ORBIT DUALITY IN IND-VARIETIES OF MAXIMAL GENERALIZED FLAGS

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To Ernest Borisovich Vinberg
on the occasion of his 80th birthday

Abstract. We extend Matsuki duality to arbitrary ind-varieties of maximal generalized flags, in other words, to any homogeneous ind-variety $\mathbf{G}/\mathbf{B}$ for a classical ind-group $\mathbf{G}$ and a splitting Borel ind-subgroup $\mathbf{B} \subset \mathbf{G}$. As a first step, we present an explicit combinatorial version of Matsuki duality in the finite-dimensional case, involving an explicit parametrization of K- and G^0-orbits on G/B. After proving Matsuki duality in the infinite-dimensional case, we give necessary and sufficient conditions on a Borel ind-subgroup $\mathbf{B} \subset \mathbf{G}$ for the existence of open and closed $\mathbf{K}$- and $\mathbf{G}^0$-orbits on $\mathbf{G}/\mathbf{B}$, where $(\mathbf{K}, \mathbf{G}^0)$ is an aligned pair of a symmetric ind-subgroup $\mathbf{K}$ and a real form $\mathbf{G}^0$ of $\mathbf{G}$.

1. Introduction

In this paper we extend Matsuki duality to ind-varieties of maximal generalized flags, i.e., to homogeneous ind-spaces of the form $\mathbf{G}/\mathbf{B}$ for $\mathbf{G} = \mathrm{GL}(\infty)$, $\mathrm{SL}(\infty)$, $\mathrm{SO}(\infty)$, $\mathrm{Sp}(\infty)$. In the case of a finite-dimensional reductive algebraic group G, Matsuki duality [6, 11, 12] is a bijection between the (finite) set of K-orbits on G/B and the set of G^0-orbits on G/B, where K is a symmetric subgroup of G and G^0 is a properly chosen real form of G. Moreover, this bijection reverses the inclusion relation between orbit closures. In particular, the remarkable theorem about the uniqueness of a closed G^0-orbit on G/B, see [19], follows via Matsuki duality from the uniqueness of a (Zariski) open K-orbit on G/B. In the monograph [7] on cycle spaces there is a self-contained treatment of Matsuki duality. In fact, the origins of Matsuki duality can be traced back to J. A. Wolf's work [19].

If $\mathbf{G} = \mathrm{GL}(\infty)$, $\mathrm{SL}(\infty)$, $\mathrm{SO}(\infty)$, $\mathrm{Sp}(\infty)$ is a classical ind-group, then its Borel ind-subgroups are neither $\mathbf{G}$-conjugate nor $\mathrm{Aut}(\mathbf{G})$-conjugate, hence there are many ind-varieties of the form $\mathbf{G}/\mathbf{B}$. We show that Matsuki duality extends to any ind-variety $\mathbf{G}/\mathbf{B}$ where $\mathbf{B}$ is a splitting Borel ind-subgroup of $\mathbf{G}$ for $\mathbf{G} = \mathrm{GL}(\infty)$, $\mathrm{SL}(\infty)$, $\mathrm{SO}(\infty)$, $\mathrm{Sp}(\infty)$. In the infinite-dimensional case, the structure of $\mathbf{G}^0$-orbits and $\mathbf{K}$-orbits on $\mathbf{G}/\mathbf{B}$ is more complicated than in the finite-dimensional case, and there are always infinitely many orbits.

A first study of the $\mathbf{G}^0$-orbits on $\mathbf{G}/\mathbf{B}$ for $\mathbf{G} = \mathrm{GL}(\infty), \mathrm{SL}(\infty)$ was done in [9] and was continued in [20]. In particular, in [9] it was shown that, for some real forms $\mathbf{G}^0$, there are splitting Borel ind-subgroups $\mathbf{B} \subset \mathbf{G}$ such that $\mathbf{G}/\mathbf{B}$ has neither an

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open nor a closed $\mathbf{G}^0$-orbit. We know of no prior studies of the structure of $\mathbf{K}$-orbits on $\mathbf{G}/\mathbf{B}$ of $\mathbf{G} = \mathrm{GL}(\infty), \mathrm{SL}(\infty), \mathrm{SO}(\infty), \mathrm{Sp}(\infty)$. The duality we establish in this paper shows that the structure of $\mathbf{K}$-orbits on $\mathbf{G}/\mathbf{B}$ is a "mirror image" of the structure of $\mathbf{G}^0$-orbits on $\mathbf{G}/\mathbf{B}$. In particular, the fact that $\mathbf{G}/\mathbf{B}$ admits at most one closed $\mathbf{G}^0$-orbit is now a corollary of the obvious statement that $\mathbf{G}/\mathbf{B}$ admits at most one Zariski-open $\mathbf{K}$-orbit.

Our main result can be stated as follows. Let $(\mathbf{G}, \mathbf{K}, \mathbf{G}^0)$ be one of the triples listed in Section 2.1 consisting of a classical (complex) ind-group $\mathbf{G}$, a symmetric ind-subgroup $\mathbf{K} \subset \mathbf{G}$, and the corresponding real form $\mathbf{G}^0 \subset \mathbf{G}$. Let $\mathbf{B} \subset \mathbf{G}$ be a splitting Borel ind-subgroup such that $\mathbf{X} := \mathbf{G}/\mathbf{B}$ is an ind-variety of maximal generalized flags (isotropic, in types B, C, D) weakly compatible with a basis of $\mathbf{V}$ adapted to the choice of $\mathbf{K}$, $\mathbf{G}^0$ in the sense of Sections 2.1, 2.3. There are natural exhaustions $\mathbf{G} = \bigcup_{n\geq 1} G_n$ and $\mathbf{X} = \bigcup_{n\geq 1} X_n$. Here G_n is a finite-dimensional algebraic group, X_n is the full flag variety of G_n, and the inclusion $X_n \subset \mathbf{X}$ is in particular G_n-equivariant. The subgroups $K_n := \mathbf{K} \cap G_n$ and $G_n^0 := \mathbf{G}^0 \cap G_n$ are respectively a symmetric subgroup and the corresponding real form of G_n. See Section 4.4 for more details.

Theorem 1. (a) *For every $n \geq 1$ the inclusion $X_n \subset \mathbf{X}$ induces embeddings of orbit sets $X_n/K_n \hookrightarrow \mathbf{X}/\mathbf{K}$ and $X_n/G_n^0 \hookrightarrow \mathbf{X}/\mathbf{G}^0$.*

(b) *There is a bijection $\varXi : \mathbf{X}/\mathbf{K} \to \mathbf{X}/\mathbf{G}^0$ such that the diagram*

$$\begin{array}{ccc} X_n/K_n & \hookrightarrow & \mathbf{X}/\mathbf{K} \\ \downarrow \varXi_n & & \downarrow \varXi \\ X_n/G_n^0 & \hookrightarrow & \mathbf{X}/\mathbf{G}^0 \end{array} \tag{1}$$

is commutative, where $\varXi_n$ stands for Matsuki duality.

(c) *For every $\mathbf{K}$-orbit $\boldsymbol{\mathcal{O}} \subset \mathbf{X}$ the intersection $\boldsymbol{\mathcal{O}} \cap \varXi(\boldsymbol{\mathcal{O}})$ consists of a single $\mathbf{K} \cap \mathbf{G}^0$-orbit.*

(d) *The bijection $\varXi$ reverses the inclusion relation of orbit closures. In particular $\varXi$ maps open (resp., closed) $\mathbf{K}$-orbits to closed (resp., open) $\mathbf{G}^0$-orbits.*

Actually our results are much more precise: in Propositions 7, 8, 9 we show that $\mathbf{X}/\mathbf{K}$ and $\mathbf{X}/\mathbf{G}^0$ admit the same explicit parametrization which is nothing but the inductive limit of suitable joint parametrizations of X_n/K_n and X_n/G_n^0. This yields the bijection $\varXi$ of Theorem 1 (b). Parts (a) and (b) of Theorem 1 are implied by our claims (39), (42), (43) below. Theorem 1 (c) follows from the corresponding statements in Propositions 7, 8, 9. Finally, Theorem 1 (d) is implied by Theorem 1 (a)–(b), the definition of the ind-topology, and the fact that the duality $\varXi_n$ reverses the inclusion relation between orbit closures.

As an example, if $\mathbf{G} = \mathbf{GL}(\infty)$ and $\mathbf{K} \subset \mathbf{G}$ is the ind-subgroup of transformations preserving an orthogonal form ω, we show that both orbit sets $\mathbf{X}/\mathbf{K}$ and $\mathbf{X}/\mathbf{G}^0$ are parametrized by the involutions $w : \mathbb{N}^* \to \mathbb{N}^*$ such that $w(\ell) = \iota(\ell)$ for all but finitely many $\ell \in \mathbb{N}^*$, where ι is the involution induced by the matrix of ω in a suitable basis of the natural representation of $\mathbf{G}$ (see Section 4.1).

Our methods are based on the classification of symmetric subgroups and real forms of the classical simple algebraic groups. Possibly one could provide a classification-free proof of our results in a future study.

Organization of the paper. In Section 2 we introduce the notation for classical ind-groups, symmetric ind-subgroups, and real forms. We recall some basic facts on finite-dimensional flag varieties, as well as the notion of ind-variety of generalized flags [4, 8]. In Section 3 we give the joint parametrization of K- and G^0-orbits in a finite-dimensional flag variety. This parametrization should be known in principle (see [13, 21]) but we have not found a reference where it would appear exactly as we present it. For the sake of completeness we provide full proofs of these results. In Section 4 we state our main results on the parametrization of $\mathbf{K}$- and $\mathbf{G}^0$-orbits in ind-varieties of generalized flags. Theorem 1 above is a consequence of these results. In Section 5 we point out some further corollaries of our main results.

In what follows $\mathbb{N}^*$ stands for the set of positive integers. $|A|$ stands for the cardinality of a set A. The symmetric group on n letters is denoted by $\mathfrak{S}_n$ and $\mathfrak{S}_\infty = \varinjlim \mathfrak{S}_n$ stands for the infinite symmetric group. Often we write w_k for the image $w(k)$ of k by a permutation w. By $(k;\ell)$ we denote the transposition that switches k and ℓ. We use boldface letters to denote ind-varieties. An index of notation can be found at the end of the paper.

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2. Notation and preliminary facts

2.1. **Classical groups and classical ind-groups.** Let $\mathbf{V}$ be a complex vector space of countable dimension, with an ordered basis $E = (e_1, e_2, \ldots) = (e_\ell)_{\ell\in\mathbb{N}^*}$. Every vector $x \in \mathbf{V}$ is identified with the column of its coordinates in the basis E, and $x \mapsto \overline{x}$ stands for complex conjugation with respect to E. We also consider the finite dimensional subspace $V = V_n := \langle e_1, \ldots, e_n\rangle_\mathbb{C}$ of $\mathbf{V}$.

The classical ind-group $\mathrm{GL}(\infty)$ is defined as

$$\mathrm{GL}(\infty) = \mathbf{G}(E) := \{g \in \mathrm{Aut}(\mathbf{V}) : g(e_\ell) = e_\ell \text{ for all } \ell \gg 1\} = \bigcup_{n\geq 1} \mathrm{GL}(V_n).$$

The real forms of $\mathrm{GL}(\infty)$ are well known and can be traced back to the work of Baranov [1]. Below we list aligned pairs $(\mathbf{K}, \mathbf{G}^0)$, where $\mathbf{G}^0$ is a real form of $\mathbf{G}$ and $\mathbf{K} \subset \mathbf{G}$ is a symmetric ind-subgroup of $\mathbf{G}$. The pairs $\big(\mathbf{K}, \mathbf{G}^0\big)$ we consider are aligned in the following way: for the exhaustion of $\mathbf{G}$ as a union $\bigcup_n \mathrm{GL}\,(V_n)$, the subgroup $K_n := \mathbf{K} \cap \mathrm{GL}\,(V_n)$ is a symmetric subgroup of $\mathrm{GL}\,(V_n)$, $G_0^n := \mathbf{G}^0 \cap \mathrm{GL}\,(V_n)$ is a real form of $\mathrm{GL}\,(V_n)$, and $K_n \cap G_n^0$ is a maximal compact subgroup of G_n^0.

2.1.1. *Types A1 and A2.* Let Ω be a $\mathbb{N}^* \times \mathbb{N}^*$-matrix of the form

$$(2)\quad \Omega = \begin{pmatrix} J_1 & & (0) \\ & J_2 & \\ (0) & & \ddots \end{pmatrix} \text{ where } \begin{cases} J_k \in \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (1) \right\} & \text{(orthogonal case, type A1)}, \\ J_k = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{(symplectic case, type A2)}. \end{cases}$$

The bilinear form

$$\omega(x, y) := {}^t x \Omega y \quad (x, y \in \mathbf{V})$$

is symmetric in type A1 and symplectic in type A2, whereas the map

$$\gamma(x) := \Omega\overline{x} \quad (x \in \mathbf{V})$$

is an involution of $\mathbf{V}$ in type A1 and satisfies $\gamma^2 = -\mathrm{id}_{\mathbf{V}}$ in type A2. Let

$$\mathbf{K} = \mathbf{G}(E,\omega) := \{g \in \mathbf{G}(E) : \omega(gx, gy) = \omega(x,y)\ \forall x, y \in \mathbf{V}\}$$

and

$$\mathbf{G}^0 := \{g \in \mathbf{G}(E) : \gamma(gx) = g\gamma(x)\ \forall x \in \mathbf{V}\}.$$

2.1.2. *Type A3.* Fix a (proper) decomposition $\mathbb{N}^* = N_+ \sqcup N_-$ and let

$$\Phi = \begin{pmatrix} \epsilon_1 & & (0) \\ & \epsilon_2 & \\ (0) & & \ddots \end{pmatrix} \tag{3}$$

where $\epsilon_\ell = 1$ for $\ell \in N_+$ and $\epsilon_\ell = -1$ for $\ell \in N_-$. Thus

$$\phi(x,y) := {}^t\overline{x}\Phi y \quad (x, y \in \mathbf{V})$$

is a Hermitian form of signature $(|N_+|, |N_-|)$ and

$$\delta(x) := \Phi x \quad (x \in \mathbf{V})$$

is an involution. Finally let

$$\mathbf{K} := \{g \in \mathbf{G}(E) : \delta(gx) = g\delta(x)\ \forall x \in \mathbf{V}\}$$

and

$$\mathbf{G}^0 := \{g \in \mathbf{G}(E) : \phi(gx, gy) = \phi(x,y)\ \forall x, y \in \mathbf{V}\}.$$

Types B, C, D. Next we describe pairs $(\mathbf{K}, \mathbf{G}^0)$ associated to the other classical ind-groups $\mathrm{SO}(\infty)$ and $\mathrm{Sp}(\infty)$. Let $\mathbf{G} = \mathbf{G}(E,\omega)$ where ω is a (symmetric or symplectic) bilinear form given by a matrix Ω as in (2). In view of (2), for every $\ell \in \mathbb{N}^*$ there is a unique $\ell^* \in \mathbb{N}^*$ such that

$$\omega(e_\ell, e_{\ell^*}) \neq 0.$$

Moreover $\ell^* \in \{\ell - 1, \ell, \ell + 1\}$. The map $\ell \mapsto \ell^*$ is an involution of $\mathbb{N}^*$.

2.1.3. *Types BD1 and C2.* Assume that ω is symmetric in type BD1 and symplectic in type C2. Fix a (proper) decomposition $\mathbb{N}^* = N_+ \sqcup N_-$ such that

$$\forall \ell \in \mathbb{N}^*,\ \ell \in N_+ \Leftrightarrow \ell^* \in N_+$$

and the restriction of ω on each of the subspaces $\mathbf{V}_+ := \langle e_\ell : \ell \in N_+ \rangle_{\mathbb{C}}$ and $\mathbf{V}_- := \langle e_\ell : \ell \in N_- \rangle_{\mathbb{C}}$ is nondegenerate. Let Φ, ϕ, δ be as in Section 2.1.2. Then we set

$$\mathbf{K} := \{g \in \mathbf{G}(E,\omega) : \delta(gx) = g\delta(x)\ \forall x \in \mathbf{V}\} \tag{4}$$

and

$$\mathbf{G}^0 := \{g \in \mathbf{G}(E,\omega) : \phi(gx, gy) = \phi(x,y)\ \forall x, y \in \mathbf{V}\}. \tag{5}$$

2.1.4. *Types C1 and D3.* Assume that ω is symmetric in type D3 and symplectic in type C1. Fix a decomposition $\mathbb{N}^* = N_+ \sqcup N_-$ satisfying

$$\forall \ell \in \mathbb{N}^*,\ \ell \in N_+ \Leftrightarrow \ell^* \in N_-.$$

Note that this forces every block J_k in (2) to be of size 2. In this situation $\mathbf{V}_+ := \langle e_\ell : \ell \in N_+\rangle_\mathbb{C}$ and $\mathbf{V}_- := \langle e_\ell : \ell \in N_-\rangle_\mathbb{C}$ are maximal isotropic subspaces for the form ω. Let Φ, ϕ, δ be as in Section 2.1.2. Finally, we define the ind-subgroups $\mathbf{K}, \mathbf{G}^0 \subset \mathbf{G}$ as in (4), (5).

Finite-dimensional case. The following table summarizes the form of the intersections $G = \mathbf{G} \cap \mathrm{GL}(V_n)$, $K = \mathbf{K} \cap \mathrm{GL}(V_n)$, $G^0 = \mathbf{G}^0 \cap \mathrm{GL}(V_n)$, where $n = 2m$ is even whenever we are in types A2, C1, C2, and D3. In types A3, BD1, and C2, we set $(p, q) = (|N_+ \cap \{1, \ldots, n\}|, |N_- \cap \{1, \ldots, n\}|)$. By $\mathbb{H}$ we denote the skew field of quaternions. In this way we retrieve the classical finite-dimensional symmetric pairs and real forms (see, e.g., [2, 15, 16]).

type	$G := \mathbf{G} \cap \mathrm{GL}(V_n)$	$K := \mathbf{K} \cap \mathrm{GL}(V_n)$	$G^0 := \mathbf{G}^0 \cap \mathrm{GL}(V_n)$
A1		$\mathrm{O}_n(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{R})$
A2	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{Sp}_n(\mathbb{C})$	$\mathrm{GL}_m(\mathbb{H})$
A3		$\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$	$\mathrm{U}_{p,q}(\mathbb{C})$
BD1	$\mathrm{O}_n(\mathbb{C})$	$\mathrm{O}_p(\mathbb{C}) \times \mathrm{O}_q(\mathbb{C})$	$\mathrm{O}_{p,q}(\mathbb{C})$
C1	$\mathrm{Sp}_n(\mathbb{C})$	$\mathrm{GL}_m(\mathbb{C})$	$\mathrm{Sp}_n(\mathbb{R})$
C2		$\mathrm{Sp}_p(\mathbb{C}) \times \mathrm{Sp}_q(\mathbb{C})$	$\mathrm{Sp}_{p,q}(\mathbb{C})$
D3	$\mathrm{O}_n(\mathbb{C}) = \mathrm{O}_{2m}(\mathbb{C})$	$\mathrm{GL}_m(\mathbb{C})$	$\mathrm{O}^*_n(\mathbb{C})$

In each case G^0 is a real form obtained from K so that $K \cap G^0$ is a maximal compact subgroup of G^0. Conversely K is obtained from G^0 as the complexification of a maximal compact subgroup.

2.2. **Finite-dimensional flag varieties.** Recall that $V = V_n$. The flag variety $X := \mathrm{GL}(V)/B = \{gB : g \in \mathrm{GL}(V)\}$ (for a Borel subgroup $B \subset \mathrm{GL}(V)$) can as well be viewed as the set of Borel subgroups $\{gBg^{-1} : g \in \mathrm{GL}(V)\}$ or as the set of complete flags

$$\big\{\mathcal{F} = (F_0 \subset F_1 \subset \ldots \subset F_n = V) : \dim F_k = k \ \text{ for all } k\big\}. \tag{6}$$

For every complete flag $\mathcal{F}$ let $B_\mathcal{F} := \{g \in \mathrm{GL}(V) : g\mathcal{F} = \mathcal{F}\}$ denote the corresponding Borel subgroup. When $(v_1, \ldots, v_n)$ is a basis of V we write

$$\mathcal{F}(v_1, \ldots, v_n) := \big(0 \subset \langle v_1\rangle_\mathbb{C} \subset \langle v_1, v_2\rangle_\mathbb{C} \subset \ldots \subset \langle v_1, \ldots, v_n\rangle_\mathbb{C}\big) \in X.$$

Bruhat decomposition. The double flag variety $X \times X$ has a finite number of $\mathrm{GL}(V)$-orbits parametrized by permutations $w \in \mathfrak{S}_n$. Specifically, given two flags $\mathcal{F} = (F_k)_{k=0}^n$ and $\mathcal{F}' = (F'_\ell)_{\ell=0}^n$ there is a unique permutation $w =: w(\mathcal{F}, \mathcal{F}')$ such that

$$\dim F_k \cap F'_\ell = \big|\{j \in \{1, \ldots, \ell\} : w_j \in \{1, \ldots, k\}\}\big|.$$

The permutation $w(\mathcal{F}, \mathcal{F}')$ is called the *relative position* of the pair $(\mathcal{F}, \mathcal{F}') \in X \times X$. Then

$$X \times X = \bigsqcup_{w \in \mathfrak{S}_n} \mathbb{O}_w \quad \text{where } \mathbb{O}_w := \big\{(\mathcal{F}, \mathcal{F}') \in X \times X : w(\mathcal{F}, \mathcal{F}') = w\big\}$$

is the decomposition of $X\times X$ into $\mathrm{GL}(V)$-orbits. The unique closed orbit is $\mathbb{O}_{\mathrm{id}}$ and the unique open orbit is $\mathbb{O}_{w_0}$ where w_0 is the involution given by $w_0(k)=n-k+1$ for all k. The map $\mathbb{O}_w\mapsto\mathbb{O}_{w_0w}$ is an involution on the set of orbits and reverses inclusions between orbit closures. Representatives of $\mathbb{O}_w$ can be obtained as follows: for every basis $(v_1,\ldots,v_n)$ of V we have

$$\big(\mathcal{F}(v_1,\ldots,v_n),\mathcal{F}(v_{w_1},\ldots,v_{w_n})\big)\in\mathbb{O}_w.$$

Variety of isotropic flags. Let V be endowed with a nondegenerate symmetric or symplectic bilinear form ω. For a subspace $F\subset V$, set $F^\perp=\{x\in V:\omega(x,y)=0\ \forall y\in F\}$. The variety of isotropic flags is the subvariety X_ω of X, where

$$X_\omega=\{\mathcal{F}=(F_k)_{k=0}^n\in X:F_k^\perp=F_{n-k}\ \forall k=0,\ldots,n\}. \tag{7}$$

It is endowed with a transitive action of the subgroup $G(V,\omega)\subset\mathrm{GL}(V)$ of automorphisms preserving ω.

Lemma 1. (a) *For every endomorphism $f\in\mathrm{End}(V)$, let $f^*\in\mathrm{End}(V)$ denote the endomorphism adjoint to f with respect to ω. Let $H\subset\mathrm{GL}(V)$ be a subgroup satisfying the condition*

$$\mathbb{C}[g^*g]\cap\mathrm{GL}(V)\subset H\quad\textit{for all } g\in H. \tag{8}$$

Assume that $\mathcal{F}\in X_\omega$ and $\mathcal{F}'\in X_\omega$ belong to the same H-orbit of X. Then they belong to the same $H\cap G(V,\omega)$-orbit of X_ω.
(b) *Let $H=\{g\in\mathrm{GL}(V):g(V_+)=V_+,\ g(V_-)=V_-\}$ where $V=V_+\oplus V_-$ is a decomposition such that $(V_+^\perp,V_-^\perp)=(V_+,V_-)$ or (V_-,V_+). Then (8) is fulfilled.*

Proof. (a) Note that $G(V,\omega)=\{g\in\mathrm{GL}(V):g^*=g^{-1}\}$. Consider $g\in H$ such that $\mathcal{F}'=g\mathcal{F}$. The equality $(gF)^\perp=(g^*)^{-1}F^\perp$ holds for all subspaces $F\subset V$. Since $\mathcal{F},\mathcal{F}'$ belong to X_ω we have $\mathcal{F}'=(g^*)^{-1}\mathcal{F}$, hence $g^*g\mathcal{F}=\mathcal{F}$. Let $g_1=g^*g$. By [10, Lemma 1.5] there is a polynomial $P(t)\in\mathbb{C}[t]$ such that $P(g_1)^2=g_1$. Set $h=P(g_1)$. Then $h\in\mathrm{GL}(V)$ (since $h^2=g_1\in\mathrm{GL}(V)$), and (8) shows that actually $h\in H$. Moreover $h^*=h$ (since $h\in\mathbb{C}[g_1]$ and $g_1^*=g_1$) and $h\mathcal{F}=\mathcal{F}$ (as each subspace in $\mathcal{F}$ is g_1-stable hence also h-stable). Set $h_1:=gh^{-1}\in H$. Then, on the one hand,

$$h_1^*=(h^*)^{-1}g^*=h^{-1}g_1g^{-1}=h^{-1}h^2g^{-1}=hg^{-1}=h_1^{-1},$$

and therefore $h_1\in H\cap G(V,\omega)$. On the other hand, we have $h_1\mathcal{F}=gh^{-1}\mathcal{F}=g\mathcal{F}=\mathcal{F}'$, and part (a) is proved.
(b) The equality $g^*(gF)^\perp=F^\perp$ (already mentioned) applied to $F=V_\pm$ yields $g^*\in H$, and thus $g^*g\in H$, whenever $g\in H$. This implies (8). ☐

Remark 1. The proof of Lemma 1 (a) is inspired by [10, §1.4]. We also refer to [14, 17] for similar results and generalizations.

2.3. **Ind-varieties of generalized flags.** Recall that $\mathbf{V}$ denotes a complex vector space of countable dimension, with an ordered basis $E=(e_\ell)_{\ell\in\mathbb{N}^*}$.

Definition 1 ([4]). Let $\mathcal{F}$ be a chain of subspaces in $\mathbf{V}$, i.e., a set of subspaces of $\mathbf{V}$ which is totally ordered by inclusion. Let $\mathcal{F}'$ (resp., $\mathcal{F}''$) be the subchain consisting of all $F\in\mathcal{F}$ with an immediate successor (resp., an immediate predecessor). By $s(F)\in\mathcal{F}''$ we denote the immediate successor of $F\in\mathcal{F}'$.

A *generalized flag* in $\mathbf{V}$ is a chain of subspaces $\mathcal{F}$ such that:

(i) each $F\in\mathcal{F}$ has an immediate successor or predecessor, i.e., $\mathcal{F}=\mathcal{F}'\cup\mathcal{F}''$;

(ii) for every $v \in \mathbf{V} \setminus \{0\}$ there is a unique $F_v \in \mathcal{F}'$ such that $v \in s(F_v) \setminus F_v$, i.e., $\mathbf{V} \setminus \{0\} = \bigcup_{F \in \mathcal{F}'} (s(F) \setminus F)$.

A generalized flag is *maximal* if it is not properly contained in another generalized flag. Specifically, $\mathcal{F}$ is maximal if and only if $\dim s(F)/F = 1$ for all $F \in \mathcal{F}'$.

Notation 1. Let $\sigma : \mathbb{N}^* \to (A, \prec)$ be a surjective map onto a totally ordered set. Let $\underline{v} = (v_1, v_2, \ldots)$ be a basis of $\mathbf{V}$. For every $a \in A$, let

$$F'_a := \langle v_\ell : \sigma(\ell) \prec a \rangle_{\mathbb{C}}, \quad F''_a := \langle v_\ell : \sigma(\ell) \preceq a \rangle_{\mathbb{C}}.$$

Then $\mathcal{F} = \mathcal{F}_\sigma(\underline{v}) := \{F'_a, F''_a : a \in A\}$ is a generalized flag such that $\mathcal{F}' = \{F'_a : a \in A\}$, $\mathcal{F}'' = \{F''_a : a \in A\}$, and $s(F'_a) = F''_a$ for all a. We call such a generalized flag *compatible with the basis* $\underline{v}$.

Moreover, $\mathcal{F}_\sigma(\underline{v})$ is maximal if and only if the map σ is bijective.

We use the abbreviation $\mathcal{F}_\sigma := \mathcal{F}_\sigma(E)$.

Note that every generalized flag admits a compatible basis [4, Proposition 4.1]. A generalized flag is *weakly compatible with* E if it is compatible with some basis $\underline{v}$ such that $E \setminus (E \cap \underline{v})$ is finite (equivalently, $\dim \mathbf{V}/\langle E \cap \underline{v} \rangle_{\mathbb{C}} < \infty$).

The group $\mathbf{G}(E)$ (as well as $\mathrm{Aut}(\mathbf{V})$) acts on generalized flags in a natural way. Let $\mathbf{P}_\mathcal{F} \subset \mathbf{G}(E)$ denote the ind-subgroup of elements preserving $\mathcal{F}$. It is a closed ind-subgroup of $\mathbf{G}(E)$. If $\mathcal{F}$ is compatible with E, then $\mathbf{P}_\mathcal{F}$ is a splitting parabolic ind-subgroup of $\mathbf{G}(E)$ in the sense that it is locally parabolic (i.e., there exists an exhaustion of $\mathbf{G}(E)$ by finite-dimensional reductive algebraic subgroups G_n such that the intersections $\mathbf{P}_\mathcal{F} \cap G_n$ are parabolic subgroups of G_n) and contains the Cartan ind-subgroup $\mathbf{H}(E) \subset \mathbf{G}(E)$ of elements diagonal with respect to E. Moreover if $\mathcal{F}$ is maximal, then $\mathbf{B}_\mathcal{F} := \mathbf{P}_\mathcal{F}$ is a splitting Borel ind-subgroup (i.e., all intersections $\mathbf{B}_\mathcal{F} \cap G_n$ as above are Borel subgroups of G_n).

Definition 2 ([4]). Two generalized flags $\mathcal{F}, \mathcal{G}$ are called *E-commensurable* if $\mathcal{F}, \mathcal{G}$ are weakly compatible with E, and there is an isomorphism $\phi : \mathcal{F} \to \mathcal{G}$ of ordered sets and a finite dimensional subspace $U \subset \mathbf{V}$ such that

(i) $\phi(F) + U = F + U$ for all $F \in \mathcal{F}$;
(ii) $\dim \phi(F) \cap U = \dim F \cap U$ for all $F \in \mathcal{F}$.

E-commensurability is an equivalence relation on the set of generalized flags weakly compatible with E. In fact, according to the following proposition, each equivalence class consists of a single $\mathbf{G}(E)$-orbit. If $\mathcal{F}$ is a generalized flag weakly compatible with E we denote by $\mathbf{X}(\mathcal{F}, E)$ the set of generalized flags which are E-commensurable with $\mathcal{F}$.

Proposition 1 ([4]). *The set $\mathbf{X} = \mathbf{X}(\mathcal{F}, E)$ is endowed with a natural structure of ind-variety. Moreover $\mathbf{X}$ is $\mathbf{G}(E)$-homogeneous and the map $g \mapsto g\mathcal{F}$ induces an isomorphism of ind-varieties $\mathbf{G}(E)/\mathbf{P}_\mathcal{F} \xrightarrow{\sim} \mathbf{X}$.*

Proposition 2 ([5]). *Let $\sigma : \mathbb{N}^* \to (A, \prec)$ and $\tau : \mathbb{N}^* \to (B, \prec)$ be maps onto two totally ordered sets.*

(a) *Each E-compatible generalized flag in $\mathbf{X}(\mathcal{F}_\sigma, E)$ is of the form $\mathcal{F}_{\sigma w}$ for $w \in \mathfrak{S}_\infty$. Moreover $\mathcal{F}_{\sigma w} = \mathcal{F}_{\sigma w'} \Leftrightarrow w'w^{-1} \in \mathrm{Stab}_\sigma := \{v \in \mathfrak{S}_\infty : \sigma v = \sigma\}$.*
(b) *Assume that $\mathcal{F}_\tau$ is maximal (i.e., τ is bijective) so that $\mathbf{B}_{\mathcal{F}_\tau}$ is a splitting Borel ind-subgroup. Then each $\mathbf{B}_{\mathcal{F}_\tau}$-orbit of $\mathbf{X}(\mathcal{F}_\sigma, E)$ contains a unique element of the form $\mathcal{F}_{\sigma w}$ for $w \in \mathfrak{S}_\infty/\mathrm{Stab}_\sigma$.*

(c) *In particular, if $\mathcal{F}_\sigma, \mathcal{F}_\tau$ are both maximal (i.e., σ, τ are both bijective), then*

$$\mathbf{X}(\mathcal{F}_\tau, E) \times \mathbf{X}(\mathcal{F}_\sigma, E) = \bigsqcup_{w \in \mathfrak{S}_\infty} (\mathbb{O}_{\tau,\sigma})_w$$

where

$$(\mathbb{O}_{\tau,\sigma})_w := \{(g\mathcal{F}_\tau, g\mathcal{F}_{\sigma w}) : g \in \mathbf{G}(E)\}$$

is a decomposition of $\mathbf{X}(\mathcal{F}_\tau, E) \times \mathbf{X}(\mathcal{F}_\sigma, E)$ into $\mathbf{G}(E)$-orbits.

Remark 2. The orbit $(\mathbb{O}_{\tau,\sigma})_w$ of Proposition 2 (c) actually consists of all couples of generalized flags $(\mathcal{F}_\tau(\underline{v}), \mathcal{F}_{\sigma w}(\underline{v}))$ weakly compatible with the basis $\underline{v} = (v_1, v_2, \ldots)$.

Assume $\mathbf{V}$ is endowed with a nondegenerate symmetric or symplectic form ω whose values on the basis E are given by the matrix Ω in (2).

Definition 3. A generalized flag $\mathcal{F}$ is called *ω-isotropic* if the map $F \mapsto F^\perp := \{x \in \mathbf{V} : \omega(x, y) = 0 \ \forall y \in F\}$ is a well-defined involution of $\mathcal{F}$.

Proposition 3 ([4])**.** *Let $\mathcal{F}$ be an ω-isotropic generalized flag weakly compatible with E. The set $\mathbf{X}_\omega(\mathcal{F}, E)$ of all ω-isotropic generalized flags which are E-commensurable with $\mathcal{F}$ is a $\mathbf{G}(E, \omega)$-homogeneous, closed ind-subvariety of $\mathbf{X}(\mathcal{F}, E)$.*

Finally, we emphasize that one of the main features of classical ind-groups is that their Borel ind-subgroups are not $\mathrm{Aut}(\mathbf{G})$-conjugate. Here are three examples of maximal generalized flags in $\mathbf{V}$, compatible with the basis E and such that their stablizers in $\mathbf{G}(E)$ are pairwise not $\mathrm{Aut}(\mathbf{G})$-conjugate. A more detailed discussion of these examples see in [4].

Example 1. (a) Let $\sigma_1 : \mathbb{N}^* \to (\mathbb{N}^*, <)$, $\ell \mapsto \ell$. The generalized flag $\mathcal{F}_{\sigma_1}$ is an ascending chain of subspaces $\mathcal{F}_{\sigma_1} = \{0 = F_0 \subset F_1 \subset F_2 \subset \ldots\}$ isomorphic to $(\mathbb{N}, <)$ as an ordered set.
(b) Let $\sigma_2 : \mathbb{N}^* \to \left(\{\frac{1}{n} : n \in \mathbb{Z}^*\}, <\right)$, $\ell \mapsto \frac{(-1)^\ell}{\ell}$. The generalized flag $\mathcal{F}_{\sigma_2}$ is a chain of the form $\mathcal{F}_{\sigma_2} = \{0 = F_0 \subset F_1 \subset \ldots \subset F_{-2} \subset F_{-1} = \mathbf{V}\}$ and is not isomorphic as ordered set to a subset of $(\mathbb{Z}, <)$.
(c) Let $\sigma_3 : \mathbb{N}^* \to (\mathbb{Q}, <)$ be a bijection. In this case no subspace $F \in \mathcal{F}_{\sigma_3}$ has both immediate successor or immediate predecessor.

3. Parametrization of orbits in the finite-dimensional case

In Sections 3.1-3.3, we state explicit parametrizations of the K- and G^0-orbits in the finite-dimensional case. All proofs are given in Section 3.5.

3.1. **Types A1 and A2.** Let the notation be as in Subsection 2.1.1. The space $V = V_n := \langle e_1, \ldots, e_n \rangle_\mathbb{C}$ is endowed with the symmetric or symplectic form $\omega(x, y) = {}^t x \cdot \Omega \cdot y$ and the conjugation $\gamma(x) = \Omega \overline{x}$ which actually stand for the restrictions to V of the maps ω, γ introduced in Section 2.1. This allows us to define two involutions of the flag variety X:

$$\mathcal{F} = (F_0, \ldots, F_n) \mapsto \mathcal{F}^\perp := (F_n^\perp, \ldots, F_0^\perp) \quad \text{and} \quad \mathcal{F} \mapsto \gamma(\mathcal{F}) := (\gamma(F_0), \ldots, \gamma(F_n))$$

where $F^\perp \subset V$ stands for the subspace orthogonal to F with respect to ω.

Let $K = \{g \in \mathrm{GL}(V) : g \text{ preserves } \omega\}$ and $G^0 = \{g \in \mathrm{GL}(V) : \gamma g = g\gamma\}$.

By $\mathfrak{I}_n \subset \mathfrak{S}_n$ we denote the subset of involutions. If $n = 2m$ is even, we let $\mathfrak{I}'_n \subset \mathfrak{I}_n$ be the subset of involutions without fixed points.

Definition 4. Let $w \in \mathfrak{I}_n$. Set $\epsilon := 1$ in type A1 and $\epsilon := -1$ in type A2. A basis $(v_1, \ldots, v_n)$ of V such that

$$\omega(v_k, v_\ell) = \begin{cases} 1 & \text{if } w_k = \ell \geq k \\ \epsilon & \text{if } w_k = \ell < k \\ 0 & \text{if } w_k \neq \ell \end{cases} \quad \text{for all } k, \ell \in \{1, \ldots, n\}$$

is said to be *w-dual*. A basis $(v_1, \ldots, v_n)$ of V such that

$$\gamma(v_k) = \begin{cases} \epsilon v_{w_k} & \text{if } w_k \geq k \\ v_{w_k} & \text{if } w_k < k \end{cases} \quad \text{for all } k \in \{1, \ldots, n\}$$

is said to be *w-conjugate*. Set

$$\mathcal{O}_w := \{\mathcal{F}(v_1, \ldots, v_n) : (v_1, \ldots, v_n) \text{ is a } w\text{-dual basis}\},$$
$$\mathfrak{O}_w := \{\mathcal{F}(v_1, \ldots, v_n) : (v_1, \ldots, v_n) \text{ is a } w\text{-conjugate basis}\}.$$

Proposition 4. *Let $\mathfrak{I}_n^\epsilon = \mathfrak{I}_n$ in type A1 and $\mathfrak{I}_n^\epsilon = \mathfrak{I}_n'$ in type A2. Recall the notation $\mathbb{O}_w$ and w_0 introduced in Section 2.2.*

(a) *For every $w \in \mathfrak{I}_n^\epsilon$ we have $\mathcal{O}_w \neq \emptyset$, $\mathfrak{O}_w \neq \emptyset$ and*

$$\mathcal{O}_w \cap \mathfrak{O}_w = \{\mathcal{F}(v_1, \ldots, v_n) : (v_1, \ldots, v_n) \text{ is both } w\text{-dual and } w\text{-conjugate}\} \neq \emptyset.$$

(b) *For every $w \in \mathfrak{I}_n^\epsilon$,*

$$\mathcal{O}_w = \{\mathcal{F} \in X : (\mathcal{F}^\perp, \mathcal{F}) \in \mathbb{O}_{w_0 w}\} \quad \text{and} \quad \mathfrak{O}_w = \{\mathcal{F} \in X : (\gamma(\mathcal{F}), \mathcal{F}) \in \mathbb{O}_w\}.$$

(c) *The subsets $\mathcal{O}_w$ ($w \in \mathfrak{I}_n^\epsilon$) are exactly the K-orbits of X. The subsets $\mathfrak{O}_w$ ($w \in \mathfrak{I}_n^\epsilon$) are exactly the G^0-orbits of X.*

(d) *The map $\mathcal{O}_w \mapsto \mathfrak{O}_w$ is Matsuki duality.*

3.2. **Type A3.** Let the notation be as in Subsection 2.1.2: the space $V = V_n = \langle e_1, \ldots, e_n \rangle_\mathbb{C}$ is endowed with the hermitian form $\phi(x, y) = {}^t\overline{x}\Phi y$ and a conjugation $\delta(x) = \Phi x$ where Φ is a diagonal matrix with entries $\epsilon_1, \ldots, \epsilon_n \in \{+1, -1\}$ (the left upper $n \times n$-corner of the matrix Φ of Section 2.1).

Set $V_+ = \langle e_k : \epsilon_k = 1 \rangle_\mathbb{C}$ and $V_- = \langle e_k : \epsilon_k = -1 \rangle_\mathbb{C}$. Then $V = V_+ \oplus V_-$. Let $K = \{g \in \mathrm{GL}(V) : \delta g = g\delta\} = \mathrm{GL}(V_+) \times \mathrm{GL}(V_-)$ and $G^0 = \{g \in \mathrm{GL}(V) : g \text{ preserves } \phi\}$.

As in Section 3.1 we get two involutions of the flag variety X:

$$\mathcal{F} = (F_0, \ldots, F_n) \mapsto \delta(\mathcal{F}) := (\delta(F_0), \ldots, \delta(F_n)) \quad \text{and} \quad \mathcal{F} \mapsto \mathcal{F}^\dagger := (F_n^\dagger, \ldots, F_0^\dagger)$$

where $F^\dagger \subset V$ stands for the orthogonal of $F \subset V$ with respect to ϕ. The hermitian form on the quotient $F/(F \cap F^\dagger)$ induced by ϕ is nondegenerate; we denote its signature by $\varsigma(\phi : F)$. Given $\mathcal{F} = (F_0, \ldots, F_n) \in X$, let

$$\varsigma(\phi : \mathcal{F}) := \big(\varsigma(\phi : F_\ell)\big)_{\ell=1}^n \in (\{0, \ldots, n\}^2)^n.$$

Then

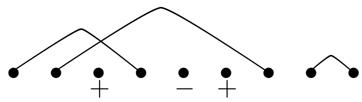
$$\varsigma(\delta : \mathcal{F}) := \big((\dim F_\ell \cap V_+, \dim F_\ell \cap V_-)\big)_{\ell=1}^n \in (\{0, \ldots, n\}^2)^n$$

records the relative position of $\mathcal{F}$ with respect to the subspaces V_+ and V_-.

Combinatorial notation. We call a *signed involution* a pair (w,ε) consisting of an involution $w \in \mathfrak{I}_n$ and signs $\varepsilon_k \in \{+1,-1\}$ attached to its fixed points $k \in \{\ell : w_\ell = \ell\}$. (Equivalently, ε is a map $\{\ell : w_\ell = \ell\} \to \{+1,-1\}$.)

It is convenient to represent w by a graph $l(w)$ (called *link pattern*) with n vertices $1,2,\ldots,n$ and an arc (k,w_k) connecting k and w_k whenever $k < w_k$. The *signed link pattern* $l(w,\varepsilon)$ is obtained from the graph $l(w)$ by marking each vertex $k \in \{\ell : w_\ell = \ell\}$ with the label $+$ or $-$ depending on whether $\varepsilon_k = +1$ or $\varepsilon_k = -1$.

For instance, the signed link pattern (where the numbering of vertices is implicit)



represents (w,ε) with $w = (1;4)(2;7)(8;9) \in \mathfrak{I}_9$ and $(\varepsilon_3,\varepsilon_5,\varepsilon_6) = (+1,-1,+1)$.

We define $\varsigma(w,\varepsilon) := \{(p_\ell,q_\ell)\}_{\ell=1}^n$ as the sequence given by

$$p_\ell \text{ (resp., } q_\ell) = (\text{number of } + \text{ signs (resp., } - \text{ signs) and arcs among the first } \ell \text{ vertices of } l(w,\varepsilon)).$$

Assuming $n = p+q$, let $\mathfrak{I}_n(p,q)$ be the set of signed involutions of signature (p,q), i.e., such that $(p_n,q_n) = (p,q)$. Note that the elements of $\mathfrak{I}_n(p,q)$ coincide with the clans of signature (p,q) in the sense of [13, 21].

For instance, for the above pair (w,ε) we have $(w,\varepsilon) \in \mathfrak{I}_9(5,4)$ and

$$\varsigma(w,\varepsilon) = \big((0,0),(0,0),(1,0),(2,1),(2,2),(3,2),(4,3),(4,3),(5,4)\big).$$

Definition 5. Given a signed involution (w,ε), we say that a basis $(v_1,\ldots,v_n)$ of V is (w,ϵ)-*conjugate* if

$$\delta(v_k) = \begin{cases} \varepsilon_k v_{w_k} & \text{if } w_k = k \\ v_{w_k} & \text{if } w_k \neq k \end{cases} \quad \text{for all } k \in \{1,\ldots,n\}.$$

A basis $(v_1,\ldots,v_n)$ such that

$$\phi(v_k,v_\ell) = \begin{cases} \varepsilon_k & \text{if } w_k = \ell = k \\ 1 & \text{if } w_k = \ell \neq k \\ 0 & \text{if } w_k \neq \ell \end{cases} \quad \text{for all } k,\ell \in \{1,\ldots,n\}$$

is said to be (w,ε)-*dual*. We set

$$\mathcal{O}_{(w,\varepsilon)} := \{\mathcal{F}(v_1,\ldots,v_n) : (v_1,\ldots,v_n) \text{ is a } (w,\varepsilon)\text{-conjugate basis}\},$$
$$\mathfrak{O}_{(w,\varepsilon)} := \{\mathcal{F}(v_1,\ldots,v_n) : (v_1,\ldots,v_n) \text{ is a } (w,\varepsilon)\text{-dual basis}\}.$$

Proposition 5. *In addition to the above notation, let* $(p,q) = (\dim V_+, \dim V_-)$. *Then:*

(a) *For every* $(w,\varepsilon) \in \mathfrak{I}_n(p,q)$ *the subsets* $\mathcal{O}_{(w,\varepsilon)}$ *and* $\mathfrak{O}_{(w,\varepsilon)}$ *are nonempty, and* $\mathcal{O}_{(w,\varepsilon)} \cap \mathfrak{O}_{(w,\varepsilon)} = \{\mathcal{F}(\underline{v}) : \underline{v} = (v_k)_{k=1}^n$ *is* (w,ε)-*dual and* (w,ε)-*conjugate*$\} \neq \emptyset$.

(b) *For every* $(w,\varepsilon) \in \mathfrak{I}_n(p,q)$,

$$\mathcal{O}_{(w,\varepsilon)} = \big\{\mathcal{F} \in X : (\delta(\mathcal{F}),\mathcal{F}) \in \mathbb{O}_w \text{ and } \varsigma(\delta : \mathcal{F}) = \varsigma(w,\varepsilon)\big\},$$
$$\mathfrak{O}_{(w,\varepsilon)} = \big\{\mathcal{F} \in X : (\mathcal{F}^\dagger,\mathcal{F}) \in \mathbb{O}_{w_0 w} \text{ and } \varsigma(\phi : \mathcal{F}) = \varsigma(w,\varepsilon)\big\}.$$

(c) *The subsets* $\mathcal{O}_{(w,\varepsilon)}$ $((w,\varepsilon) \in \mathfrak{I}_n(p,q))$ *are exactly the* K-*orbits of* X. *The subsets* $\mathfrak{O}_{(w,\varepsilon)}$ $((w,\varepsilon) \in \mathfrak{I}_n(p,q))$ *are exactly the* G^0-*orbits of* X.

(d) *The map* $\mathcal{O}_{(w,\varepsilon)} \mapsto \mathfrak{O}_{(w,\varepsilon)}$ *is Matsuki duality.*

3.3. **Types B, C, D.** In this section we assume that the space $V = V_n = \langle e_1, \dots, e_n\rangle_{\mathbb{C}}$ is endowed with a symmetric or symplectic form ω whose action on the basis $(e_1, \dots, e_n)$ is described by the matrix Ω in (2). We consider the group $G = G(V, \omega) = \{g \in \mathrm{GL}(V) : g \text{ preserves } \omega\}$ and the variety of isotropic flags $X_\omega = \{\mathcal{F} \in X : \mathcal{F}^\perp = \mathcal{F}\}$ (see Section 2.2).

In addition we assume that V is endowed with a hermitian form ϕ, a conjugation δ, and a decomposition $V = V_+ \oplus V_-$ (as in Section 3.2) such that

- in types BD1 and C2, the restriction of ω to V_+ and V_- is nondegenerate, i.e., $V_+^\perp = V_-$,
- in types C1 and D3, V_+ and V_- are Lagrangian with respect to ω, i.e., $V_+^\perp = V_+$ and $V_-^\perp = V_-$.

Set $K := \{g \in G : g\delta = \delta g\}$ and $G^0 := \{g \in G : g \text{ preserves } \phi\}$.

Combinatorial notation. Recall that $w_0(k) = n - k + 1$. Let $(\eta, \epsilon) \in \{1, -1\}^2$. A signed involution (w, ε) is called (η, ϵ)-*symmetric* if the following conditions hold

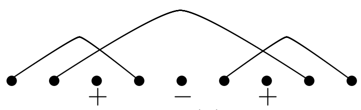
(i) $ww_0 = w_0w$ (so that the set $\{\ell : w_\ell = \ell\}$ is w_0-stable);
(ii) $\varepsilon_{w_0(k)} = \eta\varepsilon_k$ for all $k \in \{\ell : w_\ell = \ell\}$;

and in the case where $\eta \neq \epsilon$:

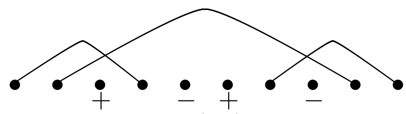
(iii) $w_k \neq w_0(k)$ for all k.

Assuming $n = p + q$, let $\mathfrak{I}_n^{\eta,\epsilon}(p, q) \subset \mathfrak{I}_n(p, q)$ denote the subset of signed involutions of signature (p, q) which are (η, ϵ)-symmetric.

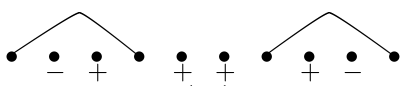
Specifically, (w, ε) is $(1, 1)$-symmetric when the signed link pattern $l(w, \varepsilon)$ is symmetric with respect to reversing the enumeration of vertices; (w, ε) is $(1, -1)$-symmetric when $l(w, \varepsilon)$ is symmetric and does not have symmetric arcs (i.e., joining k and $n - k + 1$); (w, ε) is $(-1, -1)$-symmetric when $l(w, \varepsilon)$ is antisymmetric in the sense that the mirror image of $l(w, \varepsilon)$ is a signed link pattern with the same arcs but opposite signs; (w, ε) is $(-1, 1)$-symmetric when $l(w, \varepsilon)$ is antisymmetric and does not have symmetric arcs. For instance:



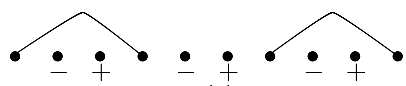
$(w, \varepsilon) \in \mathfrak{I}_9^{1,1}(5, 4)$,



$(w, \varepsilon) \in \mathfrak{I}_{10}^{-1,-1}(5, 5)$,



$(w, \varepsilon) \in \mathfrak{I}_{10}^{1,-1}(6, 4)$,



$(w, \varepsilon) \in \mathfrak{I}_{10}^{-1,1}(5, 5)$.

Proposition 6. *Let* $(p, q) = (\dim V_+, \dim V_-)$ *(so that* $p = q = \frac{n}{2}$ *in types C1 and D3). Set* $(\eta, \epsilon) = (1, 1)$ *in type BD1,* $(\eta, \epsilon) = (1, -1)$ *in type C2,* $(\eta, \epsilon) = (-1, -1)$ *in types C1, and* $(\eta, \epsilon) = (-1, 1)$ *in type D3.*

(a) *For every* $(w, \varepsilon) \in \mathfrak{I}_n^{\eta,\epsilon}(p, q)$, *considering bases* $\underline{v} = (v_1, \dots, v_n)$ *of* V *such that*

$$\omega(v_k, v_\ell) = \begin{cases} 0 & \text{if } \ell \neq n - k + 1 \\ 1 & \text{if } \ell = n - k + 1 \text{ and } w_k, w_\ell \in [k, \ell] \ (k \leq \ell) \\ \epsilon & \text{if } \ell = n - k + 1 \text{ and } w_k, w_\ell \in [\ell, k] \ (\ell \leq k) \\ \eta & \text{if } \ell = n - k + 1 \text{ and } k, \ell \in]w_k, w_\ell[\\ \eta\epsilon & \text{if } \ell = n - k + 1 \text{ and } k, \ell \in]w_\ell, w_k[, \end{cases} \tag{9}$$

we have

$$\mathcal{O}^{\eta,\epsilon}_{(w,\varepsilon)} := \mathcal{O}_{(w,\varepsilon)} \cap X_\omega = \{\mathcal{F}(\underline{v}) : \underline{v} \text{ is } (w,\varepsilon)\text{-conjugate and satisfies (9)}\} \neq \emptyset,$$
$$\mathfrak{O}^{\eta,\epsilon}_{(w,\varepsilon)} := \mathfrak{O}_{(w,\varepsilon)} \cap X_\omega = \{\mathcal{F}(\underline{v}) : \underline{v} \text{ is } (w,\varepsilon)\text{-dual and satisfies (9)}\} \neq \emptyset,$$
$$\mathcal{O}^{\eta,\epsilon}_{(w,\varepsilon)} \cap \mathfrak{O}^{\eta,\epsilon}_{(w,\varepsilon)} = \{\mathcal{F}(\underline{v}) : \underline{v} \text{ is } (w,\varepsilon)\text{-conjugate and } (w,\varepsilon)\text{-dual and satisfies (9)}\} \neq \emptyset.$$

(b) *The subsets* $\mathcal{O}^{\eta,\epsilon}_{(w,\varepsilon)}$ *($(w,\varepsilon) \in \mathfrak{I}^{\eta,\epsilon}_n(p,q)$) are exactly the K-orbits of X_ω. The subsets* $\mathfrak{O}^{\eta,\epsilon}_{(w,\varepsilon)}$ *($(w,\varepsilon) \in \mathfrak{I}^{\eta,\epsilon}_n(p,q)$) are exactly the G^0-orbits of X_ω.*

(c) *The map* $\mathcal{O}^{\eta,\epsilon}_{(w,\varepsilon)} \mapsto \mathfrak{O}^{\eta,\epsilon}_{(w,\varepsilon)}$ *is Matsuki duality.*

3.4. **Remarks.** Set $X_0 := X$ in type A and $X_0 := X_\omega$ in types B, C, D.

Remark 3. The characterization of the K-orbits in Propositions 4–6 can be stated in the following unified way. For $\mathcal{F} \in X$ we write $\sigma(\mathcal{F}) = \mathcal{F}^\perp$ in types A1–A2 and $\sigma(\mathcal{F}) = \delta(\mathcal{F})$ in types A3, BD1, C1–C2, D3. Let $P \subset G$ be a parabolic subgroup containing K and which is minimal for this property. Two flags $\mathcal{F}_1, \mathcal{F}_2 \in X_0$ belong to the same K-orbit if and only if $(\sigma(\mathcal{F}_1), \mathcal{F}_1)$ and $(\sigma(\mathcal{F}_2), \mathcal{F}_2)$ belong to the same orbit of P for the diagonal action of P on $X_0 \times X_0$.

Remark 4 (Open K-orbits)**.** With the notation of Remark 3 the map $\sigma_0 : X_0 \to X \times X$, $\mathcal{F} \mapsto (\sigma(\mathcal{F}), \mathcal{F})$ is a closed embedding.

In types A and C the flag variety X_0 is irreducible. In particular there is a unique G-orbit $\mathbb{O}_w \subset X \times X$ such that $\mathbb{O}_w \cap \sigma_0(X_0)$ is open in $\sigma_0(X_0)$; it corresponds to an element $w \in \mathfrak{S}_n$ maximal for the Bruhat order such that $\mathbb{O}_w$ intersects $\sigma_0(X_0)$. In each case one finds a unique K-orbit $\mathcal{O} \subset X_0$ such that $\sigma_0(\mathcal{O}) \subset \mathbb{O}_w$, it is therefore the (unique) open K-orbit of X_0. This yields the following list of open K-orbits in types A1–A3, C1–C2:

A1: $\mathcal{O}_{\mathrm{id}}$;
A2: $\mathcal{O}_{v_0}$ where $v_0 = (1;2)(3;4)\cdots(n-1;n)$;
A3: $\mathcal{O}_{(w_0^{(t)},\varepsilon)}$ where $t = \min\{p,q\}$, $\varepsilon \equiv \mathrm{sign}(p-q)$, and $w_0^{(t)} = \prod_{k=1}^{t}(k; n-k+1)$;
C1: $\mathcal{O}^{-1,-1}_{(w_0,\emptyset)}$;
C2: $\mathcal{O}^{1,-1}_{(\hat{w}_0^{(t)},\varepsilon)}$ where $t = \min\{p,q\}$, $\varepsilon \equiv \mathrm{sign}(p-q)$, and $\hat{w}_0^{(t)} = v_0^{(t)} w_0^{(t)} v_0^{(t)}$, where $v_0^{(t)} = (1;2)(3;4)\cdots(t-1;t)$.

If $n = \dim V$ is even and the form ω is orthogonal, then the variety X_ω has two connected components. In fact, for every isotropic flag $\mathcal{F} = (F_k)_{k=0}^n \in X_\omega$ there is a unique $\tilde{\mathcal{F}} = (\tilde{F}_k)_{k=0}^n \in X_\omega$ such that $F_k = \tilde{F}_k$ for all $k \neq m := \frac{n}{2}$, $\tilde{F}_m \neq F_m$. Then the map $\tilde{I} : \mathcal{F} \mapsto \tilde{\mathcal{F}}$ is an automorphism of X_ω which maps one component of X_ω onto the other. If $\mathcal{F} = \mathcal{F}(v_1, \ldots, v_n)$ for a basis $\underline{v} = (v_1, \ldots, v_n)$ such that

$$\omega(v_k, v_\ell) \neq 0 \Leftrightarrow \ell = n - k + 1$$

then $\tilde{I}(\mathcal{F}(\underline{v})) = \mathcal{F}(\tilde{\underline{v}})$ where $\tilde{\underline{v}}$ is the basis obtained from $\underline{v}$ by switching the two middle vectors v_m, v_{m+1}. If $\underline{v}$ is (w,ε)-conjugate then $\tilde{\underline{v}}$ is $\tilde{\imath}(w,\varepsilon)$-conjugate where $\tilde{\imath}(w,\varepsilon) := \big((m;m+1)w(m;m+1), \varepsilon \circ (m;m+1)\big)$. Hence $\tilde{I}$ maps the K-orbit $\mathcal{O}^{\eta,\epsilon}_{(w,\varepsilon)}$ onto $\mathcal{O}^{\eta,\epsilon}_{\tilde{\imath}(w,\varepsilon)}$.

In type D3, X_ω has exactly two open K-orbits. More precisely, $w = \hat{w}_0 := w_0 v_0$ is maximal for the Bruhat order such that $\mathbb{O}_w \cap \sigma_0(X_0)$ is nonempty, hence $\sigma_0^{-1}(\mathbb{O}_{\hat{w}_0})$

is open. The permutation $\hat{w}_0$ has no fixed point if $m := \frac{n}{2}$ is even; if $m := \frac{n}{2}$ is odd, $\hat{w}_0$ fixes m and $m+1$. In the former case $\sigma_0^{-1}(\mathbb{O}_{\hat{w}_0}) = \mathcal{O}^{-1,1}_{(\hat{w}_0,\emptyset)}$ is a single K-orbit, and $\tilde{I}(\mathcal{O}^{-1,1}_{(\hat{w}_0,\emptyset)}) = \mathcal{O}^{-1,1}_{\tilde{i}(\hat{w}_0,\emptyset)}$ is a second open K-orbit. In the latter case $\sigma_0^{-1}(\mathbb{O}_{\hat{w}_0^{(m-1)}}) = \mathcal{O}^{-1,1}_{(\hat{w}_0,\varepsilon)} \cup \mathcal{O}^{-1,1}_{(\hat{w}_0,\tilde{\varepsilon})}$, where $(\varepsilon_m, \varepsilon_{m+1}) = (\tilde{\varepsilon}_{m+1}, \tilde{\varepsilon}_m) = (+1,-1)$, is the union of two distinct open K-orbits which are image of each other by $\tilde{I}$.

In type BD1 the variety X_ω may be reducible but $w = w_0^{(t)}$, for $t := \min\{p,q\}$, is the unique maximal element of $\mathfrak{S}_n$ such that $\mathbb{O}_w \cap \sigma_0(X_0)$ is nonempty. Then $\sigma_0^{-1}(\mathbb{O}_w)$ consists of a single $\tilde{I}$-stable open K-orbit, namely $\mathcal{O}^{1,1}_{(w_0^{(t)},\varepsilon)}$ for $\varepsilon \equiv \mathrm{sign}(p-q)$. The flag variety X_ω has therefore a unique open K-orbit (which is not connected whenever n is even).

Remark 5 (Closed K-orbits)**.** We use the notation of Remarks 3–4. As seen from Propositions 4–6, in each case one finds a unique $w_{\min} \in \mathfrak{S}_n$ such that $\mathbb{O}_{w_{\min}} \cap \sigma_0(X_0)$ is closed; actually $w_{\min} = \mathrm{id}$ except in type BD1 for p,q odd: in that case $w_{\min} = (\frac{n}{2}; \frac{n}{2}+1)$. For every K-orbit $\mathcal{O} \subset X_0$ the following equivalence holds:

$$\mathcal{O} \text{ is closed} \quad \Leftrightarrow \quad \sigma_0(\mathcal{O}) \subset \mathbb{O}_{w_{\min}}$$

(see [3, 18]). In view of this equivalence, we deduce the following complete list of closed K-orbits of X_0 for the different types. In types A1 and A2, $\mathcal{O}_{w_0}$ is the unique closed K-orbit. In type A3 the closed K-orbits are exactly the orbits $\mathcal{O}_{(\mathrm{id},\varepsilon)}$ for all pairs of the form $(\mathrm{id},\varepsilon) \in \mathfrak{I}_n(p,q)$; there are $\binom{n}{p}$ such orbits. In types B, C, D, the closed K-orbits are the orbits $\mathcal{O}^{\eta,\epsilon}_{(\mathrm{id},\varepsilon)}$ for all pairs of the form $(\mathrm{id},\varepsilon) \in \mathfrak{I}^{\eta,\epsilon}_n(p,q)$, except in type BD1 in the case where $n =: 2m$ is even and p,q are odd; in that case the closed K-orbits are the orbits $\mathcal{O}^{1,1}_{((m;m+1),\varepsilon)}$ for all pairs of the form $((m;m+1),\varepsilon) \in \mathfrak{I}^{1,1}_n(p,q)$. There are $\binom{\lfloor \frac{p}{2}\rfloor + \lfloor \frac{q}{2}\rfloor}{\lfloor \frac{p}{2}\rfloor}$ closed orbits in types BD1 and C2, and there are $2^{\frac{n}{2}}$ closed orbits in types C1 and D3.

Remark 6. Propositions 4–6 show in particular that the *special elements of* X_0, in the sense of Matsuki [11, 12], are precisely the flags $\mathcal{F} \in X_0$ of the form $\mathcal{F} = \mathcal{F}(v_1, \ldots, v_n)$ where $(v_1, \ldots, v_n)$ is a basis of V which is both dual and conjugate, with respect to some involution $w \in \mathfrak{I}^\epsilon_n$ in types A1 and A2, and to some signed involution $(w,\varepsilon) \in \mathfrak{I}_n(p,q)$ in types A3, B–D. Indeed, in view of [11, 12] the set $\mathcal{S} \subset X_0$ of special elements equals

$$\bigcup_{\mathcal{O} \in X_0/K} \mathcal{O} \cap \Xi(\mathcal{O})$$

where the map $X_0/K \to X_0/G^0$, $\mathcal{O} \mapsto \Xi(\mathcal{O})$ stands for Matsuki duality.

3.5. **Proofs.**

Proof of Proposition 4 (a). We write $w = (a_1;b_1)\cdots(a_m;b_m)$ with $a_1 < \ldots < a_m$ and $a_k < b_k$ for all k; let $c_1 < \ldots < c_{n-2m}$ be the elements of the set $\{k : w_k = k\}$. In type A2 we have $n = 2m$, and $(e_1, \ldots, e_n)$ is both a $(1;2)(3;4)\cdots(n-1;n)$-dual basis and a $(1;2)(3;4)\cdots(n-1;n)$-conjugate basis; then the basis $\{e'_1, \ldots, e'_n\}$ given by

$$e'_{a_\ell} = e_{2\ell-1} \quad \text{and} \quad e'_{b_\ell} = e_{2\ell} \quad \text{for all } \ell \in \{1, \ldots, m\}$$

is simultaneously w-dual and w-conjugate. In type A1, up to replacing e_ℓ and e_{ℓ^*} by $\frac{e_\ell + e_{\ell^*}}{\sqrt{2}}$ and $\frac{e_\ell - e_{\ell^*}}{i\sqrt{2}}$ whenever $\ell < \ell^*$, we may assume that the basis $(e_1, \ldots, e_n)$ is both id-dual and id-conjugate. For every $\ell \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, n-2m\}$, we set

$$e'_{a_\ell} = \frac{e_{2\ell-1} + ie_{2\ell}}{\sqrt{2}}, \quad e'_{b_\ell} = \frac{e_{2\ell-1} - ie_{2\ell}}{\sqrt{2}}, \quad \text{and} \quad e'_{c_k} = e_{2m+k}.$$

Then $(e'_1, \ldots, e'_n)$ is simultaneously a w-dual and a w-conjugate basis. In both cases we conclude that

$$\emptyset \neq \{\mathcal{F}(v_1, \ldots, v_n) : (v_1, \ldots, v_n) \text{ is } w\text{-dual and } w\text{-conjugate}\} \subset \mathcal{O}_w \cap \mathfrak{O}_w. \tag{10}$$

Let us show the inverse inclusion. Assume $\mathcal{F} = (F_0, \ldots, F_n) \in \mathcal{O}_w \cap \mathfrak{O}_w$. Let $(v_1, \ldots, v_n)$ be a w-dual basis such that $\mathcal{F} = \mathcal{F}(v_1, \ldots, v_n)$. Since $\mathcal{F} \in \mathfrak{O}_w$ we have

$$w_k = \min\{\ell = 1, \ldots, n : \gamma(F_k) \cap F_\ell \neq \gamma(F_{k-1}) \cap F_\ell\}. \tag{11}$$

For all $\ell \in \{0, \ldots, n\}$ we will now construct a w-dual basis $(v_1^{(\ell)}, \ldots, v_n^{(\ell)})$ of V such that

$$F_k = \langle v_1^{(\ell)}, \ldots, v_k^{(\ell)} \rangle_{\mathbb{C}} \quad \text{for all } k \in \{1, \ldots, n\} \tag{12}$$

and

$$\gamma(v_k^{(\ell)}) = \begin{cases} \epsilon v_{w_k}^{(\ell)} & \text{if } w_k \geq k, \\ v_{w_k}^{(\ell)} & \text{if } w_k < k \end{cases} \qquad \text{for all } k \in \{1, \ldots, \ell\}. \tag{13}$$

This will then imply $\mathcal{F} = \mathcal{F}(v_1^{(n)}, \ldots, v_n^{(n)})$ for a basis $(v_1^{(n)}, \ldots, v_n^{(n)})$ both w-dual and w-conjugate, i.e., will complete the proof of (a).

Our construction is done by induction starting with $(v_1^{(0)}, \ldots, v_n^{(0)}) = (v_1, \ldots, v_n)$. Let $\ell \in \{1, \ldots, n\}$, and assume that $(v_1^{(\ell-1)}, \ldots, v_n^{(\ell-1)})$ is constructed. We distinguish three cases.

Case 1: $w_\ell < \ell$.

The inequality $w_\ell < \ell = w(w_\ell)$ implies $\gamma(v_{w_\ell}^{(\ell-1)}) = \epsilon v_\ell^{(\ell-1)}$, whence $\gamma(v_\ell^{(\ell-1)}) = v_{w_\ell}^{(\ell-1)}$ as $\gamma^2 = \epsilon\mathrm{id}$. Therefore the basis $(v_1^{(\ell)}, \ldots, v_n^{(\ell)}) := (v_1^{(\ell-1)}, \ldots, v_n^{(\ell-1)})$ fulfills conditions (12) and (13).

Case 2: $w_\ell = \ell$.

This case occurs only in type A1. On the one hand, (11) yields

$$\gamma(v_\ell^{(\ell-1)}) \in \langle v_1^{(\ell-1)}, \ldots, v_\ell^{(\ell-1)}, v_{w_1}^{(\ell-1)}, \ldots, v_{w_{\ell-1}}^{(\ell-1)} \rangle_{\mathbb{C}}.$$

On the other hand, since the basis $(v_1^{(\ell-1)}, \ldots, v_n^{(\ell-1)})$ is w-dual, we have

$$v_\ell^{(\ell-1)} \in \langle v_1^{(\ell-1)}, \ldots, v_{\ell-1}^{(\ell-1)}, v_{w_1}^{(\ell-1)}, \ldots, v_{w_{\ell-1}}^{(\ell-1)} \rangle_{\mathbb{C}}^{\perp}.$$

Hence, as γ preserves orthogonality with respect to ω,

$$\begin{aligned} \gamma(v_\ell^{(\ell-1)}) \quad &\in \quad \langle \gamma(v_1^{(\ell-1)}), \ldots, \gamma(v_{\ell-1}^{(\ell-1)}), \gamma(v_{w_1}^{(\ell-1)}), \ldots, \gamma(v_{w_{\ell-1}}^{(\ell-1)}) \rangle_{\mathbb{C}}^{\perp} \\ &= \langle v_1^{(\ell-1)}, \ldots, v_{\ell-1}^{(\ell-1)}, v_{w_1}^{(\ell-1)}, \ldots, v_{w_{\ell-1}}^{(\ell-1)} \rangle_{\mathbb{C}}^{\perp}. \end{aligned}$$

Altogether this yields a nonzero complex number λ such that $\gamma(v_\ell^{(\ell-1)}) = \lambda v_\ell^{(\ell-1)}$. Since γ is an involution, we have $\lambda \in \{+1, -1\}$. In addition we know that

$$\lambda = \omega(\gamma(v_\ell^{(\ell-1)}), v_\ell^{(\ell-1)}) = {}^t\overline{v_\ell^{(\ell-1)}} v_\ell^{(\ell-1)} \in \mathbb{R}^+.$$

Whence $\gamma(v_\ell^{(\ell-1)}) = v_\ell^{(\ell-1)}$, and we can put $(v_1^{(\ell)}, \ldots, v_n^{(\ell)}) := (v_1^{(\ell-1)}, \ldots, v_n^{(\ell-1)})$.

Case 3: $w_\ell > \ell$.

By (11) we have

$$\gamma(v_\ell^{(\ell-1)}) \in \langle v_k^{(\ell-1)} : 1 \leq k \leq w_\ell \rangle_{\mathbb{C}} + \langle v_{w_k}^{(\ell-1)} : 1 \leq k \leq \ell-1 \rangle_{\mathbb{C}}.$$

On the other hand, arguing as in Case 2 we see that

$$\gamma(v_\ell^{(\ell-1)}) \in \langle v_1^{(\ell-1)}, \ldots, v_{\ell-1}^{(\ell-1)}, v_{w_1}^{(\ell-1)}, \ldots, v_{w_{\ell-1}}^{(\ell-1)} \rangle_{\mathbb{C}}^{\perp}.$$

Hence we can write

$$\gamma(v_\ell^{(\ell-1)}) = \sum_{k \in I} \lambda_k v_k^{(\ell-1)} \quad \text{for some } \lambda_k \in \mathbb{C}, \tag{14}$$

where $I := \{k : \ell \leq k \leq w_\ell \text{ and } \ell \leq w_k\} \subset \hat{I} := \{k : \ell \leq k \text{ and } \ell \leq w_k\}$. Using (14), the fact that the basis $(v_1^{(\ell-1)}, \ldots, v_n^{(\ell-1)})$ is w-dual, and the definition of ω and γ, we see that

$$\lambda_{w_\ell} = \omega(v_\ell^{(\ell-1)}, \gamma(v_\ell^{(\ell-1)})) = \epsilon\, {}^t v_\ell^{(\ell-1)} \overline{v_\ell^{(\ell-1)}} = \epsilon\alpha \tag{15}$$

with $\alpha \in \mathbb{R}$, $\alpha > 0$. Set

$$\begin{aligned}
v_\ell^{(\ell)} &:= \frac{1}{\sqrt{\alpha}} v_\ell^{(\ell-1)}, \quad v_{w_\ell}^{(\ell)} := \frac{\epsilon}{\sqrt{\alpha}} \gamma(v_\ell^{(\ell-1)}), \\
v_k^{(\ell)} &:= v_k^{(\ell-1)} - \frac{\omega(v_k^{(\ell-1)}, \gamma(v_\ell^{(\ell-1)}))}{\lambda_{w_\ell}} v_\ell^{(\ell-1)} \quad \text{for all } k \in \hat{I} \setminus \{\ell, w_\ell\}, \\
v_k^{(\ell)} &:= v_k^{(\ell-1)} \quad \text{for all } k \in \{1, \ldots, n\} \setminus \hat{I}.
\end{aligned}$$

Using (14) and (15) it is easy to check that $(v_1^{(\ell)}, \ldots, v_n^{(\ell)})$ is a w-dual basis which satisfies (12) and (13). This completes Case 3. ☐

Proof of Proposition 4 (b)–(d). Let $\mathcal{F} \in \mathcal{O}_w$, so $\mathcal{F} = \mathcal{F}(v_1, \ldots, v_n)$ for some w-dual basis $(v_1, \ldots, v_n)$ of V. From the definition of w-dual basis we see that

$$\begin{aligned}
\langle v_1, \ldots, v_{n-k} \rangle_{\mathbb{C}}^{\perp} &= \langle v_j : w_j \notin \{1, \ldots, n-k\} \rangle_{\mathbb{C}} \\
&= \langle v_j : w_j \in \{n-k+1, \ldots, n\} \rangle_{\mathbb{C}} \\
&= \langle v_j : (w_0 w)_j \in \{1, \ldots, k\} \rangle_{\mathbb{C}}\,.
\end{aligned}$$

Therefore

$$\dim \langle v_1, \ldots, v_{n-k} \rangle_{\mathbb{C}}^{\perp} \cap \langle v_1, \ldots, v_\ell \rangle_{\mathbb{C}} = \left| \{ j \in \{1, \ldots, \ell\} : (w_0 w)_j \in \{1, \ldots, k\} \} \right|$$

for all $k, \ell \in \{1, \ldots, n\}$, which yields the equality $w(\mathcal{F}^\perp, \mathcal{F}) = w_0 w$ and hence the inclusion

$$\mathcal{O}_w \subset \{\mathcal{F} \in X : (\mathcal{F}^\perp, \mathcal{F}) \in \mathbb{O}_{w_0 w}\}. \tag{16}$$

Let $\mathcal{F} = \mathcal{F}(v_1, \ldots, v_n) \in \mathfrak{O}_w$ for a w-conjugate basis $(v_1, \ldots, v_n)$ of V. From the definition of w-conjugate basis we get

$$\gamma(\langle v_1, \ldots, v_k \rangle_{\mathbb{C}}) = \langle v_{w_j} : j \in \{1, \ldots, k\} \rangle_{\mathbb{C}}\,.$$

Therefore

$$\dim \gamma(\langle v_1, \ldots, v_k \rangle_{\mathbb{C}}) \cap \langle v_1, \ldots, v_\ell \rangle_{\mathbb{C}} = \left| \{ j \in \{1, \ldots, \ell\} : w_j^{-1} \in \{1, \ldots, k\} \} \right|$$

for all $k, \ell \in \{1, \ldots, n\}$, whence $w(\gamma(\mathcal{F}), \mathcal{F}) = w^{-1} = w$ (since w is an involution). This implies the inclusion

$$\mathfrak{O}_w \subset \{\mathcal{F} \in X : (\gamma(\mathcal{F}), \mathcal{F}) \in \mathbb{O}_w\}. \tag{17}$$

It is clear that the group K acts transitively on the set of w-dual bases, hence $\mathcal{O}_w$ is a K-orbit. Moreover (16) implies that the orbits $\mathcal{O}_w$ (for $w \in \mathfrak{I}_w^\epsilon$) are pairwise distinct. Similarly the subsets $\mathfrak{O}_w$ (for $w \in \mathfrak{I}_w^\epsilon$) are pairwise distinct G^0-orbits.

We denote by L_k the $k \times k$-matrix with 1 on the antidiagonal and 0 elsewhere. Let $\underline{v} = (v_1, \ldots, v_n)$ be a w_0-dual basis, in other words,

$$\begin{cases} \omega(v_k, v_{n+1-k}) = \begin{cases} 1 & \text{if } k \leq \frac{n+1}{2} \\ \epsilon & \text{if } k > \frac{n+1}{2} \end{cases} \\ \omega(v_k, v_\ell) = 0 \quad \text{if } \ell \neq n+1-k; \end{cases}$$

hence $L := (\omega(v_k, v_\ell))_{1 \leq k, \ell \leq n}$ is the following matrix

$$L = L_n \quad \text{(type A1)} \quad \text{or} \quad L = \begin{pmatrix} 0 & L_m \\ -L_m & 0 \end{pmatrix} \quad \text{(type A2, } n = 2m\text{)}.$$

The flag $\mathcal{F}_0 := \mathcal{F}(v_1, \ldots, v_n)$ satisfies the condition $\mathcal{F}_0^\perp = \mathcal{F}_0$. By Richardson–Springer [18] every K-orbit $\mathcal{O} \subset X$ contains an element of the form $g\mathcal{F}_0$ with $g \in G$ such that $h := L^t[g]_{\underline{v}} L^{-1} [g]_{\underline{v}} \in N$ where $[g]_{\underline{v}}$ denotes the matrix of g in the basis $\underline{v}$ and N stands for the group of invertible $n \times n$-matrices with exactly one nonzero coefficient in each row and each column. Note that $Lh = {}^t[g]_{\underline{v}} L [g]_{\underline{v}}$ also belongs to N (as L does) and is symmetric in type A1 and antisymmetric in type A2. Consequently, there are $w \in \mathfrak{I}_n$ and constants $t_1, \ldots, t_n \in \mathbb{C}^*$ such that the matrix $Lh =: (a_{k,\ell})_{1 \leq k, \ell \leq n}$ has the following entries:

$$a_{k,\ell} = 0 \ \text{ if } \ell \neq w_k, \qquad a_{k,w_k} = \begin{cases} t_k & \text{if } w_k \geq k \\ \epsilon t_k & \text{if } w_k \leq k. \end{cases}$$

Since $\epsilon = -1$ in type A2, we must have $w_k \neq k$ for all k, hence $w \in \mathfrak{I}_n'$. Therefore in both cases $w \in \mathfrak{I}_n^\epsilon$. For each $k \in \{1, \ldots, n\}$, we choose $s_k = s_{w_k} \in \mathbb{C}^*$ such that $s_k^{-2} = t_k$ (note that $t_{w_k} = t_k$). Thus

$$g\mathcal{F}_0 = \mathcal{F}(s_1 g v_1, \ldots, s_n g v_n)\,,$$

and for all $k, \ell \in \{1, \ldots, n\}$ we have

$$\omega(s_k g v_k, s_\ell g v_\ell) = s_k s_\ell \omega(g v_k, g v_\ell) = s_k s_\ell a_{k,\ell} = \begin{cases} 1 & \text{if } \ell = w_k \geq k \\ \epsilon & \text{if } \ell = w_k < k \\ 0 & \text{if } \ell \neq w_k. \end{cases}$$

Whence $g\mathcal{F}_0 \in \mathcal{O}_w$. This yields $\mathcal{O} = \mathcal{O}_w$.

We have shown that the subsets $\mathcal{O}_w$ (for $w \in \mathfrak{I}_w^\epsilon$) are precisely the K-orbits of X. In particular, $X = \bigcup_{w \in \mathfrak{I}_w^\epsilon} \mathcal{O}_w$ so that the inclusion (16) is actually an equality. By Matsuki duality the number of G^0-orbits of X is the same as the number of K-orbits, hence the subsets $\mathfrak{O}_w$ (for $w \in \mathfrak{I}_w^\epsilon$) are exactly the G^0-orbits of X. Thereby equality holds in (17). Finally we have shown parts (b) and (c) of the statement.

Part (a) implies that, for every $w \in \mathfrak{I}_n^\epsilon$, the intersection $\mathcal{O}_w \cap \mathfrak{O}_w$ is nonempty and consists of a single $K \cap G^0$-orbit. This shows that the orbit $\mathfrak{O}_w$ is the Matsuki dual of $\mathcal{O}_w$ (see [12]), and part (d) of the statement is also proved. □

Proof of Proposition 5 (a). We write w as a product of pairwise disjoint transpositions $w = (a_1; b_1) \cdots (a_m; b_m)$, and let $c_{m+1} < \ldots < c_p$ be the elements of $\{k : w_k = k,\ \varepsilon_k = +1\}$ and $d_{m+1} < \ldots < d_q$ be the elements of $\{k : w_k = k,\ \varepsilon_k = -1\}$.

Let $\{e_1,\ldots,e_n\}=\{e_1^+,\ldots,e_p^+\}\cup\{e_1^-,\ldots,e_q^-\}$ so that $V_+=\langle e_\ell^+ : \ell=1,\ldots,p\rangle_\mathbb{C}$ and $V_-=\langle e_\ell^- : \ell=1,\ldots,q\rangle_\mathbb{C}$. Setting

$$v_{a_k}:=\tfrac{e_k^++e_k^-}{\sqrt{2}},\quad v_{b_k}:=\tfrac{e_k^+-e_k^-}{\sqrt{2}}\quad\text{for all } k\in\{1,\ldots,m\},$$

$$v_{c_k}:=e_k^+\quad\text{for all } k\in\{m+1,\ldots,p\},\quad\text{and}\quad v_{d_k}:=e_k^-\quad\text{for all } k\in\{m+1,\ldots,q\},$$

it is easy to see that $(v_1,\ldots,v_n)$ is a basis of V which is (w,ε)-dual and (w,ε)-conjugate. Therefore

$$\emptyset\neq\{\mathcal{F}(\underline{v}) : \underline{v}\text{ is }(w,\varepsilon)\text{-dual and }(w,\varepsilon)\text{-conjugate}\}\subset\mathcal{O}_{(w,\varepsilon)}\cap\mathfrak{O}_{(w,\varepsilon)}. \tag{18}$$

For showing the inverse inclusion, consider $\mathcal{F}=(F_0,\ldots,F_n)\in\mathcal{O}_{(w,\varepsilon)}\cap\mathfrak{O}_{(w,\varepsilon)}$. On the one hand, since $\mathcal{F}\in\mathfrak{O}_{(w,\varepsilon)}$ there is a (w,ε)-dual basis $(v_1,\ldots,v_n)$ such that $\mathcal{F}=\mathcal{F}(v_1,\ldots,v_n)$. On the other hand, the fact that $\mathcal{F}\in\mathcal{O}_{(w,\varepsilon)}$ yields

$$w_k=\min\{\ell=1,\ldots,n : \delta(F_k)\cap F_\ell\neq\delta(F_{k-1})\cap F_\ell\}\quad\text{for all } k\in\{1,\ldots,n\}. \tag{19}$$

For all $\ell\in\{0,\ldots,n\}$ we will now construct a (w,ε)-dual basis $(v_1^{(\ell)},\ldots,v_n^{(\ell)})$ such that

$$F_k=\langle v_1^{(\ell)},\ldots,v_k^{(\ell)}\rangle_\mathbb{C}\quad\text{for all } k\in\{1,\ldots,n\} \tag{20}$$

$$\text{and}\quad\delta(v_k^{(\ell)})=\begin{cases} v_{w_k}^{(\ell)} & \text{if } w_k\neq k,\\ \varepsilon_k v_k^{(\ell)} & \text{if } w_k=k\end{cases}\qquad\text{for all } k\in\{1,\ldots,\ell\}. \tag{21}$$

This will then provide a basis $(v_1^{(n)},\ldots,v_n^{(n)})$ which is both (w,ε)-dual and (w,ε)-conjugate and such that $\mathcal{F}=\mathcal{F}(v_1^{(n)},\ldots,v_n^{(n)})$, i.e., will complete the proof of part (a).

The construction is carried out by induction on $\ell\in\{0,\ldots,n\}$, and is initialized by setting $(v_1^{(0)},\ldots,v_n^{(0)}):=(v_1,\ldots,v_n)$. Let $\ell\in\{1,\ldots,n\}$ be such that the basis $(v_1^{(\ell-1)},\ldots,v_n^{(\ell-1)})$ is already constructed. We distinguish three cases.

Case 1: $w_\ell<\ell$.

Since in this case since $w_\ell\leq\ell-1$ and $w(w_\ell)=\ell$, we get $\delta(v_{w_\ell}^{(\ell-1)})=v_\ell^{(\ell-1)}$ and hence $\delta(v_\ell^{(\ell-1)})=v_{w_\ell}^{(\ell-1)}$ (as δ is an involution). Therefore the basis $(v_1^{(\ell)},\ldots,v_n^{(\ell)}):=(v_1^{(\ell-1)},\ldots,v_n^{(\ell-1)})$ satisfies conditions (20) and (21).

Case 2: $w_\ell=\ell$.

Using (19) we have

$$\delta(v_\ell^{(\ell-1)})\in\langle v_1^{(\ell-1)},v_2^{(\ell-1)},\ldots,v_\ell^{(\ell-1)}\rangle_\mathbb{C}+\langle v_{w_1}^{(\ell-1)},\ldots,v_{w_{\ell-1}}^{(\ell-1)}\rangle_\mathbb{C}\,.$$

On the other hand, the fact that the basis $(v_1^{(\ell-1)},\ldots,v_n^{(\ell-1)})$ is (w,ε)-conjugate implies

$$v_\ell^{(\ell-1)}\in\langle v_1^{(\ell-1)},\ldots,v_{\ell-1}^{(\ell-1)},v_{w_1}^{(\ell-1)},\ldots,v_{w_{\ell-1}}^{(\ell-1)}\rangle_\mathbb{C}^\dagger. \tag{22}$$

Since δ preserves orthogonality with respect to the form ϕ and since $\delta(v_k^{(\ell-1)})=v_{w_k}^{(\ell-1)}$ for all $k\in\{1,\ldots,\ell-1\}$ (by the induction hypothesis), (22) yields

$$\delta(v_\ell^{(\ell-1)})\in\langle v_1^{(\ell-1)},\ldots,v_{\ell-1}^{(\ell-1)},v_{w_1}^{(\ell-1)},\ldots,v_{w_{\ell-1}}^{(\ell-1)}\rangle_\mathbb{C}^\dagger.$$

Altogether we deduce that

$$\delta(v_\ell^{(\ell-1)})=\lambda v_\ell^{(\ell-1)}\quad\text{for some }\lambda\in\mathbb{C}^*.$$

As δ is an involution, we conclude that $\lambda \in \{+1,-1\}$. Moreover, knowing that $\phi(v_\ell^{(\ell-1)},v_\ell^{(\ell-1)})=\varepsilon_\ell$ we see that

$$\lambda\varepsilon_\ell=\phi(v_\ell^{(\ell-1)},\delta(v_\ell^{(\ell-1)}))={}^t\overline{v_\ell^{(\ell-1)}}\Phi\Phi v_\ell^{(\ell-1)}={}^t\overline{v_\ell^{(\ell-1)}}v_\ell^{(\ell-1)}\geq 0.$$

Finally we conclude that $\lambda=\varepsilon_\ell$. It follows that the basis $(v_1^{(\ell)},\ldots,v_n^{(\ell)}):=(v_1^{(\ell-1)},\ldots,v_n^{(\ell-1)})$ satisfies (20) and (21).

Case 3: $w_\ell>\ell$.

Invoking (19), the fact that $(v_1^{(\ell-1)},\ldots,v_n^{(\ell-1)})$ is (w,ε)-dual, the induction hypothesis, and the fact that δ preserves orthogonality with respect to ϕ, we see as in Case 2 that

$$\begin{aligned}\delta(v_\ell^{(\ell-1)}) \quad\in\quad & \big(\langle v_k^{(\ell-1)}:1\leq k\leq w_\ell\rangle_{\mathbb{C}}+\langle v_{w_k}^{(\ell-1)}:1\leq k\leq \ell-1\rangle_{\mathbb{C}}\big)\\ & \cap\langle v_1^{(\ell-1)},\ldots,v_{\ell-1}^{(\ell-1)},v_{w_1}^{(\ell-1)},\ldots,v_{w_{\ell-1}}^{(\ell-1)}\rangle_{\mathbb{C}}^{\dagger}.\end{aligned}$$

Therefore

$$\delta(v_\ell^{(\ell-1)})=\sum_{k\in I}\lambda_k v_k^{(\ell-1)}\quad\text{with }\lambda_k\in\mathbb{C},\tag{23}$$

where $I:=\{k:\ell\leq k\leq w_\ell,\ \ell\leq w_k\}\subset\hat{I}:=\{k:\ell\leq k,\ \ell\leq w_k\}$. This implies

$$\lambda_{w_\ell}=\phi(v_\ell^{(\ell-1)},\delta(v_\ell^{(\ell-1)}))={}^t\overline{v_\ell^{(\ell-1)}}\Phi\Phi v_\ell^{(\ell-1)}={}^t\overline{v_\ell^{(\ell-1)}}v_\ell^{(\ell-1)}\in\mathbb{R}_+^*.$$

It is straightforward to check that the basis $(v_1^{(\ell)},\ldots,v_n^{(\ell)})$ defined by

$$\begin{aligned}& v_\ell^{(\ell)}:=\frac{1}{\sqrt{\lambda_{w_\ell}}}v_\ell^{(\ell-1)},\quad v_{w_\ell}^{(\ell)}:=\frac{1}{\sqrt{\lambda_{w_\ell}}}\delta(v_\ell^{(\ell-1)}),\\ & v_k^{(\ell)}:=v_k^{(\ell-1)}-\frac{\phi(v_k^{(\ell-1)},\delta(v_\ell^{(\ell-1)}))}{\lambda_{w_\ell}}v_\ell^{(\ell-1)}\quad\text{for all }k\in\hat{I}\setminus\{\ell,w_\ell\},\\ & v_k^{(\ell)}:=v_k^{(\ell-1)}\quad\text{for all }k\in\{1,\ldots,n\}\setminus\hat{I}\end{aligned}$$

is (w,ε)-dual and satisfies conditions (20) and (21). □

Proof of Proposition 5 (b)–(d). Let $\mathcal{F}=\mathcal{F}(v_1,\ldots,v_n)$ where $(v_1,\ldots,v_n)$ is a (w,ε)-conjugate basis. Then by definition we have

$$\delta(\langle v_1,\ldots,v_k\rangle_{\mathbb{C}})=\langle v_{w_j}:j\in\{1,\ldots,k\}\rangle_{\mathbb{C}}\,,$$

hence

$$\begin{aligned}\dim\delta(\langle v_1,\ldots,v_k\rangle_{\mathbb{C}})\cap\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}} &= |\{j\in\{1,\ldots,\ell\}:w_j^{-1}\in\{1,\ldots,k\}\}|\\ &= |\{j\in\{1,\ldots,\ell\}:w_j\in\{1,\ldots,k\}\}|\end{aligned}$$

for all $k,\ell\in\{1,\ldots,n\}$. Moreover, for $\varepsilon\in\{+1,-1\}$ we have

$$\begin{aligned}\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}}\cap\ker(\delta-\varepsilon\mathrm{id}) &= \langle v_j:1\leq w_j=j\leq\ell\text{ and }\varepsilon_j=\varepsilon\rangle_{\mathbb{C}}\\ &\quad+\langle v_j+\varepsilon v_{w_j}:1\leq w_j<j\leq\ell\rangle_{\mathbb{C}}\,.\end{aligned}$$

Therefore

$$\big(\dim\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}}\cap V_+,\dim\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}}\cap V_-\big)_{\ell=1}^n=\varsigma(w,\varepsilon).$$

Altogether this yields the inclusion

$$\mathcal{O}_{(w,\varepsilon)}\subset\big\{\mathcal{F}\in X:(\delta(\mathcal{F}),\mathcal{F})\in\mathbb{O}_w\text{ and }\varsigma(\delta:\mathcal{F})=\varsigma(w,\varepsilon)\big\}.\tag{24}$$

Now let $(v_1,\ldots,v_n)$ be a (w,ε)-dual basis. Then

$$\begin{aligned}\langle v_1,\ldots,v_{n-k}\rangle_{\mathbb{C}}^\dagger\cap\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}} &= \langle v_j : j\in\{1,\ldots,\ell\} \text{ and } w_j>n-k\rangle_{\mathbb{C}} \\ &= \langle v_j : j\in\{1,\ldots,\ell\} \text{ and } (w_0w)_j\leq k\rangle_{\mathbb{C}}\,,\end{aligned}$$

whence

$$\dim\langle v_1,\ldots,v_{n-k}\rangle_{\mathbb{C}}^\dagger\cap\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}} = |\{j\in\{1,\ldots,\ell\} : (w_0w)_j\in\{1,\ldots,k\}|$$

for all $k,\ell\in\{1,\ldots,n\}$. In particular we see that

$$\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}} = \langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}}\cap\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}}^\dagger\oplus\langle v_j : j\in\{1,\ldots,\ell\} \text{ and } w_j\leq\ell\rangle_{\mathbb{C}}.$$

It follows that the vectors v_j (for $1\leq w_j=j\leq\ell$) and $\frac{1}{\sqrt{2}}(v_j\pm v_{w_j})$ (for $1\leq w_j<j\leq\ell$) form a basis of the quotient space $\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}}/\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}}\cap\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}}^\dagger$. This basis is ϕ-orthogonal and, since $(v_1,\ldots,v_n)$ is (w,ε)-dual, we have

$$\phi(v_j,v_j)=\varepsilon_j \text{ if } w_j=j;\qquad \left\{\begin{array}{l}\phi(\frac{v_j+v_{w_j}}{\sqrt{2}},\frac{v_j+v_{w_j}}{\sqrt{2}})=1,\\ \phi(\frac{v_j-v_{w_j}}{\sqrt{2}},\frac{v_j-v_{w_j}}{\sqrt{2}})=-1\end{array}\right.\quad \text{if } w_j<j.$$

Therefore the signature of ϕ on $\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}}/\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}}\cap\langle v_1,\ldots,v_\ell\rangle_{\mathbb{C}}^\dagger$ is the pair

$$\begin{array}{ll}(& |\{j : w_j=j\leq\ell,\ \varepsilon_j=+1\}|+|\{j : w_j<j\leq\ell\}|,\\ & |\{j : w_j=j\leq\ell,\ \varepsilon_j=-1\}|+|\{j : w_j<j\leq\ell\}| \quad)\end{array}$$

which coincides with the ℓ-th term of the sequence $\varsigma(w,\varepsilon)$. Finally, we obtain the inclusion

$$\mathfrak{O}_{(w,\varepsilon)}\subset\big\{\mathcal{F}\in X : (\mathcal{F}^\dagger,\mathcal{F})\in\mathbb{O}_{w_0w} \text{ and } \varsigma(\phi:\mathcal{F})=\varsigma(w,\varepsilon)\big\}. \tag{25}$$

It is clear that K (resp., G^0) acts transitively on the set of (w,ε)-conjugate bases (resp., (w,ε)-dual bases). Hence the subsets $\mathcal{O}_{(w,\varepsilon)}$ (resp. $\mathfrak{O}_{(w,\varepsilon)}$) are K-orbits (resp., G^0-orbits). Moreover, in view of (24) and (25) these orbits are pairwise distinct.

Let $\mathcal{O}$ be a K-orbit of X. Note that the basis $(e_1,\ldots,e_n)$ of V satisfies $\delta(e_j)=\pm e_j$ for all j, hence the flag $\mathcal{F}_0:=\mathcal{F}(e_1,\ldots,e_n)$ satisfies $\delta(\mathcal{F}_0)=\mathcal{F}_0$. By [18] the K-orbit $\mathcal{O}$ contains an element of the form $g\mathcal{F}_0$ for some $g\in G$ such that $h:=\Phi g^{-1}\Phi g\in N$ where, as in the proof of Proposition 4, $N\subset G$ stands for the subgroup of matrices with exactly one nonzero entry in each row and each column. Since $\Phi\in N$ we also have $\Phi h\in N$. Hence there is a permutation $w\in\mathfrak{S}_n$ and constants $t_1,\ldots,t_n\in\mathbb{C}^*$ such that the matrix $\Phi h=:\big(a_{k,\ell}\big)_{1\leq k,\ell\leq n}$ has entries

$$a_{k,\ell}=0 \ \text{ if } \ell\neq w_k,\quad a_{k,w_k}=t_k \quad \text{for all } k,\ell\in\{1,\ldots,n\}.$$

The relation $\Phi h=g^{-1}\Phi g$ shows that $(\Phi h)^2=1_n$. This yields $w^2=\mathrm{id}$ and $t_kt_{w_k}=1$ for all k; hence

$$t_{w_k}=t_k^{-1} \ \text{ whenever } w_k\neq k \quad\text{and}\quad \varepsilon_k:=t_k\in\{+1,-1\} \ \text{ whenever } w_k=k.$$

In addition, since Φh is conjugate to Φ, its eigenvalues $+1$ and -1 have respective multiplicities p and q, which forces

$$(w,\varepsilon)\in\mathfrak{I}_n(p,q).$$

For each $k \in \{1,\dots,n\}$ with $w_k < k$, we take $s_k \in \mathbb{C}^*$ such that $t_k = s_k^2$ and set $s_{w_k} = s_k^{-1}$ (so that $s_{w_k}^2 = t_k^{-1} = t_{w_k}$). Moreove,r for each $k \in \{1,\dots,n\}$ with $w_k = k$ we set $s_k = 1$. The equality $\Phi g = g\Phi h$ yields

$$\delta(g(s_ke_k)) = s_k\Phi ge_k = s_kg(\Phi h)e_k = s_kg(t_{w_k}e_{w_k}) = s_{w_k}^{-1}g(s_{w_k}^2e_{w_k}) = g(s_{w_k}e_{w_k})$$

for all $k \in \{1,\dots,n\}$ such that $w_k \neq k$, and

$$\delta(g(s_ke_k)) = \delta(g(e_k)) = \Phi ge_k = g(\Phi h)e_k = g(\varepsilon_ke_k) = \varepsilon_kg(e_k) = \varepsilon_kg(s_ke_k)$$

for all $k \in \{1,\dots,n\}$ such that $w_k = k$. Hence the family $(g(s_1e_1),\dots,g(s_ne_n))$ is a (w,ε)-conjugate basis of V. Thus

$$g\mathcal{F}_0 = g\mathcal{F}(e_1,\dots,e_n) = g\mathcal{F}(s_1e_1,\dots,s_ne_n) = \mathcal{F}(g(s_1e_1),\dots,g(s_ne_n)) \in \mathcal{O}_{(w,\varepsilon)}.$$

Therefore $\mathcal{O} = \mathcal{O}_{(w,\varepsilon)}$.

We conclude that the subsets $\mathcal{O}_{(w,\varepsilon)}$ (for $(w,\varepsilon) \in \mathfrak{I}_n(p,q)$) are exactly the K-orbits of X. Matsuki duality then guarantees that the subsets $\mathfrak{O}_{(w,\varepsilon)}$ (for $(w,\varepsilon) \in \mathfrak{I}_n(p,q)$) are exactly the G^0-orbits of X. This fact implies in particular that equality holds in (24) and (25). Altogether we have shown parts (b) and (c) of the statement.

Finally, part (a) shows that for every $(w,\varepsilon) \in \mathfrak{I}_n(p,q)$ the intersection $\mathcal{O}_{(w,\varepsilon)} \cap \mathfrak{O}_{(w,\varepsilon)}$ consists of a single $K \cap G^0$-orbit, which guarantees that the orbits $\mathcal{O}_{(w,\varepsilon)}$ and $\mathfrak{O}_{(w,\varepsilon)}$ are Matsuki dual (see [11, 12]). This proves part (d) of the statement. The proof of Proposition 5 is complete. □

Proof of Proposition 6. The proof relies on the following two technical claims.

Claim 1: For every signed involution $(w,\varepsilon) \in \mathfrak{I}_n(p,q)$ we have $\mathcal{O}_{(w,\varepsilon)} \cap X_\omega = \emptyset$ unless $(w,\varepsilon) \in \mathfrak{I}_n^{\eta,\epsilon}(p,q)$.

Claim 2: For every $(w,\varepsilon) \in \mathfrak{I}_n^{\eta,\epsilon}(p,q)$ there is a basis $\underline{v} = (v_1,\dots,v_n)$ which is simultaneously (w,ε)-dual and (w,ε)-conjugate and satisfies (9).

Assuming Claims 1 and 2, the proof of the proposition proceeds as follows. For every $(w,\varepsilon) \in \mathfrak{I}_n(p,q)$ the inclusions

$$\{\mathcal{F}(\underline{v}) : \underline{v} \text{ is } (w,\varepsilon)\text{-conjugate and satisfies (9)}\} \subset \mathcal{O}_{(w,\varepsilon)} \cap X_\omega, \tag{26}$$

$$\{\mathcal{F}(\underline{v}) : \underline{v} \text{ is } (w,\varepsilon)\text{-dual and satisfies (9)}\} \subset \mathfrak{O}_{(w,\varepsilon)} \cap X_\omega, \tag{27}$$

$$\begin{aligned}\{\mathcal{F}(\underline{v}) : \underline{v} \text{ is } (w,\varepsilon)\text{-dual and } (w,\varepsilon)\text{-conjugate and satisfies (9)}\}\\ \subset \mathcal{O}_{(w,\varepsilon)} \cap \mathfrak{O}_{(w,\varepsilon)} \cap X_\omega\end{aligned} \tag{28}$$

clearly hold. Hence Claim 2 shows that $\mathcal{O}_{(w,\varepsilon)}^{\eta,\epsilon}$, $\mathfrak{O}_{(w,\varepsilon)}^{\eta,\epsilon}$, and $\mathcal{O}_{(w,\varepsilon)}^{\eta,\epsilon} \cap \mathfrak{O}_{(w,\varepsilon)}^{\eta,\epsilon}$ are all nonempty whenever $(w,\varepsilon) \in \mathfrak{I}_n^{\eta,\epsilon}(p,q)$. By Claim 1, Lemma 1, and Proposition 5 (c), the K-orbits of X_ω are exactly the subsets $\mathcal{O}_{(w,\varepsilon)}^{\eta,\epsilon}$. On the other hand the subsets $\mathfrak{O}_{(w,\varepsilon)} \cap X_\omega$ (for $(w,\varepsilon) \in \mathfrak{I}_n(p,q)$) are G^0-stable and pairwise disjoint. By Matsuki duality there is a bijection between K-orbits and G^0-orbits. This forces $\mathfrak{O}_{(w,\varepsilon)}^{\eta,\epsilon} = \mathfrak{O}_{(w,\varepsilon)} \cap X_\omega$ to be a single G^0-orbit whenever $(w,\varepsilon) \in \mathfrak{I}_n^{\eta,\epsilon}(p,q)$ and $\mathfrak{O}_{(w,\varepsilon)} \cap X_\omega$ to be empty if $(w,\varepsilon) \notin \mathfrak{I}_n^{\eta,\epsilon}(p,q)$. This proves Proposition 6 (b).

Since the orbits $\mathcal{O}_{(w,\varepsilon)}, \mathfrak{O}_{(w,\varepsilon)} \subset X$ are Matsuki dual (see Proposition 5 (d)), their intersection $\mathcal{O}_{(w,\varepsilon)} \cap \mathfrak{O}_{(w,\varepsilon)}$ is compact, hence such is the intersection $\mathcal{O}_{(w,\varepsilon)}^{\eta,\epsilon} \cap \mathfrak{O}_{(w,\varepsilon)}^{\eta,\epsilon}$ for all $(w,\varepsilon) \in \mathfrak{I}_n^{\eta,\epsilon}(p,q)$. This implies that $\mathcal{O}_{(w,\varepsilon)}^{\eta,\epsilon}$ and $\mathfrak{O}_{(w,\varepsilon)}^{\eta,\epsilon}$ are Matsuki dual (see [6]), and therefore part (c) of the statement.

Let $(w,\varepsilon) \in \mathfrak{I}_n^{\eta,\epsilon}(p,q)$. Since $\mathcal{O}_{(w,\varepsilon)}^{\eta,\epsilon}$ and $\mathfrak{O}_{(w,\varepsilon)}^{\eta,\epsilon}$ are Matsuki dual, their intersection is a single $K \cap G^0$-orbit. The set on the left-hand side in (28) is nonempty (by

Claim 2) and $K\cap G^0$-stable, hence equality holds in (28). Similarly, the sets on the left-hand sides in (26) and (27) are nonempty (by Claim 2) and respectively K- and G^0-stable. Since $\mathcal{O}^{\eta,\epsilon}_{(w,\varepsilon)}=\mathcal{O}_{(w,\varepsilon)}\cap X_\omega$ and $\mathfrak{O}^{\eta,\epsilon}_{(w,\varepsilon)}=\mathfrak{O}_{(w,\varepsilon)}\cap X_\omega$ are respectively a K-orbit and a G^0-orbit, equality holds in (26) and (27). This shows part (a) of the statement.

Thus the proof of Proposition 6 will be complete once we establish Claims 1 and 2.

Proof of Claim 1. Note that for two subspaces $A,B\subset V$ we have $A^\perp+B^\perp=(A\cap B)^\perp$, hence

$$\dim A^\perp\cap B^\perp+\dim A+\dim B=\dim A\cap B+\dim V. \tag{29}$$

Note also that the map δ is selfadjoint (in types BD1 and C2) or antiadjoint (in types C1 and D3) with respect to ω, hence the equality $\delta(A)^\perp=\delta(A^\perp)$ holds for any subspace $A\subset V$ in all types.

Let $(w,\varepsilon)\in\mathfrak{I}_n(p,q)$ such that $\mathcal{O}_{(w,\varepsilon)}\cap X_\omega\neq\emptyset$. Let $\mathcal{F}=(F_0,\ldots,F_n)\in\mathcal{O}_{(w,\varepsilon)}\cap X_\omega$.

By applying (29) to $A=\delta\,(F_k)$ and $B=F_\ell$ for $1\leq k,\ell\leq n$ we obtain

$$\dim\delta\,(F_{n-k})\cap F_{n-\ell}+k+\ell=\dim\delta\,(F_k)\cap F_\ell+n. \tag{30}$$

On the other hand, since $\mathcal{F}\in\mathcal{O}_{(w,\varepsilon)}$ Proposition 5 (b) gives

$$\dim\delta\,(F_{n-k})\cap F_{n-\ell}=|\{j=1,\ldots,n-\ell:1\leq w_j\leq n-k\}| \tag{31}$$

and

$$\begin{aligned}\dim\delta\,(F_k)\cap F_\ell&=|\{j=1,\ldots,\ell:1\leq w_j\leq k\}|\\&=\ell-|\{j=1,\ldots,\ell:w_j\geq k+1\}|\\&=\ell-(n-k-|\{j\geq \ell+1:w_j\geq k+1\}|)\\&=\ell+k-n+|\{j=1,\ldots,n-\ell:w_0ww_0(j)\leq n-k\}|\end{aligned} \tag{32}$$

for all $k,\ell\in\{1,\ldots,n\}$. Comparing (30)–(32) we conclude that $w=w_0ww_0$.

Let $k\in\{1,\ldots,n\}$ such that $w_k=k$. Since $ww_0=w_0w$, we have $w_{n-k+1}=n-k+1$. Applying (29) with $A=F_k$ (resp., $A=F_{k-1}$) and $B=V_+$, we get

$$1+\dim F_{k-1}\cap V_+-\dim F_k\cap V_+=\dim F_{n-k+1}\cap V_--\dim F_{n-k}\cap V_-$$

in types BD1 and C2 (where $V_+^\perp=V_-$), whence

$$\begin{aligned}\varepsilon_k=1\quad&\Leftrightarrow\quad\dim F_k\cap V_+=\dim F_{k-1}\cap V_++1\\&\Leftrightarrow\quad\dim F_{n-k+1}\cap V_-=\dim F_{n-k}\cap V_-\Leftrightarrow\varepsilon_{n-k+1}=1\end{aligned}$$

in that case. In types C1 and D3 (where $V_+^\perp=V_+$), we get

$$1+\dim F_{k-1}\cap V_+-\dim F_k\cap V_+=\dim F_{n-k+1}\cap V_+-\dim F_{n-k}\cap V_+\,,$$

whence also

$$\varepsilon_k=1\Leftrightarrow\varepsilon_{n-k+1}=-1\,.$$

At this point we obtain that the signed involution (w,ε) satisfies conditions (i)–(ii) in Section 3.3. To conclude that $(w,\varepsilon)\in\mathfrak{I}_n^{\eta,\epsilon}(p,q)$, it remains to check that in types C2 and D3 we have $w_k\neq n-k+1$ for all $k\leq\frac{n}{2}$. Arguing by contradiction, assume that $w_k=n-k+1$. Since $\mathcal{F}\in\mathcal{O}_{(w,\varepsilon)}$ there is a (w,ε)-conjugate basis $\underline{v}=(v_1,\ldots,v_n)$ such that $\mathcal{F}=\mathcal{F}(\underline{v})$. Thus $\delta(v_k)=v_{n-k+1}$ so that we can write

$v_k = v_k^+ + v_k^-$ and $v_{n-k+1} = v_k^+ - v_k^-$. In type C2 we have $V_+^\perp = V_-$ and ω is antisymmetric, hence

$$\omega(v_k^+ + v_k^-, v_k^+ - v_k^-) = \omega(v_k^+, v_k^+) - \omega(v_k^-, v_k^-) = 0 - 0 = 0.$$

In type D3 we have $V_+^\perp = V_+$, $V_-^\perp = V_-$, and ω is symmetric hence

$$\omega(v_k^+ + v_k^-, v_k^+ - v_k^-) = -\omega(v_k^+, v_k^-) + \omega(v_k^-, v_k^+) = 0.$$

In both cases we deduce

$$F_{n-k+1} = F_{n-k} + \langle v_{n-k+1}\rangle_\mathbb{C} \subset F_k^\perp + F_{k-1}^\perp \cap \langle v_k\rangle_\mathbb{C}^\perp = F_k^\perp = F_{n-k},$$

a contradiction. This completes the proof of Claim 1.

Proof of Claim 2. For $k \in \{1, \ldots, n\}$ set $k^* = n - k + 1$. We can write

$$w = (c_1; c_1') \cdots (c_s; c_s')(c_1'^*; c_1^*) \cdots (c_s'^*; c_s^*)(d_1; d_1^*) \cdots (d_t; d_t^*)$$

where $c_1 < \ldots < c_s < c_s^* < \ldots < c_1^*$, $c_j < c_j' \neq c_j^*$ for all j, $d_1 < \ldots < d_t < d_t^* < \ldots < d_1^*$. Note that $t = 0$ in types C2 and D3. Moreover, we denote

$$\{a_1 < \ldots < a_{p-t-2s}\} := \{k : w_k = k,\ \varepsilon_k = 1\},$$
$$\{b_1 < \ldots < b_{q-t-2s}\} := \{k : w_k = k,\ \varepsilon_k = -1\}.$$

We can construct a ϕ-orthonormal basis

$$x_1^+, \ldots, x_t^+, y_1^+, \ldots, y_s^+, y_s^{+*}, \ldots, y_1^{+*}, z_1^+, \ldots, z_{p-t-2s}^+$$

of V_+, and a $(-\phi)$-orthonormal basis

$$x_1^-, \ldots, x_t^-, y_1^-, \ldots, y_s^-, y_s^{-*}, \ldots, y_1^{-*}, z_1^-, \ldots, z_{q-t-2s}^-$$

of V_-, such that in types BD1 and C2 (where the restriction of ω on V_+ and V_- is nondegenerate) we have

$$\omega(x_j^+, x_j^+) = \omega(x_j^-, x_j^-) = 1,$$
$$\omega(y_j^+, y_j^{+*}) = \omega(y_j^-, y_j^{-*}) = 1, \quad \omega(y_j^{+*}, y_j^+) = \omega(y_j^{-*}, y_j^-) = \epsilon,$$
$$\omega(z_j^+, z_\ell^+) = \begin{cases} 1 & \text{if } j \leq \ell = p - t - 2s + 1 - j \\ \epsilon & \text{if } j > \ell = p - t - 2s + 1 - j, \end{cases}$$
$$\omega(z_j^-, z_\ell^-) = \begin{cases} 1 & \text{if } j \leq \ell = q - t - 2s + 1 - j \\ \epsilon & \text{if } j > \ell = q - t - 2s + 1 - j, \end{cases}$$

and the other values of ω on the basis to equal 0. In types C1 and D3 (where $V_+^\perp = V_+$, $V_-^\perp = V_-$, and in particular $p = q = \frac{n}{2}$ in this case) we require that

$$\omega(x_j^+, x_j^-) = i, \quad \omega(x_j^-, x_j^+) = \epsilon i,$$
$$\omega(y_j^+, y_j^{-*}) = \omega(y_j^-, y_j^{+*}) = 1, \quad \omega(y_j^{+*}, y_j^-) = \omega(y_j^{-*}, y_j^+) = \epsilon,$$
$$\omega(z_j^+, z_\ell^-) = \epsilon\omega(z_\ell^-, z_j^+) = \begin{cases} 1 & \text{if } \ell = \tilde{j} := \frac{n}{2} - t - 2s + 1 - j \text{ and } a_j < b_{\tilde{j}} \\ \epsilon & \text{if } \ell = \tilde{j} := \frac{n}{2} - t - 2s + 1 - j \text{ and } a_j > b_{\tilde{j}}, \end{cases}$$

while the other values of ω on the basis are 0. In contrast to the value of $\omega(z_j^\pm, z_\ell^\pm)$ in types BD1,C2, the value of $\omega(z_j^+, z_\ell^-)$ in types C1,D3 is not subject to a constraint but is chosen so that the basis $(v_1, \ldots, v_n)$ below satisfies (9).

In all cases we construct a basis $(v_1, \ldots, v_n)$ by setting

$$v_{d_j} = \frac{x_j^+ + ix_j^-}{\sqrt{2}}, \quad v_{d_j^*} = \frac{x_j^+ - ix_j^-}{\sqrt{2}},$$

$$v_{c_j} = \frac{y_j^+ + y_j^-}{\sqrt{2}}, \quad v_{c'_j} = \frac{y_j^+ - y_j^-}{\sqrt{2}}, \quad v_{c_j^*} = \frac{y_j^{+*} + y_j^{-*}}{\sqrt{2}}, \quad v_{c'^*_j} = \frac{y_j^{+*} - y_j^{-*}}{\sqrt{2}},$$
$$v_{a_j} = z_j^+, \quad \text{and} \quad v_{b_j} = z_j^-.$$

It is straightforward to check that the basis $(v_1, \ldots, v_n)$ is both (w, ε)-dual and (w, ε)-conjugate and satisfies (9). This completes the proof of Claim 2. □

4. Orbit duality in ind-varieties of generalized flags

Following the pattern of Section 3, we now present our results on orbit duality in the infinite-dimensional case. All proofs are given in Section 4.5.

4.1. **Types A1 and A2.** The notation is as Section 2.1.1. For every $\ell \in \mathbb{N}^*$ there is a unique $\ell^* \in \mathbb{N}^*$ such that $\omega(e_\ell, e_{\ell^*}) \neq 0$, and this yields a bijection $\iota : \mathbb{N}^* \to \mathbb{N}^*$, $\ell \mapsto \ell^*$.

Let $\mathfrak{I}_\infty(\iota)$ be the set of involutions $w : \mathbb{N}^* \to \mathbb{N}^*$ such that $w(\ell) = \ell^*$ for all but finitely many $\ell \in \mathbb{N}^*$. In particular we have $w\iota \in \mathfrak{S}_\infty$ for all $w \in \mathfrak{I}_\infty(\iota)$. Let $\mathfrak{I}'_\infty(\iota) \subset \mathfrak{I}_\infty(\iota)$ be the subset of involutions without fixed points (i.e., such that $w(\ell) \neq \ell$ for all $\ell \in \mathbb{N}^*$).

Let $\sigma : \mathbb{N}^* \to (A, \prec)$ be a bijection onto a totally ordered set, and let us consider the ind-variety of generalized flags $\mathbf{X}(\mathcal{F}_\sigma, E)$. In Proposition 7 below we show that the $\mathbf{K}$-orbits and the $\mathbf{G}^0$-orbits of $\mathbf{X}(\mathcal{F}_\sigma, E)$ are parametrized by the elements of $\mathfrak{I}_\infty(\iota)$ in type A1, and by elements of $\mathfrak{I}'_\infty(\iota)$ in type A2.

Definition 6. Let $w \in \mathfrak{I}_\infty(\iota)$. Let $\underline{v} = (v_1, v_2, \ldots)$ be a basis of $\mathbf{V}$ such that

$$v_\ell = e_\ell \quad \text{for all but finitely many } \ell \in \mathbb{N}^*. \tag{33}$$

We call $\underline{v}$ *w-dual* if in addition to (33) $\underline{v}$ satisfies

$$\omega(v_\ell, v_k) = \begin{cases} 0 & \text{if } \ell \neq w_k, \\ \pm 1 & \text{if } \ell = w_k \end{cases} \quad \text{for all } k, \ell \in \mathbb{N}^*,$$

and we call $\underline{v}$ *w-conjugate* if in addition to (33)

$$\gamma(v_k) = \pm v_{w_k} \quad \text{for all } k \in \mathbb{N}^*.$$

Set $\boldsymbol{\mathcal{O}}_w := \{\mathcal{F}_\sigma(\underline{v}) : \underline{v} \text{ is } w\text{-dual}\}$ and $\boldsymbol{\mathfrak{O}}_w := \{\mathcal{F}_\sigma(\underline{v}) : \underline{v} \text{ is } w\text{-conjugate}\}$, so that $\boldsymbol{\mathcal{O}}_w$ and $\boldsymbol{\mathfrak{O}}_w$ are subsets of the ind-variety $\mathbf{X}(\mathcal{F}_\sigma, E)$.

Notation. (a) We use the abbreviation $\mathbf{X} := \mathbf{X}(\mathcal{F}_\sigma, E)$.
(b) If $\mathcal{F}$ is a generalized flag weakly compatible with E, then $\mathcal{F}^\perp := \{F^\perp : F \in \mathcal{F}\}$ is also a generalized flag weakly compatible with E.

Let $(A^*, \prec^*)$ be the totally ordered set given by $A^* = A$ as a set and $a \prec^* a'$ whenever $a \succ a'$. Let $\sigma^\perp : \mathbb{N}^* \to (A^*, \prec^*)$ be defined by $\sigma^\perp(\ell) = \sigma(\ell^*)$. Then we have $\mathcal{F}_\sigma^\perp = \mathcal{F}_{\sigma^\perp}$. Note that $\mathcal{F}^\perp$ is E-commensurable with $\mathcal{F}_{\sigma^\perp}$ whenever $\mathcal{F}$ is E-commensurable with $\mathcal{F}_\sigma$. Hence the map

$$\mathbf{X} \to \mathbf{X}^\perp := \mathbf{X}(\mathcal{F}_{\sigma^\perp}, E), \ \mathcal{F} \mapsto \mathcal{F}^\perp$$

is well defined. We use the abbreviation $\mathbb{O}_w^\perp := (\mathbb{O}_{\sigma^\perp, \sigma})_w$ for all $w \in \mathfrak{S}_\infty$.
(c) We further note that $\gamma(\mathcal{F}_\sigma) = \mathcal{F}_{\sigma \circ \iota}$ and that $\gamma(\mathcal{F}) \in \mathbf{X}^\gamma := \mathbf{X}(\mathcal{F}_{\sigma\circ\iota}, E)$ whenever $\mathcal{F} \in \mathbf{X}$. We abbreviate $\mathbb{O}_w^\gamma := (\mathbb{O}_{\sigma\circ\iota, \sigma})_w$ for all $w \in \mathfrak{S}_\infty$.

Thus $\mathbf{X}^\perp \times \mathbf{X} = \bigsqcup_{w \in \mathfrak{S}_\infty} \mathbb{O}_w^\perp$ and $\mathbf{X}^\gamma \times \mathbf{X} = \bigsqcup_{w \in \mathfrak{S}_\infty} \mathbb{O}_w^\gamma$ (see Proposition 2).

Proposition 7. *Let $\mathfrak{I}^\epsilon_\infty(\iota) = \mathfrak{I}_\infty(\iota)$ in type A1 and $\mathfrak{I}^\epsilon_\infty(\iota) = \mathfrak{I}'_\infty(\iota)$ in type A2.*

(a) *For every* $w \in \mathfrak{I}^{\epsilon}_{\infty}(\iota)$,

$$\boldsymbol{\mathcal{O}}_w \cap \boldsymbol{\mathfrak{O}}_w = \{\mathcal{F}_\sigma(\underline{v}) : \underline{v} \textit{ is } w\textit{-dual and } w\textit{-conjugate}\} \neq \emptyset.$$

(b) *For every* $w \in \mathfrak{I}^{\epsilon}_{\infty}(\iota)$,

$$\boldsymbol{\mathcal{O}}_w = \{\mathcal{F} \in \mathbf{X} : (\mathcal{F}^\perp, \mathcal{F}) \in \mathbb{O}^\perp_{w\iota}\} \quad \textit{and} \quad \boldsymbol{\mathfrak{O}}_w = \{\mathcal{F} \in \mathbf{X} : (\gamma(\mathcal{F}), \mathcal{F}) \in \mathbb{O}^\gamma_{w\iota}\}.$$

(c) *The subsets* $\boldsymbol{\mathcal{O}}_w$ *(for* $w \in \mathfrak{I}^{\epsilon}_{\infty}(\iota)$*) are exactly the* $\mathbf{K}$*-orbits of* $\mathbf{X}$*. The subsets* $\boldsymbol{\mathfrak{O}}_w$ *(for* $w \in \mathfrak{I}^{\epsilon}_{\infty}(\iota)$*) are exactly the* $\mathbf{G}^0$*-orbits of* $\mathbf{X}$*. Moreover* $\boldsymbol{\mathcal{O}}_w \cap \boldsymbol{\mathfrak{O}}_w$ *is a single* $\mathbf{K} \cap \mathbf{G}^0$*-orbit.*

4.2. **Type A3.** The notation is as in Section 2.1.2. In particular, we fix a partition $\mathbb{N}^* = N_+ \sqcup N_-$ yielding Φ as in (3) and we consider the corresponding hermitian form ϕ and involution δ on $\mathbf{V}$.

Let $\mathfrak{I}_\infty(N_+, N_-)$ be the set of pairs (w, ε) consisting of an involution $w : \mathbb{N}^* \to \mathbb{N}^*$ and a map $\varepsilon : \{\ell : w_\ell = \ell\} \to \{1, -1\}$ such that the subsets

$$N'_\pm = N'_\pm(w, \varepsilon) := \{\ell \in N_\pm : (w_\ell, \varepsilon_\ell) = (\ell, \pm 1)\}$$

satisfy

$$|N_\pm \setminus N'_\pm| = |\{\ell \in N_\mp : (w_\ell, \varepsilon_\ell) = (\ell, \pm 1)\}| + \frac{1}{2}|\{\ell \in \mathbb{N}^* : w_\ell \neq \ell\}| < \infty.$$

In particular, $w \in \mathfrak{S}_\infty$.

Fix $\sigma : \mathbb{N}^* \to (A, \prec)$ a bijection onto a totally ordered set. We show in Proposition 8 that the $\mathbf{K}$-orbits and the $\mathbf{G}^0$-orbits of the ind-variety $\mathbf{X} := \mathbf{X}(\mathcal{F}_\sigma, E)$ are parametrized by the elements of $\mathfrak{I}_\infty(N_+, N_-)$.

Definition 7. Let $(w, \varepsilon) \in \mathfrak{I}_\infty(N_+, N_-)$. A basis $\underline{v} = (v_1, v_2, \ldots)$ of $\mathbf{V}$ such that $v_\ell = e_\ell$ for all but finitely many $\ell \in \mathbb{N}^*$ is (w, ε)*-conjugate* if

$$\delta(v_k) = \begin{cases} v_{w_k} & \text{if } w_k \neq k, \\ \varepsilon_k v_k & \text{if } w_k = k \end{cases} \qquad \text{for all } k \in \mathbb{N}^*,$$

and is (w, ε)*-dual* if

$$\phi(v_k, v_\ell) = \begin{cases} 0 & \text{if } \ell \neq w_k, \\ 1 & \text{if } \ell = w_k \neq k, \\ \varepsilon_k & \text{if } \ell = w_k = k \end{cases} \qquad \text{for all } k, \ell \in \mathbb{N}^*.$$

Set $\boldsymbol{\mathcal{O}}_{(w,\varepsilon)} := \{\mathcal{F}_\sigma(\underline{v}) : \underline{v} \text{ is } (w, \varepsilon)\text{-conjugate}\}$, $\boldsymbol{\mathfrak{O}}_{(w,\varepsilon)} := \{\mathcal{F}_\sigma(\underline{v}) : \underline{v} \text{ is } (w, \varepsilon)\text{-dual}\}$.

Notation. (a) Note that every subspace in the generalized flag $\mathcal{F}_\sigma$ is δ-stable, i.e., $\delta(\mathcal{F}_\sigma) = \mathcal{F}_\sigma$. The map $\mathbf{X} \to \mathbf{X}$, $\mathcal{F} \mapsto \delta(\mathcal{F})$ is well defined.
(b) Write $F^\dagger = \{x \in \mathbf{V} : \phi(x, y) = 0 \ \forall y \in F\}$ and $\mathcal{F}^\dagger := \{F^\dagger : F \in \mathcal{F}\}$, which is a generalized flag weakly compatible with E whenever $\mathcal{F}$ is so.

As in Section 4.1 we write $(A^*, \prec^*)$ for the totally ordered set such that $A^* = A$ and $a \prec^* a'$ whenever $a \succ a'$. It is readily seen that $\mathcal{F}_\sigma^\dagger = \mathcal{F}_{\sigma^\dagger}$ where $\sigma^\dagger : \mathbb{N}^* \to (A^*, \prec^*)$ is such that $\sigma^\dagger(\ell) = \sigma(\ell)$ for all $\ell \in \mathbb{N}^*$, and we get a well-defined map

$$\mathbf{X} \to \mathbf{X}^\dagger := \mathbf{X}(\mathcal{F}_{\sigma^\dagger}, E),\ \mathcal{F} \mapsto \mathcal{F}^\dagger.$$

(c) We write $\mathbb{O}_w := (\mathbb{O}_{\sigma,\sigma})_w$ and $\mathbb{O}^\dagger_w := (\mathbb{O}_{\sigma^\dagger,\sigma})_w$ so that

$$\mathbf{X} \times \mathbf{X} = \bigsqcup_{w \in \mathfrak{S}_\infty} \mathbb{O}_w \quad \text{and} \quad \mathbf{X}^\dagger \times \mathbf{X} = \bigsqcup_{w \in \mathfrak{S}_\infty} \mathbb{O}^\dagger_w$$

(see Proposition 2).

Proposition 8. (a) *For every* $(w,\varepsilon)\in\mathfrak{I}_\infty(N_+,N_-)$ *we have*

$$\boldsymbol{\mathcal{O}}_{(w,\varepsilon)}\cap\boldsymbol{\mathfrak{O}}_{(w,\varepsilon)}=\{\mathcal{F}_\sigma(\underline{v}):\underline{v}\text{ is }(w,\varepsilon)\text{-conjugate and }(w,\varepsilon)\text{-dual}\}\neq\emptyset.$$

(b) *Let* $(w,\varepsilon)\in\mathfrak{I}_\infty(N_+,N_-)$ *and* $\mathcal{F}=\{F'_a,F''_a:a\in A\}\in\mathbf{X}$. *Then* $\mathcal{F}\in\boldsymbol{\mathcal{O}}_{(w,\varepsilon)}$ *(resp.,* $\mathcal{F}\in\boldsymbol{\mathfrak{O}}_{(w,\varepsilon)}$*) if and only if*

$$(\delta(\mathcal{F}),\mathcal{F})\in\mathbb{O}_w\quad(\text{resp., }(\mathcal{F}^\dagger,\mathcal{F})\in\mathbb{O}_w^\dagger)$$

and for all $\ell\in\mathbb{N}^*$ *the following condition holds:*

$$\dim F''_{\sigma(\ell)}\cap\mathbf{V}_\pm/F'_{\sigma(\ell)}\cap\mathbf{V}_\pm=\begin{cases}1 & \text{if }\sigma(w_\ell)\prec\sigma(\ell)\text{ or }(w_\ell,\varepsilon_\ell)=(\ell,\pm1),\\ 0 & \text{otherwise}\end{cases}$$

where $\mathbf{V}_\pm=\langle e_\ell:\ell\in N_\pm\rangle_\mathbb{C}$ *(resp., for* $n\in\mathbb{N}^*$ *large enough*

$$\varsigma(\phi:F''_{\sigma(\ell)}\cap V_n)=\varsigma(\phi:F'_{\sigma(\ell)}\cap V_n)+\begin{cases}(1,1) & \text{if }\sigma(w_\ell)\prec\sigma(\ell),\\ (1,0) & \text{if }(w_\ell,\varepsilon_\ell)=(\ell,1),\\ (0,1) & \text{if }(w_\ell,\varepsilon_\ell)=(\ell,-1),\\ (0,0) & \text{if }\sigma(w_\ell)\succ\sigma(\ell),\end{cases}$$

where $V_n=\langle e_k:k\leq n\rangle_\mathbb{C}$ *and* $\varsigma(\phi:F)$ *stands for the signature of* ϕ *on* $F/F\cap F^\dagger$*).*

(c) *The subsets* $\boldsymbol{\mathcal{O}}_{(w,\varepsilon)}$ *(*$(w,\varepsilon)\in\mathfrak{I}_\infty(N_+,N_-)$*) are exactly the* $\mathbf{K}$*-orbits of* $\mathbf{X}$*. The subsets* $\boldsymbol{\mathfrak{O}}_{(w,\varepsilon)}$ *(*$(w,\varepsilon)\in\mathfrak{I}_\infty(N_+,N_-)$*) are exactly the* $\mathbf{G}^0$*-orbits of* $\mathbf{X}$*. Moreover* $\boldsymbol{\mathcal{O}}_{(w,\varepsilon)}\cap\boldsymbol{\mathfrak{O}}_{(w,\varepsilon)}$ *is a single* $\mathbf{K}\cap\mathbf{G}^0$*-orbit.*

4.3. **Types B, C, D.** Assume that $\mathbf{V}$ is endowed with a nondegenerate symmetric or symplectic form ω, determined by a matrix Ω as in (2). Let $\iota:\mathbb{N}^*\to\mathbb{N}^*$, $\ell\mapsto\ell^*$ satisfy $\omega(e_\ell,e_{\ell^*})\neq0$ for all ℓ.

Let $\mathbb{N}^*=N_+\sqcup N_-$ be a partition such that N_+,N_- are either both ι-stable or such that $\iota(N_+)=N_-$. As before, let ϕ and δ be the hermitian form and the involution of $\mathbf{V}$ corresponding to this partition. The following table summarizes the different cases.

	ω symmetric $\epsilon=1$	ω symplectic $\epsilon=-1$
$\iota(N_\pm)\subset N_\pm$ $\eta=1$	type BD1	type C2
$\iota(N_\pm)=N_\mp$ $\eta=-1$	type D3	type C1

Let $\mathfrak{I}_\infty^{\eta,\epsilon}(N_+,N_-)\subset\mathfrak{I}_\infty(N_+,N_-)$ be the subset of pairs (w,ε) such that

(i) $\iota w=w\iota$ (hence the set $\{\ell:w_\ell=\ell\}$ is ι-stable);
(ii) $\varepsilon_{\iota(k)}=\eta\varepsilon_k$ for all $k\in\{\ell:w_\ell=\ell\}$;

and if $\eta\epsilon=-1$:

(iii) $w_k\neq\iota(k)$ for all $k\in\mathbb{N}^*$.

Let $\mathcal{F}_\sigma$ be an ω-isotropic maximal generalized flag compatible with E. Thus $\sigma:\mathbb{N}^*\to(A,\prec)$ is a bijection onto a totally ordered set $(A,\prec)$ endowed with an (involutive) antiautomorphism of ordered sets $\iota_A:(A,\prec)\to(A,\prec)$ such that $\sigma\iota=\iota_A\sigma$. The following statement shows that the $\mathbf{K}$-orbits and the $\mathbf{G}^0$-orbits of the ind-variety $\mathbf{X}_\omega:=\mathbf{X}_\omega(\mathcal{F}_\sigma,E)$ are parametrized by the elements of the set $\mathfrak{I}_\infty^{\eta,\epsilon}(N_+,N_-)$.

Proposition 9. *We consider bases* $\underline{v} = (v_1, v_2, \ldots)$ *of* $\mathbf{V}$ *such that*

$$\omega(v_k, v_\ell) \neq 0 \quad \text{if and only if} \quad \ell = \iota(k). \tag{34}$$

(a) *For every* $(w, \varepsilon) \in \mathfrak{I}_\infty^{\eta,\epsilon}(N_+, N_-)$ *we have*

$$\boldsymbol{\mathcal{O}}^{\eta,\epsilon}_{(w,\varepsilon)} := \boldsymbol{\mathcal{O}}_{(w,\varepsilon)} \cap \mathbf{X}_\omega = \{\mathcal{F}_\sigma(\underline{v}) : \underline{v} \text{ is } (w,\varepsilon)\text{-conjugate and satisfies (34)}\} \neq \emptyset,$$

$$\boldsymbol{\mathfrak{O}}^{\eta,\epsilon}_{(w,\varepsilon)} := \boldsymbol{\mathfrak{O}}_{(w,\varepsilon)} \cap \mathbf{X}_\omega = \{\mathcal{F}_\sigma(\underline{v}) : \underline{v} \text{ is } (w,\varepsilon)\text{-dual and satisfies (34)}\} \neq \emptyset,$$

$$\boldsymbol{\mathcal{O}}^{\eta,\epsilon}_{(w,\varepsilon)} \cap \boldsymbol{\mathfrak{O}}^{\eta,\epsilon}_{(w,\varepsilon)} = \{\mathcal{F}_\sigma(\underline{v}) : \underline{v} \text{ is } (w,\varepsilon)\text{-conjugate, } (w,\varepsilon)\text{-dual and satisfies (34)}\} \neq \emptyset.$$

(b) *The subsets* $\boldsymbol{\mathcal{O}}^{\eta,\epsilon}_{(w,\varepsilon)}$ *(*$(w, \varepsilon) \in \mathfrak{I}_\infty^{\eta,\epsilon}(N_+, N_-)$*) are exactly the* $\mathbf{K}$*-orbits of* $\mathbf{X}_\omega$*. The subsets* $\boldsymbol{\mathfrak{O}}^{\eta,\epsilon}_{(w,\varepsilon)}$ *(*$(w, \varepsilon) \in \mathfrak{I}_\infty^{\eta,\epsilon}(N_+, N_-)$*) are exactly the* $\mathbf{G}^0$*-orbits of* $\mathbf{X}_\omega$*. Moreover* $\boldsymbol{\mathcal{O}}^{\eta,\epsilon}_{(w,\varepsilon)} \cap \boldsymbol{\mathfrak{O}}^{\eta,\epsilon}_{(w,\varepsilon)}$ *is a single* $\mathbf{K} \cap \mathbf{G}^0$*-orbit.*

4.4. **Ind-variety structure.** In this section we recall from [4] the ind-variety structure on $\mathbf{X}$ and $\mathbf{X}_\omega$.

Recall that $E = (e_1, e_2, \ldots)$ is a countable ordered basis of $\mathbf{V}$. Fix an E-compatible maximal generalized flag $\mathcal{F}_\sigma$ corresponding to a bijection $\sigma : \mathbb{N}^* \to (A, \prec)$ onto a totally ordered set, and let $\mathbf{X} = \mathbf{X}(\mathcal{F}_\sigma, E)$.

Let $V_n := \langle e_1, \ldots, e_n \rangle_\mathbb{C}$ and let X_n denote the variety of complete flags of V_n defined as in (6). There are natural inclusions $V_n \subset V_{n+1}$, and

$$\mathrm{GL}(V_n) \cong \{g \in \mathrm{GL}(V_{n+1}) : g(V_n) = V_n,\ g(e_{n+1}) = e_{n+1}\} \subset \mathrm{GL}(V_{n+1}). \tag{35}$$

We define a $\mathrm{GL}(V_n)$-equivariant embedding

$$\iota_n = \iota_n(\sigma) : X_n \to X_{n+1},\ (F_k)_{k=0}^n \mapsto (F'_k)_{k=0}^{n+1}$$

by letting

$$F'_k := \begin{cases} F_k & \text{if } a_k \prec \sigma(n+1) \\ F_{k-1} \oplus \langle e_{n+1} \rangle_\mathbb{C} & \text{if } a_k \succeq \sigma(n+1) \end{cases}$$

where $a_1 \prec a_2 \prec \ldots \prec a_{n+1}$ are the elements of the set $\{\sigma(\ell) : 1 \leq \ell \leq n+1\}$ written in increasing order. Therefore, we get a chain of embeddings (which are morphisms of algebraic varieties)

$$\cdots \hookrightarrow X_{n-1} \overset{\iota_{n-1}}{\hookrightarrow} X_n \overset{\iota_n}{\hookrightarrow} X_{n+1} \overset{\iota_{n+1}}{\hookrightarrow} \cdots,$$

and $\mathbf{X}$ is obtained as the direct limit

$$\mathbf{X} = \mathbf{X}(\mathcal{F}_\sigma, E) = \varinjlim X_n.$$

In particular, for each n we get an embedding $\hat{\iota}_n : X_n \hookrightarrow \mathbf{X}$ and up to identifying X_n with its image by this embedding we can view $\mathbf{X}$ as the union $\mathbf{X} = \bigcup_{n \geq 1} X_n$. Every generalized flag $\mathcal{F} \in \mathbf{X}$ belongs to all X_n after some rank $n_\mathcal{F}$. For instance $\mathcal{F}_\sigma \in X_n$ for all $n \geq 1$.

A basis $\underline{v} = (v_1, \ldots, v_n)$ of V_n can be completed into the basis of $\mathbf{V}$ denoted by $\hat{\underline{v}} := (v_1, \ldots, v_n, e_{n+1}, e_{n+2}, \ldots)$, and we have

$$\hat{\iota}_n(\mathcal{F}(v_{\tau_1}, \ldots, v_{\tau_n})) = \mathcal{F}_\sigma(\hat{\underline{v}}) \tag{36}$$

(using the notation of Sections 2.2–2.3) where $\tau = \tau^{(n)} \in \mathfrak{S}_n$ is the permutation such that $\sigma(\tau_1^{(n)}) \prec \ldots \prec \sigma(\tau_n^{(n)})$.

Recall that the ind-topology on $\mathbf{X}$ is defined by declaring a subset $\mathbf{Z} \subset \mathbf{X}$ open (resp., closed) if every intersection $\mathbf{Z} \cap X_n$ is open (resp., closed).

Clearly the ind-variety structure on $\mathbf{X}$ is not modified if the sequence $(X_n, \iota_n)_{n \geq 1}$ is replaced by a subsequence $(X_{n_k}, \iota'_k)_{k \geq 1}$ where $\iota'_k := \iota_{n_{k+1}-1} \circ \cdots \circ \iota_{n_k}$.

In type A3 (using the notation of Section 2.1) the subspace $V_n \subset \mathbf{V}$ is endowed with the restrictions of ϕ and δ, hence we can define $K_n, G_n^0 \subset \mathrm{GL}(V_n)$ as in Section 3.2, with the condition that the inclusion of (35) restricts to natural inclusions $K_n \subset K_{n+1}$ and $G_n^0 \subset G_{n+1}^0$.

Next assume that the space $\mathbf{V}$ is endowed with a nondegenerate symmetric or symplectic form ω determined by the matrix Ω of (2). The blocks $J_1, J_2, \ldots$ in the matrix Ω are of size 1 or 2. We set $n_k := |J_1| + \ldots + |J_k|$ so that the restriction of ω to each subspace V_{n_k} is nondegenerate. Hence in types A1, A2, BD1, C1, C2, and D3 we can define the subgroups $K_{n_k}, G_{n_k}^0 \subset \mathrm{GL}(V_{n_k})$ as in Section 3, and so that (35) yields natural inclusions

$$K_{n_k} \subset K_{n_{k+1}} \quad \text{and} \quad G_{n_k}^0 \subset G_{n_{k+1}}^0.$$

Moreover, the subvariety $(X_{n_k})_\omega \subset X_{n_k}$ of isotropic flags (with respect to ω) can be defined as in (7). Assuming that the generalized flag $\mathcal{F}_\sigma$ is ω-isotropic, the embedding $\iota'_k : X_{n_k} \hookrightarrow X_{n_{k+1}}$ maps $(X_{n_k})_\omega$ into $(X_{n_{k+1}})_\omega$ and we have

$$\mathbf{X}_\omega = \mathbf{X}_\omega(\mathcal{F}_\sigma, E) = \bigcup_{k\geq 1} (X_{n_k})_\omega \quad \text{and} \quad (X_{n_k})_\omega = \mathbf{X}_\omega \cap X_{n_k} \text{ for all } k \geq 1.$$

In particular, $\mathbf{X}_\omega$ is a closed ind-subvariety of $\mathbf{X}$ (as stated in Proposition 3).

4.5. **Proofs.**

Proof of Proposition 7. Let $\mathcal{F} = \{F'_a, F''_a : a \in A\} = \mathcal{F}_\sigma(\underline{v})$ for a basis $\underline{v} = (v_1, v_2, \ldots)$ of $\mathbf{V}$. Let $w \in \mathfrak{I}_\infty^\epsilon(\iota)$. If the basis $\underline{v}$ is w-dual, then

$$(F'_a)^\perp = \langle v_\ell : \sigma(w_\ell) \succeq a \rangle_\mathbb{C} \quad \text{and} \quad (F''_a)^\perp = \langle v_\ell : \sigma(w_\ell) \succ a \rangle_\mathbb{C}\,,$$

hence $\mathcal{F}^\perp = \mathcal{F}_{\sigma^\perp \iota w}(\underline{v})$; this yields $(\mathcal{F}^\perp, \mathcal{F}) \in \mathbb{O}_{w\iota}^\perp$. If $\underline{v}$ is w-conjugate, then

$$\gamma(F'_a) = \langle v_\ell : \sigma(w_\ell) \prec a \rangle_\mathbb{C} \quad \text{and} \quad \gamma(F''_a) = \langle v_\ell : \sigma(w_\ell) \preceq a \rangle_\mathbb{C}\,,$$

whence $\gamma(\mathcal{F}) = \mathcal{F}_{\sigma w}(\underline{v})$ and $(\gamma(\mathcal{F}), \mathcal{F}) \in \mathbb{O}_{w\iota}^\gamma$. This proves the inclusions $\subset$ in Proposition 7 (b). Note that these inclusions imply in particular that the subsets $\boldsymbol{\mathcal{O}}_w$, as well as $\boldsymbol{\mathfrak{O}}_w$, are pairwise disjoint.

For $w \in \mathfrak{I}_{n_k}^\epsilon$ we define $\hat{w} : \mathbb{N}^* \to \mathbb{N}^*$ by letting

$$\hat{w}(\ell) = \begin{cases} \tau w \tau^{-1}(\ell) & \text{if } \ell \leq n_k, \\ \iota(\ell) & \text{if } \ell \geq n_k + 1 \end{cases}$$

where $\tau = \tau^{(n_k)} : \{1, \ldots, n_k\} \to \{1, \ldots, n_k\}$ is the permutation such that $\sigma(\tau_1) \prec \ldots \prec \sigma(\tau_{n_k})$. It is easy to see that we obtain a well-defined (injective) map $j_k : \mathfrak{I}_{n_k}^\epsilon \to \mathfrak{I}_\infty^\epsilon(\iota)$, $j_k(w) := \hat{w}$, and

$$\mathfrak{I}_\infty^\epsilon(\iota) = \bigcup_{k\geq 1} j_k(\mathfrak{I}_{n_k}^\epsilon). \tag{37}$$

Moreover, given a basis $\underline{v} = (v_1, \ldots, v_{n_k})$ of V_{n_k} and the basis $\hat{\underline{v}}$ of $\mathbf{V}$ obtained by adding the vectors e_ℓ for $\ell \geq n_k + 1$, the implication

$$\begin{aligned} &(v_{\tau_1}, \ldots, v_{\tau_{n_k}}) \text{ is } w\text{-dual (resp., } w\text{-conjugate)} \\ &\qquad \Rightarrow \ \hat{\underline{v}} \text{ is } \hat{w}\text{-dual (resp., } \hat{w}\text{-conjugate)} \end{aligned} \tag{38}$$

clearly follows from our constructions. Note that

$$\boldsymbol{\mathcal{O}}_{\hat{w}} \cap X_{n_k} = \mathcal{O}_w \quad \text{and} \quad \boldsymbol{\mathfrak{O}}_{\hat{w}} \cap X_{n_k} = \mathfrak{O}_w \tag{39}$$

where $\mathcal{O}_w, \mathfrak{O}_w \subset X_{n_k}$ are the orbits from Definition 4; indeed, the inclusions $\supset$ in (39) are implied by (36) and (38), whereas the inclusions $\subset$ follow from Proposition 4 (c) and the fact that the subsets $\boldsymbol{\mathcal{O}}_{\hat{w}}$, as well as $\boldsymbol{\mathfrak{O}}_{\hat{w}}$, are pairwise disjoint. Parts (a) and (c) of Proposition 7 now follow from (37)–(39) and Proposition 4 (a), (c). By Proposition 7 (a) we deduce that equalities hold in Proposition 7 (b), and the proof is complete. $\square$

Proof of Proposition 8. For every $n \geq 1$ we set $p_n = |N_+ \cap \{1, \ldots, n\}|$ and $q_n = |N_- \cap \{1, \ldots, n\}|$.

Let $\mathcal{F} = \{F'_a, F''_a : a \in A\} = \mathcal{F}_\sigma(\underline{v})$ for some basis $\underline{v} = (v_1, v_2, \ldots)$ of $\mathbf{V}$. Let $(w, \varepsilon) \in \mathfrak{I}_\infty(N_+, N_-)$. If $\underline{v}$ is (w, ε)-conjugate, then

$$\delta(F'_a) = \langle v_\ell : \sigma(w_\ell) \prec a \rangle_\mathbb{C} \quad \text{and} \quad \delta(F''_a) = \langle v_\ell : \sigma(w_\ell) \preceq a \rangle_\mathbb{C}$$

so that $(\delta(\mathcal{F}), \mathcal{F}) = (\mathcal{F}_{\sigma w}(\underline{v}), \mathcal{F}_\sigma(\underline{v})) \in \mathbb{O}_w$. In addition,

$$\begin{cases} F''_{\sigma(\ell)} \cap \mathbf{V}_+ / F'_{\sigma(\ell)} \cap \mathbf{V}_+ = \langle v_\ell \rangle_\mathbb{C},\ F''_{\sigma(\ell)} \cap \mathbf{V}_- = F'_{\sigma(\ell)} \cap \mathbf{V}_- & \text{if } (w_\ell, \varepsilon_\ell) = (\ell, +1), \\ F''_{\sigma(\ell)} \cap \mathbf{V}_- / F'_{\sigma(\ell)} \cap \mathbf{V}_- = \langle v_\ell \rangle_\mathbb{C},\ F''_{\sigma(\ell)} \cap \mathbf{V}_+ = F'_{\sigma(\ell)} \cap \mathbf{V}_+ & \text{if } (w_\ell, \varepsilon_\ell) = (\ell, -1), \\ F''_{\sigma(\ell)} \cap \mathbf{V}_+ / F'_{\sigma(\ell)} \cap \mathbf{V}_+ = \langle v_\ell + v_{w_\ell} \rangle_\mathbb{C}, & \\ \qquad F''_{\sigma(\ell)} \cap \mathbf{V}_- / F'_{\sigma(\ell)} \cap \mathbf{V}_- = \langle v_\ell - v_{w_\ell} \rangle_\mathbb{C} & \text{if } \sigma(w_\ell) \prec \sigma(\ell), \\ F''_{\sigma(\ell)} \cap \mathbf{V}_+ = F'_{\sigma(\ell)} \cap \mathbf{V}_+,\ F''_{\sigma(\ell)} \cap \mathbf{V}_+ = F'_{\sigma(\ell)} \cap \mathbf{V}_+ & \text{if } \sigma(w_\ell) \succ \sigma(\ell), \end{cases}$$

which proves the formula for $\dim F''_{\sigma(\ell)} \cap \mathbf{V}_\pm / F'_{\sigma(\ell)} \cap \mathbf{V}_\pm$ stated in Proposition 8 (b). If $\underline{v}$ is (w, ε)-dual, then we get similarly

$$(F'_a)^\dagger = \langle v_\ell : \sigma(w_\ell) \succeq a \rangle_\mathbb{C} \quad \text{and} \quad (F''_a)^\dagger = \langle v_\ell : \sigma(w_\ell) \succ a \rangle_\mathbb{C}\,.$$

Hence $(\mathcal{F}^\dagger, \mathcal{F}) = (\mathcal{F}_{\sigma^\dagger w}(\underline{v}), \mathcal{F}_\sigma(\underline{v})) \in \mathbb{O}^\dagger_w$. For $n \geq 1$ large enough we have $(w_\ell, \varepsilon_\ell) = (\ell, \pm 1)$ for all $\ell \in N_\pm \cap \{n+1, n+2, \ldots\}$ and $v_\ell = e_\ell$ for all $\ell \geq n+1$. Thus the pair $(\check{w}, \check{\varepsilon}) := (w|_{\{1,\ldots,n\}}, \varepsilon|_{\{1,\ldots,n\}})$ belongs to $\mathfrak{I}_n(p_n, q_n)$ whereas by (36) we have

$$\mathcal{F} = \mathcal{F}(v_{\tau_1}, \ldots, v_{\tau_n}).$$

The basis $(v_{\tau_1}, \ldots, v_{\tau_n})$ of V_n is $(\tau^{-1}\check{w}\tau, \check{\varepsilon}\tau)$-dual if $\underline{v}$ is (w, ε)-dual; the last formula in Proposition 8 (b) now follows from Proposition 5 (b) and this observation. Altogether this shows the "only if" part in Proposition 8 (b), which guarantees in particular that the subsets $\boldsymbol{\mathcal{O}}_{(w,\varepsilon)}$, as well as the subsets $\boldsymbol{\mathfrak{O}}_{(w,\varepsilon)}$, are pairwise disjoint. The "if" part of Proposition 8 (b) follows once we show Proposition 8 (a).

For $(w, \varepsilon) \in \mathfrak{I}_n(p_n, q_n)$ we set

$$\hat{w}(\ell) = \begin{cases} \tau w \tau^{-1}(\ell) & \text{if } \ell \leq n, \\ \ell & \text{if } \ell \geq n+1 \end{cases} \qquad \text{for all } \ell \in \mathbb{N}^*, \tag{40}$$

where $\tau = \tau^{(n)} \in \mathfrak{S}_n$ is as in (36), and

$$\hat{\varepsilon}(\ell) = \begin{cases} \varepsilon\tau^{-1}(\ell) & \text{if } \ell \leq n, \\ 1 & \text{if } \ell \geq n+1,\ n \in N_+, \\ -1 & \text{if } \ell \geq n+1,\ n \in N_- \end{cases} \tag{41}$$

for all $\ell \in \mathbb{N}^*$ such that $\hat{w}_\ell = \ell$. It is easy to check that $(\hat{w}, \hat{\varepsilon}) \in \mathfrak{I}_\infty(N_+, N_-)$, and that the so obtained map $j_n : \mathfrak{I}_n(p_n, q_n) \to \mathfrak{I}_\infty(N_+, N_-)$ is injective and

$$\mathfrak{I}_\infty(N_+, N_-) = \bigcup_{n \geq 1} j_n(\mathfrak{I}_n(p_n, q_n)).$$

Moreover, it follows from our constructions that, given a basis $\underline{v} = (v_1, \ldots, v_n)$ of V_n and the basis $\hat{\underline{v}}$ of $\mathbf{V}$ obtained by adding the vectors e_ℓ for $\ell \geq n+1$, we have:

$$(v_{\tau_1}, \ldots, v_{\tau_n}) \text{ is } (w, \varepsilon)\text{-conjugate (resp., dual)}$$
$$\Rightarrow\ \hat{\underline{v}} \text{ is } (\hat{w}, \hat{\varepsilon})\text{-conjugate (resp., dual).}$$

As in the proof of Proposition 7 we derive the equalities

$$\boldsymbol{\mathcal{O}}_{(\hat{w},\hat{\varepsilon})} \cap X_n = \mathcal{O}_{(w,\varepsilon)} \quad \text{and} \quad \boldsymbol{\mathfrak{O}}_{(\hat{w},\hat{\varepsilon})} \cap X_n = \mathfrak{O}_{(w,\varepsilon)} \tag{42}$$

where $\mathcal{O}_{(w,\varepsilon)}, \mathfrak{O}_{(w,\varepsilon)} \subset X_n$ are as in Definition 5. Parts (a) and (c) of Proposition 8 then follow from Proposition 5 (a) and (c). $\square$

Proof of Proposition 9. Let $n \in \{n_1, n_2, \ldots\}$ (where $n_k = |J_1| + \ldots + |J_k|$ as before) and $(p_n, q_n) = (|N_+ \cap \{1, \ldots, n\}|, |N_- \cap \{1, \ldots, n\}|)$ and let $\tau = \tau^{(n)} : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be the permutation such that $\sigma(\tau_1) \prec \ldots \prec \sigma(\tau_n)$. Since the generalized flag $\mathcal{F}_\sigma$ is ω-isotropic, we must have

$$\iota(\tau_\ell) = \tau_{n-\ell+1} \quad \text{for all } \ell \in \{1, \ldots, n\}.$$

This observation easily implies that the map j_n defined in the proof of Proposition 8 restricts to a well-defined injective map

$$j_n : \mathfrak{I}_n^{\eta,\epsilon}(p_n, q_n) \to \mathfrak{I}_\infty^{\eta,\epsilon}(N_+, N_-)$$

such that

$$\mathfrak{I}_\infty^{\eta,\epsilon}(N_+, N_-) = \bigcup_{k \geq 1} j_{n_k}(\mathfrak{I}_{n_k}^{\eta,\epsilon}(p_{n_k}, q_{n_k})).$$

By (42) for $(\hat{w}, \hat{\varepsilon}) = j_n(w, \varepsilon)$ we get

$$\boldsymbol{\mathcal{O}}^{\eta,\epsilon}_{(\hat{w},\hat{\varepsilon})} \cap (X_n)_\omega = \mathcal{O}^{\eta,\epsilon}_{(w,\varepsilon)} \quad \text{and} \quad \boldsymbol{\mathfrak{O}}^{\eta,\epsilon}_{(\hat{w},\hat{\varepsilon})} \cap (X_n)_\omega = \mathfrak{O}^{\eta,\epsilon}_{(w,\varepsilon)}. \tag{43}$$

Proposition 9 easily follows from this fact and Proposition 6. $\square$

5. Corollaries

Corollary 1. *The duality map Ξ from Theorem 1(b) depends only on the choice of $\mathbf{G}$, $\mathbf{B}$, $\mathbf{K}$ and $\mathbf{G}^0$, but not on the particular choice of ordered basis E used to construct $\mathbf{G}$, $\mathbf{B}$, $\mathbf{K}$, and $\mathbf{G}^0$ as above. In particular, Ξ does not depend on the exhaustion $\mathbf{X} = \bigcup_{n \geq 1} X_n$ determined by E and referred to in Theorem 1(b).*

Proof. The statement follows immediately from the commutativity of diagram (1) and from the observation that for any two exhaustions $\mathbf{X} = \bigcup_{n\geq 1} X_n$ and $\mathbf{X} = \bigcup_{n \geq 1} X'_n$, and any n_0 and n'_0, there are n_1 and n'_1 such that $X_{n_0} \cup X'_{n'_0} \subset X_{n_1}$ and $X_{n_0} \cup X'_{n'_0} \subset X'_{n'_1}$. $\square$

Our second corollary states that the parametrization of $\mathbf{K}$- and $\mathbf{G}^0$-orbits on $\mathbf{G}/\mathbf{B}$ depends in fact only on the triple $(\mathbf{G}, \mathbf{K}, \mathbf{G}^0)$ and not on the choice of the ind-variety $\mathbf{G}/\mathbf{B}$.

Corollary 2. *Let $E, \mathbf{G}, \mathbf{K}, \mathbf{G}^0$ be as in Section 2.1. Let $\mathcal{F}_{\sigma_j}$ ($j = 1, 2$) be two E-compatible maximal generalized flags, which are ω-isotropic in types B,C,D, and let $\mathbf{X}_j = \mathbf{G}/\mathbf{B}_{\mathcal{F}_{\sigma_j}}$. Then there are natural bijections*

$$\mathbf{X}_1/\mathbf{K} \cong \mathbf{X}_2/\mathbf{K} \quad \textit{and} \quad \mathbf{X}_1/\mathbf{G}^0 \cong \mathbf{X}_2/\mathbf{G}^0$$

which commute with the duality of Theorem 1.

Next, a straightforward counting of the parameters yields:

Corollary 3. *In Corollary 2 the orbit sets* $\mathbf{X}_j/\mathbf{K}$ *and* $\mathbf{X}_j/\mathbf{G}^0$ *are always infinite.*

It is important to note that, despite Corollary 2, the topological properties of the orbits on $\mathbf{G}/\mathbf{B}$ are not the same for different choices of Borel ind-subgroups $\mathbf{B} \subset \mathbf{G}$. The following corollary establishes criteria for the existence of open and closed orbits on $\mathbf{G}/\mathbf{B} = \mathbf{X}\,(\mathcal{F}_\sigma, E)$.

Corollary 4. *Let* $E, \mathbf{G}, \mathbf{K}, \mathbf{G}^0$ *be as in Section 2.1, and let* $\mathcal{F}_\sigma$ *be an* E*-compatible maximal generalized flag,* ω*-isotropic in types B,C,D, where* $\sigma : \mathbb{N}^* \to (A, \prec)$ *is a bijection onto a totally ordered set. Let* $\mathbf{X} = \mathbf{G}/\mathbf{B}_{\mathcal{F}_\sigma}$*; i.e.,* $\mathbf{X} = \mathbf{X}(\mathcal{F}_\sigma, E)$ *in type* A *and* $\mathbf{X} = \mathbf{X}_\omega(\mathcal{F}_\sigma, E)$ *in types B,C,D.*

(a_1) *In type A1,* $\mathbf{X}$ *has an open* $\mathbf{K}$*-orbit (equivalently, a closed* $\mathbf{G}^0$*-orbit) if and only if* $\iota(\ell) = \ell$ *for all* $\ell \gg 1$ *(i.e., if the matrix* Ω *of (2) contains finitely many diagonal blocks of size 2).*

(a_2) *In type A2,* $\mathbf{X}$ *has an open* $\mathbf{K}$*-orbit (equivalently, a closed* $\mathbf{G}^0$*-orbit) if and only if for all* $\ell \gg 1$ *the elements* $\sigma(2\ell-1), \sigma(2\ell)$ *are consecutive in* A *and the number* $|\{k < 2\ell - 1 : \sigma(k) \prec \sigma(2\ell-1)\}|$ *is even.*

(a'_{12}) *In types A1 and A2,* $\mathbf{X}$ *has at most one closed* $\mathbf{K}$*-orbit (equivalently, at most one open* $\mathbf{G}^0$*-orbit).* $\mathbf{X}$ *has a closed* $\mathbf{K}$*-orbit (equivalently, an open* $\mathbf{G}^0$*-orbit) if and only if* $\mathbf{X}$ *contains* ω*-isotropic generalized flags. This latter condition is equivalent to the existence of an involutive antiautomorphism of ordered sets* $\iota_A : (A, \prec) \to (A, \prec)$ *such that* $\iota_A\sigma(\ell) = \sigma\iota(\ell)$ *for all* $\ell \gg 1$.

(a_3) *In type A3,* $\mathbf{X}$ *has always infinitely many closed* $\mathbf{K}$*-orbits (equivalently, infinitely many open* $\mathbf{G}^0$*-orbits).* $\mathbf{X}$ *has an open* $\mathbf{K}$*-orbit (equivalently, a closed* $\mathbf{G}^0$*-orbit) if and only if* $d := \min\{|N_+|, |N_-|\} < \infty$ *and* $\mathcal{F}_\sigma$ *contains a* d*-dimensional and a* d*-codimensional subspace.*

(bcd) *In types B,C,D,* $\mathbf{X}$ *has always infinitely many closed* $\mathbf{K}$*-orbits (equivalently, infinitely many open* $\mathbf{G}^0$*-orbits). In types C1 and D3,* $\mathbf{X}$ *has never an open* $\mathbf{K}$*-orbit (equivalently, no closed* $\mathbf{G}^0$*-orbit). In types BD1 and C2,* $\mathbf{X}$ *has an open* $\mathbf{K}$*-orbit (equivalently, a closed* $\mathbf{G}^0$*-orbit) if and only if* $d := \min\{|N_+|, |N_-|\} < \infty$ *and* $\mathcal{F}_\sigma$ *contains a* d*-dimensional subspace (or equivalently it has a* d*-codimensional subspace).*

Proof. This follows from Remarks 4 and 5, Propositions 7, 8, 9, and relations (39), (42), (43). □

Corollary 5. *The only situation where* $\mathbf{X}$ *has simultaneously open and closed* $\mathbf{K}$*-orbits (equivalently, open and closed* $\mathbf{G}^0$*-orbits) is in types A3, BD1, C2, in the case where* $d := \min\{|N_+|, |N_-|\} < \infty$ *and* $\mathcal{F}_\sigma$ *contains a* d*-dimensional and a* d*-codimensional subspace.*

Index of notation

§1: $\mathbb{N}^*$, $|A|$, $\mathfrak{S}_n$, $\mathfrak{S}_\infty$, $(k;\ell)$

§2.1: $\mathbf{G}(E)$, $\mathbf{G}(E,\omega)$, Ω, ω, γ, Φ, ϕ, δ

§2.2: $\mathcal{F}(v_1, \ldots, v_n)$, $\mathbb{O}_w$

§2.3: $\mathcal{F}_\sigma(\underline{v})$, $\mathcal{F}_\sigma$, $\mathbf{P}_\mathcal{F}$, $\mathbf{B}_\mathcal{F}$, $\mathbf{X}(\mathcal{F}, E)$, $(\mathbb{O}_{\tau,\sigma})_w$, $\mathbf{X}_\omega(\mathcal{F}, E)$

§3.1: $\mathcal{F}^\perp$, $\gamma(\mathcal{F})$, $\mathfrak{I}_n$, $\mathfrak{I}'_n$, $\mathcal{O}_w$, $\mathfrak{O}_w$

§3.2: $\delta(\mathcal{F})$, $\mathcal{F}^\dagger$, $\varsigma(\phi:\mathcal{F})$, $\varsigma(\delta:\mathcal{F})$, $\varsigma(w,\varepsilon)$, $\mathfrak{I}_n(p,q)$, $\mathcal{O}_{(w,\varepsilon)}$, $\mathfrak{O}_{(w,\varepsilon)}$

§3.3: $\mathfrak{I}_n^{\eta,\epsilon}(p,q)$, $\mathcal{O}_{(w,\varepsilon)}^{\eta,\epsilon}$, $\mathfrak{O}_{(w,\varepsilon)}^{\eta,\epsilon}$

§4.1: ι, $\mathfrak{I}_\infty(\iota)$, $\mathfrak{I}'_\infty(\iota)$, $\boldsymbol{\mathcal{O}}_w$, $\boldsymbol{\mathfrak{O}}_w$, $(A^*,\prec^*)$, $\sigma^\perp$, $\mathbf{X}^\perp$, $\mathbf{X}^\gamma$, $\mathbb{O}_w^\perp$, $\mathbb{O}_w^\gamma$

§4.2: $\mathfrak{I}_\infty(N_+,N_-)$, $\boldsymbol{\mathcal{O}}_{(w,\varepsilon)}$, $\boldsymbol{\mathfrak{O}}_{(w,\varepsilon)}$, $\sigma^\dagger$, $\mathbf{X}^\dagger$, $\mathbb{O}_w$, $\mathbb{O}_w^\dagger$

§4.3: $\mathfrak{I}_\infty^{\eta,\epsilon}(N_+,N_-)$, $\boldsymbol{\mathcal{O}}_{(w,\varepsilon)}^{\eta,\epsilon}$, $\boldsymbol{\mathfrak{O}}_{(w,\varepsilon)}^{\eta,\epsilon}$

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