A Koszul category of representations of finitary Lie algebras

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Abstract

We find for each simple finitary Lie algebra $\mathfrak{g}$ a category $\mathbb{T}_{\mathfrak{g}}$ of integrable modules in which the tensor product of copies of the natural and conatural modules are injective. The objects in $\mathbb{T}_{\mathfrak{g}}$ can be defined as the finite length absolute weight modules, where by absolute weight module we mean a module which is a weight module for every splitting Cartan subalgebra of $\mathfrak{g}$. The category $\mathbb{T}_{\mathfrak{g}}$ is Koszul in the sense that it is antiequivalent to the category of locally unitary finite-dimensional modules over a certain direct limit of finite-dimensional Koszul algebras. We describe these finite-dimensional algebras explicitly. We also prove an equivalence of the categories $\mathbb{T}_{o(\infty)}$ and $\mathbb{T}_{sp(\infty)}$ corresponding respectively to the orthogonal and symplectic finitary Lie algebras $o(\infty)$, $sp(\infty)$.

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1. Introduction

The classical simple complex Lie algebras $sl(n)$, $o(n)$, $sp(2n)$ have several natural infinite-dimensional versions. In this paper we concentrate on the "smallest possible" such versions: the direct limit Lie algebras $sl(\infty) := \varinjlim (sl(n))_{n\in\mathbb{Z}_{>2}}$, $o(\infty) := \varinjlim (o(n))_{n\in\mathbb{Z}_{\geq 3}}$, $sp(\infty) := \varinjlim (sp(2n))_{n\in\mathbb{Z}_{\geq 2}}$. From a traditional finite-dimensional point of view, these Lie algebras are a suitable language for various stabilization phenomena, for instance stable branching laws as studied by R. Howe, E.-C. Tan and J. Willenbring [**HTW**]. The direct limit Lie algebras $sl(\infty)$, $o(\infty)$, $sp(\infty)$ admit many characterizations: for instance, they represent (up to isomorphism) the simple finitary (locally finite) complex Lie algebras [**B**, **BSt**]. Alternatively, these Lie algebras are the only three locally simple locally finite complex Lie algebras which admit a root decomposition [**PStr**].

Several attempts have been made to build a basic representation theory for $\mathfrak{g} = sl(\infty)$, $o(\infty)$, $sp(\infty)$. As the only simple finite-dimensional representation of $\mathfrak{g}$ is the trivial one, one has to study infinite-dimensional representations. On the other hand, it is still possible to study representations which are close analogs of finite-dimensional representations. Such a representation should certainly be integrable, i.e. it should be isomorphic to a direct sum of finite-dimensional representations when restricted to any simple finite-dimensional subalgebra.

The first phenomenon one encounters when studying integrable representations of $\mathfrak{g}$ is that they are not in general semisimple. This phenomenon has been studied in [**PStyr**] and [**PS**], but it had not previously been placed within a known more general framework for non-semisimple categories. The main purpose of the present paper is to show that the notion of Koszulity for a category of modules over a graded ring, as defined by A. Beilinson, V. Ginzburg and W. Soergel in [**BGS**], provides an excellent tool for the study of integrable representations of $\mathfrak{g} = sl(\infty)$, $o(\infty)$, $sp(\infty)$.

In this paper we introduce the category $\mathbb{T}_\mathfrak{g}$ of tensor $\mathfrak{g}$-modules. The objects of $\mathbb{T}_\mathfrak{g}$ are defined at first by the equivalent abstract conditions of Theorem 3.4. Later we show in Corollary 4.6 that the objects of $\mathbb{T}_\mathfrak{g}$ are nothing but finite length submodules of a direct sum of several copies of the tensor algebra T of the natural and conatural representations. In the finite-dimensional case, i.e. for $sl(n)$, $o(n)$, or $sp(2n)$, the appropriate tensor algebra is a cornerstone of the theory of finite-dimensional representations (Schur-Weyl duality, etc.). In the infinite-dimensional case, the tensor algebra T was studied by Penkov and K. Styrkas in [**PStyr**]; nevertheless its indecomposable direct summands were not understood until now from a categorical point of view.

We prove that these indecomposable modules are precisely the indecomposable injectives in the category $\mathbb{T}_{\mathfrak{g}}$. Furthermore, the category $\mathbb{T}_{\mathfrak{g}}$ is Koszul in the following sense: $\mathbb{T}_{\mathfrak{g}}$ is antiequivalent to the category of locally unitary finite-dimensional modules over an algebra $\mathcal{A}_{\mathfrak{g}}$ which is a direct limit of finite-dimensional Koszul algebras (see Proposition 5.1 and Theorem 5.5).

Moreover, we prove in Corollary 6.4 that for $\mathfrak{g} = sl(\infty)$ the Koszul dual algebra $(\mathcal{A}_{\mathfrak{g}}^{!})^{\mathrm{opp}}$ is isomorphic to $\mathcal{A}_{\mathfrak{g}}$. This together with the main result of [**PStyr**] allows us to give an explicit formula for the Ext group between any two simple objects of $\mathbb{T}_{\mathfrak{g}}$ when $\mathfrak{g} = sl(\infty)$. For the cases of $\mathfrak{g} = o(\infty)$ and $\mathfrak{g} = sp(\infty)$ we discover another interesting fact: the algebras $\mathcal{A}_{o(\infty)}$ and $\mathcal{A}_{sp(\infty)}$ are isomorphic. This yields an equivalence of categories $\mathbb{T}_{o(\infty)} \simeq \mathbb{T}_{sp(\infty)}$, which is Corollary 6.11.

In summary, the results of the present paper show how the non-semisimplicity of tensor modules arising from the limit process $n \to \infty$ falls strikingly into the general Koszul pattern discovered by Beilinson, Ginzburg and Soergel. This enables us to uncover the structure of the category of tensor representations of $\mathfrak{g}$.

Since the present paper has been submitted there have been several developments. First, in [**SS**] the categories $\mathbb{T}_{\mathfrak{g}}$ have been studied from a different perspective. In particular, it is shown there that these categories satisfy important universality properties in the class of abelian symmetric monoidal categories.

In [**PS1**] categories of tensor modules have been introduced for a larger class of infinite-dimensional Lie algebras, and it has been shown that these categories are equivalent to $\mathbb{T}_{\mathfrak{g}}$ for appropriate $\mathfrak{g}$. In [**Sr**] results from the present paper are generalized to the case of classical Lie superalgebras.

Finally, in [**FPS**] the category $\mathbb{T}_{sl(\infty)}$ has been used to categorify the boson-fermion correspondence.

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2. Preliminaries

The ground field is $\mathbb{C}$. By S_n we denote the n-th symmetric group, and by $\mathbb{C}[S_n]$ its group algebra. The sign $\otimes$ stands for $\otimes_{\mathbb{C}}$, and the sign $\niplus$ stands for the semidirect sum of Lie algebras. We denote by $(\,\cdot\,)^*$ the algebraic dual, i.e. $\mathrm{Hom}_{\mathbb{C}}(\,\cdot\,,\mathbb{C})$.

Let $\mathfrak{g}$ be one of the infinite-dimensional simple finitary Lie algebras, $sl(\infty)$, $o(\infty)$, or $sp(\infty)$. Here $sl(\infty) = \varinjlim sl(n)$, $o(\infty) = \varinjlim o(n)$, $sp(\infty) = \varinjlim sp(2n)$, where in each direct limit the inclusions can be chosen as "left upper corner" inclusions. We consider the "exhaustion" $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ to be fixed, taking $\mathfrak{g}_n = sl(n)$ for $\mathfrak{g} = sl(\infty)$, $\mathfrak{g}_n = o(2n)$ or $\mathfrak{g}_n = o(2n+1)$ for $\mathfrak{g} = o(\infty)$, and $\mathfrak{g}_n = sp(2n)$ for $\mathfrak{g} = sp(\infty)$. By G_n we denote the adjoint group of $\mathfrak{g}_n$ It is clear that $\{G_n\}$ forms a direct system and defines an ind-group $G = \varinjlim G_n$. As mentioned in the introduction, the Lie algebras $sl(\infty)$, $o(\infty)$, and $sp(\infty)$ admit several equivalent intrinsic descriptions, see for instance [**B**, **BSt**, **PStr**].

It is clear from the definition of $\mathfrak{g} = sl(\infty)$, $o(\infty)$, $sp(\infty)$ that the notions of semisimple or nilpotent elements make sense: an element $g \in \mathfrak{g}$ is *semisimple* (respectively, *nilpotent*) if g is semisimple (resp., nilpotent) as an element of $\mathfrak{g}_n$ for some n. In [**NP**, **DPS**], Cartan subalgebras of $\mathfrak{g}$ have been studied. In the present paper we need only the notion of a *splitting Cartan subalgebra* of $\mathfrak{g}$: this is a maximal toral (where *toral* means consisting of semisimple elements) subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g}$ is an $\mathfrak{h}$-weight module, i.e.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha = \{g \in \mathfrak{g} \mid [h,g] = \alpha(h)g \text{ for all } h \in \mathfrak{h}\}$. The set $\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}^\alpha \neq 0\}$ is the *set of* $\mathfrak{h}$*-roots* of $\mathfrak{g}$. More generally, if $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$ and M is a $\mathfrak{g}$-module, M is an $\mathfrak{h}$*-weight module* if

$$M = \bigoplus_{\alpha \in \mathfrak{h}^*} M^\alpha,$$

where $M^\alpha := \{m \in M \mid h \cdot m = \alpha(h)m \text{ for all } h \in \mathfrak{h}\}$.

By V we denote the natural representation of $\mathfrak{g}$; that is, $V = \varinjlim V_n$, where V_n is the natural representation of $\mathfrak{g}_n$. We set also $V_* = \varinjlim V_n^*$; this is the conatural representation of $\mathfrak{g}$. For $\mathfrak{g} = o(\infty)$, $sp(\infty)$, $V \simeq V_*$, whereas $V \not\simeq V_*$ for $\mathfrak{g} = sl(\infty)$. Note that V_* is a submodule of the algebraic dual

$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ of V. Moreover, $\mathfrak{g} \subset V \otimes V_*$, and $sl(\infty)$ can be identified with the kernel of the contraction $\phi : V \otimes V_* \to \mathbb{C}$, while

$$\mathfrak{g} \simeq \Lambda^2(V) \subset V \otimes V = V \otimes V_* \qquad \text{for } \mathfrak{g} = o(\infty),$$
$$\mathfrak{g} \simeq S^2(V) \subset V \otimes V = V \otimes V_* \qquad \text{for } \mathfrak{g} = sp(\infty).$$

Let $\tilde{G}$ be the subgroup of $\mathrm{Aut}\, V$ consisting of those automorphisms for which the induced automorphism of V^* restricts to an automorphism of V_*. Then clearly $G \subset \tilde{G} \subset \mathrm{Aut}\, \mathfrak{g}$, and moreover $\tilde{G} = \mathrm{Aut}\, \mathfrak{g}$ for $\mathfrak{g} = o(\infty)$, $sp(\infty)$ [**BBCM**, Corollary 1.6 (b)]. For $\mathfrak{g} = sl(\infty)$, the group $\tilde{G}$ has index 2 in $\mathrm{Aut}\, \mathfrak{g}$: the quotient $\mathrm{Aut}\, \mathfrak{g}/\tilde{G}$ is represented by the automorphism

$$g \mapsto -g^t$$

for $g \in sl(\infty)$ [**BBCM**, Corollary 1.2 (a)].

It is essential to recall that if $\mathfrak{g} = sl(\infty)$, $sp(\infty)$, all splitting Cartan subalgebras of $\mathfrak{g}$ are $\tilde{G}$-conjugate, while there are two $\tilde{G}$-conjugacy classes for $\mathfrak{g} = o(\infty)$. One class comes from the exhaustion of $o(\infty)$ as $\varinjlim o(2n)$, and the other from the exhaustion of the form $\varinjlim o(2n+1)$. For further details we refer the reader to [**DPS**]. Here are the explicit forms of the root systems of $\mathfrak{g}$:

$$\{\epsilon_i - \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0}\} \qquad \text{for } \mathfrak{g} = sl(\infty),\ \mathfrak{g}_n = sl(n),$$
$$\{\pm\epsilon_i \pm \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0}\} \cup \{\pm 2\epsilon_i \mid i \in \mathbb{Z}_{>0}\} \qquad \text{for } \mathfrak{g} = sp(\infty),\ \mathfrak{g}_n = sp(2n),$$
$$\{\pm\epsilon_i \pm \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0}\} \qquad \text{for } \mathfrak{g} = o(\infty),\ \mathfrak{g}_n = o(2n),$$
$$\{\pm\epsilon_i \pm \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0}\} \cup \{\pm\epsilon_i \mid i \in \mathbb{Z}_{>0}\} \qquad \text{for } \mathfrak{g} = o(\infty),\ \mathfrak{g}_n = o(2n+1).$$

Our usage of $\epsilon_i \in \mathfrak{h}^*$ is compatible with the standard usage of ϵ_i as a linear function on $\mathfrak{h} \cap \mathfrak{g}_n$ for all $n > i$.

In the present paper we study integrable $\mathfrak{g}$-modules M for $\mathfrak{g} \simeq sl(\infty)$, $o(\infty)$, $sp(\infty)$. By definition, a $\mathfrak{g}$-module M is *integrable* if $\dim\{m, g \cdot m, g^2 \cdot m, ...\} < \infty$ for all $g \in \mathfrak{g}$, $m \in M$. More generally, if M is any $\mathfrak{g}$-module, the set $\mathfrak{g}[M]$ of M-locally finite elements in $\mathfrak{g}$, that is

$$\mathfrak{g}[M] := \{g \in \mathfrak{g} \mid \dim\{m, g \cdot m, g^2 \cdot m, \ldots\} < \infty \text{ for all } m \in M\},$$

is a Lie subalgebra of $\mathfrak{g}$. This follows from the analogous fact for finite-dimensional Lie algebras, discovered and rediscovered by several mathematicians [**GQS**, **F**, **K**]. We refer to $\mathfrak{g}[M]$ as the *Fernando-Kac subalgebra* of M.

By $\mathfrak{g}$-mod we denote the category of all $\mathfrak{g}$-modules, and following the notation of [**PS**], we let $\mathrm{Int}_{\mathfrak{g}}$ denote the category of integrable $\mathfrak{g}$-modules. We have the functor

$$\Gamma_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \to \mathrm{Int}_{\mathfrak{g}}$$

which takes an arbitrary $\mathfrak{g}$-module to its largest integrable submodule.

3. The category $\mathbb{T}_{\mathfrak{g}}$

If $\gamma \in \mathrm{Aut}\,\mathfrak{g}$ and M is a $\mathfrak{g}$-module, let M^γ denote the $\mathfrak{g}$-module twisted by γ: that is, M^γ is equal to M as a vector space, and the $\mathfrak{g}$-module structure on M^γ is given by $\gamma(g) \cdot m$ for $m \in M^\gamma$ and $g \in \mathfrak{g}$.

Definition 3.1.

1. A $\mathfrak{g}$-module M is called an *absolute weight module* if M is an $\mathfrak{h}$-weight module for every splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.
2. A $\mathfrak{g}$-module M is called *$\tilde{G}$-invariant* if for any $\gamma \in \tilde{G}$ there is a $\mathfrak{g}$-isomorphism $M^\gamma \simeq M$.
3. A subalgebra of $\mathfrak{g}$ is called *finite corank* if it contains the commutator subalgebra of the centralizer of some finite-dimensional subalgebra of $\mathfrak{g}$.

Proposition 3.2. *Any absolute weight $\mathfrak{g}$-module is integrable.*

Proof. Let M be an absolute weight $\mathfrak{g}$-module. Every semisimple element h of $\mathfrak{g}$ lies in some splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and since M is an $\mathfrak{h}$-weight module, we see that h acts locally finitely on M. As $\mathfrak{g}$ is generated by its semisimple elements, the Fernando-Kac subalgebra $\mathfrak{g}[M]$ equals $\mathfrak{g}$, i.e. M is integrable. □

We define the category of absolute weight modules as the full subcategory of $\mathfrak{g}$-mod whose objects are the absolute weight modules. Proposition 3.2 shows that the category of absolute weight modules is in fact a subcategory of $\mathrm{Int}_{\mathfrak{g}}$.

Lemma 3.3. *For each n one has $\tilde{G} = G \cdot \tilde{G}'_n$, where*

$$\tilde{G}'_n := \{\gamma \in \tilde{G} \mid \gamma(g) = g \text{ for all } g \in \mathfrak{g}_n\}.$$

Proof. Let $\mathfrak{g}$ be $o(\infty)$ or $sp(\infty)$, and let $\gamma \in \tilde{G}$. Fix a basis $\{w_i\}$ of V_n. There exists $\gamma'' \in G$ such that $(\gamma'')^{-1}(\gamma(w_i)) = w_i$ for all $1 \leq i \leq 2n$. Since $\mathfrak{g} \subset V \otimes V$ and $\mathfrak{g}_n = \mathfrak{g} \cap (V_n \otimes V_n)$, we see that $(\gamma'')^{-1}\gamma \in \tilde{G}'_n$.

For $\mathfrak{g} = sl(\infty)$, the analogous statement is as follows. In this case one has $\mathfrak{g}_n = \mathfrak{g} \cap (V_n \otimes V_n^*)$. Fix dual bases $\{w_i\}$ and $\{w_i^*\}$ of V_n and V_n^*, respectively. Then for any $\gamma \in \tilde{G}$, there is an element $\gamma'' \in G$ such that $(\gamma'')^{-1}(\gamma(w_i)) = w_i$ and $(\gamma'')^{-1}(\gamma(w_i^*)) = w_i^*$ for each $1 \leq i \leq n$. Therefore $(\gamma'')^{-1}\gamma \in \tilde{G}'_n$. □

Theorem 3.4. *The following conditions on a $\mathfrak{g}$-module M of finite length are equivalent:*

1. *M is an absolute weight module.*
2. *M is a weight module for some splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and M is $\tilde{G}$-invariant.*
3. *M is integrable and $\operatorname{Ann}_{\mathfrak{g}} m$ is finite corank for all $m \in M$.*

Proof. Let us show that (1) implies (3). We already proved in Proposition 3.2 that a $\mathfrak{g}$-module M satisfying (1) is integrable. Furthermore, it suffices to prove that $\operatorname{Ann}_{\mathfrak{g}} m$ is finite corank for all $m \in M$ under the assumption that the $\mathfrak{g}$-module M is simple. This follows from the observation that a finite intersection of finite corank subalgebras is finite corank.

Fix a splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathfrak{g}_n$ is a Cartan subalgebra of $\mathfrak{g}_n$; let $\mathfrak{b} = \mathfrak{h} \supsetplus \mathfrak{n}$ be a Borel subalgebra of $\mathfrak{g}$ whose set of roots (i.e. positive roots) is denoted by Δ^+. For each positive root α, let e_α, h_α, f_α be a standard basis for the corresponding root $sl(2)$-subalgebra. Fix additionally a nonzero $\mathfrak{h}$-weight vector $m \in M$.

Choose a set of commuting mutually orthogonal positive roots $Y \subset \Delta^+$. The set of semisimple elements $\{h_\alpha + e_\alpha \mid \alpha \in Y\}$ is $\tilde{G}$-conjugate to the set $\{h_\alpha \mid \alpha \in Y\}$, and can thus be extended to a splitting Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$. Since M is an absolute weight module, there is a nonzero $\mathfrak{h}'$-weight vector $m' \in M$. As M is simple, it must be that $m \in U(\mathfrak{g}) \cdot m'$. Moreover, one has $m \in U(\mathfrak{g}_n) \cdot m'$ for some n. For almost all $\alpha \in Y$, h_α and e_α commute with $\mathfrak{g}_n$, in which case m is an eigenvector for $h_\alpha + e_\alpha$. Thus $e_\alpha \cdot m$ is a scalar multiple of m. Since M is integrable, e_α acts locally nilpotently, and we conclude that $e_\alpha \cdot m = 0$ for all but finitely many α. By considering the set $\{h_\alpha + f_\alpha \mid \alpha \in Y\}$ in place of $\{h_\alpha + e_\alpha \mid \alpha \in Y\}$, we see that $f_\alpha \cdot m = 0$ for all but finitely many α, and hence $e_\alpha \cdot m = f_\alpha \cdot m = 0$ for all but finitely many $\alpha \in Y$.

We now consider separately each of the three possible choices of $\mathfrak{g}$. For $\mathfrak{g} = sl(\infty)$, we may assume that the simple roots of $\mathfrak{b}$ are of the form $\{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0}\}$. We first choose the set of commuting mutually orthogonal positive roots to be $Y_1 = \{\epsilon_{2i-1} - \epsilon_{2i} \mid i \in \mathbb{Z}_{>0}\}$ and obtain in this way that $e_{\epsilon_i - \epsilon_{i+1}} \cdot m = f_{\epsilon_i - \epsilon_{i+1}} \cdot m = 0$ for almost all odd indices i. By choosing the set of commuting mutually orthogonal positive roots as $Y_2 = \{\epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0}\}$, we have $e_{\epsilon_i - \epsilon_{i+1}} \cdot m = f_{\epsilon_i - \epsilon_{i+1}} \cdot m = 0$ for almost all even indices i, hence for almost all i. Since it contains $e_{\epsilon_i - \epsilon_{i+1}}$ and $f_{\epsilon_i - \epsilon_{i+1}}$ for almost all i, the subalgebra $\operatorname{Ann}_{\mathfrak{g}} m$ is a finite corank subalgebra of $\mathfrak{g} = sl(\infty)$.

For $\mathfrak{g} = o(\infty)$, one may assume that the set of simple roots of $\mathfrak{g}$ is $\{-\epsilon_1 - \epsilon_2\} \cup \{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0}\}$. In this case in addition to the two sets $Y_1 = \{\epsilon_{2i-1} - \epsilon_{2i} \mid i \in \mathbb{Z}_{>0}\}$ and $Y_2 = \{\epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0}\}$, one considers also the set of commuting mutually orthogonal positive roots $Y_3 = \{-\epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0}\}$. For $\mathfrak{g} = sp(\infty)$, the set of simple roots can be chosen as $\{-2\epsilon_1\} \cup \{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0}\}$, and one considers the following three sets of commuting mutually orthogonal positive roots:

$$\begin{aligned} Y_1 =& \{\epsilon_{2i-1} - \epsilon_{2i} \mid i \in \mathbb{Z}_{>0}\} \\ Y_2 =& \{\epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0}\} \\ Y_3 =& \{-2\epsilon_i \mid i \in \mathbb{Z}_{>0}\}. \end{aligned}$$

In both cases $\operatorname{Ann}_{\mathfrak{g}} m$ contains e_α, f_α for all but finitely many $\alpha \in Y_1 \cup Y_2 \cup Y_3$. Hence we conclude that the subalgebra $\operatorname{Ann}_{\mathfrak{g}} m$ is a finite corank subalgebra of $\mathfrak{g}$; that is, (1) implies (3).

Next we prove that (3) implies (2).

We first show that a $\mathfrak{g}$-module M satisfying (3) is a weight module for some splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Fix a finite set $\{m_1, \dots, m_s\}$ of generators of M. Let $\mathfrak{g}'_n$ be the commutator subalgebra of the centralizer in $\mathfrak{g}$ of $\mathfrak{g}_n$. There exists a finite corank subalgebra that annihilates $m_1, \dots, m_s$, and hence $\mathfrak{g}'_n$ annihilates $m_1, \dots, m_s$ for some n. Let $\mathfrak{h}'_n$ be a splitting Cartan subalgebra of $\mathfrak{g}'_n$. Obviously M is semisimple over $\mathfrak{h}'_n$. One can find k and a Cartan subalgebra $\mathfrak{h}_k \subset \mathfrak{g}_k$ such that $\mathfrak{h} = \mathfrak{h}'_n + \mathfrak{h}_k$ is a splitting Cartan subalgebra of $\mathfrak{g}$. (If $\mathfrak{g} = o(\infty)$ or $sp(\infty)$ one can choose $k = n$; if $\mathfrak{g} = sl(\infty)$, one can set $k = n + 1$). Since M is integrable, M is semisimple over $\mathfrak{h}_k$. Hence M is semisimple over $\mathfrak{h}$.

To finish the proof that (3) implies (2), we need to show that M is $\tilde{G}$-invariant. For each n one has $\tilde{G} = G \cdot \tilde{G}'_n$ by Lemma 3.3. Fix $\gamma \in \tilde{G}$

and $m \in M$. Then for some n, the vector m is fixed by $\mathfrak{g}'_n$. We choose a decomposition $\gamma = \gamma''\gamma'$ so that $\gamma' \in \tilde{G}'_n$ and $\gamma'' \in G$. We then set $\gamma(m) := \gamma''(m)$, and note that the action of G on M is well defined because M is assumed to be integrable. This yields a well-defined $\tilde{G}$-module structure on M since, for any other decomposition $\gamma = \bar{\gamma}''\bar{\gamma}'$ as above, one has $(\bar{\gamma}'')^{-1}\gamma'' = \bar{\gamma}'(\gamma')^{-1} \in \tilde{G}'_n \cap G = \{\gamma \in G \mid \gamma(g) = g \text{ for all } g \in \mathfrak{g}_n\}$ which must preserve m.

Fix now $\gamma \in \tilde{G}$ and consider the linear operator

$$\varphi_\gamma : M^\gamma \to M, \quad m \mapsto \gamma^{-1}(m).$$

We claim that φ_γ is an isomorphism. For this we need to check that $g \cdot \varphi_\gamma(m) = \varphi_\gamma(\gamma(g) \cdot m)$ for any $g \in \mathfrak{g}$ and $m \in M$. We have $g \cdot \varphi_\gamma(m) = g \cdot (\gamma^{-1}(m))) = \varphi_\gamma(\gamma(g \cdot \gamma^{-1}(m)))$, hence it suffices to check that $\gamma(g \cdot \gamma^{-1}(m)) = \gamma(g) \cdot m$ for every $g \in \mathfrak{g}$ and $m \in M$. After choosing a decomposition $\gamma = \gamma''\gamma'$ such that $\gamma'' \in G$ and γ' fixes m, g and $g \cdot m$, all that remains to check is that

$$\gamma''(g \cdot (\gamma'')^{-1}(m)) = \gamma''(g) \cdot m$$

for all $g \in \mathfrak{g}$. This latter equality is the well-known relation between the G-module structure on M and the adjoint action of G on $\mathfrak{g}$.

To complete the proof of the theorem we need to show that (2) implies (1). What is clear is that (2) implies a slightly weaker statement, namely that M is a weight module for any splitting Cartan subalgebra belonging to the same $\tilde{G}$-conjugacy class as the given splitting Cartan subalgebra $\mathfrak{h}$. For $\mathfrak{g} = sl(\infty), sp(\infty)$, this proves (1), as all splitting Cartan subalgebras are conjugate under $\tilde{G}$.

Consider now the case $\mathfrak{g} = o(\infty)$. In this case there are two $\tilde{G}$-conjugacy classes of splitting Cartan subalgebras [**DPS**]. Note that if M is semisimple over every Cartan subalgebra from one $\tilde{G}$-conjugacy class, then (3) holds as follows from the proof of the implication (1) $\Rightarrow$ (3). Furthermore, the proof that a $\mathfrak{g}$-module of finite length M satisfying (3) is a weight module for some splitting Cartan subalgebra involves a choice of $\mathfrak{g}_n$. For $\mathfrak{g} = o(\infty)$ there are two different possible choices, namely $\mathfrak{g}_n = o(2n)$ and $\mathfrak{g}_n = o(2n+1)$, which in turn produce splitting Cartan subalgebras from the two $\tilde{G}$-conjugacy classes. This shows that in each $\tilde{G}$-conjugacy class there is a splitting Cartan subalgebra of $\mathfrak{g}$ for which M is a weight module, and hence we may conclude that (2) implies (1) also for $\mathfrak{g} = o(\infty)$. □

Corollary 3.5. *Let M be a module satisfying the conditions of Theorem 3.4. Then $M = \bigcup_{n>0} M^{\mathfrak{g}'_n}$, where $\mathfrak{g}'_n$ is the commutator of the centralizer of $\mathfrak{g}_n$ in $\mathfrak{g}$.*

Proof. Any submodule of M with the property that the annihilator of each of its elements contains $\mathfrak{g}'_n$ for some n is clearly contained in $\bigcup_{n>0} M^{\mathfrak{g}'_n}$. Condition (3) of Theorem 3.4 states that $\mathrm{Ann}_{\mathfrak{g}}\, m$ is finite corank for all $m \in M$, which is to say that M has this property. □

Corollary 3.6. *Let $\mathfrak{g} = o(\infty)$ and M be a finite length $\mathfrak{g}$-module which is an $\mathfrak{h}$-weight module for all splitting Cartan subalgebras $\mathfrak{h} \subset \mathfrak{g}$ in either of the two $\tilde{G}$-conjugacy classes. Then M is an $\mathfrak{h}$-weight module for all splitting Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$.*

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let M be a finite length $\mathfrak{g}$-module which is a weight module for all splitting Cartan subalgebras in the $\tilde{G}$-conjugacy class of $\mathfrak{h}$. Then M is integrable, by the same argument as in the proof of Proposition 3.2. Finally, (3) holds by the same proof as that of (1)⇒(3) in Theorem 3.4. □

We denote by $\mathbb{T}_{\mathfrak{g}}$ the full subcategory of $\mathfrak{g}$-mod consisting of finite length modules satisfying the equivalent conditions of Theorem 3.4. Then $\mathbb{T}_{\mathfrak{g}}$ is an abelian category and a monoidal category with respect to the usual tensor product of $\mathfrak{g}$-modules, and $\mathbb{T}_{\mathfrak{g}}$ is a subcategory of the category of absolute weight modules. In addition, for $\mathfrak{g} = sl(\infty)$, $\mathbb{T}_{\mathfrak{g}}$ has an involution

$$(\,\cdot\,)_* : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{g}},$$

which one can think of as "restricted dual." Indeed, in this case any outer automorphism $w \in \mathrm{Aut}\,\mathfrak{g}$ induces the autoequivalence of categories

$$\begin{aligned} w_{\mathfrak{g}} : \mathbb{T}_{\mathfrak{g}} &\to \mathbb{T}_{\mathfrak{g}} \\ M &\mapsto M^w. \end{aligned}$$

Since, however, any object of $\mathbb{T}_{\mathfrak{g}}$ is $\tilde{G}$-invariant, the functor $w_{\mathfrak{g}}$ does not depend on the choice of w and is an involution, i.e. $w_{\mathfrak{g}}^2 = \mathrm{id}$. We denote this involution by $(\,\cdot\,)_*$ in agreement with the fact that it maps V to V_*. For $\mathfrak{g} = o(\infty)$, $sp(\infty)$, we define $(\,\cdot\,)_*$ to be the trivial involution on $\mathbb{T}_{\mathfrak{g}}$.

4. Simple objects and indecomposable injectives of $\mathbb{T}_{\mathfrak{g}}$

Next we describe the simple objects of $\mathbb{T}_{\mathfrak{g}}$. For this we need to recall some results about tensor representations from [**PStyr**].

By T we denote the tensor algebra $T(V \oplus V_*)$ for $\mathfrak{g} = sl(\infty)$, and $T(V)$ for $\mathfrak{g} = o(\infty)$, $sp(\infty)$. That is, we have

$$T := \bigoplus_{p\geq 0,\ q\geq 0} T^{p,q} \qquad\qquad \text{for } \mathfrak{g} = sl(\infty),$$

and

$$T := \bigoplus_{p\geq 0} T^{p} \qquad\qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty),$$

where $T^{p,q} := V^{\otimes p} \otimes (V_*)^{\otimes q}$ and $T^p := V^{\otimes p}$. In addition, we set

$$T^{\leq r} := \bigoplus_{p+q\leq r} T^{p,q} \qquad\qquad \text{for } \mathfrak{g} = sl(\infty),$$

and

$$T^{\leq r} := \bigoplus_{p\leq r} T^{p} \qquad\qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty).$$

By a *tensor module* we mean any $\mathfrak{g}$-module isomorphic to a subquotient of a finite direct sum of copies of $T^{\leq r}$ for some r.

By a *partition* we mean a non-strictly decreasing finite sequence of positive integers $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s$. The empty partition is denoted by 0.

Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ and a classical finite-dimensional Lie algebra $\mathfrak{g}_n$ of rank $n \geq s$, the irreducible $\mathfrak{g}_n$-module $(V_n)_\mu$ with highest weight μ is always well defined. Moreover, for a fixed μ and growing n, the modules $(V_n)_\mu$ are naturally nested and determine a unique simple ($\mathfrak{g} = \varinjlim \mathfrak{g}_n$)-module $V_\mu := \varinjlim (V_n)_\mu$. For $\mathfrak{g} = sl(n)$, there is another simple $\mathfrak{g}$-module naturally associated to μ, namely $(V_\mu)_*$.

In what follows we will consider pairs of partitions for $\mathfrak{g} = sl(\infty)$ and single partitions for $\mathfrak{g} = o(\infty)$, $sp(\infty)$. Given $\lambda = (\lambda^1, \lambda^2)$ for $\mathfrak{g} = sl(\infty)$, we set $\tilde{V}_\lambda := V_{\lambda^1} \otimes (V_{\lambda^2})_*$. For $\mathfrak{g} = o(\infty)$, $sp(\infty)$ and for a single partition λ, the

$\mathfrak{g}$-module $\tilde{V}_\lambda$ is similarly defined: we embed $\mathfrak{g}$ into $sl(\infty)$ so that both the natural $sl(\infty)$-module and the conatural $sl(\infty)$-module are identified with V as $\mathfrak{g}$-modules, and define $\tilde{V}_\lambda$ as the irreducible $sl(\infty)$-module V_λ corresponding to the partition λ as defined above. Then $\tilde{V}_\lambda$ is generally a reducible $\mathfrak{g}$-module.

It is easy to see that for $\mathfrak{g} = sl(\infty)$,

$$T = \bigoplus_\lambda d_\lambda \tilde{V}_\lambda \tag{4.1}$$

where $\lambda = (\lambda^1, \lambda^2)$, $d_\lambda := d_{\lambda^1} d_{\lambda^2}$, and d_{λ^i} is the dimension of the simple S_n-module corresponding to the partition λ^i for $n = |\lambda^i|$. For $\mathfrak{g} = o(\infty)$, $sp(\infty)$, Equation (4.1) also holds, with λ taken to stand for a single partition. Both statements follow from the obvious infinite-dimensional version of Schur-Weyl duality for the tensor algebra T considered as an $sl(\infty)$-module (see for instance [**PStyr**]). Moreover, according to [**PStyr**, Theorems 3.2, 4.2],

$$\mathrm{soc}(\tilde{V}_\lambda) = V_\lambda \tag{4.2}$$

for $\mathfrak{g} = o(\infty)$, $sp(\infty)$, while $\mathrm{soc}(\tilde{V}_\lambda)$ is a simple $\mathfrak{g}$-module for $\mathfrak{g} = sl(\infty)$ [**PStyr**, Theorem 2.3]. Here $\mathrm{soc}(\,\cdot\,)$ stands for the socle of a $\mathfrak{g}$-module. We set $V_\lambda := \mathrm{soc}(\tilde{V}_\lambda)$ also for $\mathfrak{g} = sl(\infty)$, so that (4.2) holds for any $\mathfrak{g}$. It is proved in [**PStyr**] that $\tilde{V}_\lambda$ (and consequently $T^{\leq r}$) has finite length.

It follows also from [**PStyr**] that any simple tensor module is isomorphic to V_λ for some λ. In particular, every simple subquotient of T is also a simple submodule of T.

For any partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$, we set $\#\mu := s$ and $|\mu| := \sum_{i=1}^s \mu_i$. In the case of $\mathfrak{g} = sl(\infty)$, when $\lambda = (\lambda^1, \lambda^2)$, we set $\#\lambda := \#\lambda^1 + \#\lambda^2$ and $|\lambda| := |\lambda^1| + |\lambda^2|$.

We are now ready for the following lemma.

Lemma 4.1. *Let $\mathfrak{g} = sl(\infty)$ and $\lambda = (\lambda^1, \lambda^2)$ with $\#\lambda = k > 0$. Then $(V_k)_{\lambda^1} \otimes (V_k^*)_{\lambda^2}$ generates $\tilde{V}_\lambda$.*

Let $\mathfrak{g} = o(\infty)$, $sp(\infty)$, and let λ be a partition with $\#\lambda = k > 0$. Then the $sl(V_k)$-submodule $(V_k)_\lambda$ of $\tilde{V}_\lambda$ generates $\tilde{V}_\lambda$.

Proof. Set $M := \tilde{V}_\lambda$. Let $\mathfrak{g} = sl(\infty)$. Then $M = V_{\lambda^1} \otimes (V_*)_{\lambda^2}$, and let $M_n := (V_n)_{\lambda^1} \otimes (V_n^*)_{\lambda^2}$. It is easy to check that the length of M_n as a $\mathfrak{g}_n$-module stabilizes for $n \geq k$, and moreover it coincides with the length of M;

a formula for the length of M is implied by [**PStyr**, Theorem 2.3]. Hence $(V_k)_{\lambda^1} \otimes (V_k^*)_{\lambda^2}$ generates M.

For $\mathfrak{g} = o(\infty)$, $sp(\infty)$ the length of the $sl(V_n)$-module $(V_n)_\lambda$ considered as a $\mathfrak{g}_n$-module equals the length of M as a $\mathfrak{g}$-module when $2k \leq \dim V_n$ (see Theorems 3.3 and 4.3 in [**PStyr**]). Hence $(V_k)_\lambda$ generates M. $\square$

Theorem 4.2. *A simple absolute weight* $\mathfrak{g}$*-module is a simple tensor module.*

Proof. Let M be a simple absolute weight $\mathfrak{g}$-module. Then M is integrable by Proposition 3.2, and it also satisfies Theorem 3.4 (3). Fix $0 \neq m \in M$ and choose k such that the commutator subalgebra $\mathfrak{g}'_k$ of the centralizer of $\mathfrak{g}_k$ annihilates m. In the orthogonal case we assume that $\mathfrak{g}_k = o(2k)$. We will prove that M is the unique simple quotient of a parabolically induced module for a parabolic subalgebra $\mathfrak{p}$ of the form $\mathfrak{p} = \mathfrak{l} \supsetplus \mathfrak{m}$, where $\mathfrak{m}$ is the nil-radical of $\mathfrak{p}$ and $\mathfrak{l}$ is a locally reductive subalgebra. Define $\mathfrak{p} \subset \mathfrak{g}$ as follows:

- If $\mathfrak{g} = sl(\infty)$, we identify $\mathfrak{g}$ with the subspace of traceless elements in $V_* \otimes V$. Consider the decomposition $V = V_k \oplus V'$, where V_k is the natural $\mathfrak{g}_k$-module and V' is the natural $\mathfrak{g}'_k$-module. Furthermore, $V_* = V_k^\perp \oplus (V')^\perp$, where $(V')^\perp = V_k^*$ and $V_k^\perp = V'_*$. We define the subalgebra $\mathfrak{l}$ of $\mathfrak{p}$ to be equal the traceless part of $V_k^* \otimes V_k \oplus V'_* \otimes V'$, and we set $\mathfrak{m} := V_k^* \otimes V'$.

- If $\mathfrak{g} = o(\infty)$, we use the identification $\mathfrak{g} \simeq \Lambda^2(V)$. Let $V_k \subset V$ be the copy of the natural representation of $\mathfrak{g}_k$. Consider the decomposition $V_k = W \oplus W^*$ for some maximal isotropic subspaces W, W^* of V_k and set $V' = V_k^\perp$. Then $\mathfrak{p} := \mathfrak{l} \supsetplus \mathfrak{m}$, where $\mathfrak{l} := W^* \otimes W \oplus \Lambda^2(V')$ and $\mathfrak{m} := W \otimes V' \supsetplus \Lambda^2(W)$.

- If $\mathfrak{g} = sp(\infty)$, we use the identification $\mathfrak{g} \simeq S^2(V)$. Then V_k, W, W^* and V' are defined in the same way as for $\mathfrak{g} = o(\infty)$, and $\mathfrak{p} := \mathfrak{l} \supsetplus \mathfrak{m}$, where $\mathfrak{l} := W^* \otimes W \oplus S^2(V')$ and $\mathfrak{m} := W \otimes V' \supsetplus S^2(W)$.

Note that $\mathfrak{g}'_k$ is a subalgebra of finite codimension in $\mathfrak{l}$. In the orthogonal and symplectic cases $\mathfrak{l} = gl(W) \oplus \mathfrak{g}'_k$. If $\mathfrak{g} = sl(\infty)$, then $\mathfrak{l} = sl(V_k) \oplus \tilde{\mathfrak{g}}_k$, where $\tilde{\mathfrak{g}}_k$ is the centralizer of $\mathfrak{g}_k$ in $\mathfrak{g}$:

$$\tilde{\mathfrak{g}}_k = \{-\frac{\operatorname{tr} X}{k} \operatorname{Id}_{V_k} \oplus X \mid X \in V'_* \otimes V'\}.$$

Clearly $\tilde{\mathfrak{g}}_k$ is isomorphic to the Lie algebra $V'_* \otimes V'$.

We claim that the $\mathfrak{m}$-invariant part of M, denoted $M^{\mathfrak{m}}$, is nonzero. Note that $\mathfrak{m}$ is abelian for $\mathfrak{g} = sl(\infty)$. For $\mathfrak{g} = o(\infty)$ or $sp(\infty)$, we have a decomposition $\mathfrak{m} = \mathfrak{m}_1 \supsetplus \mathfrak{m}_2$ such that $\mathfrak{m}_2 = [\mathfrak{m}_1, \mathfrak{m}_1]$ is a finite-dimensional abelian subalgebra: $\mathfrak{m}_2 = \Lambda^2(W)$ in the orthogonal case and $\mathfrak{m}_2 = S^2(W)$ in the symplectic case. Since M is integrable, $\mathfrak{m}_2$ acts locally nilpotently on M. Hence without loss of generality we may assume that $\mathfrak{m}_2 \cdot m = 0$. We put $\mathfrak{m}_1 := \mathfrak{m}$ in the case $\mathfrak{g} = sl(\infty)$.

Next observe that $U(\mathfrak{m})\cdot m = S^{\cdot}(\mathfrak{m}_1)\cdot m$ and that $S^{\cdot}(\mathfrak{m}_1)$ is isomorphic as a $\mathfrak{g}'_k$-module to a direct sum of $(\tilde{V}')_\lambda$ for some (infinite) set of λ satisfying $\#\lambda \leq k$. By Lemma 4.1, there exists a finite-dimensional subspace $X \subset \mathfrak{m}_1$ such that $S^{\cdot}(X)$ generates $S^{\cdot}(\mathfrak{m}_1)$ as a $\mathfrak{g}'_k$-module. Since M is integrable, X acts locally nilpotently on M. Hence $S^{>p}(X) \cdot m = 0$ for some p. This, together with our assumption that $\mathfrak{g}'_k \cdot m = 0$, allows us to conclude $S^{>p}(\mathfrak{m}) \cdot m = 0$, which in turn implies $M^{\mathfrak{m}} \neq 0$.

Since M is irreducible, it is generated by $M^{\mathfrak{m}}$ and is therefore the unique irreducible quotient of the parabolically induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^{\mathfrak{m}}$. Furthermore, the irreducibility of M implies the irreducibility of $M^{\mathfrak{m}}$ as an $\mathfrak{l}$-module (otherwise a proper submodule of $M^{\mathfrak{m}}$ would generate a proper submodule of M). Note also that the argument of the previous paragraph implies that as a $\mathfrak{g}'_k$-module $M^{\mathfrak{m}}$ is isomorphic to a subquotient of $S^{\cdot}(\mathfrak{m}_1)$; that is, $M^{\mathfrak{m}}$ is isomorphic to a subquotient of a finite direct sum of some tensor powers of V'.

Let us first consider the case $\mathfrak{g} = o(\infty)$ or $sp(\infty)$. Recall that $M^{\mathfrak{m}}$ is irreducible as an $\mathfrak{l}$-module and is a tensor module over $\mathfrak{g}'_k$. This, together with the integrability of $M^{\mathfrak{m}}$ as an $\mathfrak{l}$-module, implies the existence of an isomorphism of $\mathfrak{l}$-modules $M^{\mathfrak{m}} \simeq L \otimes (V')_\nu$, where L is some irreducible finite-dimensional $gl(W)$-module and $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ is some partition. Let $(\mu_1, \ldots, \mu_k)$ denote the highest weight of L with respect to some Borel subalgebra of $\mathfrak{b}_k$ of $gl(W)$. Consider a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{b}_k \subset \mathfrak{b} \subset \mathfrak{p}$. Without loss of generality, we may assume that the roots of $\mathfrak{b}$ are

$$\{\epsilon_i \pm \epsilon_j \mid i < j \in \mathbb{Z}_{>0}\} \qquad \text{for } \mathfrak{g} = o(\infty),$$
$$\{\epsilon_i \pm \epsilon_j \mid i < j \in \mathbb{Z}_{>0}\} \cup \{2\epsilon_i \mid i \in \mathbb{Z}_{>0}\} \qquad \text{for } \mathfrak{g} = sp(\infty).$$

The roots of $\mathfrak{b}_k$ will then be

$$\{\epsilon_i \pm \epsilon_j \mid 0 < i < j \leq k\} \qquad \text{for } \mathfrak{g} = o(\infty),$$
$$\{\epsilon_i \pm \epsilon_j \mid 0 < i < j \leq k\} \cup \{2\epsilon_i \mid 0 < i \leq k\} \qquad \text{for } \mathfrak{g} = sp(\infty).$$

Observe that M is a highest weight module with respect to $\mathfrak{b}$, and its highest weight equals $\lambda := \mu_1\epsilon_1 + \cdots + \mu_k\epsilon_k + \nu_1\epsilon_{k+1} + \cdots + \nu_r\epsilon_{k+r}$. Furthermore, the integrability of M as a $\mathfrak{g}$-module implies that $\mu_k \geq \nu_1$ and all μ_i are integers. In other words, the weight λ can be identified with the partition $(\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_r)$. Next, consider the simple tensor $\mathfrak{g}$-module V_λ (where λ is considered as a partition), and note that both M and V_λ are simple $\mathfrak{g}$-modules with the same highest weight with respect to $\mathfrak{b}$. Therefore M and V_λ are isomorphic as $\mathfrak{g}$-modules.

Now let $\mathfrak{g} = sl(\infty)$. Then by the same argument as above we see that $M^{\mathfrak{m}}$ is isomorphic to $L_1 \otimes L_2$, where L_1 is a simple finite-dimensional $sl(V_k)$-module and L_2 is a simple integrable $\tilde{\mathfrak{g}}_k$-module. Since, however, L_2 is isomorphic to a submodule of the tensor algebra $T(V')$ as a $\mathfrak{g}'_k$-module, we check immediately that as a $\mathfrak{g}'_k$-module L_2 must be isomorphic to $(V')_\nu$ for some partition $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$. There is a $\tilde{\mathfrak{g}}_k$-submodule of $(V')^{\otimes|\nu|}$ (i.e. a tensor module of $\tilde{\mathfrak{g}}_k \cong V'_* \otimes V'$) with the same restriction to $\mathfrak{g}'_k$ as L_2; abusing notation slightly, we denote it also by $(V')_\nu$. Next, using the inclusions

$$sl(V_k) \oplus \mathfrak{g}'_k \subset \mathfrak{l} \subset gl(V_k) \oplus \tilde{\mathfrak{g}}_k$$

and the fact that $gl(V_k) \oplus \tilde{\mathfrak{g}}_k$ is a direct sum of $\mathfrak{l}$ and the abelian one-dimensional Lie algebra (namely the center of $gl(V_k)$), we conclude that $M^{\mathfrak{m}}$ must be isomorphic to the restriction to $\mathfrak{l}$ of a $gl(V_k) \oplus \tilde{\mathfrak{g}}_k$-module of the form $L \otimes (V')_\nu$, where the $gl(V_k)$-module L is simple and uniquely determined up to isomorphism. Denote by $\mu = (\mu_1, \ldots, \mu_k)$ the highest weight of L. It is easy to check in this case that the integrability of M as a $\mathfrak{g}$-module implies that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ are nonpositive integers. Consider the pair of partitions

$$\lambda := ((\nu_1, \nu_2, \ldots, \nu_r), (-\mu_k, \ldots, -\mu_1))$$

and the corresponding tensor $\mathfrak{g}$-module V_λ. Then we clearly have an isomorphism of $\mathfrak{p}$-modules $V_\lambda^{\mathfrak{m}} \simeq M^{\mathfrak{m}}$. Therefore, being the unique irreducible quotients of the corresponding parabolically induced modules, M and V_λ are isomorphic as $\mathfrak{g}$-modules. $\square$

Remark 4.3. In [**PS**] certain categories $\mathrm{Tens}_{\mathfrak{g}}$ and $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are introduced and studied in detail. The simple objects of both $\mathrm{Tens}_{\mathfrak{g}}$ and $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are the same as the simple objects of $\mathbb{T}_{\mathfrak{g}}$, and in fact these three categories form the following chain:

$$\mathbb{T}_{\mathfrak{g}} \subset \mathrm{Tens}_{\mathfrak{g}} \subset \widetilde{\mathrm{Tens}}_{\mathfrak{g}}.$$

However, the objects of the categories $\mathrm{Tens}_{\mathfrak{g}}$ and $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ generally have infinite length. In the present paper we will not make use of the categories $\mathrm{Tens}_{\mathfrak{g}}$ and $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$, and refer the interested reader to [**PS**].

Let us denote by $\mathcal{C}$ the category of $\mathfrak{g}$-modules which satisfy Condition (3) of Theorem 3.4. Consider the functor $\mathcal{B}$ from $\mathrm{Int}_{\mathfrak{g}}$ to $\mathcal{C}$ given by

$$\mathcal{B}(M) = \bigcup_{n>0} M^{\mathfrak{g}'_n}.$$

It is clear that $\mathcal{B}$ does not depend on the choice of fixed exhaustion $\mathfrak{g} = \varinjlim \mathfrak{g}_n$.

Lemma 4.4. *For any $M \in \mathrm{Int}_{\mathfrak{g}}$, the module $\mathcal{B}(\Gamma_{\mathfrak{g}}(M^*))$ is injective in the category $\mathcal{C}$. Furthermore, any finite length injective module in the category $\mathcal{C}$ is injective in $\mathbb{T}_{\mathfrak{g}}$.*

Proof. First, let us note that $\mathcal{B}$ is a right adjoint to the inclusion functor $\mathcal{C} \subset \mathrm{Int}_{\mathfrak{g}}$. To see this, consider that the image of any homomorphism from a module $M \in \mathcal{C}$ to a module $Y \in \mathrm{Int}_{\mathfrak{g}}$ is automatically contained in $\mathcal{B}(Y)$. Since it is a right adjoint to the inclusion functor, $\mathcal{B}$ takes injective modules to injective modules, and the lemma follows from the fact that $\Gamma_{\mathfrak{g}}(M^*)$ is injective for any integrable $\mathfrak{g}$-module M, which is [**PS**, Proposition 3.2]. The second statement is clear. $\square$

Proposition 4.5. *For each r, the module $T^{\leq r}$ is injective in the category of absolute weight modules and in $\mathbb{T}_{\mathfrak{g}}$.*

Proof. We consider the case $\mathfrak{g} = sl(\infty)$, and note that the other cases are similar. It was shown in [**PS**] that $(T^{q,p})^*$ is an integrable $\mathfrak{g}$-module. We will show $\mathcal{B}\big((T^{q,p})^*\big)$ is a finite-length module, and furthermore that it has a direct summand isomorphic to $T^{p,q}$. Since any direct summand of an injective module is itself injective, it will follow immediately that $T^{p,q}$ is injective in the category $\mathbb{T}_{\mathfrak{g}}$.

We start with calculating $\left((T^{q,p})^*\right)^{\mathfrak{g}'_n}$. Consider the decomposition

$$V = V_n \oplus V', \quad V_* = V_n^* \oplus V'_*,$$

where V' and V'_* are respectively the natural and conatural $\mathfrak{g}'_n$-modules. If we use the notation

$$T_n^{r,s} := V_n^{\otimes r} \otimes (V_n^*)^{\otimes s}, \quad (T')^{r,s} := (V')^{\otimes r} \otimes (V'_*)^{\otimes s},$$

then we have the following isomorphism of $\mathfrak{g}_n \oplus \mathfrak{g}'_n$-modules

$$T^{q,p} \simeq \bigoplus_{r \le q, s \le p} \left((T')^{r,s} \otimes T_n^{q-r,p-s}\right)^{\oplus b_{r,s}},$$

where $b_{r,s} = \binom{q}{r}\binom{p}{s}$.

Therefore

$$\left((T^{q,p})^*\right)^{\mathfrak{g}'_n} \simeq \bigoplus_{r \le q, s \le p} \mathrm{Hom}_{\mathfrak{g}'_n}((T')^{r,s}, \mathbb{C}) \otimes \left(T_n^{p-s,q-r}\right)^{\oplus b_{r,s}}.$$

Since $\mathfrak{g}'_n \simeq \mathfrak{g}$, we can use the results of [**PStyr**]. In particular,

$$\mathrm{Hom}_{\mathfrak{g}'_n}((T')^{r,s}, \mathbb{C}) = \begin{cases} 0 & \text{if} \quad r \neq s \\ \mathbb{C}^{r!} & \text{if} \quad r = s. \end{cases}$$

The degree $r!$ appears for the following reason. For any $\sigma \in S_r$ we define $\varphi_\sigma \in \mathrm{Hom}_{\mathfrak{g}'_n}((T')^{r,r}, \mathbb{C})$ by

$$\varphi_\sigma(v_1 \otimes \cdots \otimes v_r \otimes u_1 \otimes \cdots \otimes u_r) = \prod_{i=1}^{r} \langle u_i, v_{\sigma(i)} \rangle.$$

Then φ_σ for all $\sigma \in S_r$ form a basis in $\mathrm{Hom}_{\mathfrak{g}'_n}((T')^{r,r}, \mathbb{C})$. Thus we obtain

$$\left((T^{q,p})^*\right)^{\mathfrak{g}'_n} \simeq \bigoplus_{r \le \min(p,q)} \left(T_n^{p-s,q-r}\right)^{\oplus b_{r,r} r!},$$

which implies

$$\mathcal{B}\left((T^{q,p})^*\right) \simeq \bigoplus_{r \le \min(p,q)} \left(T^{p-s,q-r}\right)^{\oplus b_{r,r} r!}.$$

Hence the statement. ∎

Corollary 4.6. 1. *$\tilde{V}_\lambda$ is injective in $\mathbb{T}_{\mathfrak{g}}$.*

2. *$\tilde{V}_\lambda$ is an injective hull of V_λ in $\mathbb{T}_{\mathfrak{g}}$.*
3. *Every indecomposable injective module in $\mathbb{T}_{\mathfrak{g}}$ is isomorphic to $\tilde{V}_\lambda$ for some λ.*
4. *Every module $M \in \mathbb{T}_{\mathfrak{g}}$ is isomorphic to a submodule of the direct sum of finitely many copies of $T^{\leq r}$ for some r.*
5. *A $\mathfrak{g}$-module M is a tensor module if and only if $M \in \mathbb{T}_{\mathfrak{g}}$.*

Proof. 1. Each module $\tilde{V}_\lambda$ is a direct summand of $T^{\leq r}$ for some r, and a direct summand of an injective module is injective.

2. Any indecomposable injective module is an injective hull of its socle, and $\mathrm{soc}(\tilde{V}_\lambda) = V_\lambda$ by (4.2).
3. Every indecomposable injective module in $\mathbb{T}_{\mathfrak{g}}$ has a simple socle, which must be isomorphic to V_λ for some λ by Theorem 4.2.
4. Let $M \in \mathbb{T}_{\mathfrak{g}}$. Then $\mathrm{soc}(M)$ admits an injective homomorphism into a direct sum of finitely many copies of $T^{\leq r}$ for some r. Since the latter is injective in $\mathbb{T}_{\mathfrak{g}}$, this homomorphism factors through the inclusion $\mathrm{soc}(M) \hookrightarrow M$. The resulting homomorphism must be injective because its kernel has trivial intersection with $\mathrm{soc}(M)$.
5. A tensor module is by definition a subquotient of a direct sum of finitely many copies of $T^{\leq r}$ for some r, hence it is clearly finite length. Furthermore, any subquotient of an absolute weight module must be an absolute weight module, so any tensor module must be in $\mathbb{T}_{\mathfrak{g}}$. The converse was seen in (4).

∎

5. Koszulity of $\mathbb{T}_{\mathfrak{g}}$

For $r \in \mathbb{Z}_{\geq 0}$, let $\mathbb{T}_{\mathfrak{g}}^r$ be the full abelian subcategory of $\mathbb{T}_{\mathfrak{g}}$ whose simple objects are submodules of $T^{\leq r}$. Then $\mathbb{T}_{\mathfrak{g}} = \varinjlim \mathbb{T}_{\mathfrak{g}}^r$. Moreover, $T^{\leq r}$ is an injective cogenerator of $\mathbb{T}_{\mathfrak{g}}^r$. Consider the finite-dimensional algebra $\mathcal{A}_{\mathfrak{g}}^r := \mathrm{End}_{\mathfrak{g}} T^{\leq r}$ and the direct limit algebra $\mathcal{A}_{\mathfrak{g}} = \varinjlim \mathcal{A}_{\mathfrak{g}}^r$.

Let $\mathcal{A}_{\mathfrak{g}}^r$-mof denote the category of unitary finite-dimensional $\mathcal{A}_{\mathfrak{g}}^r$-modules, and $\mathcal{A}_{\mathfrak{g}}$-mof the category of locally unitary finite-dimensional $\mathcal{A}_{\mathfrak{g}}$-modules.

Proposition 5.1. *The functors $\mathrm{Hom}_{\mathfrak{g}}(\,\cdot\,, T^{\leq r})$ and $\mathrm{Hom}_{\mathcal{A}_{\mathfrak{g}}^r}(\,\cdot\,, T^{\leq r})$ are mutually inverse antiequivalences of the categories $\mathbb{T}_{\mathfrak{g}}^r$ and $\mathcal{A}_{\mathfrak{g}}^r$-mof.*

Proof. Consider the opposite category $(\mathbb{T}^r_{\mathfrak{g}})^{\mathrm{opp}}$. It has finitely many simple objects and enough projectives, and any object has finite length. Moreover, $T^{\leq r}$ is a projective generator of $(\mathbb{T}^r_{\mathfrak{g}})^{\mathrm{opp}}$. By a well-known result of Gabriel [**G**], the functor

$$\mathrm{Hom}_{(\mathbb{T}^r_{\mathfrak{g}})^{\mathrm{opp}}}(T^{\leq r}, \,\cdot\,) = \mathrm{Hom}_{\mathfrak{g}}(\,\cdot\,, T^{\leq r}) : (\mathbb{T}^r_{\mathfrak{g}})^{\mathrm{opp}} \to \mathcal{A}^r_{\mathfrak{g}}\text{-mof}$$

is an equivalence of categories.

We claim that $\mathrm{Hom}_{\mathcal{A}^r_{\mathfrak{g}}}(\,\cdot\,, T^{\leq r})$ is an inverse to $\mathrm{Hom}_{\mathfrak{g}}(\,\cdot\,, T^{\leq r})$. For this it suffices to check that $\mathrm{Hom}_{(\mathbb{T}^r_{\mathfrak{g}})^{\mathrm{opp}}}(T^{\leq r}, \,\cdot\,)$ is a right adjoint to $\mathrm{Hom}_{\mathcal{A}^r_{\mathfrak{g}}}(\,\cdot\,, T^{\leq r})$, i.e. that

$$\mathrm{Hom}_{\mathcal{A}^r_{\mathfrak{g}}}(X, \mathrm{Hom}_{(\mathbb{T}^r_{\mathfrak{g}})^{\mathrm{opp}}}(T^{\leq r}, M)) \simeq \mathrm{Hom}_{(\mathbb{T}^r_{\mathfrak{g}})^{\mathrm{opp}}}(\mathrm{Hom}_{\mathcal{A}^r_{\mathfrak{g}}}(X, T^{\leq r}), M)$$

for any $X \in \mathcal{A}^r_{\mathfrak{g}}$-mof and any $M \in \mathbb{T}^r_{\mathfrak{g}}$. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}^r_{\mathfrak{g}}}(X, \mathrm{Hom}_{(\mathbb{T}^r_{\mathfrak{g}})^{\mathrm{opp}}}(T^{\leq r}, M)) &= \mathrm{Hom}_{\mathcal{A}^r_{\mathfrak{g}}}(X, \mathrm{Hom}_{\mathfrak{g}}(M, T^{\leq r})) \\
&\overset{\Psi}{\simeq} \mathrm{Hom}_{\mathcal{A}^r_{\mathfrak{g}} \otimes U(\mathfrak{g})}(X \otimes M, T^{\leq r}) \\
&= \mathrm{Hom}_{U(\mathfrak{g}) \otimes \mathcal{A}^r_{\mathfrak{g}}}(M \otimes X, T^{\leq r}) \\
&\overset{\Theta}{\simeq} \mathrm{Hom}_{\mathfrak{g}}(M, \mathrm{Hom}_{\mathcal{A}^r_{\mathfrak{g}}}(X, T^{\leq r})) \\
&= \mathrm{Hom}_{(\mathbb{T}^r_{\mathfrak{g}})^{\mathrm{opp}}}(\mathrm{Hom}_{\mathcal{A}^r_{\mathfrak{g}}}(X, T^{\leq r}), M),
\end{aligned}$$

where $\Psi(\varphi)(x \otimes m) = \varphi(x)(m)$ and $(\Theta(x)(m))(\psi) = \psi(m \otimes x)$ for $x \in X$, $m \in M$, $\varphi \in \mathrm{Hom}_{\mathcal{A}^r_{\mathfrak{g}}}(X, \mathrm{Hom}_{\mathfrak{g}}(M, T^{\leq r}))$, and $\psi \in \mathrm{Hom}_{U(\mathfrak{g}) \otimes \mathcal{A}^r_{\mathfrak{g}}}(M \otimes X, T^{\leq r})$. $\square$

In order to relate the category $\mathcal{A}_{\mathfrak{g}}$-mof with the categories $\mathcal{A}^r_{\mathfrak{g}}$-mof for all $r \geq 0$, we need to establish some basic facts about the algebra $\mathcal{A}_{\mathfrak{g}}$. Note first that by [**PStyr**] $\mathrm{Hom}_{sl(\infty)}(T^{p,q}, T^{r,s}) = 0$ unless $p - r = q - s \in \mathbb{Z}_{\geq 0}$, and for $\mathfrak{g} = o(\infty)$, $sp(\infty)$, $\mathrm{Hom}_{\mathfrak{g}}(T^p, T^q) = 0$ unless $p - q \in 2\mathbb{Z}_{\geq 0}$. Furthermore, put

$$(\mathcal{A}_{\mathfrak{g}})^{p,q}_i = \mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{p-i,q-i}) \qquad \text{for } \mathfrak{g} = sl(\infty)$$

and

$$(\mathcal{A}_{\mathfrak{g}})^p_i = \mathrm{Hom}_{\mathfrak{g}}(T^p, T^{p-2i}) \qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty).$$

Then one can define a $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{A}_{\mathfrak{g}}^r$ by setting

$$(\mathcal{A}_{\mathfrak{g}}^r)_i = \bigoplus_{p+q\leq r} (\mathcal{A}_{\mathfrak{g}})_i^{p,q} \qquad \text{for } \mathfrak{g} = sl(\infty)$$

and

$$(\mathcal{A}_{\mathfrak{g}}^r)_i = \bigoplus_{p\leq r} (\mathcal{A}_{\mathfrak{g}})_i^{p} \qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty).$$

It also follows from the results of [**PStyr**] that

$$(\mathcal{A}_{\mathfrak{g}}^r)_0 = \bigoplus_{p+q\leq r} \mathrm{End}_{\mathfrak{g}}(T^{p,q}) = \bigoplus_{p+q\leq r} \mathbb{C}[S_p \times S_q] \qquad \text{for } \mathfrak{g} = sl(\infty)$$

and

$$(\mathcal{A}_{\mathfrak{g}}^r)_0 = \bigoplus_{p\leq r} \mathrm{End}_{\mathfrak{g}}(T^{p}) = \bigoplus_{p\leq r} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty).$$

Hence $(\mathcal{A}_{\mathfrak{g}}^r)_0$ is semisimple.

In addition, we have

$$(\mathcal{A}_{\mathfrak{g}})_i^{p,q} (\mathcal{A}_{\mathfrak{g}})_j^{r,s} = 0 \text{ unless } p = r - j,\ q = s - j \qquad \text{for } \mathfrak{g} = sl(\infty)$$

and

$$(\mathcal{A}_{\mathfrak{g}})_i^{p} (\mathcal{A}_{\mathfrak{g}})_j^{r} = 0 \text{ unless } p = r - 2j \qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty).$$

This shows that for each r,

$$\bar{\mathcal{A}}_{\mathfrak{g}}^r := \bigoplus_{p+q>r} \bigoplus_{i\geq 0} (\mathcal{A}_{\mathfrak{g}})_i^{p,q} \qquad \text{for } \mathfrak{g} = sl(\infty)$$

or

$$\bar{\mathcal{A}}_{\mathfrak{g}}^r := \bigoplus_{p>r} \bigoplus_{i\geq 0} (\mathcal{A}_{\mathfrak{g}})_i^{p} \qquad \text{for } \mathfrak{g} = o(\infty),\ sp(\infty)$$

is a $\mathbb{Z}_{\geq 0}$-graded ideal in $\mathcal{A}_\mathfrak{g}$ such that $\mathcal{A}^r_\mathfrak{g} \oplus \bar{\mathcal{A}}^r_\mathfrak{g} = \mathcal{A}_\mathfrak{g}$. Hence each unitary $\mathcal{A}^r_\mathfrak{g}$-module X admits a canonical $\mathcal{A}_\mathfrak{g}$-module structure with $\bar{\mathcal{A}}^r_\mathfrak{g} X = 0$, and thus becomes a locally unitary $\mathcal{A}_\mathfrak{g}$-module. This allows us to claim simply that

$$\mathcal{A}_\mathfrak{g}\text{-mof} = \varinjlim (\mathcal{A}^r_\mathfrak{g}\text{-mof}).$$

Moreover, Proposition 5.1 now implies the following.

Corollary 5.2. *The functors* $\mathrm{Hom}_\mathfrak{g}(\,\cdot\,, T)$ *and* $\mathrm{Hom}_{\mathcal{A}_\mathfrak{g}}(\,\cdot\,, T)$ *are mutually inverse antiequivalences of the categories* $\mathbb{T}_\mathfrak{g}$ *and* $\mathcal{A}_\mathfrak{g}$-mof*.*

We now need to recall the definition of a Koszul ring. See [**BGS**], where this notion is studied extensively, and, in particular, several equivalent definitions are given. According to Proposition 2.1.3 in [**BGS**], a $\mathbb{Z}_{\geq 0}$-graded ring A is *Koszul* if A_0 is a semisimple ring and for any two graded A-modules M and N of pure weight $m, n \in \mathbb{Z}$ respectively, $\mathrm{ext}^i_A(M, N) = 0$ unless $i = m - n$, where ext^i_A denotes the ext-group in the category of $\mathbb{Z}$-graded A-modules.

In the rest of this section we show that $\mathcal{A}^r_\mathfrak{g}$ is a Koszul ring.

We start by introducing the following notation: for any partition μ, we set

$$\mu^+ := \{\text{partitions } \mu' \mid |\mu'| = |\mu| + 1 \text{ and } \mu'_i \neq \mu_i \text{ for exactly one } i\},$$
$$\mu^- := \{\text{partitions } \mu' \mid |\mu'| = |\mu| - 1 \text{ and } \mu'_i \neq \mu_i \text{ for exactly one } i\}.$$

For any pair of partitions $\lambda = (\lambda^1, \lambda^2)$, we define

$$\lambda^+ := \{\text{pairs of partitions } \eta \mid \eta^1 \in {\lambda^1}^+, \eta^2 = \lambda^2\},$$
$$\lambda^- := \{\text{pairs of partitions } \eta \mid \eta^1 = \lambda^1, \eta^2 \in {\lambda^2}^-\}.$$

Lemma 5.3. *For any simple object* V_λ *of* $\mathbb{T}_\mathfrak{g}$*, there is an exact sequence*

$$0 \to V^+_\lambda \to V \otimes V_\lambda \to V^-_\lambda \to 0,$$

where

$$V_\lambda^+ = \bigoplus_{\eta\in\lambda^+} V_\eta$$
$$V_\lambda^- = \bigoplus_{\eta\in\lambda^-} V_\eta.$$

Moreover, $V_\lambda^+ = \mathrm{soc}(V\otimes V_\lambda)$*.*

Proof. We will prove the statement for $\mathfrak{g} = sl(\infty)$. The other cases are similar. The fact that the semisimplification of $V\otimes V_\lambda$ is isomorphic to $V_\lambda^+\oplus V_\lambda^-$ follows from the classical Pieri rule.

To prove the equality $V_\lambda^+ = \mathrm{soc}(V\otimes V_\lambda)$, observe that

$$V\otimes V_\lambda \subset V\otimes \tilde{V}_\lambda = T^{|\lambda^1|+1,|\lambda^2|}\cap (V\otimes\tilde{V}_\lambda).$$

On the other hand [**PStyr**, Theorem 2.3] implies directly that

$$V_\lambda^+ = \mathrm{soc}(T^{|\lambda^1|+1,|\lambda^2|})\cap (V\otimes\tilde{V}_\lambda).$$

Hence $V_\lambda^+ = \mathrm{soc}(V\otimes V_\lambda)$.

It remains to show that the quotient $(V\otimes V_\lambda)/V_\lambda^+$ is semisimple. This follows again from [**PStyr**, Theorem 2.3], since all simple subquotients of $V_\lambda^- = (V\otimes V_\lambda)/V_\lambda^+$ lie in $\mathrm{soc}^1(T^{|\lambda^1|+1,|\lambda^2|})$. □

Proposition 5.4. *If* $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\mu)\neq 0$*, then*

$$|\mu^1|-|\lambda^1| = |\mu^2|-|\lambda^2| = i \qquad \textit{for } \mathfrak{g}=sl(\infty)$$

and

$$|\mu|-|\lambda| = 2i \qquad \textit{for } \mathfrak{g}=o(\infty),\ sp(\infty).$$

Proof. Let $\mathfrak{g} = sl(\infty)$. We will prove the statement by induction on $|\mu|$. The base of induction $\mu = (0,0)$ follows immediately from the fact that $V_{(0,0)} = \mathbb{C}$ is injective. We assume $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\mu)\neq 0$. Without loss of generality we may assume that $|\mu^1|>0$. Then there exists a pair of partitions η such that $\mu\in\eta^+$. Since V_μ is a direct summand of V_η^+, we have $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\eta^+)\neq 0$.

Consider the short exact sequence from Lemma 5.3

$$0 \to V_\eta^+ \to V \otimes V_\eta \to V_\eta^- \to 0.$$

The associated long exact sequence implies that either $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V \otimes V_\eta) \neq 0$ or $\mathrm{Ext}^{i-1}_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\eta^-) \neq 0$. In the latter case, the inductive hypothesis implies that

$$|\eta^1| - |\lambda_1| = (|\eta^2| - 1) - |\lambda^2| = i - 1.$$

The condition in the statement of the proposition follows, as $|\eta^1| = |\mu^1| - 1$ and $|\eta^2| = |\mu^2|$.

Now assume that $\mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V \otimes V_\eta) \neq 0$. Let

$$0 \to V_\eta \to M_0 \to M_1 \to \dots$$

be a minimal injective resolution of V_η in $\mathbb{T}_\mathfrak{g}$. By the inductive hypothesis, $\mathrm{Ext}^j_{\mathbb{T}_\mathfrak{g}}(V_\nu, V_\eta) \neq 0$ implies

$$|\eta^1| - |\nu^1| = |\eta^2| - |\nu^2| = j. \tag{5.1}$$

We claim that by the minimality of the resolution, $\tilde{V}_\nu$ appears as a direct summand of M_j only if (5.1) holds, that is $M_j = \oplus \tilde{V}_\nu$ for some set of ν such that $|\nu^1| = |\eta^1| - j$ and $|\nu^2| = |\eta^2| - j$. Indeed, otherwise the sequence

$$\mathrm{Hom}_\mathfrak{g}(V_\nu, M_{j-1}) \to \mathrm{Hom}_\mathfrak{g}(V_\nu, M_j) \to \mathrm{Hom}_\mathfrak{g}(V_\nu, M_{j+1})$$

would be exact, and replacing M_j by $M_j/\tilde{V}_\nu$, and M_{j+1} by $M_{j+1}/\tilde{V}_\nu$ or M_{j-1} by $M_{j-1}/\tilde{V}_\nu$, we would obtain a "smaller" resolution.

Furthermore, since the functor $V \otimes (\cdot)$ is obviously exact (vector spaces are flat), the complex

$$0 \to V \otimes V_\eta \to V \otimes M_0 \to V \otimes M_1 \to \dots$$

is an injective resolution of $V \otimes V_\eta$. Thus $\mathrm{Hom}_\mathfrak{g}(V_\lambda, V \otimes M_i) \neq 0$ implies $|\lambda^1| = |\eta^1| - i + 1$ and $|\lambda^2| = |\eta^2| - i$, and the proof for $\mathfrak{g} = sl(\infty)$ is complete.

The proof for $\mathfrak{g} = o(\infty)$, $sp(\infty)$ is similar, and we leave it to the reader. $\square$

Recall that any $\mathfrak{g}$-module W has a well-defined socle filtration

$$0 \subset \operatorname{soc}^0(W) = \operatorname{soc}(W) \subset \operatorname{soc}^1(W) \subset \cdots$$

where $\operatorname{soc}^i(W) := \pi_{i-1}^{-1}(\operatorname{soc}(W/\operatorname{soc}^{i-1}(W))$ and $\pi_{i-1} : W \to W/\operatorname{soc}^{i-1}(W)$ is the projection. Similarly, any $\mathcal{A}_\mathfrak{g}$-module X has a radical filtration

$$\cdots \subset \operatorname{rad}^1(X) \subset \operatorname{rad}^0(X) = \operatorname{rad}(X) \subset X$$

where $\operatorname{rad}(X)$ is the joint kernel of all surjective $\mathcal{A}_\mathfrak{g}$-homomorphisms $X \to X'$ with X' simple, and $\operatorname{rad}^i(X) = \operatorname{rad}(\operatorname{rad}^{i-1}(X))$.

Note furthermore that the Ext's in the category $\mathbb{T}_\mathfrak{g}$ differ essentially from the Ext's in $\mathfrak{g}$-mod. In particular, as shown in [**PS**], $\operatorname{Ext}^1_\mathfrak{g}(V_\lambda, V_\mu)$ is uncountable dimensional whenever nonzero, whereas $\operatorname{Ext}^1_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\mu)$ is always finite dimensional by Corollary 5.2. Here are two characteristic examples.

1. Consider the exact sequence of $\mathfrak{g}$-modules

$$0 \to V \to (V_*)^* \to (V_*)^*/V \to 0.$$

   The $\mathfrak{g}$-module $(V_*)^*/V$ is trivial, and any vector in $\operatorname{Ext}^1_{sl(\infty)}(\mathbb{C}, V)$ determines a unique 1-dimensional subspace in $(V_*)^*/V$. On the other hand, $\operatorname{Ext}^1_{\mathbb{T}_{sl(\infty)}}(\mathbb{C}, V) = 0$ by Proposition 5.4.

2. Each nonzero vector of $\operatorname{Ext}^1_{sl(\infty)}(\mathbb{C}, sl(\infty))$ corresponds to a 1-dimensional trivial quotient of $\operatorname{soc}^1((sl(\infty)_*)^*)$ (see [**PS**]). The nonzero vectors of the 1-dimensional space $\operatorname{Ext}^1_{\mathbb{T}_{sl(\infty)}}(\mathbb{C}, sl(\infty))$ on the other hand correspond to the unique 1-dimensional quotient of $\operatorname{soc}^1((sl(\infty)_*)^*)$ which determines an absolute weight module, namely $\widetilde{sl(\infty)}/sl(\infty) = (V \otimes V_*)/sl(\infty)$.

The following is the main result of this section.

Theorem 5.5. *The ring $\mathcal{A}^r_\mathfrak{g}$ is Koszul.*

Proof. According to [**BGS**, Proposition 2.1.3], it suffices to prove that unless $i = m - n$, one has $\operatorname{ext}^i_{\mathcal{A}^r_\mathfrak{g}}(M, N) = 0$ for any pure $\mathcal{A}^r_\mathfrak{g}$-modules M, N of weights m, n respectively. We will prove that unless $i = m - n$, one has $\operatorname{ext}^i_{\mathcal{A}_\mathfrak{g}}(M, N) = 0$ for any simple pure $\mathcal{A}_\mathfrak{g}$-modules M, N of weights m, n respectively. Since any $\mathcal{A}^r_\mathfrak{g}$-module admits a canonical $\mathcal{A}_\mathfrak{g}$-module structure, it will follow that $\operatorname{ext}^i_{\mathcal{A}^r_\mathfrak{g}}(M, N) = 0$ for any simple pure $\mathcal{A}^r_\mathfrak{g}$-modules M, N

of weights m, n respectively unless $i = m - n$. The analogous statement for arbitrary $\mathcal{A}_{\mathfrak{g}}^r$-modules of pure degree will also follow, since all such modules are semisimple.

Let X_λ (respectively, $\tilde{X}_\lambda$) be the $\mathcal{A}_{\mathfrak{g}}$-module which is the image of V_λ (resp., $\tilde{V}_\lambda$) under the antiequivalence of Corollary 5.2. Then $\tilde{X}_\lambda$ is a projective cover of the simple module X_λ. Proposition 5.4 implies that $\mathrm{Ext}^i_{\mathcal{A}_{\mathfrak{g}}}(X_\mu, X_\lambda) = 0$ unless $|\mu^1| - |\lambda^1| = |\mu^2| - |\lambda^2| = i$. We consider a minimal projective resolution of X_μ

$$\cdots \to P^1 \to P^0 \to 0 \tag{5.2}$$

and claim that it must have the property $P^i \simeq \oplus \tilde{X}_\nu$ for some set of ν with $|\mu^1| - |\nu^1| = |\mu^2| - |\nu^2| = i$. This follows from the similar fact for a minimal injective resolution of V_μ in $\mathbb{T}_{\mathfrak{g}}$ (see the proof of Proposition 5.4) and the antiequivalence of the categories $\mathbb{T}_{\mathfrak{g}}$ and $\mathcal{A}_{\mathfrak{g}}$-mof.

On the other hand, by [**PStyr**] if V_ν is a simple constituent of $\mathrm{soc}^i(\tilde{V}_\mu)$, or if under the antiequivalence X_ν is a simple constituent of $\mathrm{rad}^i\,\tilde{X}_\mu$, then $|\mu^1| - |\nu^1| = |\mu^2| - |\nu^2| = i$. Therefore we see that in the above resolution the image of $\mathrm{rad}^j(P^i)$ lies in $\mathrm{rad}^{j+1}(P^{i-1})$. Now it is clear that we can endow the resolution (5.2) with a $\mathbb{Z}$-grading by setting the degree of X_μ to be an arbitrary integer n. Indeed, one should assign to each simple $(\mathcal{A}_{\mathfrak{g}})_0$-constituent of P^i which lies in $\mathrm{rad}^j(P^i)$ and not in $\mathrm{rad}^{j+1}(P^i)$ the degree $n + i + j + 1$. This immediately implies that $\mathrm{ext}^i_{\mathcal{A}_{\mathfrak{g}}}(X_\mu, X_\lambda) = 0$ unless the difference between the weights of X_λ and X_μ is i. □

6. On the structure of $\mathcal{A}_{\mathfrak{g}}$

It is a result of [**BGS**] that for any r the Koszulity of $\mathcal{A}_{\mathfrak{g}}^r$ implies that $\mathcal{A}_{\mathfrak{g}}^r$ is a quadratic algebra generated by $(\mathcal{A}_{\mathfrak{g}}^r)_0$ and $(\mathcal{A}_{\mathfrak{g}}^r)_1$. That is, $\mathcal{A}_{\mathfrak{g}}^r \simeq T_{(\mathcal{A}_{\mathfrak{g}}^r)_0}((\mathcal{A}_{\mathfrak{g}}^r)_1)/(R^r)$, where (R^r) is the two-sided ideal generated by some $(\mathcal{A}_{\mathfrak{g}}^r)_0$-bimodule R^r in $(\mathcal{A}_{\mathfrak{g}}^r)_1 \otimes_{(\mathcal{A}_{\mathfrak{g}}^r)_0} (\mathcal{A}_{\mathfrak{g}}^r)_1$. Moreover, it is easy to see that $\mathcal{A}_{\mathfrak{g}}$ is isomorphic to the quotient $T_{(\mathcal{A}_{\mathfrak{g}})_0}((\mathcal{A}_{\mathfrak{g}})_1)/(R)$, where $R = \varinjlim R^r$. In this section we describe $(\mathcal{A}_{\mathfrak{g}})_1$ and R.

In what follows we fix inclusions $S_n \subset S_{n+1}$ such that S_{n+1} acts on the set $\{1, 2, \ldots, n+1\}$ and S_n is the stabilizer of $n+1$. We start with the following lemma.

Lemma 6.1. *If $\mathfrak{g} = sl(\infty)$, then $\mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{p-1,q-1})$ as a left module over*

$\mathbb{C}[S_{p-1} \times S_{q-1}]$ is generated by the contractions

$$\phi_{i,j} : T^{p,q} \to T^{p-1,q-1},$$
$$v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q \mapsto \langle v_i, w_j \rangle (v_1 \otimes \cdots \hat{v}_i \cdots \otimes v_p \otimes w_1 \otimes \cdots \hat{w}_j \cdots \otimes w_q).$$

If $\mathfrak{g} = o(\infty)$ or $sp(\infty)$, then $\mathrm{Hom}_{\mathfrak{g}}(T^p, T^{p-2})$ as a left module over $\mathbb{C}[S_{p-2}]$ is generated by the contractions

$$\psi_{i,j} : T^p \to T^{p-2},$$
$$v_1 \otimes \cdots \otimes v_p \mapsto \langle v_i, v_j \rangle (v_1 \otimes \cdots \hat{v}_i \cdots \otimes \cdots \hat{v}_j \cdots \otimes v_p),$$

where $\langle \cdot\,, \cdot \rangle$ stands for the symmetric bilinear form on V for $\mathfrak{g} = o(\infty)$, and the symplectic bilinear form on V for $\mathfrak{g} = sp(\infty)$.

Proof. Let $\mathfrak{g} = sl(\infty)$ and $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{p-1,q-1})$. Theorem 3.2 in [**PStyr**] claims that $\mathrm{soc}(T^{p,q}) = \cap_{i \le p, j \le q} \ker \phi_{i,j}$; moreover, the same result implies that $\mathrm{soc}(T^{p,q}) \subset \ker \varphi$. Define

$$\Phi : T^{p,q} \to \bigoplus_{i \le p, j \le q} T^{p-1,q-1}$$

as the direct sum $\bigoplus_{i,j} \phi_{i,j}$. Then there exists $\alpha : \bigoplus_{i \le p, j \le q} T^{p-1,q-1} \to T^{p-1,q-1}$ such that $\varphi = \alpha \circ \Phi$. But $\alpha = \bigoplus_{i,j} \alpha_{i,j}$ for some $\alpha_{i,j} \in \mathbb{C}[S_{p-1} \times S_{q-1}]$. Therefore $\varphi = \sum_{i,j} \alpha_{i,j} \phi_{i,j}$. This proves the lemma for $\mathfrak{g} = sl(\infty)$.

We leave the proof in the cases $\mathfrak{g} = o(\infty)$, $sp(\infty)$ to the reader. □

Let $\mathfrak{g} = sl(\infty)$. Recall that $(\mathcal{A}_{\mathfrak{g}})_i^{p,q} = \mathrm{Hom}_{\mathfrak{g}}(T^{p,q}, T^{p-i,q-i})$ and that $(\mathcal{A}_{\mathfrak{g}})_0^{p,q} = \mathbb{C}[S_p \times S_q]$.

Lemma 6.2. *Let $\mathfrak{g} = sl(\infty)$.*

1. *$(\mathcal{A}_{\mathfrak{g}})_1^{p,q}$ is isomorphic to $\mathbb{C}[S_p \times S_q]$ as a right $(\mathcal{A}_{\mathfrak{g}})_0^{p,q}$-module, and the structure of a left $(\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1}$-module is given by left multiplication via the fixed inclusion*

$$(\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1} = \mathbb{C}[S_{p-1} \times S_{q-1}] \subset \mathbb{C}[S_p \times S_q] = (\mathcal{A}_{\mathfrak{g}})_0^{p,q}.$$

2. *We have*

$$(\mathcal{A}_{\mathfrak{g}})_1 \otimes_{(\mathcal{A}_{\mathfrak{g}})_0} (\mathcal{A}_{\mathfrak{g}})_1 = \bigoplus_{p,q} ((\mathcal{A}_{\mathfrak{g}})_1^{p-1,q-1} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-1,q-1}} (\mathcal{A}_{\mathfrak{g}})_1^{p,q}),$$

where $(\mathcal{A}_\mathfrak{g})_1^{p-1,q-1} \otimes_{(\mathcal{A}_\mathfrak{g})_0^{p-1,q-1}} (\mathcal{A}_\mathfrak{g})_1^{p,q}$ *is isomorphic to* $\mathbb{C}[S_p \times S_q]$*. Moreover,* $(\mathcal{A}_\mathfrak{g})_1^{p-1,q-1} \otimes_{(\mathcal{A}_\mathfrak{g})_0^{p-1,q-1}} (\mathcal{A}_\mathfrak{g})_1^{p,q}$ *is a* $(\mathbb{C}[S_{p-2} \times S_{q-2}], \mathbb{C}[S_p \times S_q])$*-bimodule via the embeddings* $\mathbb{C}[S_{p-2} \times S_{q-2}] \subset \mathbb{C}[S_{p-1} \times S_{q-1}] \subset \mathbb{C}[S_p \times S_q]$*.*

Proof. It is clear that all contractions $\phi_{i,j} \in (\mathcal{A}_\mathfrak{g})_1^{p,q}$ can be obtained from $\phi_{p,q}$ via the right $\mathbb{C}[S_p \times S_q]$-module structure of $(\mathcal{A}_\mathfrak{g})_1^{p,q}$. Thus by Lemma 6.1, as a $\mathbb{C}[S_p \times S_q]$-bimodule, $(\mathcal{A}_\mathfrak{g})_1^{p,q}$ is generated by the single contraction $\phi_{p,q}$. Moreover, $(\mathcal{A}_\mathfrak{g})_1^{p,q}$ is a free right $\mathbb{C}[S_p \times S_q]$-module of rank 1. Indeed, if for some $a_\sigma \in \mathbb{C}$

$$\sum_{\sigma \in S_p \times S_q} a_\sigma \phi_{p,q} \sigma = 0,$$

then for all $v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q \in T^{p,q}$

$$\begin{aligned} 0 &= \sum_{\sigma \in S_p \times S_q} a_\sigma \phi_{p,q} \sigma (v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q) \\ &= \sum_{\substack{\sigma = (\sigma_1, \sigma_2) \\ \in S_p \times S_q}} a_\sigma \langle v_{\sigma_1(p)}, w_{\sigma_2(q)} \rangle (v_{\sigma_1(1)} \otimes \cdots \otimes v_{\sigma_1(p-1)} \otimes w_{\sigma_2(1)} \otimes \cdots \otimes w_{\sigma_2(q-1)}), \end{aligned}$$

and hence $a_\sigma = 0$ for all $\sigma \in S_p \times S_q$. Finally, for any $\sigma \in S_{p-1} \times S_{q-1}$ we have

$$\sigma \phi_{p,q} = \phi_{p,q} \sigma.$$

This implies part (1). Part (2) is a direct corollary of part (1). □

Lemma 6.3. *Let* $\mathfrak{g} = sl(\infty)$*. Let* $S \simeq S_2 \times S_2$ *denote the subgroup of* $S_p \times S_q$ *generated by* $(p, p-1)_l$ *and* $(q, q-1)_r$*, where* $(i,j)_l$ *and* $(i,j)_r$ *stand for the transpositions in* S_p *and* S_q*, respectively. Then* $R = \bigoplus_{p,q} R^{p,q}$*, where*

$$R^{p,q} = (triv \boxtimes sgn \,\oplus\, sgn \boxtimes triv) \otimes_{\mathbb{C}[S]} \mathbb{C}[S_p \times S_q],$$

and triv and sgn denote respectively the trivial and sign representations of S_2*.*

Proof. The statement is equivalent to the equality of $R^{p,q}$ and the right $\mathbb{C}[S_p \times S_q]$-module

$$(1+(p,p-1)_l)(1-(q,q-1)_r)\mathbb{C}[S_p \times S_q] \oplus (1-(p,p-1)_l)(1+(q,q-1)_r)\mathbb{C}[S_p \times S_q].$$

We have the obvious relations in $\mathcal{A}_{sl(\infty)}$

$$\phi_{p-1,q-1}\phi_{p,q} = \phi_{p-1,q-1}\phi_{p,q}(p,p-1)_l(q,q-1)_r,$$
$$\phi_{p-1,q-1}\phi_{p,q}(p,p-1)_l = \phi_{p-1,q-1}\phi_{p,q}(q,q-1)_r.$$

Therefore $R^{p,q}$ contains the module

$$(1+(p,p-1)_l)(1-(q,q-1)_r)\mathbb{C}[S_p\times S_q]\oplus(1-(p,p-1)_l)(1+(q,q-1)_r)\mathbb{C}[S_p\times S_q],$$

which has dimension $\frac{p!q!}{2}$. On the other hand, it is easy to see that

$$\begin{aligned}\dim R^{p,q} &= \dim\left((\mathcal{A}_\mathfrak{g})_1^{p-1,q-1}\otimes_{(\mathcal{A}_\mathfrak{g})_0^{p-1,q-1}}(\mathcal{A}_\mathfrak{g})_1^{p,q}\right) - \dim(\mathcal{A}_\mathfrak{g})_2^{p,q}\\ &= \frac{(p-1)!(q-1)!p!q!}{(p-1)!(q-1)!} - \frac{p!q!}{2}\\ &= \frac{p!q!}{2}.\end{aligned}$$

Hence the statement. □

Corollary 6.4. *Let $\mathfrak{g} = sl(\infty)$. Then $\mathcal{A}^r_\mathfrak{g}$ is Koszul self-dual, i.e. $\mathcal{A}^r_\mathfrak{g} \simeq ((\mathcal{A}^r_\mathfrak{g})^!)^{\mathrm{opp}}$. Furthermore, $\mathcal{A}_\mathfrak{g} \simeq (\mathcal{A}^!_\mathfrak{g})^{\mathrm{opp}}$, where $\mathcal{A}^!_\mathfrak{g} := \varinjlim (\mathcal{A}^r_\mathfrak{g})^!$.*

Proof. By definition, we have $(\mathcal{A}^r_\mathfrak{g})^! = T_{(\mathcal{A}^r_\mathfrak{g})_0}((\mathcal{A}^r_\mathfrak{g})^*_1)/(R^{r\perp})$, where $(\mathcal{A}^r_\mathfrak{g})^*_1 = \mathrm{Hom}_{(\mathcal{A}^r_\mathfrak{g})_0}((\mathcal{A}^r_\mathfrak{g})_1, (\mathcal{A}^r_\mathfrak{g})_0)$, [**BGS**]. Note that $((A_\mathfrak{g})_1^{p,q})^*$ is a $((A_\mathfrak{g})_0^{p,q}, (A_\mathfrak{g})_0^{p-1,q-1})$-bimodule. Moreover, Lemma 6.2 (1) implies an isomorphism of bimodules

$$((A_\mathfrak{g})_1^{p,q})^* \simeq \mathbb{C}[S_p\times S_q].$$

Hence we have an isomorphism of $((\mathcal{A}^r_\mathfrak{g})^!)_0^{\mathrm{opp}}$-bimodules

$$((\mathcal{A}^r_\mathfrak{g})^!)_1^{\mathrm{opp}} \simeq (\mathcal{A}^r_\mathfrak{g})_1.$$

One can check that $R^\perp = \bar{R}$, where $\bar{R} := \oplus \bar{R}^{p,q}$, and the modules $\bar{R}^{p,q}$ are defined via the decomposition of $(\mathcal{A}_\mathfrak{g})_0^{p,q}$-modules

$$(\mathcal{A}_\mathfrak{g})_1^{p-1,q-1}\otimes_{(\mathcal{A}_\mathfrak{g})_0^{p-1,q-1}}(\mathcal{A}_\mathfrak{g})_1^{p,q} = R^{p,q}\oplus\bar{R}^{p,q}.$$

Therefore $((\mathcal{A}^r)^!_\mathfrak{g})^{\mathrm{opp}} \simeq T_{(\mathcal{A}^r_\mathfrak{g})_0}((\mathcal{A}^r_\mathfrak{g})_1)/(\bar{R}^r)$. Now consider the automorphism σ of $\mathbb{C}[S_p\times S_q]$ defined for all p and q by $\sigma(s,t) = sgn(t)(s,t)$ for all $s\in S_p$,

$t \in S_q$. Recall that $(\mathcal{A}_\mathfrak{g})_0 = \bigoplus_{p,q} \mathbb{C}[S_p \times S_q]$. Extend σ to an automorphism of $T_{(\mathcal{A}_\mathfrak{g})_0}((\mathcal{A}_\mathfrak{g})_1)$ by setting $\sigma(x) = x$ for any $x \in (\mathcal{A}_\mathfrak{g})_1$. One immediately observes that $\sigma(R^{p,q}) = \bar{R}^{p,q}$, hence σ induces an isomorphism $\mathcal{A}_\mathfrak{g}^r \simeq ((\mathcal{A}_\mathfrak{g}^r)^!)^{\mathrm{opp}}$, and clearly also an isomorphism $\mathcal{A}_\mathfrak{g} \simeq (\mathcal{A}_\mathfrak{g}^!)^{\mathrm{opp}}$. □

For a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$, we set $\mu^\perp := (s = \#\mu, \#(\mu_1 - 1, \mu_2 - 1, \ldots), \ldots)$, or in terms of Young diagrams, $\mu^\perp$ is the conjugate partition obtained from μ by interchanging rows and columns.

Corollary 6.5. *Let $\mathfrak{g} = sl(\infty)$, and for a pair of partitions $\nu = (\nu^1, \nu^2)$ take $\nu^\perp := (\nu^1, (\nu^2)^\perp)$. Then $\dim \mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\mu)$ equals the multiplicity of $V_{\lambda^\perp}$ in $\mathrm{soc}^i(\tilde{V}_{\mu^\perp})/\mathrm{soc}^{i-1}(\tilde{V}_{\mu^\perp})$, as computed in [**PStyr**, Theorem 2.3].*

Proof. The statement follows from [**BGS**, Theorem 2.10.1] applied to $\mathcal{A}_\mathfrak{g}^r$ for sufficiently large r. Indeed, this result implies that $\mathrm{Ext}_{\mathcal{A}_\mathfrak{g}}((\mathcal{A}_\mathfrak{g})_0, (\mathcal{A}_\mathfrak{g})_0)$ is isomorphic to $(\mathcal{A}_\mathfrak{g}^!)^{\mathrm{opp}}$ as a graded algebra. Moreover, the simple $\mathcal{A}_\mathfrak{g}$-module X_λ (which is the image of V_λ under the antiequivalence of Corollary 5.2) is isomorphic to $(\mathcal{A}_\mathfrak{g})_0 \mathbb{Y}_\lambda$, where $\mathbb{Y}_\lambda$ is the product of Young projectors corresponding to the partitions λ^1 and λ^2. This follows immediately from the fact that $\mathbb{Y}_\lambda$ is a primitive idempotent in $(\mathcal{A}_\mathfrak{g})_0$ and hence also in $\mathcal{A}_\mathfrak{g}$, see for example [**CR**, Theorem 54.5]. The projective cover $\tilde{X}_\lambda$ of X_λ is isomorphic to $\mathcal{A}_\mathfrak{g}\mathbb{Y}_\lambda$. Therefore we have

$$\dim \mathrm{Ext}^i_{\mathbb{T}_\mathfrak{g}}(V_\lambda, V_\mu) = \dim \mathrm{Ext}^i_{\mathcal{A}_\mathfrak{g}}(X_\mu, X_\lambda) = \dim \mathbb{Y}_\lambda (\mathcal{A}_\mathfrak{g}^!)_i^{\mathrm{opp}} \mathbb{Y}_\mu.$$

By Corollary 6.4,

$$\dim \mathbb{Y}_\lambda (\mathcal{A}_\mathfrak{g}^!)_i^{\mathrm{opp}} \mathbb{Y}_\mu = \dim \mathbb{Y}_{\lambda^\perp} (\mathcal{A}_\mathfrak{g})_i \mathbb{Y}_{\mu^\perp}.$$

Furthermore, $\dim \mathbb{Y}_{\lambda^\perp}(\mathcal{A}_\mathfrak{g})_i \mathbb{Y}_{\mu^\perp}$ equals the multiplicity of $X_{\lambda^\perp}$ in the module $\mathrm{rad}^{i-1} \tilde{X}_{\mu^\perp}/\mathrm{rad}^i \tilde{X}_{\mu^\perp}$ [**CR**, Theorem 54.15]), which coincides with the multiplicity of $V_{\lambda^\perp}$ in $\mathrm{soc}^i(\tilde{V}_{\mu^\perp})/\mathrm{soc}^{i-1}(\tilde{V}_{\mu^\perp})$. □

Corollary 6.6. *The blocks of the category $\mathbb{T}_{sl(\infty)}$ are parametrized by $\mathbb{Z}$. In particular,*

1. *V_λ and V_μ belong to the block $\mathbb{T}_{sl(\infty)}(i)$ for $i \in \mathbb{Z}$ if and only if $|\lambda^1| - |\lambda^2| = |\mu^1| - |\mu^2| = i$.*
2. *Two blocks $\mathbb{T}_{sl(\infty)}(i)$ and $\mathbb{T}_{sl(\infty)}(j)$ are equivalent if and only if $i = \pm j$.*

Proof. 1. The fact that $\tilde{V}_{(\mu^1,\mu^2)}$ is an injective hull of $V_{(\mu^1,\mu^2)}$, together with Theorem 2.3 in [**PStyr**], implies that $\mathrm{Ext}^1_{\mathbb{T}_\mathfrak{g}}(V_{(\mu^1,\mu^2)}, V_{(\lambda^1,\lambda^2)}) \neq 0$ iff $\mu^1 \in (\lambda^1)^+$ and $\mu^2 \in (\lambda^2)^+$. More precisely, Theorem 2.3 in [**PStyr**] computes the multiplicities of the constituents of the socle of $\tilde{V}_\lambda/V_\lambda$, and a simple module has nonzero $\mathrm{Ext}^1_{\mathbb{T}_\mathfrak{g}}$ with V_λ if and only if it is isomorphic to a submodule of $\tilde{V}_\lambda/V_\lambda$. Consider the minimal equivalence relation on pairs of partitions for which (λ^1, λ^2) and (μ^1, μ^2) are equivalent whenever $\mu^1 \in (\lambda^1)^+$ and $\mu^2 \in (\lambda^2)^+$. It is a simple exercise to show that then $\lambda = (\lambda^1, \lambda^2)$ and $\mu = (\mu^1, \mu^2)$ are equivalent if and only if $|\lambda^1| - |\lambda^2| = |\mu^1| - |\mu^2|$. The first assertion follows.

2. The functor $(\,\cdot\,)_*$ establishes an equivalence of $\mathbb{T}_{sl(\infty)}(i)$ and $\mathbb{T}_{sl(\infty)}(-i)$. To see that $\mathbb{T}_{sl(\infty)}(i)$ and $\mathbb{T}_{sl(\infty)}(j)$ are inequivalent for $i \neq \pm j$, assume without loss of generality that $i > 0$, $j \geq 0$. Then the isomorphism classes of simple injective objects in $\mathbb{T}_{sl(\infty)}(i)$ are parametrized by the partitions of i, since $\{V_{(\lambda^1,0)} \mid |\lambda^1| = i\}$ represents the set of isomorphism classes of simple injective objects in $\mathbb{T}_{sl(\infty)}(i)$. As the sets $\{V_{(\lambda^1,0)} \mid |\lambda^1| = i\}$ and $\{V_{(\lambda^1,0)} \mid |\lambda^1| = j\}$ have different cardinalities for $i \neq j$ except the case $i = 1, j = 0$, the assertion follows in other cases. Each of the blocks $\mathbb{T}_{sl(\infty)}(0)$ and $\mathbb{T}_{sl(\infty)}(1)$ has a single simple injective module, up to isomorphism. However, V has nontrivial extensions by both $V_{((2),(1))}$ and $V_{((1,1),(1))}$, whereas $\mathbb{C}$ has a nontrivial extension only by $V_{((1),(1))}$. This completes the proof.

□

Now we proceed to describing the structure of $\mathcal{A}_\mathfrak{g}$ for $\mathfrak{g} = o(\infty)$ and $sp(\infty)$. Recall that $(\mathcal{A}_\mathfrak{g})^p_i = \mathrm{Hom}_\mathfrak{g}(T^p, T^{p-2i})$. and $(\mathcal{A}_\mathfrak{g})^p_0 = \mathbb{C}[S_p]$. Let $S_{p-2} \subset S_p$ denote the stabilizer of p and $p-1$, and let $S' \subset S_p$ be the subgroup generated by the transposition $(p-1, p)$.

Lemma 6.7. *We have*

$$(\mathcal{A}_\mathfrak{g})^p_1 \simeq triv \otimes_{\mathbb{C}[S']} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = o(\infty)$$

and

$$(\mathcal{A}_\mathfrak{g})^p_1 \simeq sgn \otimes_{\mathbb{C}[S']} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = sp(\infty).$$

In both cases left multiplication by $\mathbb{C}[S_{p-2}]$ is well defined, as S' centralizes S_{p-2}.

Proof. Lemma 6.1 implies that the contraction $\psi_{p-1,p}$ generates $(\mathcal{A}_{\mathfrak{g}})_1^p$ as a right $\mathbb{C}[S_p]$-module. Then the statement follows from the relation

$$\psi_{p-1,p} = \pm\psi_{p-1,p}(p, p-1),$$

where the sign is $+$ for $\mathfrak{g} = o(\infty)$ and $-$ for $\mathfrak{g} = sp(\infty)$. □

Corollary 6.8. *Let $\mathfrak{g} = o(\infty)$ or $sp(\infty)$. Then*

$$(\mathcal{A}_{\mathfrak{g}})_1^{p-2} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-2}} (\mathcal{A}_{\mathfrak{g}})_1^p \simeq L_{\mathfrak{g}} \otimes_{\mathbb{C}[S]} \mathbb{C}[S_p],$$

where $S \simeq S_2 \times S_2$ is the subgroup generated by $(p, p-1)$ and $(p-2, p-3)$ and

$$L_{\mathfrak{g}} = \begin{cases} triv & \text{for } \mathfrak{g} = o(\infty) \\ sgn \boxtimes sgn & \text{for } \mathfrak{g} = sp(\infty). \end{cases}$$

To describe R, write $R = \bigoplus_p R^p$, where $R^p \subset (\mathcal{A}_{\mathfrak{g}})_1^{p-2} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-2}} (\mathcal{A}_{\mathfrak{g}})_1^p$.

We will need the following decompositions of S_4-modules:

$$triv \otimes_{\mathbb{C}[S]} \mathbb{C}[S_4] = X_{(2,1,1)} \oplus X_{(2,2)} \oplus X_{(4)}, \quad (6.1)$$

$$(sgn \boxtimes sgn) \otimes_{\mathbb{C}[S]} \mathbb{C}[S_4] = X_{(3,1)} \oplus X_{(2,2)} \oplus X_{(1,1,1,1)}. \quad (6.2)$$

Lemma 6.9. *Let $S'' \subset S_p$ be the subgroup isomorphic to S_4 that fixes 1, 2,…, $p-4$. Then*

$$R^p \simeq X_{(2,1,1)} \otimes_{\mathbb{C}[S'']} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = o(\infty),$$

and

$$R^p \simeq X_{(3,1)} \otimes_{\mathbb{C}[S'']} \mathbb{C}[S_p] \qquad \text{for } \mathfrak{g} = sp(\infty).$$

Proof. Let us deal with the case of $o(\infty)$. We consider the following Young projectors in $S'' \simeq S_4$

$$\mathbb{Y}_{(2,1,1)} = (1+(p-1,p))(1-(p,p-2)-(p,p-3)-(p-2,p-3)+(p,p-2,p-3)+(p,p-3,p-2)),$$

$\mathbb{Y}_{(2,2)} = (1+(p,p-1))(1+(p-2,p-3))(1-(p-2,p))(1-(p-1,p-3)),$
and
$$\mathbb{Y}_{(4)} = \sum_{s\in S''} s.$$
By Equation (6.1) we have
$$R^p \subset (\mathcal{A}_{\mathfrak{g}})_1^{p-2} \otimes_{(\mathcal{A}_{\mathfrak{g}})_0^{p-2}} (\mathcal{A}_{\mathfrak{g}})_1^p = \mathbb{Y}_{(2,1,1)}\mathbb{C}[S_p] \oplus \mathbb{Y}_{(2,2)}\mathbb{C}[S_p] \oplus \mathbb{Y}_{(4)}\mathbb{C}[S_p].$$
By direct inspection one can check that
$$\begin{aligned} \psi_{p-3,p-2}\psi_{p-1,p}\mathbb{Y}_{(2,1,1)} &= 0, \\ \psi_{p-3,p-2}\psi_{p-1,p}\mathbb{Y}_{(2,2)} &= 2\psi_{p-3,p-2}\psi_{p-1,p} - 2\psi_{p-3,p}\psi_{p-1,p-2}, \\ \psi_{p-3,p-2}\psi_{p-1,p}\mathbb{Y}_{(4)} &= 4\psi_{p-3,p-2}\psi_{p-1,p}. \end{aligned}$$
The statement follows for $o(\infty)$.

We leave the case of $sp(\infty)$ to the reader. □

Corollary 6.10. $\mathcal{A}_{sp(\infty)} \simeq \mathcal{A}_{o(\infty)}$.

Proof. We use the automorphism σ of $\mathbb{C}[S_p]$ which sends s to $sgn(s)s$. □

Corollary 6.11. *The categories $\mathbb{T}_{o(\infty)}$ and $\mathbb{T}_{sp(\infty)}$ are equivalent.*

In [**Sr**] a tensor functor $\mathbb{T}_{o(\infty)} \to \mathbb{T}_{sp(\infty)}$ establishing an equivalence of tensor categories is constructed using the Lie superalgebra $osp(\infty,\infty)$.

Proposition 6.12. *$\mathbb{T}_{o(\infty)}$ and $\mathbb{T}_{sp(\infty)}$ have two inequivalent blocks $\mathbb{T}_{\mathfrak{g}}^{ev}$ and $\mathbb{T}_{\mathfrak{g}}^{odd}$ generated by all V_λ with $|\lambda|$ even and odd, respectively.*

Proof. Due to the previous corollary it suffices to consider the case $\mathfrak{g} = o(\infty)$. As follows from [**PStyr**], $\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g}}}(V_\mu, V_\lambda) \neq 0$ if and only if $\mu \in \lambda^{++}$, where
$$\begin{aligned} \lambda^{++} := \{\text{partitions } \lambda' \,|\lambda_i \leq \lambda'_i \text{ for all } i,\ |\lambda'| = |\lambda| + 2, \\ \lambda'_j \neq \lambda_j \text{ and } \lambda'_k \neq \lambda_k \text{ for } j \neq k \text{ implies } \lambda_j \neq \lambda_k\}. \end{aligned}$$
Note that the partitions in λ^{++} are those which arise from λ via the Pieri rule for tensoring with $S^2(V)$. Consider the minimal equivalence relation on partitions for which λ and μ are equivalent whenever $\mu \in \lambda^{++}$. One can check that there are exactly two equivalence classes which are determined by the parity of $|\lambda|$.

To show that $\mathbb{T}_{\mathfrak{g}}^{ev}$ and $\mathbb{T}_{\mathfrak{g}}^{odd}$ are not equivalent observe that all simple injective modules in $\mathbb{T}_{\mathfrak{g}}$ correspond to partitions μ with $\mu_1 = \cdots = \mu_s = 1$, or equivalently are isomorphic to the exterior powers $\Lambda^s(V)$ of the standard module. If $s \geq 1$ then $\Lambda^s(V)$ has nontrivial extensions by two non-isomorphic simple modules, namely $V_{(3,1,\ldots,1)}$ and $V_{(2,1,1,\ldots,1)}$. The trivial module on the other hand has a nontrivial extension by only $S^2(V) = V_{(2)}$. Therefore $\mathbb{T}_{\mathfrak{g}}^{ev}$ contains a simple injective module admitting a nontrivial extension with only one simple module, whereas $\mathbb{T}_{\mathfrak{g}}^{odd}$ does not contain such a simple injective module. □

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