A Koszul category of representations of finitary Lie algebras

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Abstract

We find for each simple finitary Lie algebra $\mathfrak{g}$ a category $\mathcal{T}_\mathfrak{g}$ of integrable modules in which the tensor product of copies of the natural and conatural modules are injective. The objects in $\mathcal{T}_\mathfrak{g}$ can be defined as the finite length absolute weight modules, where by absolute weight module we mean a module which is a weight module for every splitting Cartan subalgebra of $\mathfrak{g}$. The category $\mathcal{T}_\mathfrak{g}$ is Koszul in the sense that it is antiequivalent to the category of locally unitary finite-dimensional modules over a certain direct limit of finite-dimensional Koszul algebras. We describe these finite-dimensional algebras explicitly. We also prove an equivalence of the categories $\mathcal{T}_{o(\infty)}$ and $\mathcal{T}_{sp(\infty)}$ corresponding respectively to the orthogonal and symplectic finitary Lie algebras $o(\infty)$, $sp(\infty)$.

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1. Introduction

The classical simple complex Lie algebras $\text{sl}(n)$, $\text{o}(n)$, $\text{sp}(2n)$ have several natural infinite-dimensional versions. In this paper we concentrate on the “smallest possible” such versions: the direct limit Lie algebras $\text{sl}(\infty) := \varinjlim (\text{sl}(n))_{n \in \mathbb{Z}^>2}$, $\text{o}(\infty) := \varinjlim (\text{o}(n))_{n \in \mathbb{Z}^\geq3}$, $\text{sp}(\infty) := \varinjlim (\text{sp}(2n))_{n \in \mathbb{Z}^\geq2}$.

From a traditional finite-dimensional point of view, these Lie algebras are a suitable language for various stabilization phenomena, for instance stable branching laws as studied by R. Howe, E.-C. Tan and J. Willenbring [HTW]. The direct limit Lie algebras $\text{sl}(\infty)$, $\text{o}(\infty)$, $\text{sp}(\infty)$ admit many characterizations: for instance, they represent (up to isomorphism) the simple finitary (locally finite) complex Lie algebras [B, BSt]. Alternatively, these Lie algebras are the only three locally simple locally finite complex Lie algebras which admit a root decomposition [PStr].

Several attempts have been made to build a basic representation theory for $\mathfrak{g} = \text{sl}(\infty)$, $\text{o}(\infty)$, $\text{sp}(\infty)$. As the only simple finite-dimensional representation of $\mathfrak{g}$ is the trivial one, one has to study infinite-dimensional representations. On the other hand, it is still possible to study representations which are close analogs of finite-dimensional representations. Such a representation should certainly be integrable, i.e. it should be isomorphic to a direct sum of finite-dimensional representations when restricted to any simple finite-dimensional subalgebra.

The first phenomenon one encounters when studying integrable representations of $\mathfrak{g}$ is that they are not in general semisimple. This phenomenon has been studied in [PStyr] and [PS], but it had not previously been placed within a known more general framework for non-semisimple categories. The main purpose of the present paper is to show that the notion of Koszulity for a category of modules over a graded ring, as defined by A. Beilinson, V. Ginzburg and W. Soergel in [BGS], provides an excellent tool for the study of integrable representations of $\mathfrak{g} = \text{sl}(\infty)$, $\text{o}(\infty)$, $\text{sp}(\infty)$.

In this paper we introduce the category $\mathcal{T}_\mathfrak{g}$ of tensor $\mathfrak{g}$-modules. The objects of $\mathcal{T}_\mathfrak{g}$ are defined at first by the equivalent abstract conditions of Theorem 3.4. Later we show in Corollary 4.6 that the objects of $\mathcal{T}_\mathfrak{g}$ are nothing but finite length submodules of a direct sum of several copies of the tensor algebra $T$ of the natural and conatural representations. In the finite-dimensional case, i.e. for $\text{sl}(n)$, $\text{o}(n)$, or $\text{sp}(2n)$, the appropriate tensor algebra is a cornerstone of the theory of finite-dimensional representations (Schur-Weyl duality, etc.). In the infinite-dimensional case, the tensor al-
algebra $T$ was studied by Penkov and K. Styrkas in [PStyr]; nevertheless its indecomposable direct summands were not understood until now from a categorical point of view.

We prove that these indecomposable modules are precisely the indecomposable injectives in the category $T_g$. Furthermore, the category $T_g$ is Koszul in the following sense: $T_g$ is antiequivalent to the category of locally unitary finite-dimensional modules over an algebra $A_g$ which is a direct limit of finite-dimensional Koszul algebras (see Proposition 5.1 and Theorem 5.5).

Moreover, we prove in Corollary 6.4 that for $\mathfrak{g} = sl(\infty)$ the Koszul dual algebra $(A_g^!)^{opp}$ is isomorphic to $A_g$. This together with the main result of [PStyr] allows us to give an explicit formula for the Ext group between any two simple objects of $T_g$ when $\mathfrak{g} = sl(\infty)$. For the cases of $\mathfrak{g} = o(\infty)$ and $\mathfrak{g} = sp(\infty)$ we discover another interesting fact: the algebras $A_{o(\infty)}$ and $A_{sp(\infty)}$ are isomorphic. This yields an equivalence of categories $T_{o(\infty)} \simeq T_{sp(\infty)}$, which is Corollary 6.11.

In summary, the results of the present paper show how the non-semisimplicity of tensor modules arising from the limit process $n \to \infty$ falls strikingly into the general Koszul pattern discovered by Beilinson, Ginzburg and Soergel. This enables us to uncover the structure of the category of tensor representations of $\mathfrak{g}$.

Since the present paper has been submitted there have been several developments. First, in [SS] the categories $T_g$ have been studied from a different perspective. In particular, it is shown there that these categories satisfy important universality properties in the class of abelian symmetric monoidal categories.

In [PS1] categories of tensor modules have been introduced for a larger class of infinite-dimensional Lie algebras, and it has been shown that these categories are equivalent to $T_g$ for appropriate $\mathfrak{g}$. In [Sr] results from the present paper are generalized to the case of classical Lie superalgebras.

Finally, in [FPS] the category $T_{sl(\infty)}$ has been used to categorify the boson-fermion correspondence.

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2. Preliminaries

The ground field is $\mathbb{C}$. By $S_n$ we denote the $n$-th symmetric group, and by $\mathbb{C}[S_n]$ its group algebra. The sign $\otimes$ stands for $\otimes_{\mathbb{C}}$, and the sign $\oplus$ stands for the semidirect sum of Lie algebras. We denote by $(\cdot)^*$ the algebraic dual, i.e. $\text{Hom}_{\mathbb{C}}(\cdot, \mathbb{C})$.

Let $\mathfrak{g}$ be one of the infinite-dimensional simple finitary Lie algebras, $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, or $\mathfrak{sp}(\infty)$. Here $\mathfrak{sl}(\infty) = \varinjlim \mathfrak{sl}(n)$, $\mathfrak{o}(\infty) = \varinjlim \mathfrak{o}(n)$, $\mathfrak{sp}(\infty) = \varinjlim \mathfrak{sp}(2n)$, where in each direct limit the inclusions can be chosen as “left upper corner” inclusions. We consider the “exhaustion” $\mathfrak{g} = \varprojlim \mathfrak{g}_n$ to be fixed, taking $\mathfrak{g}_n = \mathfrak{sl}(n)$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$, $\mathfrak{g}_n = \mathfrak{o}(2n)$ or $\mathfrak{g}_n = \mathfrak{o}(2n+1)$ for $\mathfrak{g} = \mathfrak{o}(\infty)$, and $\mathfrak{g}_n = \mathfrak{sp}(2n)$ for $\mathfrak{g} = \mathfrak{sp}(\infty)$. By $G_n$ we denote the adjoint group of $\mathfrak{g}_n$.

It is clear from the definition of $\mathfrak{g} = \mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$ that the notions of semisimple or nilpotent elements make sense: an element $g \in \mathfrak{g}$ is semisimple (respectively, nilpotent) if $g$ is semisimple (resp., nilpotent) as an element of $\mathfrak{g}_n$ for some $n$. In [NP, DPS], Cartan subalgebras of $\mathfrak{g}$ have been studied. In the present paper we need only the notion of a splitting Cartan subalgebra of $\mathfrak{g}$: this is a maximal toral (where toral means consisting of semisimple elements) subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g}$ is an $\mathfrak{h}$-weight module, i.e.

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha = \{ g \in \mathfrak{g} \mid [h, g] = \alpha(h)g \text{ for all } h \in \mathfrak{h} \}$. The set $\Delta := \{ \alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}^\alpha \neq 0 \}$ is the set of $\mathfrak{h}$-roots of $\mathfrak{g}$. More generally, if $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$ and $M$ is a $\mathfrak{g}$-module, $M$ is an $\mathfrak{h}$-weight module if

$$M = \bigoplus_{\alpha \in \mathfrak{h}^*} M^\alpha,$$

where $M^\alpha := \{ m \in M \mid h \cdot m = \alpha(h)m \text{ for all } h \in \mathfrak{h} \}$.

By $V$ we denote the natural representation of $\mathfrak{g}$; that is, $V = \varprojlim V_n$, where $V_n$ is the natural representation of $\mathfrak{g}_n$. We set also $V_* = \varinjlim V'_n$; this is the conatural representation of $\mathfrak{g}$. For $\mathfrak{g} = \mathfrak{o}(\infty)$, $\mathfrak{sp}(\infty)$, $V \cong V_*$, whereas $V \not\cong V_*$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$. Note that $V_*$ is a submodule of the algebraic dual...
$V^* = \text{Hom}_C(V, \mathbb{C})$ of $V$. Moreover, $\mathfrak{g} \subseteq V \otimes V_*$, and $sl(\infty)$ can be identified with the kernel of the contraction $\phi : V \otimes V_* \rightarrow \mathbb{C}$, while

$$
\mathfrak{g} \simeq \Lambda^2(V) \subset V \otimes V = V \otimes V_* \quad \text{for } \mathfrak{g} = o(\infty),
$$

$$
\mathfrak{g} \simeq S^2(V) \subset V \otimes V = V \otimes V_* \quad \text{for } \mathfrak{g} = sp(\infty).
$$

Let $\tilde{G}$ be the subgroup of $\text{Aut} V$ consisting of those automorphisms for which the induced automorphism of $V^*$ restricts to an automorphism of $V_*$. Then clearly $G \subseteq \tilde{G} \subseteq \text{Aut} \mathfrak{g}$, and moreover $\tilde{G} = \text{Aut} \mathfrak{g}$ for $\mathfrak{g} = o(\infty), sp(\infty)$ \cite[Corollary 1.6 (b)]{BBCM}. For $\mathfrak{g} = sl(\infty)$, the group $\tilde{G}$ has index 2 in $\text{Aut} \mathfrak{g}$: the quotient $\text{Aut} \mathfrak{g} / \tilde{G}$ is represented by the automorphism

$$
g \mapsto -g^t
$$

for $g \in sl(\infty)$ \cite[Corollary 1.2 (a)]{BBCM}.

It is essential to recall that if $\mathfrak{g} = sl(\infty), sp(\infty)$, all splitting Cartan subalgebras of $\mathfrak{g}$ are $\tilde{G}$-conjugate, while there are two $\tilde{G}$-conjugacy classes for $\mathfrak{g} = o(\infty)$. One class comes from the exhaustion of $o(\infty)$ as $\lim_{n \rightarrow \infty} o(2n)$, and the other from the exhaustion of the form $\lim_{n \rightarrow \infty} o(2n+1)$. For further details we refer the reader to \cite{DPS}. Here are the explicit forms of the root systems of $\mathfrak{g}$:

\begin{align*}
\{ \epsilon_i - \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0} \} & \quad \text{for } \mathfrak{g} = sl(\infty), \mathfrak{g}_n = sl(n), \\
\{ \pm \epsilon_i \pm \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0} \} \cup \{ \pm 2 \epsilon_i \mid i \in \mathbb{Z}_{>0} \} & \quad \text{for } \mathfrak{g} = sp(\infty), \mathfrak{g}_n = sp(2n), \\
\{ \pm \epsilon_i \pm \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0} \} & \quad \text{for } \mathfrak{g} = o(\infty), \mathfrak{g}_n = o(2n), \\
\{ \pm \epsilon_i \pm \epsilon_j \mid i \neq j \in \mathbb{Z}_{>0} \} \cup \{ \pm \epsilon_i \mid i \in \mathbb{Z}_{>0} \} & \quad \text{for } \mathfrak{g} = o(\infty), \mathfrak{g}_n = o(2n+1).
\end{align*}

Our usage of $\epsilon_i \in \mathfrak{h}^*$ is compatible with the standard usage of $\epsilon_i$ as a linear function on $\mathfrak{h} \cap \mathfrak{g}_n$ for all $n > i$.

In the present paper we study integrable $\mathfrak{g}$-modules $M$ for $\mathfrak{g} \simeq sl(\infty), o(\infty), sp(\infty)$. By definition, a $\mathfrak{g}$-module $M$ is integrable if $\dim \{ m, g \cdot m, g^2 \cdot m, \ldots \} < \infty$ for all $g \in \mathfrak{g}, m \in M$. More generally, if $M$ is any $\mathfrak{g}$-module, the set $\mathfrak{g}[M]$ of $M$-locally finite elements in $\mathfrak{g}$, that is

$$
\mathfrak{g}[M] := \{ g \in \mathfrak{g} \mid \dim \{ m, g \cdot m, g^2 \cdot m, \ldots \} < \infty \text{ for all } m \in M \},
$$

is a Lie subalgebra of $\mathfrak{g}$. This follows from the analogous fact for finite-dimensional Lie algebras, discovered and rediscovered by several mathemati-
cians [GQS, F, K]. We refer to \( g[M] \) as the Fernando-Kac subalgebra of \( M \).

By \( g\)-mod we denote the category of all \( g \)-modules, and following the notation of [PS], we let \( \text{Int}_g \) denote the category of integrable \( g \)-modules. We have the functor
\[
\Gamma_g : g\text{-mod} \to \text{Int}_g
\]
which takes an arbitrary \( g \)-module to its largest integrable submodule.

3. The category \( \mathcal{T}_g \)

If \( \gamma \in \text{Aut} \ g \) and \( M \) is a \( g \)-module, let \( M^\gamma \) denote the \( g \)-module twisted by \( \gamma \): that is, \( M^\gamma \) is equal to \( M \) as a vector space, and the \( g \)-module structure on \( M^\gamma \) is given by \( \gamma(g) \cdot m \) for \( m \in M^\gamma \) and \( g \in g \).

**Definition 3.1.**

1. A \( g \)-module \( M \) is called an absolute weight module if \( M \) is an \( h \)-weight module for every splitting Cartan subalgebra \( h \subset g \).
2. A \( g \)-module \( M \) is called \( \tilde{G} \)-invariant if for any \( \gamma \in \tilde{G} \) there is a \( g \)-isomorphism \( M^\gamma \cong M \).
3. A subalgebra of \( g \) is called finite corank if it contains the commutator of some finite-dimensional subalgebra of \( g \).

**Proposition 3.2.** Any absolute weight \( g \)-module is integrable.

**Proof.** Let \( M \) be an absolute weight \( g \)-module. Every semisimple element \( h \) of \( g \) lies in some splitting Cartan subalgebra \( h \subset g \), and since \( M \) is an \( h \)-weight module, we see that \( h \) acts locally finitely on \( M \). As \( g \) is generated by its semisimple elements, the Fernando-Kac subalgebra \( g[M] \) equals \( g \), i.e. \( M \) is integrable. \( \square \)

We define the category of absolute weight modules as the full subcategory of \( g \)-mod whose objects are the absolute weight modules. Proposition 3.2 shows that the category of absolute weight modules is in fact a subcategory of \( \text{Int}_g \).

**Lemma 3.3.** For each \( n \) one has \( \tilde{G} = G \cdot \tilde{G}'_n \), where
\[
\tilde{G}'_n := \{ \gamma \in \tilde{G} \mid \gamma(g) = g \text{ for all } g \in g_n \}.
\]
Proof. Let \( g \) be \( o(\infty) \) or \( sp(\infty) \), and let \( \gamma \in \tilde{G} \). Fix a basis \( \{ w_i \} \) of \( V_n \). There exists \( \gamma'' \in G \) such that \( (\gamma'')^{-1}(\gamma(w_i)) = w_i \) for all \( 1 \leq i \leq 2n \). Since \( g \subseteq V \otimes V \) and \( g_n = g \cap (V_n \otimes V_n) \), we see that \( (\gamma'')^{-1}\gamma \in \tilde{G}''_n \).

For \( g = sl(\infty) \), the analogous statement is as follows. In this case one has \( g_n = g \cap (V_n \otimes V_n^*) \). Fix dual bases \( \{ w_i \} \) and \( \{ w^*_i \} \) of \( V_n \) and \( V_n^* \), respectively. Then for any \( \gamma \in \tilde{G} \), there is an element \( \gamma'' \in G \) such that \( (\gamma'')^{-1}(\gamma(w_i)) = w_i \) and \( (\gamma'')^{-1}(\gamma(w^*_i)) = w^*_i \) for each \( 1 \leq i \leq n \). Therefore \( (\gamma'')^{-1}\gamma \in \tilde{G}''_n \). \( \square \)

**Theorem 3.4.** The following conditions on a \( g \)-module \( M \) of finite length are equivalent:

1. \( M \) is an absolute weight module.
2. \( M \) is a weight module for some splitting Cartan subalgebra \( h \subset g \) and \( M \) is \( \tilde{G} \)-invariant.
3. \( M \) is integrable and \( \text{Ann}_{\tilde{G}} m \) is finite corank for all \( m \in M \).

Proof. Let us show that (1) implies (3). We already proved in Proposition 3.2 that a \( g \)-module \( M \) satisfying (1) is integrable. Furthermore, it suffices to prove that \( \text{Ann}_{\tilde{G}} m \) is finite corank for all \( m \in M \) under the assumption that the \( g \)-module \( M \) is simple. This follows from the observation that a finite intersection of finite corank subalgebras is finite corank.

Fix a splitting Cartan subalgebra \( h \) of \( g \) such that \( h \cap g_n \) is a Cartan subalgebra of \( g_n \); let \( b = h \oplus n \) be a Borel subalgebra of \( g \) whose set of roots (i.e. positive roots) is denoted by \( \Delta^+ \). For each positive root \( \alpha \), let \( e_\alpha \), \( h_\alpha \), \( f_\alpha \) be a standard basis for the corresponding root \( sl(2) \)-subalgebra. Fix additionally a nonzero \( b \)-weight vector \( m \in M \).

Choose a set of commuting mutually orthogonal positive roots \( Y \subseteq \Delta^+ \). The set of semisimple elements \( \{ h_\alpha + e_\alpha \mid \alpha \in Y \} \) is \( \tilde{G} \)-conjugate to the set \( \{ h_\alpha \mid \alpha \in Y \} \), and can thus be extended to a splitting Cartan subalgebra \( h' \) of \( g \). Since \( M \) is an absolute weight module, there is a nonzero \( h' \)-weight vector \( m' \in M \). As \( M \) is simple, it must be that \( m \in U(g) \cdot m' \). Moreover, one has \( m \in U(g_n) \cdot m' \) for some \( n \). For almost all \( \alpha \in Y \), \( h_\alpha \) and \( e_\alpha \) commute with \( g_n \), in which case \( m \) is an eigenvector for \( h_\alpha + e_\alpha \). Thus \( e_\alpha \cdot m \) is a scalar multiple of \( m \). Since \( M \) is integrable, \( e_\alpha \) acts locally nilpotently, and we conclude that \( e_\alpha \cdot m = 0 \) for all but finitely many \( \alpha \). By considering the set \( \{ h_\alpha + f_\alpha \mid \alpha \in Y \} \) in place of \( \{ h_\alpha + e_\alpha \mid \alpha \in Y \} \), we see that \( f_\alpha \cdot m = 0 \) for all but finitely many \( \alpha \), and hence \( e_\alpha \cdot m = f_\alpha \cdot m = 0 \) for all but finitely many \( \alpha \in Y \).
We now consider separately each of the three possible choices of \( g \). For \( g = \mathfrak{sl}(\infty) \), we may assume that the simple roots of \( b \) are of the form \( \{ \epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0} \} \). We first choose the set of commuting mutually orthogonal positive roots to be \( Y_1 = \{ \epsilon_{2i-1} - \epsilon_{2i} \mid i \in \mathbb{Z}_{>0} \} \) and obtain in this way that \( e_{\epsilon_i - \epsilon_{i+1}} \cdot m = f_{\epsilon_i - \epsilon_{i+1}} \cdot m = 0 \) for almost all odd indices \( i \). By choosing the set of commuting mutually orthogonal positive roots as \( Y_2 = \{ \epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0} \} \), we have \( e_{\epsilon_i - \epsilon_{i+1}} \cdot m = f_{\epsilon_i - \epsilon_{i+1}} \cdot m = 0 \) for almost all even indices \( i \), hence for almost all \( i \). Since it contains \( e_{\epsilon_i - \epsilon_{i+1}} \) and \( f_{\epsilon_i - \epsilon_{i+1}} \) for almost all \( i \), the subalgebra \( \text{Ann}_g m \) is a finite corank subalgebra of \( g = \mathfrak{sl}(\infty) \).

For \( g = o(\infty) \), one may assume that the set of simple roots of \( g \) is \( \{-\epsilon_1 - \epsilon_2 \} \cup \{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0} \} \). In this case in addition to the two sets \( Y_1 = \{ \epsilon_{2i-1} - \epsilon_{2i} \mid i \in \mathbb{Z}_{>0} \} \) and \( Y_2 = \{ \epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0} \} \), one considers also the set of commuting mutually orthogonal positive roots \( Y_3 = \{-\epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0} \} \). For \( g = \mathfrak{sp}(\infty) \), the set of simple roots can be chosen as \( \{-2\epsilon_1 \} \cup \{\epsilon_i - \epsilon_{i+1} \mid i \in \mathbb{Z}_{>0} \} \), and one considers the following three sets of commuting mutually orthogonal positive roots:

\[
Y_1 = \{ \epsilon_{2i-1} - \epsilon_{2i} \mid i \in \mathbb{Z}_{>0} \}
\]
\[
Y_2 = \{ \epsilon_{2i} - \epsilon_{2i+1} \mid i \in \mathbb{Z}_{>0} \}
\]
\[
Y_3 = \{-2\epsilon_i \mid i \in \mathbb{Z}_{>0} \}.
\]

In both cases \( \text{Ann}_g m \) contains \( e_\alpha, f_\alpha \) for all but finitely many \( \alpha \in Y_1 \cup Y_2 \cup Y_3 \). Hence we conclude that the subalgebra \( \text{Ann}_g m \) is a finite corank subalgebra of \( g \); that is, (1) implies (3).

Next we prove that (3) implies (2).

We first show that a \( g \)-module \( M \) satisfying (3) is a weight module for some splitting Cartan subalgebra \( h \subset g \). Fix a finite set \( \{m_1, \ldots, m_s\} \) of generators of \( M \). Let \( g'_n \) be the commutator subalgebra of the centralizer in \( g \) of \( g_n \). There exists a finite corank subalgebra that annihilates \( m_1, \ldots, m_s \), and hence \( g'_n \) annihilates \( m_1, \ldots, m_s \) for some \( n \). Let \( h'_n \) be a splitting Cartan subalgebra of \( g'_n \). Obviously \( M \) is semisimple over \( h'_n \). One can find \( k \) and a Cartan subalgebra \( h_k \subset g_k \) such that \( h = h'_n + h_k \) is a splitting Cartan subalgebra of \( g \). (If \( g = o(\infty) \) or \( \mathfrak{sp}(\infty) \) one can choose \( k = n \); if \( g = \mathfrak{sl}(\infty) \), one can set \( k = n + 1 \).) Since \( M \) is integrable, \( M \) is semisimple over \( h_k \). Hence \( M \) is semisimple over \( h \).

To finish the proof that (3) implies (2), we need to show that \( M \) is \( \tilde{G} \)-invariant. For each \( n \) one has \( \tilde{G} = G \cdot \tilde{G}'_n \) by Lemma 3.3. Fix \( \gamma \in \tilde{G} \)
and \( m \in M \). Then for some \( n \), the vector \( m \) is fixed by \( g'_n \). We choose a decomposition \( \gamma = \gamma''\gamma' \) so that \( \gamma' \in \tilde{G}' \) and \( \gamma'' \in G \). We then set \( \gamma(m) := \gamma''(m) \), and note that the action of \( G \) on \( M \) is well defined because \( M \) is assumed to be integrable. This yields a well-defined \( \tilde{G} \)-module structure on \( M \) since, for any other decomposition \( \gamma = \bar{\gamma}'\bar{\gamma}'' \) as above, one has \((\bar{\gamma}'')^{-1}\gamma'' = \gamma'(\gamma')^{-1} \in \tilde{G}' \cap G = \{ \gamma \in G \mid \gamma(g) = g \text{ for all } g \in g_n \}\) which must preserve \( m \).

Fix now \( \gamma \in \tilde{G} \) and consider the linear operator
\[
\varphi_\gamma : M^\gamma \to M, \quad m \mapsto \gamma^{-1}(m).
\]
We claim that \( \varphi_\gamma \) is an isomorphism. For this we need to check that \( g \cdot \varphi_\gamma(m) = \varphi_\gamma(g \cdot m) \) for any \( g \in g \) and \( m \in M \). We have \( g \cdot \varphi_\gamma(m) = g \cdot (\gamma^{-1}(m)) = \varphi_\gamma(g \cdot \gamma^{-1}(m)) \), hence it suffices to check that \( \gamma(g \cdot \gamma^{-1}(m)) = \gamma(g) \cdot m \) for every \( g \in g \) and \( m \in M \). After choosing a decomposition \( \gamma = \gamma''\gamma' \) such that \( \gamma'' \in G \) and \( \gamma' \) fixes \( m \), \( g \) and \( g \cdot m \), all that remains to check is that
\[
\gamma''(g \cdot (\gamma'')^{-1}(m)) = \gamma''(g) \cdot m
\]
for all \( g \in g \). This latter equality is the well-known relation between the \( G \)-module structure on \( M \) and the adjoint action of \( G \) on \( g \).

To complete the proof of the theorem we need to show that (2) implies (1). What is clear is that (2) implies a slightly weaker statement, namely that \( M \) is a weight module for any splitting Cartan subalgebra belonging to the same \( \tilde{G} \)-conjugacy class as the given splitting Cartan subalgebra \( h \). For \( g = sl(\infty), sp(\infty) \), this proves (1), as all splitting Cartan subalgebras are conjugate under \( \tilde{G} \).

Consider now the case \( g = o(\infty) \). In this case there are two \( \tilde{G} \)-conjugacy classes of splitting Cartan subalgebras [DPS]. Note that if \( M \) is semisimple over every Cartan subalgebra from one \( \tilde{G} \)-conjugacy class, then (3) holds as follows from the proof of the implication (1) \( \Rightarrow \) (3). Furthermore, the proof that a \( g \)-module of finite length \( M \) satisfying (3) is a weight module for some splitting Cartan subalgebra involves a choice of \( g_n \). For \( g = o(\infty) \) there are two different possible choices, namely \( g_n = o(2n) \) and \( g_n = o(2n + 1) \), which in turn produce splitting Cartan subalgebras from the two \( \tilde{G} \)-conjugacy classes. This shows that in each \( \tilde{G} \)-conjugacy class there is a splitting Cartan subalgebra of \( g \) for which \( M \) is a weight module, and hence we may conclude that (2) implies (1) also for \( g = o(\infty) \). \( \square \)
Corollary 3.5. Let $M$ be a module satisfying the conditions of Theorem 3.4. Then $M = \bigcup_{n>0} M^{g_n}$, where $g'_n$ is the commutator of the centralizer of $g_n$ in $g$.

Proof. Any submodule of $M$ with the property that the annihilator of each of its elements contains $g'_n$ for some $n$ is clearly contained in $\bigcup_{n>0} M^{g_n}$. Condition (3) of Theorem 3.4 states that Ann$_g m$ is finite corank for all $m \in M$, which is to say that $M$ has this property. □

Corollary 3.6. Let $g = o(\infty)$ and $M$ be a finite length $g$-module which is an $h$-weight module for all splitting Cartan subalgebras $h \subset g$ in either of the two $\tilde{G}$-conjugacy classes. Then $M$ is an $h$-weight module for all splitting Cartan subalgebras $h$ of $g$.

Proof. Let $h$ be a Cartan subalgebra of $g$, and let $M$ be a finite length $g$-module which is a weight module for all splitting Cartan subalgebras in the $\tilde{G}$-conjugacy class of $h$. Then $M$ is integrable, by the same argument as in the proof of Proposition 3.2. Finally, (3) holds by the same proof as that of (1)$\Rightarrow$(3) in Theorem 3.4. □

We denote by $T_g$ the full subcategory of $g$-mod consisting of finite length modules satisfying the equivalent conditions of Theorem 3.4. Then $T_g$ is an abelian category and a monoidal category with respect to the usual tensor product of $g$-modules, and $T_g$ is a subcategory of the category of absolute weight modules. In addition, for $g = sl(\infty)$, $T_g$ has an involution

$$(\cdot)_*: T_g \rightarrow T_g,$$

which one can think of as “restricted dual.” Indeed, in this case any outer automorphism $w \in$ Aut $g$ induces the autoequivalence of categories

$$w_g : T_g \rightarrow T_g,$$

$$M \mapsto M^w.$$

Since, however, any object of $T_g$ is $\tilde{G}$-invariant, the functor $w_g$ does not depend on the choice of $w$ and is an involution, i.e. $w_g^2 = id$. We denote this involution by $(\cdot)_*$ in agreement with the fact that it maps $V$ to $V_\ast$. For $g = o(\infty)$, $sp(\infty)$, we define $(\cdot)_*$ to be the trivial involution on $T_g$. 10
4. Simple objects and indecomposable injectives of $T_\mathfrak{g}$

Next we describe the simple objects of $T_\mathfrak{g}$. For this we need to recall some
results about tensor representations from $[\text{PSsty}]$.

By $T$ we denote the tensor algebra $T(V \oplus V_*)$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$, and $T(V)$ for $\mathfrak{g} = o(\infty), sp(\infty)$. That is, we have

$$T := \bigoplus_{p \geq 0, q \geq 0} T^{p,q} \quad \text{for } \mathfrak{g} = \mathfrak{sl}(\infty),$$

and

$$T := \bigoplus_{p \geq 0} T^p \quad \text{for } \mathfrak{g} = o(\infty), sp(\infty),$$

where $T^{p,q} := V^{\otimes p} \otimes (V_*)^{\otimes q}$ and $T^p := V^{\otimes p}$. In addition, we set

$$T^{\leq r} := \bigoplus_{p+q \leq r} T^{p,q} \quad \text{for } \mathfrak{g} = \mathfrak{sl}(\infty),$$

and

$$T^{\leq r} := \bigoplus_{p \leq r} T^p \quad \text{for } \mathfrak{g} = o(\infty), sp(\infty).$$

By a tensor module we mean any $\mathfrak{g}$-module isomorphic to a subquotient of a
finite direct sum of copies of $T^{\leq r}$ for some $r$.

By a partition we mean a non-strictly decreasing finite sequence of positive integers $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s$. The empty
partition is denoted by 0.

Given a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ and a classical finite-dimensional
Lie algebra $\mathfrak{g}_n$ of rank $n \geq s$, the irreducible $\mathfrak{g}_n$-module $(V_n)_\mu$ with highest
weight $\mu$ is always well defined. Moreover, for a fixed $\mu$ and growing $n$, the modules $(V_n)_\mu$ are naturally nested and determine a unique simple ($\mathfrak{g} = \lim\downarrow \mathfrak{g}_n$)-module $V_\mu := \lim\downarrow (V_n)_\mu$. For $\mathfrak{g} = \mathfrak{sl}(n)$, there is another simple
$\mathfrak{g}$-module naturally associated to $\mu$, namely $(V_\mu)_*$. In what follows we will consider pairs of partitions for $\mathfrak{g} = \mathfrak{sl}(\infty)$ and single partitions for $\mathfrak{g} = o(\infty), sp(\infty)$. Given $\lambda = (\lambda^1, \lambda^2)$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$, we set $V_\lambda := V_{\lambda^1} \otimes (V_{\lambda^2})_*$. For $\mathfrak{g} = o(\infty), sp(\infty)$ and for a single partition $\lambda$, the
$g$-module $\tilde{V}_\lambda$ is similarly defined: we embed $g$ into $sl(\infty)$ so that both the natural $sl(\infty)$-module and the conatural $sl(\infty)$-module are identified with $V$ as $g$-modules, and define $\tilde{V}_\lambda$ as the irreducible $sl(\infty)$-module $V_\lambda$ corresponding to the partition $\lambda$ as defined above. Then $\tilde{V}_\lambda$ is generally a reducible $g$-module.

It is easy to see that for $g = sl(\infty)$,

$$T = \bigoplus_{\lambda} d_\lambda \tilde{V}_\lambda$$

(4.1)

where $\lambda = (\lambda^1, \lambda^2)$, $d_\lambda := d_{\lambda^1} d_{\lambda^2}$, and $d_{\lambda^i}$ is the dimension of the simple $S_n$-module corresponding to the partition $\lambda^i$ for $n = |\lambda^i|$. For $g = o(\infty)$, $sp(\infty)$, Equation (4.1) also holds, with $\lambda$ taken to stand for a single partition. Both statements follow from the obvious infinite-dimensional version of Schur-Weyl duality for the tensor algebra $T$ considered as an $sl(\infty)$-module (see for instance [PStyr]). Moreover, according to [PStyr, Theorems 3.2, 4.2],

$$soc(\tilde{V}_\lambda) = V_\lambda$$

(4.2)

for $g = o(\infty)$, $sp(\infty)$, while $soc(\tilde{V}_\lambda)$ is a simple $g$-module for $g = sl(\infty)$ [PStyr, Theorem 2.3]. Here $soc(\cdot)$ stands for the socle of a $g$-module. We set $V_\lambda := soc(\tilde{V}_\lambda)$ also for $g = sl(\infty)$, so that (4.2) holds for any $g$. It is proved in [PStyr] that $\tilde{V}_\lambda$ (and consequently $T^{\leq r}$) has finite length.

It follows also from [PStyr] that any simple tensor module is isomorphic to $V_\lambda$ for some $\lambda$. In particular, every simple subquotient of $T$ is also a simple submodule of $T$.

For any partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$, we set $#\mu := s$ and $|\mu| := \sum_{i=1}^s \mu_i$. In the case of $g = sl(\infty)$, when $\lambda = (\lambda^1, \lambda^2)$, we set $#\lambda := #\lambda^1 + #\lambda^2$ and $|\lambda| := |\lambda^1| + |\lambda^2|.$

We are now ready for the following lemma.

**Lemma 4.1.** Let $g = sl(\infty)$ and $\lambda = (\lambda^1, \lambda^2)$ with $#\lambda = k > 0$. Then $(V_k)_{\lambda^1} \otimes (V_k^*)_{\lambda^2}$ generates $\tilde{V}_\lambda$.

Let $g = o(\infty)$, $sp(\infty)$, and let $\lambda$ be a partition with $#\lambda = k > 0$. Then the $sl(V_k)$-submodule $(V_k)_{\lambda}$ of $\tilde{V}_\lambda$ generates $\tilde{V}_\lambda$.

**Proof.** Set $M := \tilde{V}_\lambda$. Let $g = sl(\infty)$. Then $M = V_{\lambda^1} \otimes (V^*)_{\lambda^2}$, and let $M_n := (V_n)_{\lambda^1} \otimes (V_n^*)_{\lambda^2}$. It is easy to check that the length of $M_n$ as a $g_n$-module stabilizes for $n \geq k$, and moreover it coincides with the length of $M$.
a formula for the length of $M$ is implied by [PStyr, Theorem 2.3]. Hence $(V_k)_{\lambda_1} \otimes (V_k^*)_{\lambda_2}$ generates $M$.

For $g = o(\infty)$, $sp(\infty)$ the length of the $sl(V_n)$-module $(V_n)_\lambda$ considered as a $g_n$-module equals the length of $M$ as a $g$-module when $2k \leq \dim V_n$ (see Theorems 3.3 and 4.3 in [PStyr]). Hence $(V_k)_\lambda$ generates $M$. \[\square\]

**Theorem 4.2.** A simple absolute weight $g$-module is a simple tensor module.

**Proof.** Let $M$ be a simple absolute weight $g$-module. Then $M$ is integrable by Proposition 3.2, and it also satisfies Theorem 3.4 (3). Fix $0 \neq m \in M$ and choose $k$ such that the commutator subalgebra $g_k$ of the centralizer of $g_k$ annihilates $m$. In the orthogonal case we assume that $g_k = o(2k)$. We will prove that $M$ is the unique simple quotient of a parabolically induced module for a parabolic subalgebra $p$ of the form $p = I \supset m$, where $m$ is the nil-radical of $p$ and $I$ is a locally reductive subalgebra. Define $p \subset g$ as follows:

- If $g = sl(\infty)$, we identify $g$ with the subspace of traceless elements in $V_s \otimes V$. Consider the decomposition $V = V_k \oplus V'$, where $V_k$ is the natural $g_k$-module and $V'$ is the natural $g'_k$-module. Furthermore, $V_s = V_k^\perp \oplus (V')^\perp$, where $(V')^\perp = V'_k^*$ and $V_k^\perp = V'_s$. We define the subalgebra $I$ of $p$ to be equal the traceless part of $V_k^* \otimes V_k \oplus V'_s \otimes V'$, and we set $m := V_k^* \otimes V'$.

- If $g = o(\infty)$, we use the identification $g \simeq \Lambda^2(V)$. Let $V_k \subset V$ be the copy of the natural representation of $g_k$. Consider the decomposition $V_k = W \oplus W^*$ for some maximal isotropic subspaces $W$, $W^*$ of $V_k$ and set $V' = V_k^\perp$. Then $p := I \supset m$, where $I := W^* \otimes W \oplus \Lambda^2(V')$ and $m := W \otimes V' \supset \Lambda^2(W)$.

- If $g = sp(\infty)$, we use the identification $g \simeq S^2(V)$. Then $V_k$, $W$, $W^*$ and $V'$ are defined in the same way as for $g = o(\infty)$, and $p := I \supset m$, where $I := W^* \otimes W \oplus S^2(V')$ and $m := W \otimes V' \supset S^2(W)$.

Note that $g'_k$ is a subalgebra of finite codimension in $I$. In the orthogonal and symplectic cases $I = gl(W) \oplus g'_k$. If $g = sl(\infty)$, then $I = sl(V_k) \oplus \tilde{g}_k$, where $\tilde{g}_k$ is the centralizer of $g_k$ in $g$: 

$$\tilde{g}_k = \{-\frac{\text{tr} X}{k}, \text{Id}_{V_k} \oplus X \mid X \in V'_s \otimes V'\}.$$ 

Clearly $\tilde{g}_k$ is isomorphic to the Lie algebra $V'_s \otimes V'$. 

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We claim that the \( m \)-invariant part of \( M \), denoted \( M^m \), is nonzero. Note that \( m \) is abelian for \( g = sl(\infty) \). For \( g = o(\infty) \) or \( sp(\infty) \), we have a decomposition \( m = m_1 \oplus m_2 \) such that \( m_2 = [m_1, m_1] \) is a finite-dimensional abelian subalgebra: \( m_2 = \Lambda^2(W) \) in the orthogonal case and \( m_2 = S^2(W) \) in the symplectic case. Since \( M \) is integrable, \( m_2 \) acts locally nilpotently on \( M \). Hence without loss of generality we may assume that \( m_2 \cdot m = 0 \). We put \( m_1 := m \) in the case \( g = sl(\infty) \).

Next observe that \( U(m) \cdot m = S(m_1) \cdot m \) and that \( S(m_1) \) is isomorphic as a \( \mathfrak{g}'_k \)-module to a direct sum of \((V')_\lambda \) for some (infinite) set of \( \lambda \) satisfying \( \# \lambda \leq k \). By Lemma 4.1, there exists a finite-dimensional subspace \( X \subset m_1 \) such that \( S(X) \) generates \( S(m_1) \) as a \( \mathfrak{g}'_k \)-module. Since \( M \) is integrable, \( X \) acts locally nilpotently on \( M \). Hence \( S^{>p}(X) \cdot m = 0 \) for some \( p \). This, together with our assumption that \( \mathfrak{g}'_k \cdot m = 0 \), allows us to conclude \( S^{>p}(m) \cdot m = 0 \), which in turn implies \( M^m \neq 0 \).

Since \( M \) is irreducible, it is generated by \( M^m \) and is therefore the unique irreducible quotient of the parabolically induced module \( U(g) \otimes_{U(p)} M^m \). Furthermore, the irreducibility of \( M \) implies the irreducibility of \( M^m \) as an \( l \)-module (otherwise a proper submodule of \( M^m \) would generate a proper submodule of \( M \)). Note also that the argument of the previous paragraph implies that as a \( \mathfrak{g}'_k \)-module \( M^m \) is isomorphic to a subquotient of \( S(m_1) \); that is, \( M^m \) is isomorphic to a subquotient of a finite direct sum of some tensor powers of \( V' \).

Let us first consider the case \( g = o(\infty) \) or \( sp(\infty) \). Recall that \( M^m \) is irreducible as an \( l \)-module and is a tensor module over \( \mathfrak{g}'_k \). This, together with the integrability of \( M^m \) as an \( l \)-module, implies the existence of an isomorphism of \( l \)-modules \( M^m \cong L \otimes (V')_\nu \), where \( L \) is some irreducible finite-dimensional \( gl(W) \)-module and \( \nu = (\nu_1, \nu_2, \ldots, \nu_r) \) is some partition. Let \( (\mu_1, \ldots, \mu_k) \) denote the highest weight of \( L \) with respect to some Borel subalgebra of \( \mathfrak{b}_k \) of \( gl(W) \). Consider a Borel subalgebra \( \mathfrak{b} \) of \( g \) such that \( \mathfrak{b}_k \subset \mathfrak{b} \subset p \). Without loss of generality, we may assume that the roots of \( \mathfrak{b} \) are

\[
\{\epsilon_i \pm \epsilon_j \mid i < j \in \mathbb{Z}_{>0}\}
\quad \text{for } g = o(\infty),
\]

\[
\{\epsilon_i \pm \epsilon_j \mid i < j \in \mathbb{Z}_{>0}\} \cup \{2\epsilon_i \mid i \in \mathbb{Z}_{>0}\}
\quad \text{for } g = sp(\infty).
\]
The roots of $b_k$ will then be
\[
\{\epsilon_i \pm \epsilon_j \mid 0 < i < j \leq k\} \quad \text{for } g = o(\infty),
\]
\[
\{\epsilon_i \pm \epsilon_j \mid 0 < i < j \leq k\} \cup \{2\epsilon_i \mid 0 < i \leq k\} \quad \text{for } g = sp(\infty).
\]
Observe that $M$ is a highest weight module with respect to $b$, and its highest weight equals $\lambda := \mu_1 \epsilon_1 + \cdots + \mu_k \epsilon_k + \nu_1 \epsilon_{k+1} + \cdots + \nu_r \epsilon_{k+r}$. Furthermore, the integrability of $M$ as a $g$-module implies that $\mu_k \geq \nu_1$ and all $\mu_i$ are integers. In other words, the weight $\lambda$ can be identified with the partition $(\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_r)$. Next, consider the simple tensor $g$-module $V_\lambda$ (where $\lambda$ is considered as a partition), and note that both $M$ and $V_\lambda$ are simple $g$-modules with the same highest weight with respect to $b$. Therefore $M$ and $V_\lambda$ are isomorphic as $g$-modules.

Now let $g = sl(\infty)$. Then by the same argument as above we see that $M^m$ is isomorphic to $L_1 \otimes L_2$, where $L_1$ is a simple finite-dimensional $sl(V_k)$-module and $L_2$ is a simple integrable $\tilde{g}_k$-module. Since, however, $L_2$ is isomorphic to a submodule of the tensor algebra $T(V')$ as a $g'_k$-module, we check immediately that as a $g'_k$-module $L_2$ must be isomorphic to $(V')_\nu$ for some partition $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$. There is a $\tilde{g}_k$-submodule of $(V')^{|\nu|}$ (i.e. a tensor module of $\tilde{g}_k \cong V'_s \otimes V'$) with the same restriction to $g'_k$ as $L_2$; abusing notation slightly, we denote it also by $(V')_\nu$. Next, using the inclusions
\[
sl(V_k) \oplus g'_k \subset I \subset gl(V_k) \oplus \tilde{g}_k
\]
and the fact that $gl(V_k) \oplus \tilde{g}_k$ is a direct sum of $I$ and the abelian one-dimensional Lie algebra (namely the center of $gl(V_k)$), we conclude that $M^m$ must be isomorphic to the restriction to $I$ of a $gl(V_k) \oplus \tilde{g}_k$-module of the form $L \otimes (V')_\nu$, where the $gl(V_k)$-module $L$ is simple and uniquely determined up to isomorphism. Denote by $\mu = (\mu_1, \ldots, \mu_k)$ the highest weight of $L$. It is easy to check in this case that the integrability of $M$ as a $g$-module implies that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$ are nonpositive integers. Consider the pair of partitions
\[
\lambda := ((\nu_1, \nu_2, \ldots, \nu_r), (-\mu_k, \ldots, -\mu_1))
\]
and the corresponding tensor $g$-module $V_\lambda$. Then we clearly have an isomorphism of $p$-modules $V^m_\lambda \cong M^m$. Therefore, being the unique irreducible quotients of the corresponding parabolically induced modules, $M$ and $V_\lambda$ are isomorphic as $g$-modules.

\[\square\]
Remark 4.3. In [PS] certain categories $\text{Tens}_g$ and $\text{\wedge}\text{Tens}_g$ are introduced and studied in detail. The simple objects of both $\text{Tens}_g$ and $\text{\wedge}\text{Tens}_g$ are the same as the simple objects of $\mathbb{T}_g$, and in fact these three categories form the following chain:

$$\mathbb{T}_g \subset \text{Tens}_g \subset \text{\wedge}\text{Tens}_g.$$  

However, the objects of the categories $\text{Tens}_g$ and $\text{\wedge}\text{Tens}_g$ generally have infinite length. In the present paper we will not make use of the categories $\text{Tens}_g$ and $\text{\wedge}\text{Tens}_g$, and refer the interested reader to [PS].

Let us denote by $\mathcal{C}$ the category of $g$-modules which satisfy Condition (3) of Theorem 3.4. Consider the functor $\mathcal{B}$ from $\text{Int}_g$ to $\mathcal{C}$ given by

$$\mathcal{B}(M) = \bigcup_{n>0} M^{g_n}.$$  

It is clear that $\mathcal{B}$ does not depend on the choice of fixed exhaustion $g = \lim \mathbb{g}_n$.

Lemma 4.4. For any $M \in \text{Int}_g$, the module $\mathcal{B}(\Gamma_g(M^*))$ is injective in the category $\mathcal{C}$. Furthermore, any finite length injective module in the category $\mathcal{C}$ is injective in $\mathbb{T}_g$.

Proof. First, let us note that $\mathcal{B}$ is a right adjoint to the inclusion functor $\mathcal{C} \subset \text{Int}_g$. To see this, consider that the image of any homomorphism from a module $M \in \mathcal{C}$ to a module $Y \in \text{Int}_g$ is automatically contained in $\mathcal{B}(Y)$. Since it is a right adjoint to the inclusion functor, $\mathcal{B}$ takes injective modules to injective modules, and the lemma follows from the fact that $\Gamma_g(M^*)$ is injective for any integrable $g$-module $M$, which is [PS, Proposition 3.2]. The second statement is clear.

Proposition 4.5. For each $r$, the module $T^{\leq r}$ is injective in the category of absolute weight modules and in $\mathbb{T}_g$.

Proof. We consider the case $g = \mathfrak{sl}(\infty)$, and note that the other cases are similar. It was shown in [PS] that $(T^{q,p})^*$ is an integrable $\mathfrak{g}$-module. We will show $\mathcal{B}((T^{q,p})^*)$ is a finite-length module, and furthermore that it has a direct summand isomorphic to $T^{p,q}$. Since any direct summand of an injective module is itself injective, it will follow immediately that $T^{p,q}$ is injective in the category $\mathbb{T}_g$.  

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We start with calculating $\left( (T^q,p)^* \right)^{g'_n}$. Consider the decomposition

\[ V = V_n \oplus V', \quad V^*_s = V^*_n \oplus V^*_s, \]

where $V'$ and $V'_s$ are respectively the natural and conatural $g'_n$-modules. If we use the notation

\[ T^{r,s}_n := V_n \otimes (V^*_n)^{\otimes s}, \quad (T')^{r,s} := (V')^{\otimes r} \otimes (V^*_s)^{\otimes s}, \]

then we have the following isomorphism of $g_n \oplus g'_n$-modules

\[ T^{q,p} \simeq \bigoplus_{r \leq q,s \leq p} (T')^{r,s} \otimes T^{q-p,s-r}_n \oplus b_{r,s}, \]

where $b_{r,s} = \binom{r}{p} \binom{s}{r}$.

Therefore

\[ \left( (T^q,p)^* \right)^{g'_n} \simeq \bigoplus_{r \leq q,s \leq p} \text{Hom}_{g'_n}((T')^{r,s}, \mathbb{C}) \otimes (T^{p-s,q-r}_n)^{\oplus b_{r,s}}. \]

Since $g'_n \simeq g$, we can use the results of [PStyr]. In particular,

\[ \text{Hom}_{g'_n}((T')^{r,s}, \mathbb{C}) = \begin{cases} 0 & \text{if } r \neq s \\ \mathbb{C}^r! & \text{if } r = s. \end{cases} \]

The degree $r!$ appears for the following reason. For any $\sigma \in S_r$ we define $\varphi_\sigma \in \text{Hom}_{g'_n}((T')^{r,r}, \mathbb{C})$ by

\[ \varphi_\sigma(v_1 \otimes \cdots \otimes v_r \otimes u_1 \otimes \cdots \otimes u_r) = \prod_{i=1}^r (u_i, v_{\sigma(i)}). \]

Then $\varphi_\sigma$ for all $\sigma \in S_r$ form a basis in $\text{Hom}_{g'_n}((T')^{r,r}, \mathbb{C})$. Thus we obtain

\[ \left( (T^q,p)^* \right)^{g'_n} \simeq \bigoplus_{r \leq \min(p,q)} (T^{p-s,q-r}_n)^{\oplus b_{r,r}!}, \]

which implies

\[ B\left( (T^q,p)^* \right) \simeq \bigoplus_{r \leq \min(p,q)} (T^{p-s,q-r}_n)^{\oplus b_{r,r}!}. \]
Hence the statement. \hfill \Box

**Corollary 4.6.** 1. \( \tilde{V}_\lambda \) is injective in \( T_g \).

2. \( \tilde{V}_\lambda \) is an injective hull of \( V_\lambda \) in \( T_g \).

3. Every indecomposable injective module in \( T_g \) is isomorphic to \( \tilde{V}_\lambda \) for some \( \lambda \).

4. Every module \( M \in T_g \) is isomorphic to a submodule of the direct sum of finitely many copies of \( T^{\leq r} \) for some \( r \).

5. A \( \mathfrak{g} \)-module \( M \) is a tensor module if and only if \( M \in T_g \).

**Proof.**

1. Each module \( \tilde{V}_\lambda \) is a direct summand of \( T^{\leq r} \) for some \( r \), and a direct summand of an injective module is injective.

2. Any indecomposable injective module is an injective hull of its socle, and \( \text{soc}(\tilde{V}_\lambda) = V_\lambda \) by (4.2).

3. Every indecomposable injective module in \( T_g \) has a simple socle, which must be isomorphic to \( V_\lambda \) for some \( \lambda \) by Theorem 4.2.

4. Let \( M \in T_g \). Then \( \text{soc}(M) \) admits an injective homomorphism into a direct sum of finitely many copies of \( T^{\leq r} \) for some \( r \). Since the latter is injective in \( T_g \), this homomorphism factors through the inclusion \( \text{soc}(M) \hookrightarrow M \). The resulting homomorphism must be injective because its kernel has trivial intersection with \( \text{soc}(M) \).

5. A tensor module is by definition a subquotient of a direct sum of finitely many copies of \( T^{\leq r} \) for some \( r \), hence it is clearly finite length. Furthermore, any subquotient of an absolute weight module must be an absolute weight module, so any tensor module must be in \( T_g \). The converse was seen in (4).

\hfill \Box

5. **Koszulity of \( T_g \)**

For \( r \in \mathbb{Z}_{\geq 0} \), let \( T^r_g \) be the full abelian subcategory of \( T_g \) whose simple objects are submodules of \( T^{\leq r} \). Then \( T_g = \varprojlim T^r_g \). Moreover, \( T^{\leq r} \) is an injective cogenerator of \( T^r_g \). Consider the finite-dimensional algebra \( A^r_g := \text{End}_g T^{\leq r} \) and the direct limit algebra \( A_g = \varinjlim A^r_g \).

Let \( A^r_g \)-mof denote the category of unitary finite-dimensional \( A^r_g \)-modules, and \( A_g \)-mof the category of locally unitary finite-dimensional \( A_g \)-modules.

**Proposition 5.1.** The functors \( \text{Hom}_g(\cdot, T^{\leq r}) \) and \( \text{Hom}_{A^r_g}(\cdot, T^{\leq r}) \) are mutually inverse antiequivalences of the categories \( T^r_g \) and \( A^r_g \)-mof.
Proof. Consider the opposite category \((T^r_g)^{\text{opp}}\). It has finitely many simple objects and enough projectives, and any object has finite length. Moreover, \(T^{\leq r}\) is a projective generator of \((T^r_g)^{\text{opp}}\). By a well-known result of Gabriel [G], the functor

\[
\text{Hom}_{(T^r_g)^{\text{opp}}} (T^{\leq r}, \cdot) = \text{Hom}_g (\cdot, T^{\leq r}) : (T^r_g)^{\text{opp}} \to \mathcal{A}_g^r\text{-mof}
\]

is an equivalence of categories.

We claim that \(\text{Hom}_g (\cdot, T^{\leq r})\) is an inverse to \(\text{Hom}_{(T^r_g)^{\text{opp}}} (T^{\leq r}, \cdot)\). For this it suffices to check that \(\text{Hom}_{(T^r_g)^{\text{opp}}} (T^{\leq r}, \cdot)\) is a right adjoint to \(\text{Hom}_g (\cdot, T^{\leq r})\), i.e. that

\[
\text{Hom}_g (X, \text{Hom}_{(T^r_g)^{\text{opp}}} (T^{\leq r}, M)) \simeq \text{Hom}_{(T^r_g)^{\text{opp}}} (\text{Hom}_g (X, T^{\leq r}), M)
\]

for any \(X \in \mathcal{A}_g^r\text{-mof}\) and any \(M \in T^r_g\). We have

\[
\begin{align*}
\text{Hom}_{(T^r_g)^{\text{opp}}} (X, \text{Hom}_{(T^r_g)^{\text{opp}}} (T^{\leq r}, M)) &= \text{Hom}_{(T^r_g)^{\text{opp}}} (X, \text{Hom}_g (M, T^{\leq r})) \\
&\simeq \text{Hom}_{A^r_g} (X \otimes M, T^{\leq r}) \\
&\simeq \text{Hom}_{U(g) \otimes A^r_g} (M \otimes X, T^{\leq r}) \\
&\simeq \text{Hom}_g (M, \text{Hom}_{A^r_g} (X, T^{\leq r})) \\
&\simeq \text{Hom}_{(T^r_g)^{\text{opp}}} (\text{Hom}_{A^r_g} (X, T^{\leq r}), M),
\end{align*}
\]

where \(\Psi(\varphi)(x \otimes m) = \varphi(x)(m)\) and \((\Theta(x)(m))(\psi) = \psi(m \otimes x)\) for \(x \in X, m \in M, \varphi \in \text{Hom}_{A^r_g} (X, \text{Hom}_g (M, T^{\leq r}))\), and \(\psi \in \text{Hom}_{U(g) \otimes A^r_g} (M \otimes X, T^{\leq r})\). \(\square\)

In order to relate the category \(\mathcal{A}_g^r\text{-mof}\) with the categories \(\mathcal{A}_g^r\text{-mof}\) for all \(r \geq 0\), we need to establish some basic facts about the algebra \(\mathcal{A}_g\). Note first that by [PStyr] \(\text{Hom}_{\mathfrak{sl}(\infty)} (T^{p,q}, T^{r,s}) = 0\) unless \(p - r = q - s \in \mathbb{Z}_{\geq 0}\), and for \(g = o(\infty), sp(\infty)\), \(\text{Hom}_g (T^r, T^s) = 0\) unless \(p - q \in 2\mathbb{Z}_{\geq 0}\). Furthermore, put

\[
(A_g)^{p,q}_i = \text{Hom}_g (T^{p,q}, T^{p-i,q-i})
\]

for \(g = \mathfrak{sl}(\infty)\)

and

\[
(A_g)_i^p = \text{Hom}_g (T^p, T^{p-2i})
\]

for \(g = o(\infty), sp(\infty)\).
Then one can define a $\mathbb{Z}_{\geq 0}$-grading on $\mathcal{A}_{\mathfrak{g}}^r$ by setting

$$(\mathcal{A}_{\mathfrak{g}}^r)_i = \bigoplus_{p+q=r} (\mathcal{A}_{\mathfrak{g}})^{p,q}_i$$

for $\mathfrak{g} = \mathfrak{sl}(\infty)$

and

$$(\mathcal{A}_{\mathfrak{g}}^r)_i = \bigoplus_{p \leq r} (\mathcal{A}_{\mathfrak{g}})_i^p$$

for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$.

It also follows from the results of [PStyr] that

$$(\mathcal{A}_{\mathfrak{g}}^r)_0 = \bigoplus_{p+q \leq r} \text{End}_{\mathfrak{g}}(T^{p,q}) = \bigoplus_{p+q \leq r} \mathbb{C}[S_p \times S_q]$$

for $\mathfrak{g} = \mathfrak{sl}(\infty)$

and

$$(\mathcal{A}_{\mathfrak{g}}^r)_0 = \bigoplus_{p \leq r} \text{End}_{\mathfrak{g}}(T^p) = \bigoplus_{p \leq r} \mathbb{C}[S_p]$$

for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$.

Hence $(\mathcal{A}_{\mathfrak{g}}^r)_0$ is semisimple.

In addition, we have

$$(\mathcal{A}_{\mathfrak{g}})^{p,q}_i (\mathcal{A}_{\mathfrak{g}})^{r,s}_j = 0$$

unless $p = r - j, q = s - j$ for $\mathfrak{g} = \mathfrak{sl}(\infty)$

and

$$(\mathcal{A}_{\mathfrak{g}})_i^p (\mathcal{A}_{\mathfrak{g}})_j^r = 0$$

unless $p = r - 2j$ for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$.

This shows that for each $r$,

$$\mathcal{A}_{\mathfrak{g}}^r := \bigoplus_{p+q>r} \bigoplus_{i \geq 0} (\mathcal{A}_{\mathfrak{g}})^{p,q}_i$$

for $\mathfrak{g} = \mathfrak{sl}(\infty)$

or

$$\mathcal{A}_{\mathfrak{g}}^r := \bigoplus_{p>r} \bigoplus_{i \geq 0} (\mathcal{A}_{\mathfrak{g}})^p_i$$

for $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$.
is a $\mathbb{Z}_{\geq 0}$-graded ideal in $\mathcal{A}_g$ such that $\mathcal{A}_g^r \oplus \bar{\mathcal{A}}_g^r = \mathcal{A}_g$. Hence each unitary $\mathcal{A}_g^r$-module $X$ admits a canonical $\mathcal{A}_g^r$-module structure with $\mathcal{A}_g^r X = 0$, and thus becomes a locally unitary $\mathcal{A}_g$-module. This allows us to claim simply that

$$\mathcal{A}_g\text{-mof} = \lim_{\longrightarrow} (\mathcal{A}_g^r\text{-mof}).$$

Moreover, Proposition 5.1 now implies the following.

**Corollary 5.2.** The functors $\text{Hom}_g(\cdot, T)$ and $\text{Hom}_{\mathcal{A}_g}(\cdot, T)$ are mutually inverse antiequivalences of the categories $\mathcal{T}_g$ and $\mathcal{A}_g\text{-mof}$.

We now need to recall the definition of a Koszul ring. See [BGS], where this notion is studied extensively, and, in particular, several equivalent definitions are given. According to Proposition 2.1.3 in [BGS], a $\mathbb{Z}_{\geq 0}$-graded ring $A$ is **Koszul** if $A_0$ is a semisimple ring and for any two graded $A$-modules $M$ and $N$ of pure weight $m, n \in \mathbb{Z}$ respectively, $\text{ext}^i_A(M, N) = 0$ unless $i = m - n$, where $\text{ext}^i_A$ denotes the ext-group in the category of $\mathbb{Z}$-graded $A$-modules.

In the rest of this section we show that $\mathcal{A}_g^r$ is a Koszul ring.

We start by introducing the following notation: for any partition $\mu$, we set

$$\mu^+ := \{\text{partitions } \mu' \mid |\mu'| = |\mu| + 1 \text{ and } \mu'_i \neq \mu_i \text{ for exactly one } i\},$$

$$\mu^- := \{\text{partitions } \mu' \mid |\mu'| = |\mu| - 1 \text{ and } \mu'_i \neq \mu_i \text{ for exactly one } i\}.$$  

For any pair of partitions $\lambda = (\lambda^1, \lambda^2)$, we define

$$\lambda^+ := \{\text{pairs of partitions } \eta \mid \eta^1 \in \lambda^1^+, \eta^2 = \lambda^2\},$$

$$\lambda^- := \{\text{pairs of partitions } \eta \mid \eta^1 = \lambda^1, \eta^2 \in \lambda^2^-\}.$$  

**Lemma 5.3.** For any simple object $V_\lambda$ of $\mathcal{T}_g$, there is an exact sequence

$$0 \to V_\lambda^+ \to V \otimes V_\lambda \to V_\lambda^- \to 0,$$
where

\[ V_\lambda^+ = \bigoplus_{\eta \in \lambda^+} V_\eta \]
\[ V_\lambda^- = \bigoplus_{\eta \in \lambda^-} V_\eta. \]

Moreover, \( V_\lambda^+ = \text{soc}(V \otimes V_\lambda). \)

**Proof.** We will prove the statement for \( g = \mathfrak{sl}(\infty). \) The other cases are similar. The fact that the semisimplification of \( V \otimes V_\lambda \) is isomorphic to \( V_\lambda^+ \oplus V_\lambda^- \) follows from the classical Pieri rule.

To prove the equality \( V_\lambda^+ = \text{soc}(V \otimes V_\lambda), \) observe that

\[ V \otimes V_\lambda \subset V \otimes \tilde{V}_\lambda = T^{[\lambda^1]+1,[\lambda^2]} \cap (V \otimes \tilde{V}_\lambda). \]

On the other hand [PStyr, Theorem 2.3] implies directly that

\[ V_\lambda^+ = \text{soc}(T^{[\lambda^1]+1,[\lambda^2]} \cap (V \otimes \tilde{V}_\lambda). \]

Hence \( V_\lambda^+ = \text{soc}(V \otimes V_\lambda). \)

It remains to show that the quotient \( (V \otimes V_\lambda) / V_\lambda^+ \) is semisimple. This follows again from [PStyr, Theorem 2.3], since all simple subquotients of \( V_\lambda^- = (V \otimes V_\lambda) / V_\lambda^+ \) lie in \( \text{soc}(T^{[\lambda^1]+1,[\lambda^2]}). \)

**Proposition 5.4.** If \( \text{Ext}_T^i(V_\lambda, V_\mu) \neq 0, \) then

\[ |\mu^1| - |\lambda^1| = |\mu^2| - |\lambda^2| = i \quad \text{for } g = \mathfrak{sl}(\infty) \]

and

\[ |\mu| - |\lambda| = 2i \quad \text{for } g = \mathfrak{o}(\infty), \mathfrak{sp}(\infty). \]

**Proof.** Let \( g = \mathfrak{sl}(\infty). \) We will prove the statement by induction on \(|\mu|\). The base of induction \( \mu = (0,0) \) follows immediately from the fact that \( V_{(0,0)} = \mathbb{C} \) is injective. We assume \( \text{Ext}_T^i(V_\lambda, V_\mu) \neq 0. \) Without loss of generality we may assume that \( |\mu^1| > 0. \) Then there exists a pair of partitions \( \eta \) such that \( \mu \in \eta^+. \) Since \( V_\mu \) is a direct summand of \( V_{\eta^+}, \) we have \( \text{Ext}_T^i(V_\lambda, V_{\eta^+}) \neq 0. \)
Consider the short exact sequence from Lemma 5.3
\[ 0 \rightarrow V_{\eta}^+ \rightarrow V \otimes V_{\eta} \rightarrow V_{\eta}^- \rightarrow 0. \]

The associated long exact sequence implies that either \( \Ext^j_{T_g}(V_\lambda, V \otimes V_{\eta}) \neq 0 \) or \( \Ext^{j-1}_{T_g}(V_\lambda, V_{\eta}^-) \neq 0 \). In the latter case, the inductive hypothesis implies that
\[ |\eta^1| - |\lambda|_1 = (|\eta^2| - 1) - |\lambda^2| = i - 1. \]
The condition in the statement of the proposition follows, as \(|\eta^1| = |\mu^1| - 1\) and \(|\eta^2| = |\mu^2|\).

Now assume that \( \Ext^j_{T_g}(V_\lambda, V \otimes V_{\eta}) \neq 0 \). Let
\[ 0 \rightarrow V_{\eta} \rightarrow M_0 \rightarrow M_1 \rightarrow \ldots \]
be a minimal injective resolution of \( V_{\eta} \) in \( T_g \). By the inductive hypothesis, \( \Ext^j_{T_g}(V_\nu, V_{\eta}) \neq 0 \) implies
\[ |\eta^1| - |\nu| = |\eta^2| - |\nu^2| = j. \tag{5.1} \]
We claim that by the minimality of the resolution, \( \tilde{V}_\nu \) appears as a direct summand of \( M_j \) only if (5.1) holds, that is \( M_j = \oplus V_\nu \) for some set of \( \nu \) such that \(|\nu^1| = |\eta^1| - j\) and \(|\nu^2| = |\eta^2| - j\). Indeed, otherwise the sequence
\[ \Hom_g(V_\nu, M_{j-1}) \rightarrow \Hom_g(V_\nu, M_j) \rightarrow \Hom_g(V_\nu, M_{j+1}) \]
would be exact, and replacing \( M_j \) by \( M_j/\tilde{V}_\nu \), and \( M_{j+1} \) by \( M_{j+1}/\tilde{V}_\nu \) or \( M_{j-1} \) by \( M_{j-1}/\tilde{V}_\nu \), we would obtain a “smaller” resolution.

Furthermore, since the functor \( V \otimes (\cdot) \) is obviously exact (vector spaces are flat), the complex
\[ 0 \rightarrow V \otimes V_{\eta} \rightarrow V \otimes M_0 \rightarrow V \otimes M_1 \rightarrow \ldots \]
is an injective resolution of \( V \otimes V_{\eta} \). Thus \( \Hom_g(V_\lambda, V \otimes M_i) \neq 0 \) implies \(|\lambda^1| = |\eta^1| - i + 1\) and \(|\lambda^2| = |\eta^2| - i\), and the proof for \( g = sl(\infty) \) is complete.

The proof for \( g = o(\infty) \), \( sp(\infty) \) is similar, and we leave it to the reader.  \( \square \)
Recall that any $\mathfrak{g}$-module $W$ has a well-defined socle filtration

$$0 \subset \text{soc}^0(W) = \text{soc}(W) \subset \text{soc}^1(W) \subset \cdots$$

where $\text{soc}^i(W) := \pi_{i-1}^{-1}(\text{soc}(W/\text{soc}^{i-1}(W))$ and $\pi_{i-1} : W \to W/\text{soc}^{i-1}(W)$ is the projection. Similarly, any $A_\mathfrak{g}$-module $X$ has a radical filtration

$$\cdots \subset \text{rad}^1(X) \subset \text{rad}^0(X) = \text{rad}(X) \subset X$$

where $\text{rad}(X)$ is the joint kernel of all surjective $A_\mathfrak{g}$-homomorphisms $X \to X'$ with $X'$ simple, and $\text{rad}^i(X) = \text{rad}(\text{rad}^{i-1}(X))$.

Note furthermore that the Ext’s in the category $\mathcal{T}_\mathfrak{g}$ differ essentially from the Ext’s in $\mathfrak{g}$-mod. In particular, as shown in \cite{PS}, $\text{Ext}^1_{\mathfrak{g}}(V_\lambda, V_\mu)$ is uncountable dimensional whenever nonzero, whereas $\text{Ext}^1_{A_\mathfrak{g}}(V_\lambda, V_\mu)$ is always finite dimensional by Corollary 5.2. Here are two characteristic examples.

1. Consider the exact sequence of $\mathfrak{g}$-modules

$$0 \to V \to (V^*)_\vee \to (V^*)_\vee / V \to 0.$$  

The $\mathfrak{g}$-module $(V^*)_\vee / V$ is trivial, and any vector in $\text{Ext}^1_{\mathfrak{g}}(\mathbb{C}, V)$ determines a unique 1-dimensional subspace in $(V^*)_\vee / V$. On the other hand, $\text{Ext}^1_{\mathfrak{s}(\infty)}(\mathbb{C}, V) = 0$ by Proposition 5.4.

2. Each nonzero vector of $\text{Ext}^1_{\mathfrak{g}}(\mathbb{C}, \mathfrak{s}(\infty))$ corresponds to a 1-dimensional trivial quotient of $\text{soc}^1((\mathfrak{s}(\infty)_\vee)^*)$ (see \cite{PS}). The nonzero vectors of the 1-dimensional space $\text{Ext}^1_{\mathfrak{g}}(\mathbb{C}, \mathfrak{s}(\infty))$ on the other hand correspond to the unique 1-dimensional quotient of $\text{soc}^1((\mathfrak{s}(\infty)_\vee)^*)$ which determines an absolute weight module, namely $\mathfrak{s}(\infty)/\mathfrak{s}(\infty) = (V \otimes V^*)/\mathfrak{s}(\infty)$.

The following is the main result of this section.

**Theorem 5.5.** The ring $A_\mathfrak{g}$ is Koszul.

*Proof.* According to \cite[Proposition 2.1.3]{BGS}, it suffices to prove that unless $i = m - n$, one has $\text{ext}^i_{A_\mathfrak{g}}(M,N) = 0$ for any pure $A_\mathfrak{g}$-modules $M$, $N$ of weights $m$, $n$ respectively. We will prove that unless $i = m - n$, one has $\text{ext}^i_{A_\mathfrak{g}}(M,N) = 0$ for any simple pure $A_\mathfrak{g}$-modules $M$, $N$ of weights $m$, $n$ respectively. Since any $A_\mathfrak{g}$-module admits a canonical $A_\mathfrak{g}$-module structure, it will follow that $\text{ext}^i_{A_\mathfrak{g}}(M,N) = 0$ for any simple pure $A_\mathfrak{g}$-modules $M$, $N$. 

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of weights $m, n$ respectively unless $i = m - n$. The analogous statement for arbitrary $A^r_g$-modules of pure degree will also follow, since all such modules are semisimple.

Let $X_\lambda$ (respectively, $\tilde{X}_\lambda$) be the $A_g$-module which is the image of $V_\lambda$ (resp., $\tilde{V}_\lambda$) under the antiequivalence of Corollary 5.2. Then $\tilde{X}_\lambda$ is a projective cover of the simple module $X_\lambda$. Proposition 5.4 implies that $\text{Ext}^i_{A_g}(X_\mu, X_\lambda) = 0$ unless $|\mu^1| - |\lambda^1| = |\mu^2| - |\lambda^2| = i$. We consider a minimal projective resolution of $X_\mu$

$$\cdots \to P^1 \to P^0 \to 0$$

and claim that it must have the property $P^i \simeq \oplus \tilde{X}_\nu$ for some set of $\nu$ with $|\mu^1| - |\nu^1| = |\mu^2| - |\nu^2| = i$. This follows from the similar fact for a minimal injective resolution of $V_\mu$ in $\mathcal{T}_g$ (see the proof of Proposition 5.4) and the antiequivalence of the categories $\mathcal{T}_g$ and $A_g$-mo.

On the other hand, by $[\text{PStyr}]$ if $V_\nu$ is a simple constituent of $\text{soc}^i(\tilde{V}_\mu)$, or if under the antiequivalence $X_\nu$ is a simple constituent of $\text{rad}^i \tilde{X}_\mu$, then $|\mu^1| - |\nu^1| = |\mu^2| - |\nu^2| = i$. Therefore we see that in the above resolution the image of $\text{rad}^i(P^i)$ lies in $\text{rad}^{i+1}(P^{i-1})$. Now it is clear that we can endow the resolution (5.2) with a $\mathbb{Z}$-grading by setting the degree of $X_\mu$ to be an arbitrary integer $n$. Indeed, one should assign to each simple $(A_g)_0$-constituent of $P^i$ which lies in $\text{rad}^i(P^i)$ and not in $\text{rad}^{i+1}(P^i)$ the degree $n + i + j + 1$. This immediately implies that $\text{ext}^i_{A_g}(X_\mu, X_\lambda) = 0$ unless the difference between the weights of $X_\lambda$ and $X_\mu$ is $i$. □

6. On the structure of $A_g$

It is a result of $[\text{BGS}]$ that for any $r$ the Koszulity of $A^r_g$ implies that $A^r_g$ is a quadratic algebra generated by $(A^r_g)_0$ and $(A^r_g)_1$. That is, $A^r_g \simeq T_{(A^r_g)_0}((A^r_g)_1)/ (R^r)$, where $(R^r)$ is the two-sided ideal generated by some $(A^r_g)_0$-bimodule $R^r$ in $(A^r_g)_1 \otimes (A^r_g)_0 (A^r_g)_1$. Moreover, it is easy to see that $A_g$ is isomorphic to the quotient $T_{(A_g)_0}((A_g)_1)/ (R)$, where $R = \lim \to R^r$. In this section we describe $(A_g)_1$ and $R$.

In what follows we fix inclusions $S_n \subset S_{n+1}$ such that $S_{n+1}$ acts on the set $\{1, 2, \ldots, n+1\}$ and $S_n$ is the stabilizer of $n+1$. We start with the following lemma.

**Lemma 6.1.** If $g = sl(\infty)$, then $\text{Hom}_g(T^{p,q}, T^{p-1,q-1})$ as a left module over
$\mathbb{C}[S_{p-1} \times S_{q-1}]$ is generated by the contractions

$$\phi_{i,j} : T^{p,q} \to T^{p-1,q-1},$$

$$v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q \mapsto \langle v_1, w_j \rangle (v_1 \otimes \cdots \otimes \hat{v}_i \cdots \otimes v_p \otimes w_1 \otimes \cdots \hat{w}_j \cdots \otimes w_q).$$

If $g = o(\infty)$ or $sp(\infty)$, then $\text{Hom}_g(T^p, T^{p-2})$ as a left module over $\mathbb{C}[S_{p-2}]$ is generated by the contractions

$$\psi_{i,j} : T^p \to T^{p-2},$$

$$v_1 \otimes \cdots \otimes v_p \mapsto \langle v_i, v_j \rangle (v_1 \otimes \cdots \otimes \hat{v}_i \cdots \otimes \hat{v}_j \cdots \otimes v_p),$$

where $\langle \cdot, \cdot \rangle$ stands for the symmetric bilinear form on $V$ for $g = o(\infty)$, and the symplectic bilinear form on $V$ for $g = sp(\infty)$.

**Proof.** Let $g = sl(\infty)$ and $\varphi \in \text{Hom}_g(T^{p,q}, T^{p-1,q-1})$. Theorem 3.2 in [PS] claims that $\text{soc}(T^{p,q}) = \bigcap_{i \leq p, j \leq q} \ker \phi_{i,j}$; moreover, the same result implies that $\text{soc}(T^{p,q}) \subset \ker \varphi$. Define

$$\Phi : T^{p,q} \to \bigoplus_{i \leq p, j \leq q} T^{p-1,q-1}$$

as the direct sum $\bigoplus_{i,j} \phi_{i,j}$. Then there exists $\alpha : \bigoplus_{i \leq p, j \leq q} T^{p-1,q-1} \to T^{p-1,q-1}$ such that $\varphi = \alpha \circ \Phi$. But $\alpha = \bigoplus_{i,j} \alpha_{i,j}$ for some $\alpha_{i,j} \in \mathbb{C}[S_{p-1} \times S_{q-1}]$. Therefore $\varphi = \sum_{i,j} \alpha_{i,j} \phi_{i,j}$. This proves the lemma for $g = sl(\infty)$.

We leave the proof in the cases $g = o(\infty)$, $sp(\infty)$ to the reader. \ \Box

Let $g = sl(\infty)$. Recall that $(\mathcal{A}_g)^{p,q}_i = \text{Hom}_g(T^{p,q}, T^{p-i,q-i})$ and that $(\mathcal{A}_g)^{p,q}_0 = \mathbb{C}[S_p \times S_q]$.

**Lemma 6.2.** Let $g = sl(\infty)$.

1. $(\mathcal{A}_g)^{p,q}_i$ is isomorphic to $\mathbb{C}[S_p \times S_q]$ as a right $(\mathcal{A}_g)^{p,q}_0$-module, and the structure of a left $(\mathcal{A}_g)^{p-1,q-1}_0$-module is given by left multiplication via the fixed inclusion

$$(\mathcal{A}_g)^{p-1,q-1}_0 = \mathbb{C}[S_{p-1} \times S_{q-1}] \subset \mathbb{C}[S_p \times S_q] = (\mathcal{A}_g)^{p,q}_0.$$ 

2. We have

$$(\mathcal{A}_g)_1 \otimes_{(\mathcal{A}_g)_0} (\mathcal{A}_g)_1 = \bigoplus_{p,q} ((\mathcal{A}_g)^{p-1,q-1}_1 \otimes_{(\mathcal{A}_g)^{p-1,q-1}_0} (\mathcal{A}_g)^{p,q}_1),$$
where \((\mathcal{A}_g)^{p-1,q-1}_1 \otimes (\mathcal{A}_g)^{p-1,q-1}_0\) is isomorphic to \(\mathbb{C}[S_p \times S_q]\). Moreover, \((\mathcal{A}_g)^{p-1,q-1}_1 \otimes (\mathcal{A}_g)^{p-1,q-1}_0\) is a \((\mathbb{C}[S_{p-2} \times S_{q-2}], \mathbb{C}[S_p \times S_q])\)-bimodule via the embeddings \(\mathbb{C}[S_{p-2} \times S_{q-2}] \subset \mathbb{C}[S_{p-1} \times S_{q-1}] \subset \mathbb{C}[S_p \times S_q]\).

Proof. It is clear that all contractions \(\phi_{i,j} \in (\mathcal{A}_g)^{p,q}_1\) can be obtained from \(\phi_{p,q}\) via the right \(\mathbb{C}[S_p \times S_q]\)-module structure of \((\mathcal{A}_g)^{p,q}_1\). Thus by Lemma 6.1, as a \(\mathbb{C}[S_p \times S_q]\)-bimodule, \((\mathcal{A}_g)^{p,q}_1\) is generated by the single contraction \(\phi_{p,q}\). Moreover, \((\mathcal{A}_g)^{p,q}_1\) is a free right \(\mathbb{C}[S_p \times S_q]\)-module of rank 1. Indeed, if for some \(a_\sigma \in \mathbb{C}\)

\[
\sum_{\sigma \in S_p \times S_q} a_\sigma \phi_{p,q} \sigma = 0,
\]

then for all \(v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q \in T^{p,q}\)

\[
0 = \sum_{\sigma \in S_p \times S_q} a_\sigma \phi_{p,q} \sigma (v_1 \otimes \cdots \otimes v_p \otimes w_1 \otimes \cdots \otimes w_q)
\]

\[
= \sum_{\sigma = (\sigma_1, \sigma_2) \in S_p \times S_q} a_\sigma \langle v_{\sigma_1(p)}, w_{\sigma_2(q)} \rangle (v_{\sigma_1(1)} \otimes \cdots \otimes v_{\sigma_1(p-1)} \otimes w_{\sigma_2(1)} \otimes \cdots \otimes w_{\sigma_2(q-1)}),
\]

and hence \(a_\sigma = 0\) for all \(\sigma \in S_p \times S_q\). Finally, for any \(\sigma \in S_{p-1} \times S_{q-1}\) we have

\[
\sigma \phi_{p,q} = \phi_{p,q} \sigma.
\]

This implies part (1). Part (2) is a direct corollary of part (1). \(\Box\)

Lemma 6.3. Let \(g = sl(\infty)\). Let \(S \cong S_2 \times S_2\) denote the subgroup of \(S_p \times S_q\) generated by \((p, p-1)_l\) and \((q, q-1)_r\), where \((i, j)_l\) and \((i, j)_r\) stand for the transpositions in \(S_p\) and \(S_q\), respectively. Then \(R = \bigoplus_{p,q} R^{p,q}\), where

\[
R^{p,q} = (\text{triv} \boxtimes \text{sgn} \oplus \text{sgn} \boxtimes \text{triv}) \otimes_{\mathbb{C}[S]} \mathbb{C}[S_p \times S_q],
\]

and \text{triv} and \text{sgn} denote respectively the trivial and sign representations of \(S_2\).

Proof. The statement is equivalent to the equality of \(R^{p,q}\) and the right \(\mathbb{C}[S_p \times S_q]\)-module

\[
(1+(p, p-1)_l)(1-(q, q-1)_r)\mathbb{C}[S_p \times S_q] \oplus (1-(p, p-1)_l)(1+(q, q-1)_r)\mathbb{C}[S_p \times S_q] = (1+(p, p-1)_l)(1-(q, q-1)_r)\mathbb{C}[S_p \times S_q].
\]

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We have the obvious relations in $A_{sl(\infty)}$

$$
\phi_{p-1,q-1} \phi_{p,q} = \phi_{p-1,q-1} \phi_{p,q}(p,p-1)_l(q,q-1)_r,
$$

$$
\phi_{p-1,q-1} \phi_{p,q}(p,p-1)_l = \phi_{p-1,q-1} \phi_{p,q}(q,q-1)_r.
$$

Therefore $R^{p,q}$ contains the module

$$(1+(p,p-1)_l)(1-(q,q-1)_r) \mathbb{C}[S_p \times S_q] \oplus (1-(p,p-1)_l)(1+(q,q-1)_r) \mathbb{C}[S_p \times S_q],$$

which has dimension $\frac{p!q!}{2}$. On the other hand, it is easy to see that

$$
\dim R^{p,q} = \dim \left( (A_g)^{p-1,q-1} \otimes (A_g)^{p-1,q-1} \right) - \dim (A_g)_2^{p,q}
$$

$$
= \frac{(p-1)!(q-1)!p!q!}{(p-1)!(q-1)!} - \frac{p!q!}{2}
$$

$$
= \frac{p!q!}{2}.
$$

Hence the statement. \(\square\)

**Corollary 6.4.** Let $g = sl(\infty)$. Then $A_g^*$ is Koszul self-dual, i.e. $A_g^* \simeq ((A_g^*)^!)^{opp}$. Furthermore, $A_g \simeq (A_g^!)^{opp}$, where $A_g^! := \lim (A_g^!)^!$.

**Proof.** By definition, we have $(A_g^*)^! = T_{(A_g^*)0}((A_g^*)^!) / (R^{p,q})$, where $(A_g^*)^! = \text{Hom}_{(A_g^*)0}((A_g^*)^!, (A_g^!)^0), [BGS]$. Note that $((A_g^*)^!)^!$ is a $(A_g^0)^{p,q}, (A_g^0)^{p-1,q-1}$-bimodule. Moreover, Lemma 6.2 (1) implies an isomorphism of bimodules

$$
((A_g^!)^!)^* \simeq \mathbb{C}[S_p \times S_q].
$$

Hence we have an isomorphism of $((A_g^*)_0)^{opp}$-bimodules

$$
((A_g^*)^!)^! \simeq (A_g^*)^!.
$$

One can check that $R^L = \bar{R}$, where $\bar{R} := \oplus R^{p,q}$, and the modules $R^{p,q}$ are defined via the decomposition of $(A_g^0)^{p,q}$-modules

$$(A_g^0)^{p-1,q-1} \otimes (A_g^0)^{p-1,q-1} (A_g^0)^{p,q} = R^{p,q} \oplus \bar{R}^{p,q}.$$

Therefore $((A_g^*)_0)^{opp} \simeq T_{(A_g^*)0}((A_g^*)^! / (\bar{R}^*)$. Now consider the automorphism $\sigma$ of $\mathbb{C}[S_p \times S_q]$ defined for all $p$ and $q$ by $\sigma(s,t) = sgn(t)(s,t)$ for all $s \in S_p$. 28
Corollary 6.5. Let \( g = \mathfrak{g}(\infty) \), and for a pair of partitions \( \nu = (\nu^1, \nu^2) \) take \( \nu^\perp := (\nu^1, (\nu^2)^\perp) \). Then \( \dim \text{Ext}_T^i(V_\lambda, V_\mu) \) equals the multiplicity of \( V_{\lambda^\perp} \) in \( \text{soc}^i(\tilde{V}_{\mu^\perp})/\text{soc}^{i-1}(\tilde{V}_{\mu^\perp}) \), as computed in \([PStyr, \text{Theorem 2.3}]\).

Proof. The statement follows from \([BGS, \text{Theorem 2.10.1}]\) applied to \( \mathcal{A}^r_g \) for sufficiently large \( r \). Indeed, this result implies that \( \text{Ext}_{\mathcal{A}^r_g}(\mathcal{A}_g^0, (\mathcal{A}_g^0)_{\mu}) \) is isomorphic to \( (\mathcal{A}^r_g)^{\text{opp}} \) as a graded algebra. Moreover, the simple \( \mathcal{A}_g \)-module \( X_\lambda \) (which is the image of \( V_\lambda \) under the antiequivalence of Corollary 5.2) is isomorphic to \( (\mathcal{A}_g^0)_{\mathbb{Y}_\lambda} \), where \( \mathbb{Y}_\lambda \) is the product of Young projectors corresponding to the partitions \( \lambda^1 \) and \( \lambda^2 \). This follows immediately from the fact that \( \mathbb{Y}_\lambda \) is a primitive idempotent in \( (\mathcal{A}_g^0)_{\mu} \) and hence also in \( \mathcal{A}_g \), see for example \([CR, \text{Theorem 54.5}]\). The projective cover \( \tilde{X}_\lambda \) of \( X_\lambda \) is isomorphic to \( \mathcal{A}_g \mathbb{Y}_\lambda \). Therefore we have

\[
\dim \text{Ext}_T^i(V_\lambda, V_\mu) = \dim \text{Ext}_{\mathcal{A}_g}(X_\mu, X_\lambda) = \dim \mathbb{Y}_\lambda (\mathcal{A}_g^0)^{\text{opp}} \mathbb{Y}_\mu.
\]

By Corollary 6.4,

\[
\dim \mathbb{Y}_\lambda (\mathcal{A}_g^0)^{\text{opp}} \mathbb{Y}_\mu = \dim \mathbb{Y}_{\lambda^\perp} (\mathcal{A}_g^0)_{\mathbb{Y}_{\mu^\perp}}.
\]

Furthermore, \( \dim \mathbb{Y}_{\lambda^\perp} (\mathcal{A}_g^0)_{\mathbb{Y}_{\mu^\perp}} \) equals the multiplicity of \( X_{\lambda^\perp} \) in the module \( \text{rad}^{i-1} \tilde{X}_{\mu^\perp}/\text{rad}^i \tilde{X}_{\mu^\perp} \) \([CR, \text{Theorem 54.15}]\), which coincides with the multiplicity of \( V_{\lambda^\perp} \) in \( \text{soc}^i(\tilde{V}_{\mu^\perp})/\text{soc}^{i-1}(\tilde{V}_{\mu^\perp}) \). 

Corollary 6.6. The blocks of the category \( \mathbb{T}_{\mathfrak{g}(\infty)} \) are parametrized by \( \mathbb{Z} \). In particular,

1. \( V_\lambda \) and \( V_\mu \) belong to the block \( \mathbb{T}_{\mathfrak{g}(\infty)}(i) \) for \( i \in \mathbb{Z} \) if and only if \( |\lambda^1| - |\lambda^2| = |\mu^1| - |\mu^2| = i \).
2. Two blocks \( \mathbb{T}_{\mathfrak{g}(\infty)}(i) \) and \( \mathbb{T}_{\mathfrak{g}(\infty)}(j) \) are equivalent if and only if \( i = \pm j \).
Lemma 6.7. We have

\[(\mathcal{A}_g)_1^p \simeq \text{triv} \otimes_{\mathbb{C}[S_p]} \mathbb{C}[S_p]\]

for \(g = o(\infty)\)

and

\[(\mathcal{A}_g)_1^p \simeq \text{sgn} \otimes_{\mathbb{C}[S_p]} \mathbb{C}[S_p]\]

for \(g = sp(\infty)\).
In both cases left multiplication by $\mathbb{C}[S_{p-2}]$ is well defined, as $S'$ centralizes $S_{p-2}$.

Proof. Lemma 6.1 implies that the contraction $\psi_{p-1,p}$ generates $(A_g)_i^p$ as a right $\mathbb{C}[S_p]$-module. Then the statement follows from the relation

$$\psi_{p-1,p} = \pm \psi_{p-1,p}(p,p-1),$$

where the sign is + for $g = o(\infty)$ and − for $g = sp(\infty)$. \qed

Corollary 6.8. Let $g = o(\infty)$ or $sp(\infty)$. Then

$$(A_g)_i^{p-2} \otimes (A_g)_0^{p-2} (A_g)_1^p \simeq L_g \otimes \mathbb{C}[S] \mathbb{C}[S_p],$$

where $S \simeq S_2 \times S_2$ is the subgroup generated by $(p,p-1)$ and $(p-2,p-3)$ and

$$L_g = \begin{cases} \text{triv} & \text{for } g = o(\infty) \\ \text{sgn} \otimes \text{sgn} & \text{for } g = sp(\infty). \end{cases}$$

To describe $R$, write $R = \bigoplus_p R^p$, where $R^p \subset (A_g)_i^{p-2} \otimes (A_g)_0^{p-2} (A_g)_1^p$.

We will need the following decompositions of $S_4$-modules:

$$\text{triv} \otimes \mathbb{C}[S] \mathbb{C}[S_4] = X_{(2,1,1)} \oplus X_{(2,2)} \oplus X_{(4)}, \quad (6.1)$$

$$\text{sgn} \otimes \text{sgn} \otimes \mathbb{C}[S] \mathbb{C}[S_4] = X_{(3,1)} \oplus X_{(2,2)} \oplus X_{(1,1,1,1)}. \quad (6.2)$$

Lemma 6.9. Let $S'' \subset S_p$ be the subgroup isomorphic to $S_4$ that fixes $1, 2, \ldots, p-4$. Then

$$R^p \simeq X_{(2,1,1)} \otimes \mathbb{C}[S''] \mathbb{C}[S_p] \quad \text{for } g = o(\infty),$$

and

$$R^p \simeq X_{(3,1)} \otimes \mathbb{C}[S''] \mathbb{C}[S_p] \quad \text{for } g = sp(\infty).$$

Proof. Let us deal with the case of $o(\infty)$. We consider the following Young projectors in $S'' \simeq S_4$

$$Y_{(2,1,1)} = (1+(p-1,p))(1-(p,p-2)-(p,p-3)-(p-2,p-3)+(p,p-2,p-3)+(p,p-3,p-2)),$$
\[ Y_{(2,2)} = (1 + (p, p - 1))(1 + (p - 2, p - 3))(1 - (p - 2, p))(1 - (p - 1, p - 3)), \]
and
\[ Y_{(4)} = \sum_{s \in S''} s. \]

By Equation (6.1) we have
\[ R^p_1 \subseteq (A_g)^{p-2} \otimes (A_g)^{p-2} = Y_{(2,1,1)} \mathbb{C}[S_p] \oplus Y_{(2,2)} \mathbb{C}[S_p] \oplus Y_{(4)} \mathbb{C}[S_p]. \]

By direct inspection one can check that
\[
\begin{align*}
\psi_{p-3,p-2,1} Y_{(2,1,1)} &= 0, \\
\psi_{p-3,p-2,1} Y_{(2,2)} &= 2 \psi_{p-3,p-2,1} - 2 \psi_{p-3,p-2,2}, \\
\psi_{p-3,p-2,1} Y_{(4)} &= 4 \psi_{p-3,p-2,1}.
\end{align*}
\]

The statement follows for \( o(\infty) \).

We leave the case of \( sp(\infty) \) to the reader. \( \square \)

**Corollary 6.10.** \( A_{sp(\infty)} \simeq A_{o(\infty)}. \)

**Proof.** We use the automorphism \( \sigma \) of \( \mathbb{C}[S_p] \) which sends \( s \) to \( \text{sgn}(s)s \). \( \square \)

**Corollary 6.11.** The categories \( T_{o(\infty)} \) and \( T_{sp(\infty)} \) are equivalent.

In \([Sr]\) a tensor functor \( T_{o(\infty)} \to T_{sp(\infty)} \) establishing an equivalence of tensor categories is constructed using the Lie superalgebra \( osp(\infty, \infty) \).

**Proposition 6.12.** \( T_{o(\infty)} \) and \( T_{sp(\infty)} \) have two inequivalent blocks \( T^{ev}_g \) and \( T^{odd}_g \) generated by all \( V_\lambda \) with \( |\lambda| \) even and odd, respectively.

**Proof.** Due to the previous corollary it suffices to consider the case \( g = o(\infty). \)

As follows from \([PStyr]\), \( \text{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda) \neq 0 \) if and only if \( \mu \in \lambda^{++} \), where
\[
\lambda^{++} := \{ \text{partitions } \lambda' \mid \lambda_i \leq \lambda'_i \text{ for all } i, \ |\lambda'| = |\lambda| + 2, \lambda'_j \neq \lambda_j \text{ and } \lambda'_k \neq \lambda_k \text{ for } j \neq k \text{ implies } \lambda_j \neq \lambda_k \}.\]

Note that the partitions in \( \lambda^{++} \) are those which arise from \( \lambda \) via the Pieri rule for tensoring with \( S^2(V) \). Consider the minimal equivalence relation on partitions for which \( \lambda \) and \( \mu \) are equivalent whenever \( \mu \in \lambda^{++} \). One can check that there are exactly two equivalence classes which are determined by the parity of \( |\lambda| \).
To show that $T^{ev}_g$ and $T^{odd}_g$ are not equivalent observe that all simple injective modules in $T_g$ correspond to partitions $\mu$ with $\mu_1 = \cdots = \mu_s = 1$, or equivalently are isomorphic to the exterior powers $\Lambda^s(V)$ of the standard module. If $s \geq 1$ then $\Lambda^s(V)$ has nontrivial extensions by two non-isomorphic simple modules, namely $V_{(3,1,\ldots,1)}$ and $V_{(2,1,1,\ldots,1)}$. The trivial module on the other hand has a nontrivial extension by only $S^2(V) = V_{(2)}$. Therefore $T^{ev}_g$ contains a simple injective module admitting a nontrivial extension with only one simple module, whereas $T^{odd}_g$ does not contain such a simple injective module.

\[\square\]


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