On ideals in certain Mackey algebras

Dmytro Rudenko

Thesis supervisor: Prof. Ivan Penkov

Abstract

We study the ideals in certain Mackey algebras $\mathrm{End}_W V$ and $\mathfrak{gl}^M(V, W)$, where the complex vector space V is countable dimensional, but the complex vector space W is not necessarily countable dimensional. We show the existence of a 1-codimensional subspace H of V^*, for which the associative Mackey algebra $\mathrm{End}_H V$ has infinite length, see subsection 4.3. In addition, we compute the lengths of the Mackey algebras $\mathrm{End}_{(V^* \oplus V_*)}(V \oplus V)$ and $\mathfrak{gl}^M((V^* \oplus V_*), (V \oplus V))$, see subsection 4.4.

CONSTRUCTOR UNIVERSITY

Statutory Declaration

Family Name, Given/First Name	Rudenko, Dmytro
Matriculation number	30004246
What kind of thesis are you submitting: Bachelor-, Master- or PhD-Thesis	Bachelor

English: Declaration of Authorship

I hereby declare that the thesis submitted was created and written solely by myself without any external support. Any sources, direct or indirect, are marked as such. I am aware of the fact that the contents of the thesis in digital form may be revised with regard to usage of unauthorized aid as well as whether the whole or parts of it may be identified as plagiarism. I do agree my work to be entered into a database for it to be compared with existing sources, where it will remain in order to enable further comparisons with future theses. This does not grant any rights of reproduction and usage, however.

This document was neither presented to any other examination board nor has it been published.

German: Erklärung der Autorenschaft (Urheberschaft)

Ich erkläre hiermit, dass die vorliegende Arbeit ohne fremde Hilfe ausschließlich von mir erstellt und geschrieben worden ist. Jedwede verwendeten Quellen, direkter oder indirekter Art, sind als solche kenntlich gemacht worden. Mir ist die Tatsache bewusst, dass der Inhalt der Thesis in digitaler Form geprüft werden kann im Hinblick darauf, ob es sich ganz oder in Teilen um ein Plagiat handelt. Ich bin damit einverstanden, dass meine Arbeit in einer Datenbank eingegeben werden kann, um mit bereits bestehenden Quellen verglichen zu werden und dort auch verbleibt, um mit zukünftigen Arbeiten verglichen werden zu können. Dies berechtigt jedoch nicht zur Verwendung oder Vervielfältigung.

Diese Arbeit wurde noch keiner anderen Prüfungsbehörde vorgelegt noch wurde sie bisher veröffentlicht.

14.05.2023,

Date, Signature

Contents

1 Introduction

For a finite-dimensional vector space V over $\mathbb{C}$, Wedderburn's theorem claims that the associative algebra End$V$ has no nonzero proper two-sided ideals. Furthermore, the Lie algebra $\mathfrak{gl}(V)$ associated with End$V$ has only two nonzero proper ideals – $\mathbb{C}$id and $\mathfrak{sl}(V)$ – of scalars and traceless elements, respectively. It is one of the basic facts in Lie theory that the Lie algebra $\mathfrak{sl}(V)$ is simple.

For an infinite-dimensional vector space V, the two-sided ideals of End$V$ have been described by N. Jacobson, for example in [J]. The characterization of ideals in the Lie algebra $\mathfrak{gl}(V)$ has appeared in [S] and [BHO].

Mackey algebras, which we now define, generalize the endomorphism algebra End$V$. Let V, W be infinite-dimensional vector spaces over $\mathbb{C}$ and let $V \times W \to \mathbb{C}$ be a non-degenerate pairing. This fixes an embedding of W into the dual space V^*. The *associative Mackey algebra* of this pairing is the algebra

$$\mathrm{End}_W V := \{\varphi \in \mathrm{End}V \mid \varphi^*(W) \subset W\},$$

where φ^* denotes the operator dual to φ. The *Mackey Lie algebra* of the same pairing is denoted by $\mathfrak{gl}^M(V, W)$, and is defined to be the vector space $\mathrm{End}_W V$ with Lie bracket $[\phi, \psi] = \phi\psi - \psi\phi$. Mackey algebras have been introduced in [M] and have been considered as Lie algebras in [PS]. The representation theory of infinite-dimensional Lie algebras has been developing very actively in the last four decades, nevertheless the theory has made only its first steps for Mackey Lie algebras, see [CP], [PS]. In particular, no reasonably general categories of representations containing the adjoint representation of a Mackey Lie algebra have been explored. In fact, the structure of the adjoint representation of a general Mackey Lie algebra has not yet been understood.

The aim of this thesis is to study ideals in Mackey algebras $\mathrm{End}_W V$ and $\mathfrak{gl}^M(V, W)$, beyond the case of End$V$. Apart from our work, to the best of our knowledge, the ideals of general Mackey algebras have been studied only in [PS], [PT], and in a recent unpublished manuscript [C] by A. Chirvasitu. The paper [PS] addresses the characterization of ideals in the Mackey Lie algebra $\mathfrak{gl}^M(V, V_*)$ for a countable-dimensional vector space V, where V_* is a subspace of V^* defined as follows.

For any infinite-dimensional vector space V and a basis $\{v_b\}_{b\in B} \subset V$ indexed by a set B, define $V_* := \mathrm{span}\,\{v_b^*\}_{b\in B}$ where $v_b^* \in V^*$ is given by $v_b^*(v_{b'}) = \delta_b^{b'}$. We call the family $\{v_b^*\}_{b\in B}$ the *dual system* of $\{v_b\}_{b\in B}$ for brevity. We say that a non-degenerate pairing $V \times W \to \mathbb{C}$ *splits* if there exists a basis of V whose dual system is a basis of W (W being viewed as a subspace of V^*). For a countable-dimensional V, the non-degenerate pairing $V \times V_* \to \mathbb{C}$ becomes an important special case because of the following proposition from [M]:

Proposition 1.1 (G. Mackey)**.** *Let $V \times W \to \mathbb{C}$ be a non-degenerate pairing. If V, W are both countable dimensional, then the pairing splits.*

The paper [PT] (in progress) contains the characterization of ideals in the Mackey Lie algebras $\mathfrak{gl}^M(V, V_*)$, where $\dim V = \aleph_k$ for any natural number k.

In our work, we focus on non-degenerate pairings $V \times W \to \mathbb{C}$, where V is countable-dimensinal, but W is not necessarily countable dimensional. Even with this restriction, there exist Mackey algebras $\mathrm{End}_W V$ that have infinite length (an infinite chain of distinct two-sided ideals), as was shown by A. Chirvasitu in [C]. As a first main point of this thesis, we show that there exists a *hyperplane* Mackey algebra $\mathrm{End}_H V$ that has infinite length, where H is a 1-codimensional subspace of V^*, see subsection 4.3. As a second main point, we compute the lengths of the Mackey algebras $\mathrm{End}_{(V^* \oplus V_*)}(V \oplus V)$ and $\mathfrak{gl}^M((V^* \oplus V_*), (V \oplus V))$, described in subsection 4.4.

We state some needed preliminaries in section 2, and we give examples of different Mackey algebras in section 3. We discuss ideals of Mackey algebras in section 4.

2 Preliminaries

In this section we gather a couple of already known propositions about Mackey algebras, as well as some general facts about linear (co)filters, all to be used later on.

2.1 Generalities on Mackey algebras

Throughout this thesis, the ideals of an associative Mackey algebra $\mathrm{End}_W V$ are always meant to be two-sided, while the ideals of a Mackey Lie algebra $\mathfrak{gl}^M(V, W)$ are always meant to be Lie ideals. A *Mackey algebra* is meant to be any of the two algebras, and is referred to in places where we make no distinction between the associative and Lie cases. Finally, from now on, V is a fixed countable-dimensional vector space over $\mathbb{C}$, unless explicitly stated otherwise.

For our convenience, we say that a subspace W of V^* is non-degenerate if the corresponding pairing $V \times W \to \mathbb{C}$ is non-degenerate. With respect to a basis $\{v_b\}_{b \in B}$ of V, the elements of V^* become (countable-)infinite rows of complex numbers, and $W \subset V^*$

becomes a subspace of such infinite rows. We state a criterion of non-degeneracy of such a subspace W.

Lemma 2.1. *Let $\{v_b\}_{b\in B}$ be a basis of V with its dual system $\{v_b^*\}_{b\in B} \subset V^*$, and let W be a subspace of V^* viewed as a subspace of the space of infinite row-vectors indexed by B. For any finite subset of indices $A \subset B$, let $\pi_A \colon V^* \to \operatorname{span}\{v_a^*\}_{a\in A}$ be the projection that sends a row to its finite sub-row indexed by A. Then W is non-degenerate if and only if for every finite $A \subset B$ the restriction $\pi_{A|W} \colon W \to \operatorname{span}\{v_a^*\}_{a\in A}$ is surjective.*

Proof. Assume first that for every finite subset $A \subset B$ the restriction $\pi_{A|W} \colon W \to \operatorname{span}\{v_a^*\}_{a\in A}$ is surjective. Any nonzero vector $v \in V$ is a linear combination of vectors $\{v_a\}_{a\in A}$ for some finite $A \subset B$. Choose a covector $\alpha \in \operatorname{span}\{v_a^*\}_{a\in A}$ with $\alpha(v) \neq 0$. Because $\pi_{A|W} \colon W \to \operatorname{span}\{v_a^*\}_{a\in A}$ is surjective, there exists $w \in W$ with $\pi_{A|W}(w) = \alpha$. We get $w(v) = \alpha(v) \neq 0$, and so W is non-degenerate. On the other hand, if W is non-degenerate, then for every nonzero vector $v \in V$ there exists $w \in W$ with $w(v) \neq 0$. Take any finite subset $A \subset B$ and consider $\operatorname{span}\{v_a\}_{a\in A} \subset V$. Since every $w \in W$ acts on every $v \in \operatorname{span}\{v_a\}_{a\in A}$ with the indices A only, it must hold that $\forall v \in \operatorname{span}\{v_a\}_{a\in A}\ \exists \alpha \in \pi_{A|W}(W) \colon \alpha(v) \neq 0$. This is only possible when $|A| = \dim(\operatorname{span}\{v_a\}_{a\in A}) \leqslant \dim(\pi_{A|W}(W))$. Since $\dim(\pi_{A|W}(W))$ is at most $|A|$, the map $\pi_{A|W} \colon W \to \operatorname{span}\{v_a^*\}_{a\in A}$ must be surjective. $\square$

Let $V \times W \to \mathbb{C}$ be a non-degenerate pairing. Then the classification of ideals of the Mackey algebra $\mathrm{End}_W V$ yields the classification of ideals of the Mackey algebra $\mathrm{End}_V W$, because of the following proposition.

Proposition 2.2 ([PS])**.** *There is an anti-isomorphism of associative algebras*

$$\mathrm{End}_W V \longrightarrow \mathrm{End}_V W, \quad \varphi \longmapsto \varphi^*_{|W}.$$

We say that two pairings $V_1 \times W_1 \to \mathbb{C}$ and $V_2 \times W_2 \to \mathbb{C}$ are *isomorphic* (respectively, *anti-isomorphic*) if there exist isomorphisms $f \colon V_1 \to V_2$ and $g \colon W_1 \to W_2$ (respectively, isomorphisms $f \colon V_1 \to W_2$ and $g \colon W_1 \to V_2$) that commute with the two pairings. Another important proposition is the following.

Proposition 2.3 ([PS], [Z])**.** *Let $V_1 \times W_1 \to \mathbb{C}$ and $V_2 \times W_2 \to \mathbb{C}$ be two non-degenerate pairings. The following statements are equivalent:*

1. *The pairings $V_1 \times W_1 \to \mathbb{C}$ and $V_2 \times W_2 \to \mathbb{C}$ are isomorphic or anti-isomorphic.*

2. *The Mackey algebras* $\mathrm{End}_{W_1}V_1$ *and* $\mathrm{End}_{W_2}V_2$ *are isomorphic or anti-isomorphic as associative algebras.*

3. *The Mackey algebras* $\mathfrak{gl}^M(V_1, W_1)$ *and* $\mathfrak{gl}^M(V_2, W_2)$ *are isomorphic as Lie algebras.*

Moreover, for $V_1 = V_2 = V$*, the pairings* $V \times W_1 \to \mathbb{C}$ *and* $V \times W_2 \to \mathbb{C}$ *are isomorphic if and only if there exists an isomoprhism* $f \in GL(V)$ *with* $f^*(W_2) = W_1$*, where the spaces* W_1 *and* W_2 *are viewed as subspaces of* V^**.*

Proof. By using proposition 1.1. from [PS], one can prove the equivalence 1. $\iff$ 3., see proposition 3.4. in [Z]. Since, clearly, 1. $\implies$ 2. $\implies$ 3., all three statements are equivalent. The main part of the proof of the remaining statement can be found in the proof of proposition 3.4. in [Z]. □

2.2 Linear (co)filters and one-sided ideals of $\mathrm{End}V$

Definition 2.4. Let $\mathrm{Gr}V$ denote the set of all vector subspaces of V. A subset $F \subset \mathrm{Gr}V$ is a *linear filter* if F satisfies the following two conditions:

i) F is closed under finite intersections: $w_1, w_2 \in F \implies w_1 \cap w_2 \in F$.

ii) F is upward closed: if $w_1 \in F, w_2 \in \mathrm{Gr}V$ and $w_1 \subset w_2$ then $w_2 \in F$.

A subset $F \subset \mathrm{Gr}V$ is a *linear cofilter* if it satisfies the following two conditions:

i) F is closed under finite sums: $w_1, w_2 \in F \implies w_1 + w_2 \in F$.

ii) F is downward closed: if $w_1 \in F, w_2 \in \mathrm{Gr}V$ and $w_2 \subset w_1$ then $w_2 \in F$.

Lemma 2.5. *A subset* $M \subset \mathrm{End}V$ *is a left* $\mathrm{End}V$*-submodule (i.e., a left ideal) if and only if there exists a linear filter* $F \subset \mathrm{Gr}V$ *with* $M = \{f \in \mathrm{End}V \mid \mathrm{ker}f \in F\}$*. Such a filter* F *is unique.*

Proof. The uniqueness of such a filter F, whenever it exists, is obvious. The direction ($\Longleftarrow$) is clear. For the other direction, assume that $M \subset \mathrm{End}V$ is a left $\mathrm{End}V$-submodule. It suffices to show that the set $\mathrm{ker}M := \{\mathrm{ker}m \colon m \in M\}$ is a filter, since then this will be the filter we are looking for.

By definition of M, $\forall a \in \mathrm{End}V$ and $\forall m \in M$ holds $am \in M$. It is not hard to check that $\mathrm{End}V \cdot m = \{f \in \mathrm{End}V \mid \mathrm{ker}m \subset \mathrm{ker}f\}$. Because $\mathrm{End}V \cdot m \subset M$, the set $\mathrm{ker}M$ is upward closed. Moreover, let $\mathrm{ker}m_1, \mathrm{ker}m_2 \in \mathrm{ker}M$. Then $\mathrm{ker}m_1 \cap \mathrm{ker}m_2 \in \mathrm{ker}M$. Indeed, one just has to find $m_3 \in M$ with $\mathrm{ker}m_3 = \mathrm{ker}m_1 \cap \mathrm{ker}m_2$, and this can be achieved by adding some appropriate m_1 and m_2. Hence, $\mathrm{ker}M$ is a filter. □

Lemma 2.6. *A subset $M \subset \mathrm{End}V$ is a right $\mathrm{End}V$-submodule if and only if there exists a linear cofilter $F \subset \mathrm{Gr}V$ with $M = \{f \in \mathrm{End}V \mid \mathrm{im}f \in F\}$. Such a cofilter F is unique.*

Proof. The proof is analogous to the proof of the previous lemma. □

Corollary 2.7. *The set of all left $\mathrm{End}V$-submodules of $\mathrm{End}V$ forms a complete lattice under inclusion, isomorphic to the complete lattice formed by linear filters on $\mathrm{Gr}V$ under inclusion. Intersection of left submodules corresponds to the intersection of the corresponding linear filters, and summation of left submodules corresponds to the join of the corresponding linear filters. The correspondence is established by the map*

$$\ker\colon \mathrm{End}V \to \mathrm{Gr}V.$$

Analogous statements hold for the right submodules and linear cofilters, the correspondence being established by the map

$$\mathrm{im}\colon \mathrm{End}V \to \mathrm{Gr}V.$$

2.3 Linear filters and non-degenerate subspaces of V^*

Here we show how linear filters on V relate to vector subspaces W of V^*.

We note first that every vector subspace $W \subset V^*$ is the sum of the annihilators that it contains.

Lemma 2.8. *Let $W \subset V^*$ be any subspace of V^*, and let $F_W \subset \mathrm{Gr}V$ be the linear filter of subspaces of V whose annihilators are contained in W:*

$$F_W := \{U \in \mathrm{Gr}V \mid U^\perp \subset W\}.$$

Then

$$W = \mathop{\Large +}_{U \in F_W} U^\perp.$$

Proof. The inclusion $+_{U \in F_W} U^\perp \subset W$ is obvious. For the other inclusion it is enough to notice that each line contained in W is actually the annihilator of some hyperplane in V, so the sum $+_{U \in F_W} U^\perp$ contains all lines of W. □

On the other hand, any subset $S \subset \mathrm{Gr}V$ of subspaces of V yields a subspace W of V^* by means of taking the sum of the annihilators of elements of S.

Lemma 2.9. *Let $S \subset \mathrm{Gr}V$ be a subset of subspaces of V. Then the subspace*

$$W_S := \sum_{U \in S} U^{\perp}$$

of V^* is non-degenerate if and only if $\bigcap_{U \in S} U = 0$.

Proof. The subspace $W_S \subset V^*$ is non-degenerate if and only if there is no nonzero subspace $U' \subset V$ whose annihilator contains W_S. Since $W_S = \sum_{U \in S} U^{\perp}$, this is equivalent to saying that there is no nonzero subspace $U' \subset V$ whose annihilator contains each $U^{\perp}$ for $U \in S$, or that there is no nonzero $U' \subset V$ contained in each $U \in S$. The latter holds if and only if $\bigcap_{U \in S} U = 0$. $\square$

Lemma 2.9 implies that any linear filter $F \subset \mathrm{Gr}V$ with nonzero intersection yields a non-degenerate subspace W_F of V^*.

The above two lemmas establish a correspondence between subspaces $W \subset V^*$ of V^* and linear filters $F \subset \mathrm{Gr}V$ on V. It also follows that $W_{F_W} = W$ and $F_{W_F} \supset F$.

Let $F_{fc} \subset \mathrm{Gr}V$ denote the linear filter of finite-codimensional subspaces of V. The fact that each W is determined by the lines it contains is reflected in the following proposition.

Proposition 2.10. *Let $W \subset V^*$. Under the above correspondence, consider the set of all filters $\mathfrak{F}_W$ corresponding to W, which is a poset under inclusion. Then the maximum element of $\mathfrak{F}_W$ is F_W, the minimum element of $\mathfrak{F}_W$ is $F_W \cap F_{fc}$.*

Proof. F_W is clearly the greatest filter corrsponding to W. It is also clear that $F_W \cap F_{fc}$ corresponds to W. It is left to show that $F_W \cap F_{fc}$ is minimum. Assume some filter $F \in \mathfrak{F}_W$ corresponds to W. Then $\sum_{U \in F} U^{\perp} = W$, and so any line $l \subset W$ is contained in some finite sum $U_1^{\perp} + \cdots + U_n^{\perp} = (U_1 \cap \cdots \cap U_n)^{\perp}$, where each $U^i \in F$. But then $U_1 \cap \cdots \cap U_n$ is contained in H, where $l = H^{\perp}$, and so $H \in F$. Therefore, F contains every hyperplane from $F_W \cap F_{fc}$, yielding the result. $\square$

Definition 2.11. Let $F \subset \mathrm{Gr}V$ be a linear filter on V. Following [BH], define

$$R(F) := \left\{ f \in \mathrm{End}V \mid \forall U \in F \colon f^{-1}(U) \in F \right\}.$$

It is not hard to check that $R(F)$ is a subalgebra of $\mathrm{End}V$.

The next proposition links this work with the paper[BH].

Proposition 2.12. *Let $W \subset V^*$ be any subspace. Then* $\mathrm{End}_W V = R(F_W)$.

Proof. Let $f \in \mathrm{End}_W V$. We have $f^*(U^\perp) \subset W$, whenever $U^\perp \subset W$. It is not hard to check that

$$f^*(U^\perp) = U^\perp \circ f = (f^{-1}(U))^\perp.$$

Hence, $f^{-1}(U) \in F_W \ \forall U \in F_W$, and so $f \in R(F_W)$. On the other hand, if $f \in R(F_W)$, then $f^*(U^\perp) \subset W$ for any $U \in F_W$. By lemma 2.8, we have $W = +_{U \in F_W} U^\perp$, implying $f^*(W) \subset W$. $\square$

We do not know much about the structure of the algebras $R(F)$ for general linear filters F, and in what follows we restrict ourselves to the study of Mackey algebras $\mathrm{End}_W V$ and $\mathfrak{gl}^M(V, W)$.

3 Examples of Mackey algebras

Any non-degenerate subspace W of V^* defines a non-degenerate pairing $V \times W \to \mathbb{C}$, thus defining the Mackey algebra of this pairing. In this section we give examples of Mackey algebras by specifying such non-degenerate subspaces W.

3.1 First examples of Mackey algebras

Let $B = (v_1, v_2, v_3, \dots)$ be an ordered basis of V. With respect to B, the elements of V can be viewed as finitary column-vectors, the elements of V^* – as (countably-)infinite row-vectors. Apart from V^* and V_*, following [C], we define $V^*_{fv} \subset V^*$ to be the subspace of rows with finitely many different entries:

$$V^*_{fv} := \{f : B \to \mathbb{C} \mid f \text{ has finite range}\}.$$

We have the inclusions $V_* \subset V^*_{fv} \subset V^*$, the three spaces are non-degenerate, and it is easy to check that the three Mackey algebras $\mathrm{End}_{V^*}V, \mathrm{End}_{V_*}V, \mathrm{End}_{V^*_{fv}}V$ are pairwise non-isomorphic.

Here are some non-degenerate subspaces of V^* whose definitions use the fact that $\mathbb{C}$ is a normed field. A first example is given by the subspace $V^*_{B,b}$ of V^* of bounded sequences, and a second one – by the subspace $V^*_{B,c0}$ of V^* of sequences convergent to 0. A third example is given by the subspace $V^*_{B,s}$ of V^* of summable sequences, and a fourth one – by the subspace $V^*_{B,as}$ of V^* of absolutely summable sequences. Note that each of the four subspaces

$$V^*_{B,as} \subset V^*_{B,s} \subset V^*_{B,c0} \subset V^*_{B,b}$$

is non-degenerate. Finally, for any real number $p \in (1, \infty)$, the space of pth-power-summable sequences l_p is also a non-degenerate subspace of V^*, with inclusions $l_p \subset l_q$ for any $p < q$.

3.2 Hyperplane Mackey algebras

One of the main results of this thesis deals with hyperplanes in V^*, i.e., subspaces of codimension 1 in V^*. Each hyperplane $H \subset V^*$ is given by an element $\phi \in V^{**}$ via $H = \ker\phi$. The corresponding pairing $V \times H \to \mathbb{C}$ is non-degenerate if and only if $\phi \notin V \hookrightarrow V^{**}$. We call the corresponding Mackey algebra $\mathrm{End}_H V$ a *hyperplane* Mackey algebra. By proposition 2.3, two hyperplane Mackey algebras $\mathrm{End}_{H_1} V$ and $\mathrm{End}_{H_2} V$ are isomorphic if and only if there exists $f \in GL(V)$ with $f^*(H_2) = H_1$. Therefore, there are $2^{2^{\aleph_0}}$ pairwise non-isomoprhic hyperplane Mackey algebras as $|V^{**}| = 2^{2^{\aleph_0}}$ and $|GL(V)| = 2^{\aleph_0}$.

3.3 Mixed Mackey algebras

For any two Mackey algebras $\mathrm{End}_{W_1} V_1$ and $\mathrm{End}_{W_2} V_2$, consider the *mixed* Mackey algebra

$$\mathrm{End}_{(W_1 \oplus W_2)}(V_1 \oplus V_2),$$

the corresponding non-degenerate pairing $(W_1 \oplus W_2) \times (V_1 \oplus V_2) \to \mathbb{C}$ being defined as the direct sum of non-degenerate pairings $W_1 \times V_1 \to \mathbb{C}$ and $W_2 \times V_2 \to \mathbb{C}$. This operation enables us to construct new Mackey algebras out of already known ones.

Definition 3.1. We call a Mackey algebra $\mathrm{End}_W V$ *Mackey-idempotent* if $\mathrm{End}_{(W \oplus W)}(V \oplus V)$ is isomorphic to $\mathrm{End}_W V$.

Proposition 3.2. *The Mackey algebras* $\mathrm{End}_{V^*} V$*,* $\mathrm{End}_{V_*} V$*, and* $\mathrm{End}_{V^*_{fv}} V$ *are Mackey-idempotent.*

Proof. Recall that V^*, V_*, V^*_{fv} are subspaces of infinite rows with certain properties, described in subsection 3.1, and do not depend on the order of the basis B. Each of the three subspaces $V^* \oplus V^*, V_* \oplus V_*, V^*_{fv} \oplus V^*_{fv}$ of $V \oplus V$ is, again, a subspace of infinite rows with respect to the concatenated basis $B \sqcup B$. Since the defining properties of infinite rows from V^*, V_*, V^*_{fv} are preserved under concatenation of bases, the claim follows. $\square$

Definition 3.3. Let $\mathrm{End}_{W_1}V_1$ be Mackey-idempotent. We say that a Mackey algebra $\mathrm{End}_{W_2}V_2$ is $\mathrm{End}_{W_1}V_1$-*saturated* if $\mathrm{End}_{(W_1\oplus W_2)}(V_1 \oplus V_2)$ is isomorphic to $\mathrm{End}_{W_2}V_2$.

To explain the term "saturated" we have a simple lemma.

Lemma 3.4. *Let* $\mathrm{End}_{W_2}V_2$ *be a Mackey algebra, and let* $\mathrm{End}_{W_1}V_1$ *be Mackey-idempotent. Then* $\mathrm{End}_{W_2}V_2$ *is* $\mathrm{End}_{W_1}V_1$*-saturated if and only if there exists a Mackey algebra* $\mathrm{End}_{W_3}V_3$ *such that* $\mathrm{End}_{W_2}V_2$ *is isomorphic to* $\mathrm{End}_{(W_1\oplus W_3)}(V_1 \oplus V_3)$.

Proof. One direction is obvious. For the other one, if $\mathrm{End}_{W_2}V_2 \cong \mathrm{End}_{(W_1\oplus W_3)}(V_1 \oplus V_3)$, then clearly

$$\mathrm{End}_{(W_1\oplus W_2)}(V_1\oplus V_2) \cong \mathrm{End}_{(W_1\oplus W_1\oplus W_3)}(V_1\oplus V_1\oplus V_3) \cong \mathrm{End}_{(W_1\oplus W_3)}(V_1\oplus V_3) \cong \mathrm{End}_{W_2}V_2,$$

so, $\mathrm{End}_{(W_1\oplus W_2)}(V_1 \oplus V_2)$ is isomorphic to $\mathrm{End}_{W_2}V_2$. $\square$

In other words, $\mathrm{End}_{W_2}V_2$ is $\mathrm{End}_{W_1}V_1$-saturated if the pairing $V_1 \times W_1 \to \mathbb{C}$ splits off within the pairing $V_2 \times W_2 \to \mathbb{C}$ as a direct summand. Can it split off infinitely many times? We simultaneously give an example of an $\mathrm{End}_{V^*}V$-saturated Mackey algebra and answer this question.

Consider the Mackey algebra

$$\mathrm{End}_{V^*_\infty}V := \mathrm{End}_{(V^*\oplus V^*\oplus V^*\oplus\dots)}(V \oplus V \oplus V \oplus \dots),$$

where the direct sums have countably many summands. In this case, $V \oplus V \oplus V \oplus \dots$ is still isomorphic to V, while $V^* \oplus V^* \oplus V^* \oplus \dots$ is not isomorphic to $(V \oplus V \oplus V \oplus \dots)^*$. This example is similar to $\mathrm{End}_{V_*}V$ in the way that the latter is given by a direct-sum decomposition of V into one-dimensional subspaces (via the choice of a basis of V), while the new example is given by a direct-sum decomposition of V into countable-dimensional subspaces (each thus being isomorphic to V itself).

One can generalize this further by considering

$$A := \mathrm{End}_{(V_1^*\oplus V_2^*\oplus V_3^*\oplus\dots)}(V_1 \oplus V_2 \oplus V_3 \oplus \dots),$$

where each $V_i \neq 0$ is either finite dimensional or countable dimensional, and the direct sums have countably many summands.

Proposition 3.5.

- *If among the V_is there are infinitely many countable-dimensional spaces, then* $A \cong \mathrm{End}_{V^*_\infty} V$.
- *If among the V_is there are finitely many countable-dimensional spaces and at least one finite-dimensional space, then* $A \cong \mathrm{End}_{V^*} V \oplus \mathrm{End}_{V_*} V$.
- *If each of the V_is is finite dimensional, then* $A \cong \mathrm{End}_{V_*} V$.

Proof. We will only prove the first point, in the case when among the V_is there are infinitely many countable dimensional and infinitely-many finite dimensional spaces. All other cases are similar. We write V for each countable-dimensional summand, U – for each finite dimensional. We use the fact that the order of summation does not matter and, without loss of generality, assume that we are dealing with

$$\mathrm{End}_{(V^* \oplus U^* \oplus V^* \oplus U^* \oplus \dots)}(V \oplus U \oplus V \oplus U \oplus \dots).$$

We observe that each pair $V \oplus U$ is countable dimensional with $V^* \oplus U^* = (V \oplus U)^*$ and write instead

$$\mathrm{End}_{((V \oplus U)^* \oplus (V \oplus U)^* \oplus \dots)}((V \oplus U) \oplus (V \oplus U) \oplus \dots),$$

which is isomorphic to $\mathrm{End}_{V^*_\infty} V$. □

Proposition 3.6. $\mathrm{End}_{V^*_\infty} V$ *is Mackey-idempotent,* $\mathrm{End}_{V^*} V$*-saturated, and* $\mathrm{End}_{V_*} V$*-saturated.*

Proof. Since the first two statements are quite obvious, we move to the third one. As we have shown above,

$$\mathrm{End}_{V^*_\infty} V \cong \mathrm{End}_{(V^* \oplus U^* \oplus V^* \oplus U^* \oplus \dots)}(V \oplus U \oplus V \oplus U \oplus \dots).$$

We again use the fact that order of summation does not matter and rewrite as

$$\begin{aligned}
\mathrm{End}_{V^*_\infty} V &\cong \big(\mathrm{End}_{(V^* \oplus V^* \oplus \dots)}(V \oplus V \oplus \dots)\big) \oplus \big(\mathrm{End}_{(U^* \oplus U^* \oplus \dots)}(U \oplus U \oplus \dots)\big) \\
&\cong \big(\mathrm{End}_{V^*_\infty} V\big) \oplus \big(\mathrm{End}_{(U^* \oplus U^* \oplus \dots)}(U \oplus U \oplus \dots)\big) \\
&\cong \big(\mathrm{End}_{V^*_\infty} V\big) \oplus \big(\mathrm{End}_{(U_* \oplus U_* \oplus \dots)}(U \oplus U \oplus \dots)\big) \\
&\cong \mathrm{End}_{V^*_\infty} V \oplus \mathrm{End}_{V_*} V,
\end{aligned}$$

where in the last two transitions we used the fact that U is finite dimensional. $\square$

4 Ideals in certain Mackey algebras

In this section we state the main results of this thesis regarding the ideals of Mackey algebras. We start with some general properties of Mackey algebras and then proceed to characterizing the ideals of certain Mackey algebras.

4.1 On ideals in general Mackey algebras

Every ideal of $\mathrm{End}_W V$ is automatically an ideal in $\mathfrak{gl}^M(V, W)$.

Each Mackey algebra $\mathrm{End}_W V$ posseses the ideal $V \otimes W = \mathrm{End}_W V \cap I_{\aleph_0}$, where $I_{\aleph_0}$ is the ideal of $\mathrm{End} V$ of elements of finite rank (i.e., of rank $< \aleph_0$). In addition, each Mackey algebra $\mathfrak{gl}^M(V, W)$ posseses the ideal $\mathfrak{sl}(V, W) \subset V \otimes W$ of traceless elements of $V \otimes W$. It was shown in [PS] that every nonzero ideal $I \neq \mathbb{C}\mathrm{id}$ of $\mathfrak{gl}^M(V, W)$ contains $\mathfrak{sl}(V, W)$. We briefly prove an analogous statement for the associative case.

Proposition 4.1. *$V \otimes W$ is the unique simple ideal in* $\mathrm{End}_W V$*.*

Proof. We will prove that every nonzero ideal I of $\mathrm{End}_W V$ contains $V \otimes W$. Since I and $V \otimes W$ are also ideals in $\mathfrak{gl}^M(V, W)$, the intersection $I \cap V \otimes W$ contains $\mathfrak{sl}(V, W)$. Choose $0 \neq v \in V$ and $0 \neq w \in W$ with $w(v) = 0$, so that $v \otimes w \in \mathfrak{sl}(V, W) \subset I \cap V \otimes W$. Since $I \cap V \otimes W$ is two-sided, $f \cdot v \otimes w \in I \cap V \otimes W$ for any $f \in \mathrm{End}_W V$. Choose $v' \in V$ for which $w(v') \neq 0$, as well as $w' \in W$ for which $w'(v) \neq 0$ (the latter is possible because $V \times W \to \mathbb{C}$ is non-degenerate). Put $f := v' \otimes w'$ and obtain $v' \otimes w' \cdot v \otimes w = w'(v) v' \otimes w \in I \cap V \otimes W$, where $\mathrm{tr}(w'(v) v' \otimes w) = w'(v) w(v') \neq 0$. Because $\mathfrak{sl}(V, W)$ is 1-codimensional in $V \otimes W$, we get $V \otimes W \subset I \cap V \otimes W \subset I$. $\square$

Next, here is the subspace of $\mathrm{End}_W V$

$$\mathrm{End}_{V^* \to W} V := \{\varphi \in \mathrm{End} V \mid \varphi^*(V^*) \subset W\},$$

that is easily seen to be an ideal of $\mathrm{End}_W V$. Note that the ideal $\mathrm{End}_{V^* \to W} V$ is not always different from the ideal $V \otimes W$.

Proposition 4.2. *Let* $\mathrm{End}_W V$ *be a Mackey algebra with the ideals* $V \otimes W$ *and* $\mathrm{End}_{V^* \to W} V$ *defined as above. The following statements are equivalent:*

1. *$V \otimes W = \mathrm{End}_{V^* \to W} V$.*

2. $\mathrm{End}_W V$ *is not* $\mathrm{End}_{V^*} V$*-saturated.*

3. W *does not contain any infinite-dimensional annihilator* $U^\perp$ *of* $U \subset V$*.*

Proof. We will first show 1. $\iff$ 3., then 3. $\iff$ 2..

(1. $\iff$ 3.) First assume that W contains an infinite-dimensional annihilator $U^\perp$ of a subspace $U \subset V$. Choose any endomorphism $f \in \mathrm{End}V$ with $\mathrm{ker}f = U$. Since $W \supset U^\perp = (\mathrm{ker}f)^\perp = \mathrm{im}f^*$, we conclude that $f \in \mathrm{End}_{V^* \to W} V$. Moreover, since $U^\perp = (\mathrm{ker}f)^\perp$ was assumed to be infinite dimensional, it follows that $\mathrm{ker}f$ is infinite codimensional, equivalently, that $\mathrm{im}f$ is infinite dimensional. Hence, $\mathrm{End}_{V^* \to W} V \neq V \otimes W$, as the ideal $\mathrm{End}_{V^* \to W} V$ contains an endomorphism of infinite rank. On the other hand, if the ideal $\mathrm{End}_{V^* \to W} V$ contains an endomorphism of infinite rank f, then the annihilator of its kernel $(\mathrm{ker}f)^\perp \subset W$ is an infinite-dimensional annihilator in W.

(3. $\iff$ 2.) First assume that there exists an infinite-dimensional $U^\perp \subset W$ for some infinite-codimensional $U \subset V$. Consider any direct-sum decomposition $U \oplus U_0 = V$ of V (where U_0 is infinite dimensional), so that $V^* = U^* \oplus U_0^* = U^* \oplus U^\perp$. Since W contains $U^\perp$, the image of the projection of W to U^* equals $W \cap U^*$, and we denote this subspace of U^* by W_1. Then $W = W_1 \oplus U^\perp = W_1 \oplus U_0^*$, and

$$\mathrm{End}_W V = \mathrm{End}_{(W_1 \oplus U_0^*)}(U \oplus U_0) \cong \mathrm{End}_{(W_1 \oplus V^*)}(U \oplus V).$$

By lemma 3.4, $\mathrm{End}_W V$ is $\mathrm{End}_{V^*} V$-saturated. Note that non-degeneracy of the pairing $U \times W_1 \to \mathbb{C}$ follows from the non-degeneracy of the pairing $(U \oplus U_0) \times (W_1 \times U_0^*) \to \mathbb{C}$. On the other hand, if $\mathrm{End}_W V$ is $\mathrm{End}_{V^*} V$-saturated, then

$$\mathrm{End}_W V \cong \mathrm{End}_{(W \oplus V^*)}(V \oplus V).$$

Since Mackey algebras are isomorphic if and only if their corresponding pairings are isomorphic, $W \oplus V^*$ is isomorphically sent to W, and hence W contains an infinite-dimensional annihilator, which is the image of V^* under this isomorphism. □

We may now conclude that for $\mathrm{End}_{V^*} V$-saturated Mackey algebras the ideals $V \otimes W$ and $\mathrm{End}_{V^* \to W} V$ are distinct.

Proposition 4.3. *Let* $\mathrm{End}_W V$ *be a Mackey algebra. Let* $F_W \subset \mathrm{Gr}V$ *be the linear filter corresponding to* W*, and let* $F_{fc} \subset \mathrm{Gr}V$ *be the linear filter of finite-codimensional subspaces of* V*. Then the left* $\mathrm{End}V$*-submodules* $\mathrm{End}_{V^* \to W} V$ *and* $V \otimes W$ *of* $\mathrm{End}V$ *correspond respectively to the linear filters* F_W *and* $F_W \cap F_{fc}$*.*

Proof. Obvious. □

Let us digress to the most general Mackey algebra $\mathrm{End}_W V$, where V is not necessarily countable dimensional. For some cardinal number $\aleph \geqslant \aleph_0$, define $I_\aleph$ to be the two-sided ideal in $\mathrm{End}V$ of elements with rank $< \aleph$. Then the Mackey algebra $\mathrm{End}_W V$ contains the ideals $\mathrm{End}_W V \cap I_\aleph$ for all cardinal numbers $\aleph \geqslant \aleph_0$. By the definition of $\mathrm{End}_W V$, each $\phi \in \mathrm{End}_W V$ induces and element $\phi^*_{|W}$ of $\mathrm{End}W$. Moreover, φ also induces an element of $\mathrm{End}(V^*/W)$, and an element of $\mathrm{End}(W^*/V)$. We thus have four maps: $1\colon \mathrm{End}_W V \to \mathrm{End}V$, $2\colon \mathrm{End}_W V \to \mathrm{End}W$, $3\colon \mathrm{End}_W V \to \mathrm{End}(V^*/W)$, $4\colon \mathrm{End}_W V \to \mathrm{End}(W^*/V)$, each of which sends an element ϕ to the element it induces. The first map is the inclusion map, and the second map is the composition of the first one with the anti-isomorphism from proposition 2.2. It is also not hard to check that the third map is an anti-homomorphism of algebras, while the fourth map is a homomorphism of algebras. Each of the four endomorphism spaces – $\mathrm{End}V$, $\mathrm{End}W$, $\mathrm{End}(V^*/W)$, $\mathrm{End}(W^*/V)$ – has the above-defined ideals $I_\aleph$. By picking a quadruple of cardinal numbers $(\alpha, \beta, \gamma, \delta)$, we pick four ideals in the respective four algebras, and each of the four ideals yields an ideal of $\mathrm{End}_W V$ by pullback along the corresponding map. We denote the intersection of these four ideals in $\mathrm{End}_W V$ by $I_{(\alpha,\beta,\gamma,\delta)} \subset \mathrm{End}_W V$. For an ideal $I_\aleph \subset \mathrm{End}X$, let I_0 denote the zero ideal $\{0\}$, and let I_∞ denote the entire algebra $\mathrm{End}X$. It is then easy to see that

$$V \otimes W = I_{(\aleph_0,\infty,\infty,\infty)}, \text{ while } \mathrm{End}_{V^* \to W} V = I_{(\infty,\infty,0,\infty)}.$$

The anti-isomorphism between $\mathrm{End}_W V$ and $\mathrm{End}_V W$ sends the $(\alpha, \beta, \gamma, \delta)$-ideal on the left to the $(\beta, \alpha, \delta, \gamma)$-ideal on the right.

We would like to note that two different quadruples may yield the same ideal, and that not every ideal of $\mathrm{End}_W V$ is given by some quadruple.

Finally, we point out that, while it is not clear when an associative Mackey algebra $\mathrm{End}_W V$ has infinitely many ideals, *every* Mackey Lie algebra $\mathfrak{gl}^M(V, W)$ has uncountably many ideals. Indeed, as it is shown in [S], [BHO], every vector space U between $\mathfrak{sl}(V, W)$ and $\mathfrak{gl}(V, W) \oplus \mathbb{C}\mathrm{id}$ is an ideal of $\mathfrak{gl}^M(V, W)$. Since $\mathfrak{sl}(V, W)$ has codimension 2 in $\mathfrak{gl}(V, W) \oplus \mathbb{C}\mathrm{id}$, there are uncountably many such vector spaces U.

4.2 Ideals in Mackey Lie algebras $\mathfrak{gl}^M(V, V^*), \mathfrak{gl}^M(V, V_*), \mathfrak{gl}^M(V, V^*_{fv})$

The characterizations of ideals in the Mackey Lie algebras $\mathfrak{gl}^M(V, V^*)$, $\mathfrak{gl}^M(V, V_*)$, $\mathfrak{gl}^M(V, V^*_{fv})$ can be deduced from [BHO], [PS], [C], respectively. More precisely, [BHO] presents the complete characterization of ideals in $\mathfrak{gl}^M(V, V^*)$, while the characterizations for $\mathfrak{gl}^M(V, V_*)$ and $\mathfrak{gl}^M(V, V^*_{fv})$ follow easily from [PS] and [C], respectively. All these characterizations are similar, as we shall now describe.

Let W be any of the three subspaces V^*, V_*, V^*_{fv} of V^*. Then every ideal $I \neq \mathbb{C}\mathrm{id}$ in $\mathfrak{gl}^M(V, W)$ is one of the following:

$$(0) \subset \mathfrak{sl}(V, W) \subset U \subset \mathfrak{gl}(V, W) \oplus \mathbb{C}\mathrm{id} \subset \mathfrak{gl}^M(V, W),$$

where U is any vector subspace between $\mathfrak{sl}(V, W)$ and $\mathfrak{gl}(V, W) \oplus \mathbb{C}\mathrm{id}$. In particular, each of the three Mackey Lie algebras has uncountably many ideals, and, moreover, has length 4. Among the above Lie ideals, only $(0), \mathfrak{gl}(V, W) = V \otimes W$, and $\mathfrak{gl}^M(V, W) = \mathrm{End}_W V$ are two-sided ideals. Thus, the corresponding associative Mackey algebras have length 2.

4.3 Infinite-length hyperplane associative Mackey algebra

Here we present an associative hyperplane Mackey algebra that has uncountably many ideals, as well as infinite length.

Let $\mathrm{End}_H V$ be a hyperplane Mackey algebra, and let $\phi \in V^{**}$ be such that $\ker\phi = H$. We first prove some general statements about hyperplane Mackey algebras.

Lemma 4.4 ([C]). *V^{**} is a left $\mathrm{End}V$-module given by the map*

$$\mathrm{End}V \times V^{**} \to V^{**}$$
$$(a, \phi) \mapsto \phi \circ a^*$$

Proof. Indeed,

$$(ba)\phi = \phi \circ (ba)^* = \phi \circ a^* \circ b^* = b(a\phi).$$

□

Proposition 4.5 ([C]). *Let $\mathrm{End}_H V$ be a hyperplane Mackey algebra, let $\phi \in V^{**}$ be such that $\ker\phi = H$, and let $a \in \mathrm{End}V$. Then*

$$a \in \mathrm{End}_{V^* \to H} V \iff a\phi = 0,$$

and

$$a \in \mathrm{End}_H V \iff \exists \lambda \in \mathbb{C} \text{ such that } (a - \lambda \mathrm{id})\phi = 0.$$

Proof. The first equivalence follows from the definition of ϕ and lemma 4.4. In particular, we see that $\mathrm{End}_{V^*\to H}V$ is exactly the annihilator (with respect to the module structure $\mathrm{End}V \times V^{**} \to V^{**}$) of ϕ. For the second equivalence, let $a \in \mathrm{End}_H V$. Then $a^*(H) \subset H$, which is equivalent to $\ker(a\phi) = \ker\phi$ or $a\phi = 0$. In other words, there exists $\lambda \in \mathbb{C}$ such that $a\phi = \lambda\phi$, or $(a - \lambda\mathrm{id})\phi = 0$. In particular,

$$\mathrm{End}_H V = \mathrm{End}_{V^*\to H}V \oplus \mathbb{C}\mathrm{id}.$$

□

This is already interesting, because it means that the ideal $\mathbb{C}\mathrm{id}$ is a direct summand in $\mathfrak{gl}^M(V,H)$. This property does not hold for a general Mackey Lie algebra $\mathfrak{gl}^M(V,W)$.

The following result is due to A. Chirvasitu.

Theorem 4.6 ([C])**.** *There exists a non-degenerate hyperplane H in V^* for which*

$$\mathrm{End}_{V^*\to H}V = V \otimes H.$$

This means that there exists a hyperplane Mackey algebra with associative length 2 and Lie length 3:

$$(0) \subset \mathfrak{sl}(V,H) \subset (V \otimes H = \mathrm{End}_{V^*\to H}V) \subset \left(\mathrm{End}_{V^*\to H}V \oplus \mathbb{C}\mathrm{id} = \mathfrak{gl}^M(V,H)\right).$$

We call such a hyperplane (pairing) *Mackey-minimal.*

Theorem 4.7. *Let H be a Mackey-minimal hyperplane. The Mackey algebra $\mathrm{End}_{(V^*\oplus H)}(V\oplus V)$ has infinite length and uncountably many ideals.*

Proof. We start by considering the block matrix form of the algebra $\mathrm{End}_{(V^*\oplus H)}(V\oplus V)$:

$$\mathrm{End}_{(V^*\oplus H)}(V\oplus V) = \begin{array}{c|c} \mathrm{End}_{V^*\to V^*}V & \mathrm{End}_{V^*\to H}V \\ \hline \mathrm{End}_{H\to V^*}V & \mathrm{End}_{H\to H}V \end{array} = \begin{array}{c|c} \mathrm{End}V & V\otimes H \\ \hline \mathrm{End}V & V\otimes H \oplus \mathbb{C}\mathrm{id} \end{array}.$$

Let $f \subset \mathrm{End}V$ be the ideal of endomorphisms of finite rank. Consider the following chain of subspaces in $\mathrm{End}_{(V^*\oplus H)}(V\oplus V)$:

$$(0) \subset \begin{array}{c|c} f & V\otimes H \\ \hline f & V\otimes H \end{array} \subset \begin{array}{c|c} f & V\otimes H \\ \hline \mathrm{End}V & V\otimes H \end{array} \subset \begin{array}{c|c} \mathrm{End}V & V\otimes H \\ \hline \mathrm{End}V & V\otimes H \end{array} \subset \begin{array}{c|c} \mathrm{End}V & V\otimes H \\ \hline \mathrm{End}V & V\otimes H \oplus \mathbb{C}\mathrm{id} \end{array}.$$

One can check that these subspaces are, in fact, two-sided ideals. To make the checking easier, one should keep in mind two facts:

- $V \otimes H$ is a left $\mathrm{End}V$-submodule of $\mathrm{End}V$, as it is the annihilator of ϕ, see proposition 4.5;
- $V \otimes H$ is a subset of f, that is, elements of $V \otimes H$ have finite rank.

We enumerate the above ideals as 0, 1, 2, 3, and 4. Note that 1 is just the ideal $(V \oplus V) \otimes (V^* \oplus H)$, and 3 is just the ideal $\mathrm{End}_{(V^*\oplus V^*)\to(V^*\oplus H)}(V \oplus V)$.

We will show that the quotient 2/1 of the ideals is not simple (in fact, that it has infinite length), meaning that there exists a subspace $f \subsetneq M \subsetneq \mathrm{End}V$ for which

$$\frac{f \;\big|\; V \otimes H}{M \;\big|\; V \otimes H} \text{ is an ideal in } \frac{\mathrm{End}V \;\big|\; V \otimes H}{\mathrm{End}V \;\big|\; V \otimes H \oplus \mathbb{C}\mathrm{id}}.$$

By straightforwardly multiplying the ideal with the whole algebra from the left and from the right, we obtain that M must be a right $\mathrm{End}V$-submodule and a left $\mathrm{End}_H V$-submodule, respectively. Clearly, this condition is also sufficient for the subalgebra to be an ideal.

Hence, it is enough to find some $(\mathrm{End}_H V, \mathrm{End}V)$-sub-bimodule M of $\mathrm{End}V$ strictly between f and $\mathrm{End}V$. For this we need a lemma.

Lemma 4.8. *M is a $(\mathrm{End}_H V, \mathrm{End}V)$-sub-bimodule of $\mathrm{End}V$ if and only if it is a $(\mathrm{End}_{V^*\to H} V, \mathrm{End}V)$-sub-bimodule of $\mathrm{End}V$.*

Proof. We will prove the non-obvious direction. Assume $\forall k \in \mathrm{End}_{V^*\to H} V$ and $m \in M$ holds $km \in M$. Since $\mathrm{End}_H V = \mathrm{End}_{V^*\to H} V \oplus \mathbb{C}\mathrm{id}$, it suffices to check $(k + c \cdot id)m \in M$. But

$$(k + c \cdot id)m = km + c \cdot id \cdot m = km + m \cdot c \cdot id \in M,$$

because $c \cdot id$ commutes with everything and because M is an $\mathrm{End}V$-module from the right. $\square$

We are now ready to construct a $(\mathrm{End}_H V, \mathrm{End}V)$-sub-bimodule M of $\mathrm{End}V$ with $f \subsetneq M \subsetneq \mathrm{End}V$. Take the cofilter $F_{fd} \subset \mathrm{Gr}V$ of all finite-dimensional subspaces of V and adjoin to it any infinite-dimensional and infinite-codimensional subspace $w_1 \in \mathrm{Gr}V$. That is, take the cofilter (F_{fd}, w_1) generated by F_{fd} and w_1, and call it F_1.

Proposition 4.9. *F_1 is strictly between F_{fd} and $\mathrm{Gr}V$.*

Proof. The right End$V$-submodule of End$V$ corresponding to F_{fd} is f. Let the right End$V$-submodule corresponding to (w_1) (which is the cofilter of subspaces of w_1) be f_1. Since the lattice of cofilters is isomorphic to the lattice of right End$V$-submodules, the join (F_{fd}, w_1) corresponds to $f + f_1$. Because all elements of f have finite rank and all elements of f_1 have infinite corank, $f + f_1$ is not End$V$. Therefore, (F_{fd}, w_1) is strictly between F_{fd} and GrV. □

Take $M := \{m \in \text{End}V \mid \text{im}(m) \in F_1\}$. By lemma 2.6, M is a right End$V$-submodule. It is also a left $\text{End}_{V^* \to H}V$-submodule, because every element k of $\text{End}_{V^* \to H}V = V \otimes H$ has finite image, and so any composition km has finite image, which is included in F_1. By lemma 4.8, M is a left $\text{End}_H V$-submodule. Finally, by proposition 4.9, M is strictly between f and End$V$.

It is not hard to show that by choosing different $w_1 \in \text{Gr}V$ we can achieve uncountably many distinct linear cofilters F_1. We now sketch a proof of this. Let $B \subset V$ be any basis, so that B is a countable set. Define an equivalence relation on the power set $\mathcal{P}(B)$ of B by saying that two subsets B_1, B_2 of B are equivalent if their symmetric difference $B_1 \Delta B_2$ is finite. Each equivalence class is then countable, because, to obtain a subset equivalent to B_1, one has to choose a finite subset outside of B_1 and a finite subset inside of B_1. Since the set $\mathcal{P}(B)$ in uncountable, there are uncountably many equivalence classes. Choose their representatives $\{B_i\}_{i \in I}$, so that the symmetric difference of any pair of them is countable. Some of these B_is might be finite or cofinite in B – we ignore such. Thus we ignore countably many among $\{B_i\}_{i \in I}$, and the rest is still an uncountable family $\{B_j\}_{j \in J}$. Consider the family of subspaces $\{\text{span}B_j\}_{j \in J}$ of V^*. Adjoining F_{fd} to each of the linear cofilters $\{(\text{span}B_j)\}_{j \in J}$, we obtain an uncountable family of distinct linear cofilters $\{F_j\}_{j \in J}$. Hence, $\text{End}_{(V^* \oplus H)}(V \oplus V)$ has uncountably many distinct ideals, which is the first part of theorem 4.7.

For the second part of theorem 4.7, take an infinite chain of subspaces $w_1 \subset w_2 \subset \dots$ such that each of them is infinite codimensional in its successor, obtaining an infinite chain of linear cofilters $F_1 \subset F_2 (= (F_{fd}, w_2)) \subset \dots$. Again, because of infinite codimensionality, the inclusions in the chain of linear cofilters are strict. Hence, $\text{End}_{(V^* \oplus H)}(V \oplus V)$ has infinite length.

□

4.4 The ideals of the Mackey algebra $\mathrm{End}_{(V^*\oplus V_*)}(V\oplus V)$

In this subsection we characterize the ideals of the Mackey algebra $\mathrm{End}_{(V^*\oplus V_*)}(V\oplus V)$. We start by considering the matrix form of $\mathrm{End}_{(V^*\oplus V_*)}(V\oplus V)$:

$$\mathrm{End}_{(V^*\oplus V_*)}(V\oplus V) = \frac{\mathrm{End}_{V^*\to V^*}V \;\Big|\; \mathrm{End}_{V^*\to V_*}V}{\mathrm{End}_{V_*\to V^*}V \;\Big|\; \mathrm{End}_{V_*\to V_*}V} = \frac{\mathrm{End}V \;\Big|\; V\otimes V_*}{\mathrm{End}V \;\Big|\; \mathrm{End}_{V_*}V},$$

where $\mathrm{End}_{V^*\to V_*}V = V\otimes V_*$ by proposition 4.2.

Let $f\subset \mathrm{End}V$ be the ideal of endomorphisms of finite rank. Consider the following chain of ideals in $\mathrm{End}_{(V^*\oplus V_*)}(V\oplus V)$:

$$(0)\subset \frac{f \;\Big|\; V\otimes V_*}{f \;\Big|\; V\otimes V_*} \subset \frac{f \;\Big|\; V\otimes V_*}{\mathrm{End}V \;\Big|\; V\otimes V_*} \subset \frac{\mathrm{End}V \;\Big|\; V\otimes V_*}{\mathrm{End}V \;\Big|\; V\otimes V_*} \subset \frac{\mathrm{End}V \;\Big|\; V\otimes V_*}{\mathrm{End}V \;\Big|\; \mathrm{End}_{V_*}V}.$$

We enumerate these ideals as 0, 1, 2, 3, and 4. Note that 1 is just the ideal $(V\oplus V)\otimes(V^*\oplus V_*)$, and 3 is just the ideal $\mathrm{End}_{(V^*\oplus V^*)\to(V^*\oplus V_*)}(V\oplus V)$. Notice also that the chain above is almost the same chain as we had for $\mathrm{End}_{(V^*\oplus H)}(V\oplus V)$ in the previous section. However, the overall situation with ideals is different from that of the previous section, as the following theorem suggests.

Theorem 4.10 ([C])**.** *The quotient $2/1$ of the ideals is simple, meaning that there is no subspace $f\subsetneq M\subsetneq \mathrm{End}V$ for which*

$$\frac{f \;\Big|\; V\otimes V_*}{M \;\Big|\; V\otimes V_*} \textit{ is an ideal in } \frac{\mathrm{End}V \;\Big|\; V\otimes V_*}{\mathrm{End}V \;\Big|\; \mathrm{End}_{V_*}V}.$$

Proof. As we have seen earlier, this amounts to showing that there is no $(\mathrm{End}_{V_*}V,\ \mathrm{End}V)$-sub-bimodule M of $\mathrm{End}V$ strictly between f and $\mathrm{End}V$. This is shown by A. Chirvasitu in [C]. $\square$

We draw attention to yet another ideal of $\mathrm{End}_{(V^*\oplus V_*)}(V\oplus V)$ that could be placed in the chain instead of the ideal 3:

$$\frac{f \;\Big|\; V\otimes V_*}{\mathrm{End}V \;\Big|\; \mathrm{End}_{V_*}V}.$$

We name it $3'$.

Theorem 4.11. *All nonzero ideals of* $\mathrm{End}_{(V^*\oplus V_*)}(V\oplus V)$ *are* $1, 2, 3, 3'$, *and* 4. *The length of* $\mathrm{End}_{(V^*\oplus V_*)}(V\oplus V)$ *is thus* 4.

Proof. We first need a lemma.

Lemma 4.12. *Let* $\mathrm{End}_{(W_1\oplus W_2)}(V\oplus V)$ *be a Mackey algebra and let*

$$\begin{array}{c|c} \mathrm{End}_{W_1}V & \mathrm{End}_{W_1\to W_2}V \\ \hline \mathrm{End}_{W_2\to W_1}V & \mathrm{End}_{W_2}V \end{array}$$

be its block matrix form. Then every ideal I *of* $\mathrm{End}_{(W_1\oplus W_2)}(V\oplus V)$ *has the form*

$$\begin{array}{c|c} A & B \\ \hline C & D \end{array} := \left\{ \begin{array}{c|c} a & b \\ \hline c & d \end{array} \mid a\in A, b\in B, c\in C, d\in D \right\},$$

where A, B, C, D *are some vector subspaces of* $\mathrm{End}_{W_1}V$, $\mathrm{End}_{W_1\to W_2}V$, $\mathrm{End}_{W_2\to W_1}V$, $\mathrm{End}_{W_2}V$, *respectively.*

Proof. Let

$$\begin{array}{c|c} a & b \\ \hline c & d \end{array}$$

be and element of I. Because I is a two-sided ideal in $\mathrm{End}_{(W_1\oplus W_2)}(V\oplus V)$, the following four elements also belong to I:

$$\begin{array}{c|c} a & 0 \\ \hline 0 & 0 \end{array} = \begin{array}{c|c} \mathrm{id}_V & 0 \\ \hline 0 & 0 \end{array} \cdot \begin{array}{c|c} a & b \\ \hline c & d \end{array} \cdot \begin{array}{c|c} \mathrm{id}_V & 0 \\ \hline 0 & 0 \end{array}, \qquad \begin{array}{c|c} 0 & b \\ \hline 0 & 0 \end{array} = \begin{array}{c|c} \mathrm{id}_V & 0 \\ \hline 0 & 0 \end{array} \cdot \begin{array}{c|c} a & b \\ \hline c & d \end{array} \cdot \begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathrm{id}_V \end{array},$$

$$\begin{array}{c|c} 0 & 0 \\ \hline c & 0 \end{array} = \begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathrm{id}_V \end{array} \cdot \begin{array}{c|c} a & b \\ \hline c & d \end{array} \cdot \begin{array}{c|c} \mathrm{id}_V & 0 \\ \hline 0 & 0 \end{array}, \qquad \begin{array}{c|c} 0 & 0 \\ \hline 0 & d \end{array} = \begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathrm{id}_V \end{array} \cdot \begin{array}{c|c} a & b \\ \hline c & d \end{array} \cdot \begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathrm{id}_V \end{array}.$$

The claim easily follows. □

We start the proof of theorem 4.11 by showing that the chain

$$(0) \subset \begin{array}{c|c} f & V\otimes V_* \\ \hline f & V\otimes V_* \end{array} \subset \begin{array}{c|c} f & V\otimes V_* \\ \hline \mathrm{End}V & V\otimes V_* \end{array} \subset \begin{array}{c|c} \mathrm{End}V & V\otimes V_* \\ \hline \mathrm{End}V & V\otimes V_* \end{array} \subset \begin{array}{c|c} \mathrm{End}V & V\otimes V_* \\ \hline \mathrm{End}V & \mathrm{End}_{V_*}V \end{array}.$$

is a composition series, i.e., that the quotients $1/0, 2/1, 3/2, 4/3$ are simple. By proposition 4.1 and theorem 4.10, the quotients $1/0$ and $2/1$ are simple.

Assume there is a two-sided ideal I between the ideals 2 and 3, so it has the form

$$\begin{array}{c|c} M & V \otimes V_* \\ \hline \mathrm{End}V & V \otimes V_* \end{array}$$

for some vector subspace M between f and End$V$. Multiplication by $\mathrm{End}_{(V^* \oplus V_*)}(V \oplus V)$ from both sides

$$\begin{array}{c|c} \mathrm{End}V & V \otimes V_* \\ \hline \mathrm{End}V & \mathrm{End}_{V_*}V \end{array} \cdot \begin{array}{c|c} M & V \otimes V_* \\ \hline \mathrm{End}V & V \otimes V_* \end{array} \cdot \begin{array}{c|c} \mathrm{End}V & V \otimes V_* \\ \hline \mathrm{End}V & \mathrm{End}_{V_*}V \end{array}$$

shows that then M is a (End$V$, End$V$)-sub-bimodule of End$V$, and hence M is either f or End$V$ (here we use the assumption $\mathrm{dim}V = \aleph_0$). Therefore, the quotient $3/2$ is simple.

Assume there is a two-sided ideal I between the ideals 3 and 4, so it has the form

$$\begin{array}{c|c} \mathrm{End}V & V \otimes V_* \\ \hline \mathrm{End}V & M \end{array}$$

for some vector subspace M between $V \otimes V_*$ and $\mathrm{End}_{V_*}V$. Using the same strategy, we deduce that M is a ($\mathrm{End}_{V_*}V$, $\mathrm{End}_{V_*}V$)-sub-bimodule of $\mathrm{End}_{V_*}V$, and hence M is either $V \otimes V_*$ or $\mathrm{End}_{V_*}V$. Therefore, the quotient $4/3$ is simple.

As a result, the chain is a composition series, and so the length of $\mathrm{End}_{(V^* \oplus V_*)}(V \oplus V)$ is 4.

We are left showing that there is no other ideal except of $0, 1, 2, 3, 3', 4$. Keeping lemma 4.12 in mind, let

$$I = \begin{array}{c|c} A & B \\ \hline C & D \end{array} = \begin{array}{c|c} A & V \otimes V_* \\ \hline C & D \end{array}$$

be a two-sided ideal in $\mathrm{End}_{(V^* \oplus V_*)}(V \oplus V)$. Then there are five cases:

- case 0: I equals 0.
- case 1: I contains 1 but not 2;
- case 2: I contains 2 but not 3;
- case 3: I contains 3 but not 4;
- case 4: I equals 4.

CASE 1: Consider the ideal $I \cap 2$. Since the quotient $2/1$ is simple, the ideal $I \cap 2$ equals 1, which implies $C = f$. Multiplication by $\mathrm{End}_{(V^* \oplus V_*)}(V \oplus V)$ from both sides

$$\frac{\mathrm{End}V \;\big|\; V \otimes V_*}{\mathrm{End}V \;\big|\; \mathrm{End}_{V_*}V} \cdot \frac{A \;\big|\; V \otimes V_*}{f \;\big|\; D} \cdot \frac{\mathrm{End}V \;\big|\; V \otimes V_*}{\mathrm{End}V \;\big|\; \mathrm{End}_{V_*}V}$$

shows that A is a $(\mathrm{End}V, \mathrm{End}V)$-sub-bimodule of $\mathrm{End}V$, and that D is a $(\mathrm{End}_{V_*}V, \mathrm{End}_{V_*}V)$-sub-bimodule of $\mathrm{End}_{V_*}V$. Hence, $A = f$ or $\mathrm{End}V$, and $D = V \otimes V_*$ or $\mathrm{End}_{V_*}V$. It is easy to check that only $A = f$ and $D = V \otimes V_*$ forms an ideal. We conclude that $I = 1$.

CASE 2: Consider the ideal $I \cap 3$. Since the quotient $3/2$ is simple, the ideal $I \cap 3$ equals 2, which implies $C = f$ and $A = \mathrm{End}V$. Multiplication by $\mathrm{End}_{(V^* \oplus V_*)}(V \oplus V)$ from both sides

$$\frac{\mathrm{End}V \;\big|\; V \otimes V_*}{\mathrm{End}V \;\big|\; \mathrm{End}_{V_*}V} \cdot \frac{f \;\big|\; V \otimes V_*}{\mathrm{End}V \;\big|\; D} \cdot \frac{\mathrm{End}V \;\big|\; V \otimes V_*}{\mathrm{End}V \;\big|\; \mathrm{End}_{V_*}V}$$

shows that D is a $(\mathrm{End}_{V_*}V, \mathrm{End}_{V_*}V)$-sub-bimodule of $\mathrm{End}_{V_*}V$. Hence, $D = V \otimes V_*$ or $\mathrm{End}_{V_*}V$. We conclude that $I = 2$ or $I = 3'$.

CASE 3: Consider the ideal $I \cap 4$. Since the quotient $4/3$ is simple, the ideal $I \cap 4$ equals 3, which implies $I = 3$.

Therefore, $I = 0, 1, 2, 3, 3'$, or 4. □

Theorem 4.13. *The Mackey Lie algebra* $\mathfrak{gl}^M((V \oplus V), (V^* \oplus V_*))$ *has length* 7.

Proof. The two-sided ideals $1, 2, 3, 3'$, and 4 are also Lie ideals of our Lie algebra, and we denote them by $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_{3'}$, and $\mathfrak{g}_4 (= \mathfrak{g})$, respectively. Let $\mathfrak{sl}$ denote the ideal of traceless elements of $\mathfrak{g}$. In addition, consider the following two Lie ideals:

$$\mathfrak{g}_3^{sc} := \frac{\mathrm{End}V \;\big|\; V \otimes V_*}{\mathrm{End}V \;\big|\; V \otimes V_* \oplus \mathbb{C}\mathrm{id}} = \mathfrak{g}_3 \oplus \mathbb{C}\mathrm{id}, \qquad \mathfrak{g}_{3'}^{sc} := \frac{f \oplus \mathbb{C}\mathrm{id} \;\big|\; V \otimes V_*}{\mathrm{End}V \;\big|\; \mathrm{End}_{V_*}V} = \mathfrak{g}_{3'} \oplus \mathbb{C}\mathrm{id},$$

where "sc" stands for semi-complex. Consider the following chain of ideals:

$$(0) \subset \mathfrak{sl} \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathfrak{g}_3^{sc} \cap \mathfrak{g}_{3'} \subset \mathfrak{g}_3 \subset \mathfrak{g}_3^{sc} \subset \mathfrak{g}_4 = \mathfrak{g}.$$

The proof that the above chain is a composition series is similar to the proof of the respective part of theorem 4.11. The length of the Lie algebra $\mathfrak{gl}^M((V \oplus V), (V^* \oplus V_*))$ is thus 7.

□

5 Outlook

As we could see, there are many Mackey algebras $\mathrm{End}_W V$ whose ideals it would be interesting to determine, even for the case of a countable-dimensional vector space V. At this point, we do not know much about the ideals in several examples mentioned in subsection 3.1 that were not considered later on in this work.

Knowing the ideals of two Mackey algebras $\mathrm{End}_{W_1} V_1$ and $\mathrm{End}_{W_2} V_2$, we still do not know a general procedure characterizing the ideals of the mixed Mackey algebra $\mathrm{End}_{(W_1 \oplus W_2)}(V_1 \oplus V_2)$. Theorems 4.7 and 4.11 suggest that the length of a mixed Mackey algebra behaves rather unpredictably. It would be quite interesting if, under some additional conditions, the length of a mixed Mackey algebra could be deduced from the lengths of the constituent Mackey algebras.

There are many further questions, in particular the following. What Mackey Lie algebras, apart from $\mathfrak{gl}^M(V, H)$ for a Mackey-minimal hyperplane H, have length 3? Can one characterize all Mackey Lie algebras of length 4? What are the possible lengths of Mackey Lie algebras $\mathfrak{gl}^M(V, H)$ for a hyperplane H? For what Mackey Lie algebras is the ideal $\mathbb{C}\mathrm{id}$ a direct summand?

References

[BH] G. Bergman, E. Hrushovski, "Linear ultrafilters", Communications in Algebra 26 (12) (1998), 4079–4113.

[BHO] O. Bezushchak, W. Hołubowski, B. Oliynyk, "Ideals of general linear Lie algebras of infinite-dimensional vector spaces", Proceedings of the American Mathematical Society 151 (2023), 467-473.

[C] A. Chirvasitu, unpublished manuscript (2023).

[CP] A. Chirvasitu, I. Penkov, "Universal tensor categories generated by dual pairs", Applied Categorical Structures 29(5) (2021), 915-950.

[J] N. Jacobson, "Lectures in abstract algebra", Graduate Texts in Mathematics, Vol.2. Linear algebra, Springer-Verlag, Berlin-Heidelberg-New York, (1975).

[M] G. Mackey, "On infinite dimensional linear spaces", Trans. AMS 57 (1945), 155-207.

[PS] I. Penkov, V. Serganova, "Tensor representations of Mackey Lie algebras and their dense subalgebras", Developments and Retrospectives in Lie Theory: Algebraic Methods, Developments in Mathematics, vol. 38, Springer Verlag, 291-330.

[PT] I. Penkov, V. Tsanov, unpublished manuscript (2023).

[S] I. Stewart, "The Lie algebra of endomorphisms of an infinite-dimensional vector space", Compositio Mathematica 25, no 1 (1972), 79-86.

[Z] M. Zhang, "Lie algebras of linear systems and their automorphisms", bachelor's thesis at Jacobs University Bremen (2014), http://math.jacobs-university.de/penkov/papers/mzhangbsc.pdf.