrow \mathfrak{k}_{n+1}$ which map a fixed Cartan subalgebra $\mathfrak{h}_n$ of $\mathfrak{k}_n$ into a fixed Cartan subalgebra $\mathfrak{h}_{n+1}$ of $\mathfrak{k}_{n+1}$, and map any root space of $\mathfrak{k}_n$ into a single root space of $\mathfrak{k}_{n+1}$. I. Dimitrov and the author showed in [DP] that any root-reductive Lie algebra is isomorphic to a split extension of an abelian Lie algebra by a direct sum of copies of the simple Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ and of simple finite dimensional Lie algebras. Remarkably, the simple infinite dimensional root-reductive Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ can be described alternatively as the simple infinite dimensional finitary Lie algebras, [B], as well as the infinite dimensional locally finite split Lie algebras, [NS]. Furthermore, H. Strade and the author showed in [PS] that $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$ are the only simple locally finite countably dimensional Lie algebras which admit a generalized root decomposition. More generally, it is proved in [PS] that any countably dimensional locally finite Lie algebra which admits a generalized root decomposition, and whose locally solvable radical is zero, is root-reductive.

It is an interesting problem to find a Lie superalgebra generalization of the results in [PS]. The finite dimensional simple Lie superalgebras are of two types, those with a reductive even part (i.e. the classical series and the exceptional Lie superalgebras) and those whose even part is non-reductive (the Cartan-type series). Our object of study is a fairly general class of Lie superalgebras which extends the class of direct sums of simple Lie superalgebras with reductive even part. More precisely, we define a countably dimensional locally finite Lie superalgebra $\mathfrak{g}$ over K to be *classically semisimple* if it is semisimple, i.e. its locally solvable radical equals zero, and in addition it admits a generalized root decomposition which satisfies the following two natural conditions:

(i) the inherited (generalized) root decomposition on the even part $\mathfrak{g}_0$ makes $\mathfrak{g}_0$ root-reductive;

(ii) $\mathfrak{g}$ is generated by its (generalized) root spaces.

Condition (i) singles out the "classical" ones among all semisimple locally finite Lie superalgebras with a generalized root decomposition, and condition (ii) is a minor technical assumption which allows us not to deal with semisimple Lie superalgebras which admit non-trivial homomorphisms into abelian superalgebras.

Our main result shows that any classically semisimple locally finite Lie superalgebra $\mathfrak{g}$ over K is isomorphic to a direct sum of classical or exceptional simple finite dimensional Lie superalgebras and of copies of the following infinite dimensional Lie superalgebras $\mathfrak{sl}(\infty|n)$, $\mathfrak{sl}(\infty|\infty)$, $\mathfrak{osp}(m|\infty)$, $\mathfrak{osp}(\infty|\infty)$, $\mathfrak{osp}(\infty|2k)$, $\mathfrak{sp}(\infty)$, and $\mathfrak{sq}(\infty)$. Therefore, up to isomorphism, the latter are the only countably dimensional locally finite simple Lie superalgebras $\mathfrak{s}$ which admit a generalized root decomposition such that $\mathfrak{s}_0$ is root-reductive. Actually we prove more: we prove that $\mathfrak{g}$ always admits a root isomorphism with a direct sum as above where in each simple component there is a fixed root decomposition, and we describe all root decompositions of any simple component. In particular, we show that $\mathfrak{osp}(\infty|2k)$ and $\mathfrak{osp}(\infty|\infty)$ each admit precisely two different structures of a classically semisimple Lie superalgebra.

The reason why the above statements are not straightforward corollaries of the results of [PS] is that a finite dimensional semisimple Lie superalgebra is not necessarily isomorphic to a direct sum of simple Lie superalgebras, see [K] and [C]. The main difficulty in the proof is the analysis of certain direct limits of finite dimensional semisimple Lie algebras.

A problem, which remains open, is to study semisimple locally finite Lie superalgebras which admit a generalized root decomposition and whose even part is not necessarily root-reductive. The direct limits of Cartan-type superalgebras provide examples of simple Lie superalgebras with this property.

1. Generalities on locally finite Lie superalgebras with root decomposition. The ground field K is algebraically closed of characteristic 0. All vector spaces V (including Lie superalgebras) are defined over K and are assumed to be $\mathbb{Z}_2$-graded, i.e. $V = V_0 \oplus V_1$. The sign $\otimes$ stands for tensor product of $\mathbb{Z}_2$-graded vector spaces over K, $\in\!\!\!+$ stands for semidirect sum of Lie superalgebras, and the superscript * denotes dual space. A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is *locally finite* if any finite set of vectors $g_1, \ldots, g_k \in \mathfrak{g}$ generates a finite dimensional subsuperalgebra of $\mathfrak{g}$. All Lie superalgebras considered below are assumed to be locally finite and countably (or finite) dimensional. By $\mathfrak{g}$ we denote a fixed such Lie superalgebra. A subsuperalgebra $\mathfrak{k} \subset \mathfrak{g}$ is *locally solvable* (respectively *locally nilpotent*) if every finite dimensional subsuperalgebra of $\mathfrak{k}$ is solvable (resp. nilpotent). The (*locally solvable*) *radical* $R_{\mathfrak{g}}$ is defined as the largest locally solvable ideal in $\mathfrak{g}$. As pointed out in the Introduction, $\mathfrak{g}$ is *semisimple* if $R_{\mathfrak{g}} = 0$. A chain of Lie superalgebras

$$\mathfrak{g}^1 \subset ... \subset \mathfrak{g}^n \subset \ldots \tag{1}$$

is a *local system* for $\mathfrak{g}$ if all $\mathfrak{g}^i$ are finite dimensional and $\cup_i \mathfrak{g}^i = \mathfrak{g}$. A *subsystem* of a local system for $\mathfrak{g}$ is a subchain which itself is a local system for $\mathfrak{g}$.

If $\mathfrak{k}$ is a Lie algebra, M is a $\mathfrak{k}$-module, and $\lambda \in \mathfrak{k}^*$ is a linear function on $\mathfrak{k}$, set

$$M^\lambda_{\mathfrak{k}} := \{m \in M | \exists n : (k - \lambda(k))^n \cdot m = 0 \quad \forall k \in \mathfrak{k}\}.$$

We define M to be a $\mathfrak{k}$-*weight module*, if

$$M = \bigoplus_{\lambda \in \mathrm{supp}_{\mathfrak{k}} M} M^\lambda_{\mathfrak{k}}, \tag{2}$$

where $\mathrm{supp}_{\mathfrak{k}} M := \{\lambda \in \mathfrak{k}^* | M^\lambda_{\mathfrak{k}} \neq 0\}$. Usually $M^\lambda_{\mathfrak{k}}$ are called generalized weight spaces, and modules satisfying (2) are called generalized weight modules, but we will use the shorter terms weight space and weight module.

In what follows we consider subsuperalgebras $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ for which $\mathfrak{h} = \mathfrak{g}^0_{\mathfrak{h}_0}$ (here $\mathfrak{g}$ is considered as an $\mathfrak{h}_0$-module and $\lambda = 0$), or equivalently, $\mathfrak{h}_0 = (\mathfrak{g}_0)^0_{\mathfrak{h}_0}$ and $\mathfrak{h}_1 = (\mathfrak{g}_1)^0_{\mathfrak{h}_0}$. Then $\mathfrak{h}$ is automatically locally nilpotent. Indeed, since $\mathfrak{h}_0 = (\mathfrak{g}^0_0)^0_{\mathfrak{h}_0}$, $\mathfrak{h}_0$ acts locally nilpotently on itself (via the adjoint representation) and is therefore a locally nilpotent Lie algebra. Furthermore, it is well known that a finite dimensional Lie superalgebra is nilpotent if and only if its even part is nilpotent. Therefore, as $\mathfrak{h}$ is locally finite, the local nilpotency of $\mathfrak{h}_0$ implies the local nilpotency of $\mathfrak{h}$.

If now M is a $\mathfrak{g}$-module which is an $\mathfrak{h}_0$-weight module, we claim that each weight space $M^\lambda_{\mathfrak{h}_0}$ is an $\mathfrak{h}$-submodule of M. Indeed, clearly $M^\lambda_{\mathfrak{h}_0}$ is an $\mathfrak{h}_0$-module. To check that $M^\lambda_{\mathfrak{h}_0}$ is an $\mathfrak{h}$-module it is sufficient to check that $M^\lambda_{\mathfrak{h}_0}$ is an $\mathfrak{h}'$-submodule for any finite dimensional Lie subsuperalgebra $\mathfrak{h}'$ of $\mathfrak{h}$. But it is well known that the $\mathfrak{h}'_0$-weight spaces of an $\mathfrak{h}'_0$-weight $\mathfrak{h}'$-module are $\mathfrak{h}'$-modules. Furthermore, for any $\lambda, \lambda' \in \mathfrak{h}_0^*$, $\lambda \neq \lambda'$, there exists a large enough finite dimensional subsuperalgebra $\mathfrak{h}' \subset \mathfrak{h}$ such that $\lambda_{|\mathfrak{h}'_0} \neq \lambda'_{|\mathfrak{h}'_0}$. Therefore each $M^\lambda_{\mathfrak{h}_0}$ is an $\mathfrak{h}'$-submodule for any $\mathfrak{h}'$, i.e. $M^\lambda_{\mathfrak{h}_0}$ is an $\mathfrak{h}$-module.

The following proposition summarizes the above discussion.

Proposition 1. *Let $\mathfrak{h}$ be a subsuperalgebra of $\mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{g}^0_{\mathfrak{h}_0}$. Then*

a) $\mathfrak{h}$ is locally nilpotent;

b) if M is an $\mathfrak{h}_0$-weight $\mathfrak{g}$-module, each $\mathfrak{h}_0$-weight space is an $\mathfrak{h}$-submodule of M.

If $\mathfrak{h}$ is a subsuperalgebra of $\mathfrak{g}$, we say that $\mathfrak{g}$ admits an $\mathfrak{h}$-*root decomposition* if $\mathfrak{h} = \mathfrak{g}^0_{\mathfrak{h}_0}$ and $\mathfrak{g}$ is an $\mathfrak{h}_0$-weight module. (In particular $\mathfrak{h}$ is locally nilpotent.) Then the set of *roots* $\Delta_{\mathfrak{h}}$ is defined as $\{\mathrm{supp}_{\mathfrak{h}_0} \mathfrak{g}\} \setminus \{0\}$, and each $\mathfrak{g}^\alpha_{\mathfrak{h}_0}$, $\alpha \in \Delta_{\mathfrak{h}}$, is called a *root space of* $\mathfrak{g}$. Although the action of $\mathfrak{h}_0$ on $\mathfrak{g}$ is not assumed semisimple, there is the usual vector space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathfrak{h}}} \mathfrak{g}^\alpha_{\mathfrak{h}_0} \right).$$

Let $\mathfrak{g}'$ and $\mathfrak{g}''$ be two Lie superalgebras with fixed subsuperalgebras $\mathfrak{h}' \subset \mathfrak{g}'$ and $\mathfrak{h}'' \subset \mathfrak{g}''$ such that $\mathfrak{g}'$ and $\mathfrak{g}''$ admit respective $\mathfrak{h}'$- and $\mathfrak{h}''$-root decompositions. If $\varphi : \mathfrak{g}' \to \mathfrak{g}''$ is a Lie algebra homomorphism, we say that φ is a *root homomorphism* if $\varphi(\mathfrak{h}') \subset \mathfrak{h}''$ and, for every $\alpha' \in \Delta_{\mathfrak{h}'}$, $\varphi\left(\mathfrak{g}'^{\alpha'}_{\mathfrak{h}'_0}\right) \subset \mathfrak{g}''^{\alpha''}_{\mathfrak{h}''_0}$ for some $\alpha'' \in \Delta_{\mathfrak{h}''}$.

In what follows we will denote root spaces simply by $\mathfrak{g}^\alpha$ instead of $\mathfrak{g}^\alpha_{\mathfrak{h}_0}$, and we will make use of the following trivial observation: if $\mathfrak{l}$ is the image of a root homomorphism into a finite dimensional semisimple Lie algebra $\mathfrak{k}$

(we call $\mathfrak{l}$ a *root subalgebra* of $\mathfrak{k}$), then the dimension of each root space of $\mathfrak{l}$ equals 1.

A Lie algebra $\mathfrak{f}$ is *root-reductive* if $\mathfrak{f}$ admits a local system of Lie subalgebras

$$(3) \qquad \mathfrak{f}^1 \subset ... \subset \mathfrak{f}^n \subset \mathfrak{f}^{n+1} \subset \dots$$

such that all $\mathfrak{f}^n$ are reductive and all inclusions $\mathfrak{f}^n \subset \mathfrak{f}^{n+1}$ are root homomorphisms.

Up to isomorphism, there are only three infinite dimensional simple root reductive Lie algebras: $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ and $\mathfrak{sp}(\infty)$, see [BB], [DP], [PS].They are defined as the direct limits of injective root homomorphisms of simple Lie algebras of types respectively $\mathfrak{sl}(n)$, $\mathfrak{o}(m)$ and $\mathfrak{sp}(2k)$ for growing n, m, or k. It is a non-difficult but important observation that up to isomorphism, in these three cases the direct limit Lie algebras do not depend on the actual injective root homomorhisms, see [DP], [PS] and also [BB]. The Lie algebra $\mathfrak{o}(\infty)$ can be obtained as a direct limit of $\mathfrak{o}(m)$ for m odd, or $\mathfrak{o}(m)$ for m even. The resulting Lie algebras, called respectively $B(\infty)$ and $D(\infty)$, are isomorphic but obviously do not admit a root isomorphism, i.e. an isomorphism which is a root homomorphism. (In this paper we do not denote $\mathfrak{sp}(\infty)$ by $C(\infty)$ as we reserve the notation $C(\infty)$ for a Lie superalgebra, see (16) below.)

Theorem 1 in [DP], together with Theorem 3.2 in [PS] imply the following general description of a root-reductive Lie algebra.

Theorem 1. *Let $\mathfrak{f}$ be a root-reductive Lie algebra and $\mathfrak{h} \subset \mathfrak{f}$ be any subalgebra such that $\mathfrak{f}$ admits an $\mathfrak{h}$-root decomposition*

$$\mathfrak{f} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathfrak{h}}} \mathfrak{f}^{\alpha} \right).$$

Then

a) $\mathfrak{h}$ is the union of Cartan subalgebras $\mathfrak{h}^n \subset \mathfrak{f}^n$ for a suitable local system (3) *of reductive Lie algebras $\mathfrak{f}^n$;*

b) the subalgebra $\mathfrak{f}^{\mathcal{L}} \subset \mathfrak{f}$ generated by all $\mathfrak{f}^{\alpha}$ for $\alpha \in \Delta_{\mathfrak{h}}$ is isomorphic to a countable (or finite) direct sum of simple Lie algebras $\mathfrak{s}$, finite dimensional *or isomorphic to* $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$*; furthermore, the intersection* $\mathfrak{h}\cap\mathfrak{s}\cap\mathfrak{f}^n$ *is a Cartan subalgebra in* $\mathfrak{s}\cap\mathfrak{f}^n$ *for every* $\mathfrak{s}$ *and every* n*;*

c) $\mathfrak{f}^L$ *is an ideal in* $\mathfrak{f}$ *and equals the intersection of all ideals* $\mathfrak{i}\subset\mathfrak{f}$ *for which the quotient* $\mathfrak{f}/\mathfrak{i}$ *is abelian;*

d) the extension

$$0\to\mathfrak{f}^L\hookrightarrow\mathfrak{f}\to\mathfrak{a}:=\mathfrak{f}/\mathfrak{f}^L\to 0$$

is split, i.e. $\mathfrak{f}$ *is isomorphic to the semidirect sum of* $\mathfrak{f}^L$ *and an abelian subalgebra of* $\mathfrak{f}$ *isomorphic to* $\mathfrak{a}$.

Corollary 1. *Let* $\mathfrak{f}$ *be a root-reductive Lie algebra.*

a) Any locally solvable ideal in $\mathfrak{f}$ *belongs to the center of* $\mathfrak{f}$.

b) If $\mathfrak{f}$ *is simple, then* $\mathfrak{f}$ *is isomorphic to a direct sum of copies of* $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$*, and of finite dimensional simple Lie algebras.*

2. Background on finite dimensional Lie superalgebras. Let $\mathfrak{k}$ be a finite dimensional Lie superalgebra and $\mathfrak{h}_{\mathfrak{k}}$ be a Cartan subalgebra of $\mathfrak{k}$, i.e. by definition $\mathfrak{h}_{\mathfrak{k}}=\mathfrak{k}^0_{(\mathfrak{h}_{\mathfrak{k}})_0}$. There is always an $\mathfrak{h}$-root decomposition

$$\mathfrak{k}=\mathfrak{h}_{\mathfrak{k}}\oplus\left(\bigoplus_{0\neq\alpha\in(\mathfrak{h}_{\mathfrak{k}})_0^*}\mathfrak{k}^\alpha\right).$$

This follows from the equality $\mathfrak{h}_{\mathfrak{k}}=\mathfrak{k}^0_{(\mathfrak{h}_{\mathfrak{k}})_0}$ and from the fact that an indecomposable finite dimensional module over a nilpotent Lie algebra has isomorphic 1-dimensional composition factors.

Furthermore, the image of $\mathfrak{h}_{\mathfrak{k}}$ under any surjective homomorphism $\mathfrak{k}\to\mathfrak{k}'$ is a Cartan subsuperalgebra of $\mathfrak{k}'$, [P, App. B]. This applies in particular to the case when $\mathfrak{k}'=\mathfrak{k}_{ss}:=\mathfrak{k}/R_{\mathfrak{k}}$ and the map is the canonical projection $p:\mathfrak{k}\to\mathfrak{k}_{ss}$. In the latter case we have also

Lemma 1. ([P]) *Let* $\mathfrak{k}$ *be a finite dimensional Lie superalgebra, and* $\mathfrak{h}_{\mathfrak{k}}$ *be a Cartan subsuperalgebra of* $\mathfrak{k}$. *Then* $p:\mathfrak{k}\to\mathfrak{k}_{ss}$ *is a root homomorphism and* $i(\Delta_{p(\mathfrak{h}_{\mathfrak{k}})})\subset\Delta_{\mathfrak{h}_{\mathfrak{k}}}$*, where* $i:p(\mathfrak{h}_{\mathfrak{k}})_0^*\hookrightarrow(\mathfrak{h}_{\mathfrak{k}})_0^*$ *is the canonical injection.*

Proof. If $\mathfrak{k}^\alpha$ is a root space with $p(\mathfrak{k}^\alpha) \neq 0$, then we claim that $\alpha \in p(\mathfrak{h}_\mathfrak{k})_0^*$. Indeed, let $k^\alpha \in \mathfrak{k}^\alpha$ with $p(k^\alpha) \neq 0$ be such that $Kp(k^\alpha)$ is an $(\mathfrak{h}_\mathfrak{k})_0$-submodule of $p(\mathfrak{k}^\alpha)$ (it is obvious that such a k^α always exists). Then, for any $h \in (\mathfrak{h}_\mathfrak{k})_0 \cap R_\mathfrak{k}$, $0 = p([h, k^\alpha]) = \alpha(h)p(k^\alpha)$. Therefore $\alpha(h) = 0$, i.e. $\alpha \in p(\mathfrak{h}_\mathfrak{k})_0^*$. The fact that p is a root homomorphism is now obvious.

In contrast with Lie algebras over K, a finite dimensional semisimple Lie superalgebra is not necessarily isomorphic to a direct sum of simple Lie superalgebras. In order to be able to state the main result describing finite dimensional semisimple Lie superalgebras, we need to introduce some notation. If m is a non-negative integer, let $\Lambda_m = \bigoplus_{0\leq t\leq m} \Lambda_m^t$ denote the Grassmann (exterior) algebra of m variables over K. By $\mathbf{W}(m)$ we denote the Lie superalgebra of superderivations of the associative algebra Λ_m. $\mathbf{W}(m)$ is a simple Lie superalgebra for $m \geq 2$, and for $m \geq 3$, $\mathbf{W}(m)_0$ is a non-reductive Lie algebra with semisimple part $\mathfrak{sl}(m)$. Furthermore, any Cartan subalgebra of $\mathbf{W}(m)$ belongs to $\mathbf{W}(m)_0$ and projects isomorphically into the semisimple part of $\mathbf{W}(m)_0$.

Theorem 2. (Kac, [K], Cheng [Ch]). *A finite dimensional Lie superalgebra $\mathfrak{k}$ is semisimple, i.e. $R_\mathfrak{k} = 0$, if*

$$\bigoplus_{i=1}^{l} (\mathfrak{s}_i \otimes \Lambda_{m_i}) \ni\!\!\!\!\!+ \, \mathfrak{d} \subset \mathfrak{k} \subset \bigoplus_{i=1}^{l} (\mathrm{der}\,\mathfrak{s}_i \otimes \Lambda_{m_i}) \ni\!\!\!\!\!+ \, \mathfrak{d}$$

for some simple Lie superalgebras $\mathfrak{s}_i$, $i = 1, \dots, l$, some $\mathfrak{d}$, and some non-negative integers m_i, where $\mathfrak{s}_i$ are simple finite dimensional Lie superalgebras, der $\mathfrak{s}_i$ *stands for the Lie superalgebra of superderivations of $\mathfrak{s}_i$, and $\mathfrak{d}$ is a subsuperalgebra of $\bigoplus_{i=1}^{l} \mathbf{W}(m_i)$ such that each $\mathfrak{s}_i \otimes \Lambda_{m_i}$ has no $\mathfrak{d}$-invariant ideals ($\mathfrak{d}$ acts non-trivially only on Λ_{m_i} via the projection $\mathfrak{d} \to \mathbf{W}(m_i)$).*

3. Classically semisimple Lie superalgebras. The class of semisimple locally finite Lie superalgebras is large and not yet explored. In this paper we define and explicitly describe its subclass of classically semisimple Lie superalgebras. According to the Introduction, $\mathfrak{g}$ is classically semisimple if, in addition to being semisimple, $\mathfrak{g}$ admits an $\mathfrak{h}$-root decomposition (for some fixed $\mathfrak{h} \subset \mathfrak{g}$) such that $\mathfrak{g}$ is generated by all root spaces $\mathfrak{g}^\alpha$, and $\mathfrak{g}_0$ is a root-reductive Lie algebra. If $\mathfrak{g}$ is finite dimensional and simple, $\mathfrak{g}$ is classically semisimple unless $\mathfrak{g}$ is a Cartan-type simple Lie superalgebra, i.e. belongs to the series $\mathbf{W}$, $\mathbf{S}$, $\tilde{\mathbf{S}}$, or $\mathbf{H}$, see [K] or [P]. In what follows we reserve the term Cartan-type superalgebra only for those Lie superalgebras of the series $\mathbf{W}$, $\mathbf{S}$, $\tilde{\mathbf{S}}$ and $\mathbf{H}$ which are not isomorphic to a classical finite dimensional Lie superalgebra as introduced below.

In this paper we refer to the following Lie superalgebras as to the *classical* finite dimensional Lie superalgebras:

$$\begin{gathered}\mathfrak{gl}(m|n), \mathfrak{sl}(m|n), \mathfrak{pgl}(m|m), \mathfrak{psl}(m|m), \mathfrak{osp}(m|n), \mathbf{p}(m),\\ \mathfrak{sp}(m), \mathfrak{q}(m), \mathfrak{sq}(m), \mathfrak{pq}(m), \mathfrak{psq}(m).\end{gathered} \tag{4}$$

Their definitions are well known and we recall them here merely for the purpose of fixing the notations. By definition, $\mathfrak{gl}(m|n)$ is the Lie superalgebra of endomorphisms of an $m|n$-dimensional ($\mathbb{Z}_2$-graded) vector space V, i.e. $V = V_0 \oplus V_1$, $\dim V_0 = m$, $\dim V_1 = n$, and $\mathfrak{sl}(m|n)$ is the subsuperalgebra of $\mathfrak{gl}(m|n)$ which consists of endomorphisms of supertrace zero. In matrix form, $\mathfrak{gl}(m|n)$ consists of square $(m+n) \times (m+n)$ block matrices

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right),$$

where A is of size $m \times n$ and D is of size $n \times n$, and $\mathfrak{sl}(m|n)$ is singled out by the condition $\operatorname{tr} A = \operatorname{tr} D$. The Lie superalgebra $\mathfrak{osp}(m|n)$ is the subsuperalgebra of $\mathfrak{sl}(m|n)$ which leaves invariant a supersymmetric even non-degenerate bilinear form on V (here necessarily $n = 2k$), see [K], and $\mathbf{p}(m)$ is the subsuperalgebra of $\mathfrak{gl}(m|m)$ which leaves invariant a superantisymmetric odd bilinear form on V. The intersection $\mathbf{p}(m) \cap \mathfrak{sl}(m|m)$ is by definition the Lie superalgebra $\mathfrak{sp}(m)$. The Lie superalgebra $\mathfrak{q}(m)$ is the subsuperalgebra of $\mathfrak{gl}(m|m)$ which consists of all block matrices of the form

$$\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right),$$

and $\mathfrak{sq}(m)$ is singled out by the equation $\operatorname{tr} B = 0$. Finally, $\mathfrak{pgl}(m|m)$, $\mathfrak{psl}(m|m)$, $\mathfrak{pq}(m)$ and $\mathfrak{psq}(m)$ are respectively the quotients of $\mathfrak{gl}(m|m)$,

$\mathfrak{sl}(m|m)$, $\mathfrak{q}(m)$ and $\mathfrak{sq}(m)$ by the 1-dimensional ideal generated by the identity matrix.

The Lie superalgebras $\mathfrak{sl}(m|n)$ for $m \neq n$, $m|n \neq 1|0, 0|1, 1|1$, $\mathfrak{psl}(m|m)$ for $m \geq 3$, $\mathfrak{osp}(m|n)$ for n even, $\mathfrak{sp}(m)$ for $m \geq 3$, and $\mathfrak{psq}(m)$ for $m \geq 3$ are the simple classical finite dimensional Lie superalgebras. Their roots are described explicitly in [K] and [P], and in fact this provides a description of the roots of all classical finite dimensional Lie superalgebras

It is natural to construct infinite dimensional simple Lie superalgebras (which will automatically be classically semisimple) as unions of classical finite dimensional Lie superalgebras embedded in each under via root injections, i.e. via injective root homomorphisms. Note first that if a Lie superalgebra is represented as a union of root injections $\mathfrak{g}^j \subset \mathfrak{g}^{j+1}$, it admits an $\mathfrak{h}$-root decomposition where $\mathfrak{h}$ is the union of the fixed subsuperalgebras $\mathfrak{h}^j \subset \mathfrak{g}^j$. Fix now infinite chains of root injections as follows:

(5) $$\mathfrak{sl}(j|n) \subset \mathfrak{sl}(j+1|n),$$

(6) $$\mathfrak{sl}(j|j) \subset \mathfrak{sl}(j+1|j+1),$$

(7) $$\mathfrak{osp}(2j+1|2k) \subset \mathfrak{osp}(2j+3|2k),$$

(8) $$\mathfrak{osp}(2j+1|2j) \subset \mathfrak{osp}(2j+3|2j+2),$$

(9) $$\mathfrak{osp}(m|2j) \subset \mathfrak{osp}(m|2j+2),\ m \text{ odd},$$

(10) $$\mathfrak{osp}(2|2j) \subset \mathfrak{osp}(2|2j+2),$$

(11) $$\mathfrak{osp}(2j|2k) \subset \mathfrak{osp}(2j+2|2k),$$

(12) $$\mathfrak{osp}(2j|2j) \subset \mathfrak{osp}(2j+2|2j+2),$$

$$\mathfrak{osp}(m|2j) \subset \mathfrak{osp}(m|2j+2), \; m \text{ even}, \; m \neq 2, \tag{13}$$

$$\mathfrak{sp}(j) \subset \mathfrak{sp}(j+1), \tag{14}$$

$$\mathfrak{sq}(j) \subset \mathfrak{sq}(j+1), \tag{15}$$

where $j \in \mathbb{Z}_+$, $j \geq 2$, and $m \geq 0$, $n \geq 0$ and $k \geq 0$ are fixed. Denote the unions of the chains (5)–(15) respectively by

$$\begin{gathered} \mathfrak{sl}(\infty|n), \mathfrak{sl}(\infty|\infty), B(\infty|2k), B(\infty|\infty), B(m|\infty), C(\infty), \\ D(\infty|2k), D(\infty|\infty), D(m|\infty), \mathfrak{sp}(\infty), \mathfrak{sq}(\infty). \end{gathered} \tag{16}$$

Each of these Lie superalgebras is infinite dimensional, simple, and has an $\mathfrak{h}$-root decomposition.

C. Oseledets' thesis [O] studies the above Lie superalgebras, as well as more general Lie superalgebras which admit a local system of finite direct sums of classical finite dimensional Lie superalgebras. The following theorem is a slightly weaker version of the main result of [O] (Oseledets establishes also a universality result which we do not use).

Theorem 3.

a) Any infinite dimensional simple Lie superalgebra, which admits a local system of root injections of classical finite dimensional Lie superalgebras, is isomorphic via a root isomorphism to one of the Lie superalgebras (16).

b) Let $\mathfrak{g}$ be a Lie superalgebra which admits a local system of root injections of finite direct sums of classical or exceptional finite dimensional Lie superalgebras. Consider $\mathfrak{g}$ as a Lie superalgebra with root decomposition, and let $\mathfrak{g}^L$ denote the subsuperalgebra of $\mathfrak{g}$ generated by all root spaces. Then $\mathfrak{g}^L$ is the smallest ideal in $\mathfrak{g}$ with an abelian quotient, and $\mathfrak{g}^L$ admits a root isomorphism with a countable or exceptional (or finite) direct sum of copies of the superalgebras (16), of simple classical finite dimensional superalgebras, and possibly of $\mathfrak{sl}(m|m)$ and $\mathfrak{sq}(m)$.

Sketch of proof. a) The first key observation is that in a chain (1) of root injections of classical Lie superalgebras $\mathfrak{g}^j$, for large enough j, all $\mathfrak{g}^j$ must be of the same type. More precisely, if $\mathfrak{g}^j$ equals $\mathfrak{osp}(2l+1|2k)$ or is of type $\mathbf{p}$ or $\mathfrak{q}$ (in the latter case $\mathfrak{g}^j$ equals $\mathbf{p}(m)$, $\mathfrak{s}\mathbf{p}(m)$, or $\mathfrak{q}(m)$, $\mathfrak{sq}(m)$, $\mathfrak{pq}(m)$, $\mathfrak{psq}(m)$), for all $j' > j$ $\mathfrak{g}^{j'}$ also equals $\mathfrak{osp}(2l'+1|2k')$ for $l' \geq l$, $k' \geq k$, or is respectively of type $\mathbf{p}$ or $\mathfrak{q}$. Note also that $\mathfrak{pgl}(m|m)$, $\mathfrak{psl}(m|m)$, $\mathfrak{pq}(m)$ and $\mathfrak{psq}(m)$ do not appear in any infinite chain (1) of proper root injections. If for no j $\mathfrak{g}^j$ equals $\mathfrak{osp}(2l+1|2k)$ or is type $\mathbf{p}$ or $\mathfrak{q}$, one checks if $\mathfrak{g}^j$ equals $\mathfrak{osp}(2l|2k)$ for some i. If this is the case, then all $\mathfrak{g}^{j'}$ for $j' \geq j$ also equal $\mathfrak{osp}(2l'|2k')$. Finally, if no $\mathfrak{g}^j$ is of type $\mathfrak{osp}$, $\mathbf{p}$ or $\mathfrak{q}$, all $\mathfrak{g}^j$ equal $\mathfrak{sl}(m^j|n^j)$ or $\mathfrak{gl}(m^j|n^j)$.

The next step is to exclude the possibility that infinitely many $\mathfrak{g}^j$ equal $\mathfrak{gl}(m^j|n^j)$, $\mathbf{p}(m^j)$, or $\mathfrak{q}(m^j)$. This follows easily from the simplicity of $\cup_j \mathfrak{g}^j$.

The final important observation is that the remaining possibilities are in bijective correspondence with the cases (5) – (15) and that in each case $\cup_j \mathfrak{g}^j$ can be identified with one of the Lie superalgebras (16). This is a tedious but essentially straightforward case by case verification and is carried out in [O].

b) The proof uses a) and is similar to the proof of Theorem 1 in [DP].

4. Main result. The Lie superalgebra $\mathfrak{g}^L$ in Theorem 3 b) is clearly classically semisimple. The main result of this paper claims that the explicit description of $\mathfrak{g}^L$ given in Theorem 3 b) holds, in a stronger form, for any classically semisimple Lie superalgebra.

Theorem 4. *Let $\mathfrak{g}$ be a classically semisimple Lie superalgebra. Then there is a root isomorphism between $\mathfrak{g}$ and a countable (or finite) direct sum of copies of the Lie superalgebras (16) and of simple classical or exceptional finite dimensional Lie superalgebras.*

We start with a lemma which is a direct extension of Lemma 3.1 in [PS] to the Lie superalgebra case.

Lemma 2. *Let $\mathfrak{g}'$ be a semisimple Lie superalgebra, and*

(17) $$(\mathfrak{g}')^1 \subset ... \subset (\mathfrak{g}')^n \subset \dots$$

be a local system for $\mathfrak{g}'$. Then (17) admits a subsystem

$$(\mathfrak{g}')^{n_1} \subset ... \subset (\mathfrak{g}')^{n_s} \subset \dots$$

with $(\mathfrak{g}')^{n_s} \cap R_{(\mathfrak{g}')^{n_s+1}} = 0$ *for every* s.

Proof. Since, for each n, $\cap_{t>n} R_{(\mathfrak{g}')^t}$ is a solvable ideal in $\mathfrak{g}$, we have $\cap_{t>n} R_{(\mathfrak{g}')^t} \subset R_{\mathfrak{g}'} = 0$. Therefore, given n, there exists $N(n) \geq n$ with $(\mathfrak{g}')^n \cap R_{(\mathfrak{g}')^{N(n)}} = 0$, and this enables us to construct by induction a local system as desired.

Proof of Theorem 4.

Step 1. Let $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta_{\mathfrak{h}}} \mathfrak{g}^\alpha)$ be the fixed root decomposition of $\mathfrak{g}$. We first choose a local system for $\mathfrak{g}$,

$$\tilde{\mathfrak{g}}^1 \subset ... \subset \tilde{\mathfrak{g}}^n \subset \dots, \tag{18}$$

compatible with the root decomposition, i.e. such that $\mathfrak{h} \cap \tilde{\mathfrak{g}}^n$ is a Cartan subsuperalgebra of $\tilde{\mathfrak{g}}^n$ for each n and all inclusions $\tilde{\mathfrak{g}}^n \subset \tilde{\mathfrak{g}}^{n+1}$ are root homomorphisms. This is done in the same way as in the proof of Theorem 3.2 in [PS]. Indeed, consider an ordered Z_2-homogeneous basis $\tilde{g}^1, \dots, \tilde{g}^n, \dots$ of $\mathfrak{g}$, such that each $\tilde{g}^n$ belongs to $\mathfrak{h}$ or to $\mathfrak{g}_{\mathfrak{h}_0}^{\alpha_t}$ for some $\alpha_t \in \Delta_{\mathfrak{h}_0}$. Let $(\tilde{\mathfrak{g}}')^n \subset \mathfrak{g}$ be the subsuperalgebra generated by $\tilde{g}^1, \dots, \tilde{g}^n$. Then, for each n, fix a finite dimensional subspace $(\tilde{\mathfrak{h}}^n)_0 \subset \mathfrak{h}_0$ such that the subspace of $\mathfrak{h}_0^*$ generated by all $\alpha \in \Delta_{\mathfrak{h}}$ with $\mathfrak{g}^\alpha \cap (\tilde{\mathfrak{g}}')^n \neq 0$ maps injectively into $(\tilde{\mathfrak{h}}^n)_0^*$. Define now $(\tilde{\mathfrak{g}})^n$ as the subsuperalgebra of $\mathfrak{g}$ generated by $(\tilde{\mathfrak{h}}^1)_0, \dots, (\tilde{\mathfrak{h}}^n)_0$ and by $(\tilde{\mathfrak{g}}')^n$. An immediate checking shows that $\tilde{\mathfrak{g}}^n$ form a local system (18) as desired.

Step 2. All Lie algebras $(\tilde{\mathfrak{g}}^n)_0$ are root subalgebras of $\mathfrak{g}_0$, and as $\mathfrak{g}_0$ is root-reductive, each $(\tilde{\mathfrak{g}}^n)_0$ is a root subalgebra of a finite dimensional reductive subalgebra of $\mathfrak{g}_0$. By our earlier remark, any quotient of $(\tilde{\mathfrak{g}}^n)_0$ has 1-dimensional root spaces. Furthermore, $R_{\mathfrak{g}_0} \subset \mathfrak{h}_0$ by Corllary 1 a). Therefore Lemma 2, applied to $\mathfrak{g}_0/R_{\mathfrak{g}_0}$, implies that for each n there exists N, such that the map

$$(\tilde{\mathfrak{g}}^n)_0/((\tilde{\mathfrak{g}}^n)_0 \cap R_{\mathfrak{g}_0}) \hookrightarrow (\tilde{\mathfrak{g}}^N)_0/R_{(\tilde{\mathfrak{g}}^N)_0} \tag{19}$$

is a root injection.

Step 3. According to Theorem 2, for each n we have

$$\bigoplus_i(\mathfrak{s}_i^n\otimes\Lambda_{m_i})\subset\!\!\!\!+\ \mathfrak{d}^n\subset(\tilde{\mathfrak{g}}^n)_{ss}=\tilde{\mathfrak{g}}^n/R_{\tilde{\mathfrak{g}}^n}\subset\bigoplus_i(\operatorname{der}\mathfrak{s}_i^n\otimes\Lambda_{m_i})\subset\!\!\!\!+\ \mathfrak{d}^n$$

for some finite dimensional simple Lie superalgebras $\mathfrak{s}_i^n$, some m_i, and for an appropriate subsuperalgebra $\mathfrak{d}^n\subset\bigoplus_i\mathbf{W}(m_i)$.

We are now ready to settle the case when $\mathfrak{g}$ is finite dimensional. Indeed, if $\mathfrak{g}=\tilde{\mathfrak{g}}^n$, then (as $R_{\mathfrak{g}}=0$) $\mathfrak{g}=(\tilde{\mathfrak{g}}^n)_{ss}$. Clearly, for any $\mathfrak{s}_i^n$, $(\mathfrak{s}_i^n)_0\otimes\Lambda^2_{m_i}\subset R_{\mathfrak{g}_0}$, $(\mathfrak{s}_i^n)_1\otimes\Lambda^1_{m_i}\subset R_{\mathfrak{g}_0}$, and, if nonzero, neither $(\mathfrak{s}_i^n)_0\otimes\Lambda^2_{m_i}$ nor $(\mathfrak{s}_i^n)_1\otimes\Lambda^1_{m_i}$ belong to the center of $\tilde{\mathfrak{g}}_0^n$. This shows that $m_i=0$ for all i with $(\mathfrak{s}_i^n)_1\neq0$ and $m_i\leq1$ for all i with $\mathfrak{s}_i^n=(\mathfrak{s}_i^n)_0$. Note next that $\mathfrak{d}_0^n\subset\mathfrak{h}$, and as $(\mathfrak{g}\cap(\bigoplus_i(\operatorname{der}\mathfrak{s}_i^n\otimes\Lambda_{m_i})))\subset\!\!\!\!+\ \mathfrak{d}_1^n$ is a subsuperalgebra of $\mathfrak{g}$, the condition that $\mathfrak{g}$ is generated by its root spaces gives $\mathfrak{d}_0^n=0$. Consequently $\mathfrak{d}_1^n\subset\mathfrak{h}_1$, and, since $\mathfrak{g}\cap(\bigoplus_i(\operatorname{der}\mathfrak{s}_i^n\otimes\Lambda^1_{m_i}))$ is a subsuperalgebra of $\mathfrak{g}$, we have $\mathfrak{d}_1^n=0$. Theorem 2 implies now that $m_i=0$ for all i.

We have proved that $\mathfrak{g}\simeq\bigoplus_i(\mathfrak{g}\cap\operatorname{der}\mathfrak{s}_i^n)$. All Cartan-type simple Lie superalgebras have non-reductive even part, so $\mathfrak{s}_i^n$ must necessarily be classical or exceptional. Furthermore, it is well known that for a classical or exceptional simple Lie superalgebra, see [K], $\operatorname{der}\mathfrak{s}=\mathfrak{s}$ unless $\mathfrak{s}\simeq\mathfrak{psl}(m|m)$, $\mathfrak{sp}(m)$, $\mathfrak{psq}(m|m)$. In the latter cases $\operatorname{der}\mathfrak{s}$ equals $\mathfrak{pgl}(m|m)$, $\mathbf{p}(m)$ and $\mathfrak{pq}(m|m)$ respectively. Since none of the latter Lie superalgebras is generated by its root spaces, we obtain finally that $\mathfrak{g}\simeq\bigoplus_i\mathfrak{s}_i^n$, and the case of a finite dimensional $\mathfrak{g}$ is settled.

¿From now on $\dim\mathfrak{g}=\infty$, and the remainder of the proof is an infinite dimensional version of the above argument in the finite dimensional case.

Note first that by Lemma 1, for every n, there is N such that the composition

$$\tilde{\mathfrak{g}}^n\hookrightarrow\tilde{\mathfrak{g}}^N\to(\tilde{\mathfrak{g}}^N)_{ss}\tag{20}$$

is a root injection. In what follows we will assume that we have replaced (18) with a subsystem for which both maps (19) and (20) are root injections with $N = n+1$. This is obviously possible. And we will make one more assumption. The condition that $\mathfrak{g}$ is generated by its root spaces does not imply that each $\tilde{\mathfrak{g}}^n$ is generated by its root spaces, but implies that for each n there is N such that the Cartan subsuperalgebra $\mathfrak{h} \cap \tilde{\mathfrak{g}}^n$ of $\tilde{g}^n$ is mapped into the subsuperalgebra of $\tilde{\mathfrak{g}}^N$ generated by the root spaces of $\tilde{\mathfrak{g}}^N$. Our third assumption on the local system (18) is that (after a possible replacement by a subsystem) the latter condition holds also for $N = n+1$.

Step 4. We claim that $m_i \leq 1$ for any $\mathfrak{s}_i^n$. Indeed, let $m := m_i \geq 2$ for some $\mathfrak{s} := \mathfrak{s}_i^n$, and let for some root α of $\tilde{\mathfrak{g}}^n$ $\mathfrak{f}(\alpha)$ be an $\mathfrak{sl}(2)$-subalgebra of $\tilde{\mathfrak{g}}_0^n$ which maps injectively into $\mathfrak{s}$. Denote the image of $\mathfrak{f}(\alpha)$ in $(\mathfrak{s}_i^n)_0$ by $\mathfrak{s}(\alpha)$. Consider the simple component $\mathfrak{c}$ of $\tilde{\mathfrak{g}}_0^{n+1}$ into which $\mathfrak{f}(\alpha)$ maps. Since as an $\mathfrak{s}(\alpha)$-module $\mathfrak{s}(\alpha) \otimes \Lambda_m^2$ is isomorphic to the direct sum of $\dim \Lambda_m^2$ copies of the adjoint representation, $\mathfrak{c}$ has a root β whose α-string through β is of length 3. Therefore the rank two subalgebra of $\mathfrak{c}$ generated by the image of $\mathfrak{f}(\alpha)$ and $\mathfrak{c}^{\pm\beta}$ is isomorphic to $\mathfrak{o}(5)$. We can also assume that $\mathfrak{c}^\beta$ belongs to the image of $\tilde{\mathfrak{g}}_0^n$, and that as a subspace of $\tilde{\mathfrak{g}}_0^n$ $\mathfrak{c}^\beta$ maps into $\mathfrak{s}(\alpha)^\alpha \otimes \Lambda_m^2$.

Consider now the preimage of $\mathfrak{s}(\alpha) \otimes \Lambda_m^1$ in $\tilde{\mathfrak{g}}_1^n$. The relation $[\mathfrak{s}(\alpha)^\alpha \otimes \Lambda_m^1, \mathfrak{s}(\alpha)^{-\alpha} \otimes \Lambda_m^1] = [\mathfrak{s}(\alpha)^\alpha, \mathfrak{s}(\alpha)^{-\alpha}] \otimes \Lambda_m^2$ implies (via Lemma 1) that $x + (x - 2\alpha) = \beta - \alpha$ for some root x of $\mathfrak{g}$ such that $\mathfrak{g}_1^x$ in contained in the preimage of $\mathfrak{s}(\alpha)^\alpha \otimes \Lambda_m^1$ in $\tilde{\mathfrak{g}}_1^n$. This means that $x = \dfrac{\alpha+\beta}{2}$, i.e. x is not an integral $\mathfrak{o}(5)$-weight. This is a contradiction as all roots of $\mathfrak{g}$ are integral weights of $\mathfrak{c}$, in particular of $\mathfrak{o}(5)$.

Step 5. Step 4 implies that $\mathfrak{d}^n$ is necessarily solvable for every n, as $\bigoplus_i \mathbf{W}(m_i)$ is solvable for $m_i \leq 1$. Consider any simple component $\mathfrak{s} = \mathfrak{s}_i^n$ and let $\tilde{\mathfrak{s}}$ be a preimage of $\mathfrak{s}$ in $\tilde{\mathfrak{g}}^n$ admitting no solvable quotients. It is clear that $\tilde{\mathfrak{s}}$ exists. As $\mathfrak{d}^{n+1}$ is solvable, the image of $\tilde{\mathfrak{s}}$ in $(\tilde{\mathfrak{g}}^{n+1})_{ss}$ belongs to the ideal $\bigoplus_j (\mathfrak{s}_j^{n+i} \otimes \Lambda_{m_j})$. Since the semisimple part of this ideal is isomorphic to $\bigoplus_j \mathfrak{s}_j^{n+1}$, projection onto $\bigoplus_j \mathfrak{s}_j^{n+1}$ yields an injective homomorphism $\tilde{\mathfrak{s}} \hookrightarrow \mathfrak{s}'$ for some $\mathfrak{s}' = \mathfrak{s}_j^{n+1}$. When iterating this procedure, the arising sequence of simple components either stabilizes or does not stabilize.

We will now modify the local system (18). Fix an integer N greater then the rank of the even part of all exceptional simple superalgebras and large enough so that all Cartan-type simple Lie superalgebras whose even part has semisimple rank N or higher have even root spaces of dimension strictly higher than 1. (It is obvious that all sufficiently "large" simple Cartan-type superalgebras have this property). Using an inductive process we will now replace (18) by a local system

$$(\mathfrak{g}')^1 \subset ... \subset (\mathfrak{g}')^n \subset ... \tag{21}$$

for $\mathfrak{g}$, such that, for every n, any simple component $\mathfrak{s}$ of $(\mathfrak{g}')^n$ has rank N or higher[1] unless the sequence of simple components corresponding to $\mathfrak{s}$ is stable. Let l^1 be the smallest index such that at least one simple component $\mathfrak{s}_i^{l^1}$ has rank N or higher, or the sequence corresponding to $\mathfrak{s}_i^{l^1}$ is already stable. Then we let $(\mathfrak{g}')^1$ be the preimage in $\tilde{\mathfrak{g}}^{l^1}$ of $(\tilde{\mathfrak{g}}^{l^1})_{ss} \cap (\mathrm{der}\, \mathfrak{s}_i^{l^1} \otimes \Lambda_{m_i} \ni (\mathfrak{d}_0^{l^1} + (\bigoplus_j \mathbf{W}(m_i))))$. This is the base of induction. If $(\mathfrak{g}')^{n-1}$ is given, we consider an index l for which $(\mathfrak{g}')^{n-1} \subset \tilde{\mathfrak{g}}^l$ and define $(\mathfrak{g}')^n$ as the preimage of $(\tilde{\mathfrak{g}}^l)_{ss} \cap (\bigoplus_j (\mathrm{der}\, \mathfrak{s}_j^l \otimes \Lambda_{m_j}) \ni (\mathfrak{d}_0^{l^1} + (\bigoplus_j \mathbf{W}(m_i))))$ in $\tilde{\mathfrak{g}}^l$, where $\mathfrak{s}_j^l$ are all simple components of $(\tilde{\mathfrak{g}}^l)_{ss}$ which have rank at least N or whose respective sequence of simple components is stable. A straightforward checking shows that the so obtained sequence (21) is a well-defined local system for $\mathfrak{g}$, compatible with the fixed root decomposition of $\mathfrak{g}$ in the same sense as the local system (18). We will continue to use the notations $\mathfrak{s}_i^n$ and $\mathfrak{d}^n$ in the context of this new local system.

Three observations are in order.

If $\mathfrak{s}$ is a simple component of $(\mathfrak{g}')^n_{ss}$, then $\mathfrak{s}$ is necessarily of classical or exceptional type. Indeed, assume the contrary, i.e. that $\mathfrak{s}$ is of Cartan-type. The sequence of simple components corresponding to $\mathfrak{s}$ cannot be stable, as then the preimages in $(\mathfrak{g}')^l$ for $l \geq n$ of the radicals of $((\mathfrak{g}')^l_{ss})_0$ would yield a non-abelian locally solvable ideal in $\mathfrak{g}_0$, which would contradict Corollary 1 a). Hence the rank of $\mathfrak{s}$ is at least N, and $\mathfrak{s}_0$ has at least one root space of

[1]Under rank of a superalgebra we mean the semisimple rank of its even part.

dimension 2 or higher. This is impossible too, as then $\mathfrak{g}_0$ would have a root space of dimension at least 2.

If $\mathfrak{s}$ is exceptional, the sequence of simple components corresponding to $\mathfrak{s}$ is stable according to the construction of (21).

Finally, if $\mathfrak{s} = \mathfrak{s}_i^n$ is a simple component of $(\mathfrak{g}')^n$ with $\mathfrak{s}_1 \neq 0$ and such that the corresponding sequence of simple components is stable, then $m_i = 0$. Indeed $m_i = 1$ would imply immediately that the preimage of $\mathfrak{s}_1 \otimes \Lambda^1_{m_i}$ in $\mathfrak{g}$ generates a non-abelian locally solvable ideal in $\mathfrak{g}_0$, which is impossible.

Step 6. We will prove now that the local system (21) can be further modified so that $m_i = 0$ for any simple component $\mathfrak{s}_i^n$ of the new local system. Fix a simple component $\mathfrak{s} = \mathfrak{s}_i^n$ and let $m_i = 1$. Put $\Xi := \Lambda^1_{m_i}$ and $\mathfrak{h}^n := \mathfrak{h} \cap (\mathfrak{g}')^n$. Consider first the case when $\mathfrak{s} \simeq \mathfrak{psq}(r)$, $r \geq 3$. Note that $p^{-1}((p(\mathfrak{h}^n) \cap \mathfrak{s}_1) \otimes \Xi)$ is an $\mathfrak{h}_0^n$-invariant subspace of $(\mathfrak{g}')^n$, $p : (\mathfrak{g}')^n \to (\mathfrak{g}')^n_{ss}$ denoting the canonical projection (p is a root homomorphism by Lemma 1). Moreover, there are two alternatives: $p^{-1}((p(\mathfrak{h}^n) \cap \mathfrak{s}_1) \otimes \Xi)$ is a single root space of $(\mathfrak{g}')^n$ (this is equivalent to the projection of $\mathfrak{h}^n$ onto $\mathbf{W}(m_i)_0$ being non-zero) or $p^{-1}((p(\mathfrak{h}^n) \cap \mathfrak{s}_1) \otimes \Xi) \subset \mathfrak{h}_0^n$. Both alternatives are contradictory. Indeed, if $p^{-1}((p(\mathfrak{h}^n) \cap \mathfrak{s}_1) \otimes \Xi)$ is a root space, its dimension is greater or equal $r - 1$, which is impossible as $r - 1 \geq 2$. Assuming that $p^{-1}((p(\mathfrak{h}^n) \cap \mathfrak{s}_1) \otimes \Xi) \subset \mathfrak{h}_0^n$, we have $[\mathfrak{s}_0, (p(\mathfrak{h}^n) \cap \mathfrak{s}_1) \otimes \Xi] \neq 0$, which contradicts the fact that $p^{-1}((p(\mathfrak{h}^n) \cap \mathfrak{s}_1) \otimes \Xi)$ belongs to $\mathfrak{h}_0^n$ intersected with the radical of $(\mathfrak{g}')^n_0$ and must therefore commute with the semisimple part of $(\mathfrak{g}')^n_0$. This implies that $m_i = 0$ whenever $\mathfrak{s} \simeq \mathfrak{psq}(r)$.

Assume now that $\mathfrak{s} \not\simeq \mathfrak{psq}(r)$ and $\mathfrak{s}_1 \neq 0$. Consider the inclusion of $p^{-1}(\mathfrak{s}_0) \cap \mathfrak{h}^n$-weight modules $p^{-1}((p(\mathfrak{h}^n) \cap \mathfrak{s}_0) \otimes \Xi) \subset [p^{-1}(\mathfrak{s}_1), p^{-1}(\mathfrak{s} \otimes \Xi)]$ and note that $p^{-1}((p(\mathfrak{h}^n) \cap \mathfrak{s}_0) \otimes \Xi)$ is a single $\mathfrak{h}_0^n$-weight space. Let $\mathfrak{s}'$ be the simple component of $(\mathfrak{g}')^{n+1}_{ss}$ into which $\mathfrak{s}$ is being mapped, see Step 5. As $p^{-1}(\mathfrak{s}_0)$ and $p^{-1}(\mathfrak{s}_1 \otimes \Xi)$ are being mapped to $\mathfrak{s}'_0$, and $p^{-1}(\mathfrak{s}_1)$ is being mapped to $\mathfrak{s}'_1$, $p^{-1}(\mathfrak{s}_0 \otimes \Xi)$ is necessarily being mapped to $\mathfrak{s}'_1$. Therefore $p^{-1}((p(\mathfrak{h}^n) \cap \mathfrak{s}_0) \otimes \Xi)$ is being mapped to $p(\mathfrak{h}^{n+1}) \cap \mathfrak{s}'_1$ or to a single root space in $\mathfrak{s}'_1$. The second alternative is impossible. Indeed, it implies that $\dim p(\mathfrak{h}^n) \cap \mathfrak{s}_0 = 1$, i.e.

$\mathfrak{s} \simeq \mathfrak{osp}(1|2)$. This implies in turn that the sequence of simple components corresponding to $\mathfrak{s}$ is stable, hence $m_i = 0$. Assume now that $p^{-1}((p(\mathfrak{h}^n) \cap \mathfrak{s}_0) \otimes \Xi)$ is being mapped to $p(\mathfrak{h}^{n+1}) \cap \mathfrak{s}'_1$. Then $\mathfrak{s}'_1 \simeq \mathfrak{psq}(r)$ for some r, and $p^{-1}(\mathfrak{s}_1)$ and $p^{-1}(\mathfrak{s}_1 \otimes \Xi)$ are being mapped to $p(\mathfrak{h}^{n+1}) \cap \mathfrak{s}'_0$-submodules, respectively of $\mathfrak{s}'_1$ and $\mathfrak{s}'_0$, with the same weights. This is impossible as $\mathfrak{s}_1 \otimes \Xi$ is abelian while $\mathfrak{s}_1$ is not. Therefore the first alternative is also contradictory, hence $m_i = 0$.

It remains to consider the case when $\mathfrak{s}_1 = 0$. Here there are the following alternatives: $p^{-1}(\mathfrak{s} \otimes \Xi)$ is being mapped to a simple component $\mathfrak{s}'$ as in Step 5, or $p^{-1}(\mathfrak{s} \otimes \Xi)$ is being mapped to $\mathfrak{s}'_0 \otimes \Lambda^1_{m_j}$. In the first case we undertake a further modification of the local system (21). By induction on n, we replace $(\mathfrak{g}')^n$ by the preimage in $(\mathfrak{g}')^n$ of $(\mathfrak{g}')^n_{ss} \cap (\bigoplus_t (\mathrm{der}\, \mathfrak{s}^n_t \otimes \Lambda_{m_t}) \in (\mathfrak{d}^n_0 + (\bigoplus_t \mathbf{W}(m_t))))$, where $\mathfrak{s}^n_t$ runs over all simple components of $(\mathfrak{g}')^n$ except those with zero odd part for which the first alternative holds. This is a well-defined new local system of $\mathfrak{g}$, which we will continue to denote by (21).

The second alternative is contradictory. Indeed, we have to consider the possibilities that the projection of $\mathfrak{h}^{n+1}$ onto $\mathbf{W}(m_i)_0$ is zero, or that it is nonzero. To rule out the first possibility, note that in this case the projection of all root spaces of $(\mathfrak{g}')^{n+1}$ into $\mathbf{W}(m_i)_1$ is zero, and consequently the projection of $(\mathfrak{g}')^{n+1}$ into $\mathbf{W}(m_i)_1$ equals zero. This is in direct contradiction with Theorem 2 as then $\mathfrak{s}'$ is a $\mathfrak{d}^{n+1}$-invariant ideal in $\mathfrak{s}' \otimes \Lambda_{m_i}$. To rule out the second possibility, note that in that case the projection of the subsuperalgebra of $(\mathfrak{g}')^{n+1}$ generated by all root spaces onto $\mathbf{W}(m_i)_0$ equals zero, while the image of $(\mathfrak{g}')^n$ in $(\mathfrak{g}')^{n+1}$ projects non-trivially onto $\mathbf{W}(m_i)_0$. This is in contradiction with the last assumption of Step 3. Hence, finally $m_i = 0$.

Step 7. We have proved that, for each n, $(\mathfrak{g}')^n_{ss} = \bigoplus_i ((\mathfrak{g}')^n_{ss} \cap \mathrm{der}\, \mathfrak{s}^n_i)$. This enables us to make one final modification to the local system (21). We denote the new system by

$$\mathfrak{g}^1 \subset ... \subset \mathfrak{g}^n \subset ... \quad . \tag{22}$$

Note that Step 6 implies that each root space $\mathfrak{g}^\alpha$ is $1|0$-, $0|1$-, or $1|1$-dimensional and is identified with a root space in $(\mathfrak{g}')^n_{ss}$ for all n greater than an appropriate n'. Order all root spaces of $\mathfrak{g}$ in a manner compatible with the local system (21) (i.e. all root spaces intersecting non-trivially with $(\mathfrak{g}')^n$ come before those intersecting non-trivially with $(\mathfrak{g}')^{n+1}$ but not with $(\mathfrak{g}')^n$) and let $\mathfrak{g}^n$ be the subalgebra of $\mathfrak{g}$ generated by all root spaces which intersect non-trivially with $(\mathfrak{g}')^n$. It is immediately clear that $\mathfrak{g}^n$ form a local system of root injections for $\mathfrak{g}$. It is also clear that $\mathfrak{g}^n$ is a direct sum of classical or exceptional simple Lie superalgebras, as those are all possible subsuperalgebras generated by the root spaces of der $\mathfrak{s}$ for a classical simple $\mathfrak{s}$.

Step 8. To complete the proof it remains to apply Theorem 3 to the local system (22). The fact that $\mathfrak{g}$ is semisimple eliminates the possibility of direct summands of the form $\mathfrak{sl}(m|m)$ or $\mathfrak{sq}(m)$.

5. Corollaries of the main result.

Corollary 2. *Let $\mathfrak{s}$ be an infinite dimensional simple (countably dimensional locally finite) Lie superalgebra with root-reductive even part $\mathfrak{s}_0$, and let $\mathfrak{h}_0 \subset \mathfrak{s}_0$ be a subalgebra such that $\mathfrak{s}_0$ admits an $\mathfrak{h}_0$-root decomposition. If $\mathfrak{s}_1$ is an $\mathfrak{h}_0$-weight module, then $\mathfrak{s}$ admits an $\mathfrak{h}$-root decomposition for $\mathfrak{h} := \mathfrak{h}_0 \oplus (\mathfrak{s}_1)^0_{\mathfrak{h}_0}$, and there is a root isomorphism between $\mathfrak{s}$ and precisely one of the Lie algebras in (16).*

Proof. Clearly $\mathfrak{h}$ is a Lie subsuperalgebra of $\mathfrak{s}$ with $\mathfrak{s}^0_{\mathfrak{h}_0} = \mathfrak{h}$, and Proposition 1 implies that $\mathfrak{h}$ is locally nilpotent. The fact that $\mathfrak{s}_0$ and $\mathfrak{s}_1$ are both $\mathfrak{h}_0$-weight modules means that $\mathfrak{s}$ admits an $\mathfrak{h}$-root decomposition. Furthermore, $\mathfrak{s}$ is automatically classically semisimple as $\mathfrak{s}$ is simple and is thus generated by its root spaces. (The contrary would yield a proper ideal in $\mathfrak{s}$). Therefore the claim is an immediate corollary of Theorem 4.

Here are two remarks. First, if $\mathfrak{s}$ is as in Corollary 2 and $\mathfrak{h}' \subset \mathfrak{s}$ is a subsuperalgebra with $\mathfrak{s}^0_{\mathfrak{h}'_0} = \mathfrak{h}'$, then $\mathfrak{s}$ does not necessarily admit an $\mathfrak{h}'$-root decomposition. This is already true when $\mathfrak{s}$ is a Lie algebra, and in [PS] there is an example of an abelian (selfnormalizing) subalgebra $\mathfrak{h}' \subset \mathfrak{sl}(\infty)$ with $\mathfrak{sl}(\infty)^0_{\mathfrak{h}'} = \mathfrak{h}'$, such that $\mathfrak{sl}(\infty)$ admits no $\mathfrak{h}'$-root decomposition.

Second, it is easy to construct simple (countably dimensional locally finite) Lie superalgebras $\mathfrak{s}$ which do not satisfy the condition of Corollary 2. Consider the following Lie superalgebra inclusion

$$\mathfrak{sl}(j|j) \subset \mathfrak{sl}(2j|2j),$$

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \mapsto \left(\begin{array}{c|c|c|c} A & 0 & B & 0 \\ \hline 0 & A & 0 & B \\ \hline C & 0 & D & 0 \\ \hline 0 & C & 0 & D \end{array}\right),$$

and let $\mathfrak{g} = \bigcup_{j\geq 2} \mathfrak{sl}(j|j)$. There is no subsuperalgebra $\mathfrak{h} \subset \mathfrak{g}$ for which $\mathfrak{g}$ admits an $\mathfrak{h}$-root decomposition. This follows from the exact sequence of Lie algebras

$$0 \to \mathfrak{g}_0' \oplus \mathfrak{g}_0' \hookrightarrow \mathfrak{g}_0 \to \mathbb{C} \to 0,$$

where $\mathfrak{g}_0'$ is the simplest "diagonal" Lie algebra admitting no root decomposition. The fact that $\mathfrak{g}_0'$ admits no root decomposition is a consequence of the known fact that $\mathfrak{g}_0'$ is simple and is not isomorphic to $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$ or $\mathfrak{sp}(\infty)$, see [BZ].

We now address the question of which pairs of superalgebras in (16) are isomorphic.

Lemma 3. *The following are the only Lie superalgebra isomorphisms between pairs of Lie superalgebras in (16):*

$$\begin{array}{c} B(\infty|2k) \simeq D(\infty|2k) \\ B(\infty|\infty) \simeq D(\infty|\infty). \end{array} \tag{23}$$

Proof. The case when $k = 0$ is crucial. Here we clam that $B(\infty|0) = B(\infty)$ and $D(\infty|0) = D(\infty)$ are isomorphic as Lie algebras. To construct an isomorphism, note that there is an alternative description of $B(\infty)$ and $D(\infty)$. For $B(\infty)$ our definition via the chain (7) is equivalent to the following. Let V be a vector space with basis $\dots, v_{-1}, v_0, v_1, \dots$, equipped with a bilinear form (,) such that $(v_i, v_j) = \delta_{i,-j}$. Then $B(\infty)$ is the subalgebra of $\mathfrak{sl}(\infty)$ ($\mathfrak{sl}(\infty)$ being identified with the automorphisms of V whose matrices are finite in the above basis) which consists of endomorphisms φ: $V \to V$ with $(\varphi v, v') + (v, \varphi v') = 0$. Similarly, $D(\infty)$ is described in terms of a basis $\dots, v'_{-1}, v'_1, \dots$ and a form (,) such that $(v_i, v_j) = \delta_{i,-j}$. The point is that a simple base change on V (which we leave to the reader to work out) identifies the two bilinear forms on V and consequently yields an isomorphism $B(\infty) \simeq D(\infty)$.

A straightforward generalization of this argument produces isomorphisms (23) as desired. We omit the details.

Finally, to check that (23) are the only isomorphisms among the Lie superalgebras (16) it is enough to know that $\mathsf{sl}(\infty)$, $B(\infty) \simeq D(\infty)$, and $\mathsf{sp}(\infty)$ are pairwise non-isomorphic. This follows from the much stronger Theorem 5.2 in [BZ]. Then the general statement follows from a straightforward inspection of the even parts of the Lie superalgebras (16) which verifies that the even parts of (16) are isomorphic precisely for the pairs $(B(\infty|2k), D(\infty|2k))$, $(B(\infty|\infty), D(\infty|\infty))$, and $((B(1|\infty), B(0|\infty))$. As $B(0|\infty)$ and $B(1|\infty)$ are obviously non-isomorphic, this yields the statement.

Lemma 3 enables us to set now

$$\mathsf{osp}(\infty|2k) := B(\infty|2k) \simeq D(\infty|2k),$$

$$\mathsf{osp}(\infty, \infty) := B(\infty|\infty) \simeq D(\infty|\infty).$$

Furthermore, Corollary 2 and Lemma 3 have the following direct corollary.

Corollary 3. *Let* s *be a simple Lie superalgebra as in Corollary 2.*

a) Then s *is isomorphic to one of the following Lie superalgebras:*

$$\begin{gathered}\mathsf{sl}(\infty|n), \mathsf{sl}(\infty|\infty), \mathsf{osp}(m|\infty), \mathsf{osp}(\infty|2k), \\ \mathsf{osp}(\infty|\infty), \mathsf{sp}(\infty), \mathsf{sq}(\infty).\end{gathered} \tag{23}$$

b) Let $\mathsf{h} \subset \mathsf{s}$ *be subsuperalgebra for which* s *admits an* h*-root decomposition. Then, for* $\mathsf{s} \simeq \mathsf{sl}(\infty|n)$*,* $\mathsf{sl}(\infty|\infty)$*,* $\mathsf{osp}(m|\infty)$*,* $\mathsf{sp}(\infty)$*,* $\mathsf{sq}(\infty)$ *there exists an automorphism* $\varphi : \mathsf{s} \to \mathsf{s}$ *such that* $\varphi(\mathsf{h})$ *is the union of fixed Cartan subsuperalgebras of the respective local system (5), (6), (9), (14) or (15). If* $\mathsf{s} \simeq \mathsf{osp}(\infty|2k)$ *(respectively* $\mathsf{s} \simeq \mathsf{osp}(\infty|\infty)$*, there exists an automorphism*

$\varphi : \mathsf{s} \to \mathsf{s}$ *which maps* $\varphi(\mathsf{h})$ *respectively into the union of fixed Cartan subsuperalgebras of the local system (7) or (11) (resp. (8) or (12)).*

Finally, note that Theorem 4 together with Corollary 3 imply that any classically semisimple Lie superalgebra g is isomorphic to a direct sum of copies of the Lie superalgebras (23) and of classical or exceptional simple finite dimensional Lie superalgebras. Theorem 4 and Corollary 3 provide also a complete description of all subsuperalgebras $\mathsf{h} \subset \mathsf{g}$ for which g admits an h-root decomposition.

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