Geometry, $\mathfrak{n}$-homology and (limits of) discrete series

Dragan Miličić

Department of Mathematics
University of Utah

Jacobs University Bremen
June 16-18, 2014

Formulation of the problem

Let

- G_0 – connected real semisimple Lie group with finite center;
- K_0 – a maximal compact subgroup of G_0;
- T_0 – a maximal torus in K_0;
- $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{t}$ – complexified Lie algebras of G_0, K_0 and T_0 respectively.

We assume that G_0 and K_0 have equal rank, i.e., T_0 is a compact Cartan subgroup of G_0.
Let

- $\mathfrak{b}$ – a Borel subalgebra containing $\mathfrak{t}$;
- $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ - nilpotent radical of $\mathfrak{b}$.

Let V be an irreducible Harish-Chandra module for G_0. Then the Lie algebra homology $H_p(\mathfrak{n}, V)$, $0 \leq p \leq \dim \mathfrak{n}$, are finite-dimensional representations of T_0.

Let

- R – the root system of $(\mathfrak{g}, \mathfrak{t})$ in $\mathfrak{t}^*$;
- R^+ – the set of positive roots such that their root subspaces span $\mathfrak{n}$;
- ρ – half-sum of roots in R^+;
- W – the Weyl group of R.

Then, by a result of Casselman-Osborne, we have

$$H_p(\mathfrak{n}, V) = \bigoplus_{w \in W} H_p(\mathfrak{n}, V)_{(w\lambda+\rho)}$$

for some $\lambda \in \mathfrak{t}^*$. Hence, one has to determine just $\dim H_p(\mathfrak{n}, V)_{(w\lambda+\rho)}$ for $w \in W$ and $p \in \mathbb{Z}_+$.

Special cases

- If G_0 is compact and V a finite-dimensional representation, this is a famous result of Kostant;
- if V is a discrete series representation, this is a result of Schmid;
- if V is a *nondegenerate* limit of discrete series, Williams observed that Schmid's argument still works;
- if $\mathfrak{n}$ is *holomorphic*, and V arbitrary limit of discrete series, Mirković proved that $\mathfrak{n}$-homology vanishes;
- Soergel published a proof of a result for arbitrary limits of discrete series, but his published proof is incorrect.

We are going to discuss ongoing joint work with Wilfried Schmid on $\mathfrak{n}$-homology of limits of discrete series. Since the theme of this conference is geometric, we will discuss the approach based on D-module theory, which should contain all above results as special cases.
This work was inspired by a question by Phillip Griffiths to Schmid.

Localization of modules

Let

- $\mathcal{U}(\mathfrak{g})$ – the enveloping algebra of $\mathfrak{g}$;
- $\mathcal{Z}(\mathfrak{g})$ – the center of $\mathcal{U}(\mathfrak{g})$.

Let X be the flag variety of $\mathfrak{g}$, i.e., the variety of all Borel subalgebras of $\mathfrak{g}$. Then X is a smooth projective variety.
For any $x \in X$ we denote by $\mathfrak{b}_x$ the corresponding Borel subalgebra in $\mathfrak{g}$.
Let $\mathcal{B}$ be the vector subbundle of the trivial bundle $X \times \mathfrak{g}$ over X such that the fiber over x is equal to $\{x\} \times \mathfrak{b}_x$.
Let $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ and $\mathcal{N}$ the vector subbundle of $\mathcal{B}$ such that the fiber over x is equal to $\{x\} \times \mathfrak{n}_x$.
The vector bundle $\mathcal{B}/\mathcal{N}$ is trivial, therefore it has the form $X \times \mathfrak{h}$.
We call $\mathfrak{h}$ the *abstract Cartan algebra* of $\mathfrak{g}$.

Let $x \in X$. Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{c} \subset \mathfrak{b}_x$. Then we have the natural isomorphism $\mathfrak{c} \longrightarrow \mathfrak{h}$. The dual map $\mathfrak{h}^* \longrightarrow \mathfrak{c}^*$ is called the *specialization* at x.
There exists a reduced root system Σ in $\mathfrak{h}^*$ which by specialization corresponds to the root system of the pair $(\mathfrak{g}, \mathfrak{c})$.
Also, there exists a set of positive roots Σ^+ in Σ such that the span of root subspaces for the corresponding positive roots of $(\mathfrak{g}, \mathfrak{c})$ spans $\mathfrak{n}_x$.
The pair (Σ, Σ^+) is independent of the choice of x. We call Σ the *abstract root system* of $\mathfrak{g}$.
Let W be the Weyl group of Σ.

Harish-Chandra homomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ establishes an isomorphism of $\mathcal{Z}(\mathfrak{g})$ with the Weyl group invariants $S(\mathfrak{h})^W$ of $S(\mathfrak{h})$.
We can view elements of $S(\mathfrak{h})^W$ as W-invariant polynomials on $\mathfrak{h}^*$. Therefore, to any orbit θ of W in $\mathfrak{h}^*$ we attach a unique maximal ideal I_θ of $\mathcal{Z}(\mathfrak{g})$ which corresponds to invariant polynomials which vanish on θ. We put

$$\mathcal{U}_\theta = \mathcal{U}(\mathfrak{g})/I_\theta\mathcal{U}(\mathfrak{g}).$$

Denote by $\mathcal{M}(\mathcal{U}_\theta)$ the category of $\mathcal{U}_\theta$-modules.

Beilinson and Bernstein constructed a family $\mathcal{D}_\lambda$, $\lambda \in \mathfrak{h}^*$, of twisted sheaves of differential operators on X together with natural homomorphisms $\mathcal{U}(\mathfrak{g}) \longrightarrow \Gamma(X, \mathcal{D}_\lambda)$ which induce isomorphisms

$$\mathcal{U}_\theta = \Gamma(X, \mathcal{D}_\lambda)$$

for $\theta = W \cdot \lambda$.
Denote by $\mathcal{M}(\mathcal{D}_\lambda)$ the category of (quasicoherent) $\mathcal{D}_\lambda$-modules.
We define the functors

$$\mathcal{M}(\mathcal{D}_\lambda) \underset{\Delta_\lambda}{\overset{\Gamma(X,-)}{\rightleftarrows}} \mathcal{M}(\mathcal{U}_\theta)$$

where $\Gamma(X, -)$ is the functor of global sections and

$$\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$$

for a module V in $\mathcal{M}(\mathcal{U}_\theta)$.
The functor Δ_λ is called the *localization at* λ. The functor Δ_λ is a left adjoint of $\Gamma(X, -)$.

Let $\Sigma^\vee$ be the dual root system of Σ. Let $\alpha^\vee$ be the dual root of α.
We say that $\lambda \in \mathfrak{h}^*$ is *antidominant* if $\alpha^\vee(\lambda) \notin \{1, 2, \dots\}$ for all $\alpha \in \Sigma^+$.
We say that $\lambda \in \mathfrak{h}^*$ is *regular* if $\alpha^\vee(\lambda) \neq 0$ for any $\alpha \in \Sigma$.

Theorem (Equivalence of Categories)

Let $\lambda \in \mathfrak{h}^*$ be antidominant and regular. Then the functors

$$\mathcal{M}(\mathcal{D}_\lambda) \underset{\Delta_\lambda}{\overset{\Gamma(X,-)}{\rightleftarrows}} \mathcal{M}(\mathcal{U}_\theta)$$

are (mutually quasi-inverse) equivalences of categories.

This is a vast generalization of the Borel-Weil theorem.

A formula for $\mathfrak{n}$-homology

The geometric fibers of the localization of a $\mathcal{U}_\theta$-module V are

$$T_x(\Delta_\lambda(V)) = H_0(\mathfrak{n}_x, V)_{(\lambda+\rho)}$$

for any $x \in X$.

Let B be the Borel subgroup of $\mathrm{Int}(\mathfrak{g})$ corresponding to $\mathfrak{b}$. The Bruhat cells, i.e. B-orbits in X, are parametrized by the Weyl group W.

- $C(w)$ – the Bruhat cell attached to $w \in W$;
- $\ell : W \longrightarrow \mathbb{Z}$ – the *length function* on W; we have $\dim C(w) = \ell(w)$.
- $i_w : C(w) \longrightarrow X$ – the inclusion of $C(w)$ into X;
- $\pi_w : C(w) \longrightarrow \{pt\}$ – the projection of $C(w)$ into a point.

The above formula translates in

$$H_0(\mathfrak{n}, V)_{(\lambda+\rho)} = i_1^+(\Delta_\lambda(V)).$$

This generalizes to

Lemma

Let λ be antidominant and regular. Let $\theta = W \cdot \lambda$. For any $\mathcal{U}_\theta$-module V we have

$$H_p(\mathfrak{n}, V)_{(w\lambda+\rho)} = H^{-p}(\pi_{w,+}(Li_w^+(D(\Delta_\lambda(V)))))$$

for all $p \in \mathbb{Z}$.

Here,

- $D : \mathcal{M}(\mathcal{D}_\lambda) \longrightarrow D^b(\mathcal{D}_\lambda)$ – the natural functor attaching to a module $\mathcal{U}$ the complex which is $\mathcal{U}$ in degree 0 and 0 elsewhere;
- $Li_w^+ : D^b(\mathcal{D}_\lambda) \longrightarrow D^b(\mathcal{D}_{C(w)})$ – derived D-module inverse image corresponding to i_w;
- $\pi_{w,+} : D^b(\mathcal{D}_{C(w)}) \longrightarrow D^b(\mathbb{C})$ – the D-module direct image of π_w.

An example

Let F be the irreducible finite-dimensional representation of $\mathfrak{g}$ with lowest weight λ. Then its infinitesimal character is $\chi_{\lambda-\rho}$ and

$$\Delta_{\lambda-\rho}(F) = \mathcal{O}(\lambda).$$

We have

$$Li_w^+(D(\mathcal{O}(\lambda))) = D(\mathcal{O}_{C(w)})$$

and

$$\pi_{w,+}(D(\mathcal{O}_{C(w)})) = D(\mathbb{C})[\ell(w)].$$

Hence, we have

$$H_p(\mathfrak{n}, F)_{(w(\lambda-\rho)+\rho)} = \begin{cases} \mathbb{C} & \text{if } p = \ell(w); \\ 0 & \text{if } p \neq \ell(w). \end{cases}$$

for any $w \in W$. This is the theorem of Kostant.

The $\mathfrak{n}$-homology of discrete series

Now we discuss the case of discrete series. Let

- K – complexification of K_0. Acts on X with finitely many orbits.

Let λ be antidominant and regular. Let

- Q – a closed orbit of K in X;
- τ – an irreducible K-equivariant connection on Q compatible with $\lambda + \rho$;
- $i_Q : Q \longrightarrow X$ – the inclusion of Q into X;
- $\mathcal{I}(Q, \tau) = i_{Q,+}(\tau)$ – the *standard Harish-Chandra sheaf* on Q – D-module direct image of τ.

Localizations of discrete series with infinitesimal character χ_λ are exactly standard Harish-Chandra sheaves $\mathcal{I}(Q, \tau)$.

Let $x \in Q$. Then $\mathfrak{b}_x \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$.
The map $x \longmapsto \mathfrak{b}_x \cap \mathfrak{k}$ is an isomorphism of the orbit Q and the flag variety X_K of $\mathfrak{k}$.
Let B_K be the subgroup of B corresponding to the Borel subalgebra $\mathfrak{b} \cap \mathfrak{k}$.
A root $\alpha \in R$ is called *compact*, if the root subspace corresponding to it is in $\mathfrak{k}$. Compact roots form a root subsystem of the root system R.

- W_K – the Weyl group of K, i.e., the subgroup of W generated by compact reflections;
- $\ell_K : W_K \longrightarrow \mathbb{Z}$ – the length function on W_K;
- $C_K(w)$ – Bruhat cell, i.e., B_K-orbit in X_K corresponding to $w \in W_K$;
- $D_Q(w)$ – the corresponding B_K-orbit in Q under above K-equivariant isomorphism.

Clearly, for any $w \in W$, $Q \cap C(w)$ is B_K-invariant. Hence it is either empty or a union of $D_Q(v)$.

Lemma (Geometric Lemma)

There exists $v \in W$ such that $Q \cap C(v) = D_Q(1)$.
Moreover, we have

$$D_Q(w) = Q \cap C(wv)$$

for all $w \in W_K$.

This allows the calculation of $\mathfrak{n}$-homology for the discrete series representation $V = \Gamma(X, \mathcal{I}(Q, \tau))$.

Theorem (Schmid)

We have

$$H_p(\mathfrak{n}, V)_{(u\lambda+\rho)} = 0$$

if $u \notin W_K v$.
For $w \in W_K$, *we have*

$$H_p(\mathfrak{n}, V)_{(wv\lambda+\rho)} = \begin{cases} 0 & \text{if } p \neq \dim X - \dim X_K - \ell(wv) + 2\ell_K(w); \\ \mathbb{C} & \text{if } p = \dim X - \dim X_K - \ell(wv) + 2\ell_K(w). \end{cases}$$

If G_0 is compact this result specializes to the result of Kostant. We see that the homology classes correspond to the elements of W_K. We want a geometric explanation of this.

Trauber resolution

Let

- N_K – the unipotent radical of B_K.

Then $D_Q(w)$ admits unique irreducible N_K-equivariant connection $\mathcal{O}_{D_Q(w)}$.

- $\mathcal{J}(w, \lambda)$ – the standard $\mathcal{D}_\lambda$-module attached to $D_Q(w)$;
- $W_K(q)$ – the subset of W_K consisting of elements of length $\ell_K(w) = q$.

The *Cousin resolution* of $\mathcal{I}(Q, \tau)$ is a complex $\mathcal{D}^\cdot$ such that

$$\mathcal{D}^p = \bigoplus_{w \in W_K(\dim Q - p)} \mathcal{J}(w, \lambda)$$

for any $0 \leq p \leq \dim X_K$, with explicitly given differentials.
In $D^b(\mathcal{D}_\lambda)$, the complex $D(\mathcal{I}(Q, \tau))$ is isomorphic to

$$\cdots \to 0 \to \mathcal{D}^0 \to \mathcal{D}^1 \to \cdots \to \mathcal{D}^{\dim X_K} \to 0 \to \ldots$$

Since the functor Γ is exact for antidominant λ, $D^\cdot = \Gamma(X, \mathcal{D}^\cdot)$ is isomorphic to $D(V)$ in $D^b(\mathcal{U}_\theta)$, i.e., we get a resolution of V by modules D^p, $0 \leq p \leq \dim X_K$.
This is the *Trauber resolution* of the discrete series V.
If G_0 is compact, this is just the dual of the BGG-resolution of an irreducible finite-dimensional representation.
Put

- $J(w, \lambda) = \Gamma(X, \mathcal{J}(w, \lambda))$ for any $w \in W_K$.

Then we have

Lemma

Let λ be an antidominant and regular. Then

$$H_p(\mathfrak{n}, J(w, \lambda)) = \begin{cases} \mathbb{C}_{wv\lambda+\rho} & \textit{if } p = \dim X - \ell(wv) + \ell_K(w); \\ 0 & \textit{if } p \neq \dim X - \ell(wv) + \ell_K(w). \end{cases}$$

Therefore, the $\mathfrak{n}$-homology of V is given by the hypercohomology of the $\mathfrak{n}$-homology functor for the complex $D^\cdot$. Since λ is regular, all weights $wv\lambda + \rho$, for $w \in W_K$, are different. Hence, all differentials in E^1 term of the hypercohomology spectral sequence for $D^\cdot$ vanish — the spectral sequence collapses. This immediately implies Schmid's theorem.
Moreover, this calculation shows that each module $J(w, \lambda)$ in the Trauber resolution contributes exactly one cohomology class in $\mathfrak{n}$-homology of V.

Limits of discrete series

If λ is singular but still antidominant, the Trauber resolution $D^{\cdot}$ is still a resolution of the limits of discrete series V. Since the $\mathfrak{n}$-homology of each summand $J(w, \lambda)$ in D^q for regular λ is concentrated in one degree and one-dimensional, the tensoring spectral sequence collapses. This implies that $\mathfrak{n}$-homology is concentrated in one degree and one-dimensional even in singular case.

If λ is W_K-regular, the $\mathfrak{n}$-homologies of D^q have different weights for different q. Hence the above argument still works. This implies that Schmid's result holds in this case too. This is the result of Williams.

If λ is not W_K-regular, the differentials in E^1 term can be nontrivial. Still, the complex has additional structure which could lead to a precise result in general.

The example of SU(2, 1)

- $G_0 = \mathrm{SU}(2,1)$;
- $K = \mathrm{GL}(2,\mathbb{C})$.

Three closed orbits – three families of discrete series.

- holomorphic;
- antiholomorphic;
- nonholomorphic.

For first two and third for $\lambda \neq 0$, we have only nondegenerate limits of discrete series.
The third one has a degenerate discrete series for $\lambda = 0$. This is also the corresponding spherical principal series.
Pick $\mathfrak{n}$ so that $\mathfrak{b}$ is in Q (other two choices correspond to holomorphic $\mathfrak{n}$).
In this case, the positive compact root γ is not simple.

$$Q = D_Q(1) \cup D_Q(s_\gamma).$$

Trauber resolution:

$$0 \to V \to J(s_\gamma, 0) \to J(1,0) \to 0.$$

We get

$$H_p(\mathfrak{n}, J(1,0)) = \begin{cases} 0 & \text{for } p \neq 3; \\ \mathbb{C}_\gamma & \text{for } p = 3, \end{cases}$$

and

$$H_p(\mathfrak{n}, J(s_\gamma,0)) = \begin{cases} 0 & \text{for } p \neq 1; \\ \mathbb{C}_\gamma & \text{for } p = 1. \end{cases}$$

Using long exact sequence of $\mathfrak{n}$-homology we get

$$H_0(\mathfrak{n}, V) = H_3(\mathfrak{n}, V) = 0 \text{ and } H_1(\mathfrak{n}, V) = H_2(\mathfrak{n}, V) = \mathbb{C}_\gamma.$$

This was found by Carayol by brutal computation.