Jacobs University Bremen

Thesis for Bachelor of Science in Mathematics

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Lie algebras of linear systems

and

their automorphisms

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Abstract

The objective of this thesis is to study the automorphism groups of the Lie algebras attached to linear systems. A linear system is a pair of vector spaces (U, W) with a nondegenerate pairing $\langle\cdot,\cdot\rangle\colon U\otimes W\to\mathbb{C}$, to which we attach three Lie algebras $\mathfrak{sl}_{U,W}\subset\mathfrak{gl}_{U,W}\subset\mathfrak{gl}^M_{U,W}$. If both U and W are countable dimensional, then, up to isomorphism, there is a unique linear system (V, V_*). In this case $\mathfrak{sl}_{V,V_*}$ and $\mathfrak{gl}_{V,V_*}$ are the well-known Lie algebras $\mathfrak{sl}_\infty$ and $\mathfrak{gl}_\infty$, while the Lie algebra $\mathfrak{gl}^M_{V,V_*}$ is the Mackey Lie algebra introduced in [PS13].

We review results about the monoidal categories $\mathbb{T}_{\mathfrak{sl}_{U,W}}$ and $\mathbb{T}_{\mathfrak{gl}^M_{U,W}}$ of tensor modules, both of which turn out to be equivalent as monoidal categories to the category $\mathbb{T}_{\mathfrak{sl}_\infty}$ introduced earlier in [DCPS11]. Using the relations between the categories $\mathbb{T}_{\mathfrak{sl}_\infty}$ and $\mathbb{T}_{\mathfrak{gl}^M_\infty}$, we compute the automorphism group of $\mathfrak{gl}^M_\infty$.

Contents

1 Preliminaries

The ground field is $\mathbb{C}$. $(\;\cdot\;)^*$ denotes the contravariant functor which maps a vector space M to $\mathrm{Hom}(M, \mathbb{C})$ and maps a linear map $f\colon M \to N$ to its dual $f^*\colon N^* \to M^*$. In this paper, countable dimensional means infinite countable dimensional. $T(X)$ denotes the tensor algebra of a vector space X. $\mathfrak{g}$ denotes an arbitrary Lie algebra, and $\mathfrak{g}$-mod denotes the category of $\mathfrak{g}$-modules, where morphisms are $\mathfrak{g}$-homomorphisms.

Throughout this paper, let V be a fixed vector space with a fixed countable basis $\mathcal{B} = \{e_i\}_{i\in\mathbb{N}}$ and let e^i be the linear functional dual to e_i, i.e. $e^i(e_j) = \delta_{ij} \quad \forall i, j$. Let $\mathcal{B}^* := \{e^i\}_{i\in\mathbb{N}}$ and $V_* := \mathrm{span}\ \mathcal{B}^*$. We also set $\mathcal{B}_n := \{e_i\}_{i=1}^n$ and $\mathcal{B}_n^* := \{e^i\}_{i=1}^n$. Then clearly $V = \varinjlim V_n$ and $V_* = \varinjlim V_n^*$, where $V_n := \mathrm{span}\ \mathcal{B}_n$ and $V_n^* := \mathrm{span}\ \mathcal{B}_n^*$.

Mat_n denotes the Lie algebra of $n \times n$ matrices. For $n \le m$, consider the embedding $\mathrm{Mat}_n \overset{\iota_{nm}}{\hookrightarrow} \mathrm{Mat}_m$ by upper left corner identity inclusion and filling zeros elsewhere. Then $(\{\mathrm{Mat}_n\}, \{\iota_{nm}\})$ forms a direct system of Lie algebras, and $\mathrm{Mat}_\mathbb{N} := \varinjlim \mathrm{Mat}_n$ is its direct limit. $\mathrm{Mat}_\mathbb{N}$ is nothing but the Lie algebra of matrices $(A_{ij})_{i,j\in\mathbb{N}}$ that have finitely many nonzero entries.

We define the Lie algebra $\mathfrak{gl}_n := V_n \otimes V_n^*$, with Lie bracket given by

$$[e_i \otimes e^j, e_k \otimes e^l] = \delta_{jk} e_i \otimes e^l - \delta_{il} e_k \otimes e^j \quad \forall e_i, e_k \in \mathcal{B}_n, e^j, e^l \in \mathcal{B}_n^*. \tag{1}$$

Observe that $\mathfrak{gl}_n$ and Mat_n are isomorphic as Lie algebras. An isomorphism is given by sending $e_i \otimes e^j$ to the matrix E_{ij} having 1 at (i, j) and 0 elsewhere. For any $n \le m$, the embedding ι_{nm} induces an imbedding $\mathfrak{gl}_n \overset{\iota'_{nm}}{\hookrightarrow} \mathfrak{gl}_m$ such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{gl}_n & \overset{\iota'_{nm}}{\hookrightarrow} & \mathfrak{gl}_m \\ \updownarrow & & \updownarrow \\ \mathrm{Mat}_n & \overset{\iota_{nm}}{\hookrightarrow} & \mathrm{Mat}_m, \end{array}$$

making $(\{\mathfrak{gl}_n\}, \{\iota'_{nm}\})$ a direct system with direct limit

$$\mathfrak{gl}_\infty := V \otimes V_* = \varinjlim (V_n \otimes V_n^*) = \varinjlim \mathfrak{gl}_n.$$

The Lie bracket on $\mathfrak{gl}_\infty$ is given by (1), where now $e_i, e_k \in \mathcal{B}, e^j, e^l \in \mathcal{B}^*$.

Clearly, since the two direct systems $(\{\mathrm{Mat}_n\}, \{\iota_{nm}\})$ and $(\{\mathfrak{gl}_n\}, \{\iota'_{nm}\})$ are isomorphic, their direct limits $\mathrm{Mat}_\mathbb{N}$ and $\mathfrak{gl}_\infty$ are isomorphic.

The trace homomorphism $\mathrm{tr}_n\colon \mathfrak{gl}_n \to \mathbb{C}$ can be extended to a homomorphism $\mathrm{tr}\colon \mathfrak{gl}_\infty \to \mathbb{C}$. Define $\mathfrak{sl}_\infty := \ker \mathrm{tr}$. Then $\mathfrak{sl}_\infty = \varinjlim \ker \mathrm{tr}_n$. Similarly, $\mathfrak{sl}_n \cong \mathrm{Mat}_n^0$ and $\mathfrak{sl}_\infty \cong \mathrm{Mat}_\mathbb{N}^0$, where Mat_n^0 denotes the Lie subalgebra of Mat_n consisting of traceless $n \times n$ matrices and $\mathrm{Mat}_\mathbb{N}^0$ denotes the Lie subalgebra of $\mathrm{Mat}_\mathbb{N}$ consisting of traceless finitary matrices.

2 The Lie algebras $\mathfrak{sl}_{U,W}$ and $\mathfrak{gl}_{U,W}$

In this section we generalize the construction of $\mathfrak{sl}_\infty$ and $\mathfrak{gl}_\infty$ to arbitrary linear systems (U, W). A pair of vector spaces of arbitrary dimensions (U, W) is called a *linear system* if they are equipped with a nondegenerate bilinear form $\langle\cdot,\cdot\rangle\colon U \times W \to \mathbb{C}$. We say that (U', W') is a *subsystem* if U', W' are subspaces of U, W respectively and $\langle\cdot,\cdot\rangle$ is nondegenerate when restricted to $U' \times W'$.

Proposition 2.1. Let (U, W) be a linear system and let $U_f \subset U$ be a finite-dimensional subspace of U. Let W_f be a direct complement to $U_f^\perp$ in W, i.e. $W = U_f^\perp \oplus W_f$. Then (U_f, W_f) is a subsystem and $\dim U_f = \dim W_f$.

Proof. Suppose $v \in V_f$ is such that $\langle v, W_f\rangle = 0$. Then $\langle v, W\rangle = 0$, and we must have $v = 0$. Suppose $w \in W_f$ is such that $\langle V_f, w\rangle = 0$. Then $w \in V_f^\perp \cap W_f$, thus $w = 0$. We conclude that $\langle\cdot,\cdot\rangle$ is nondegenerate on $V_f \times W_f$. Therefore the nondegenerate form $\langle\cdot,\cdot\rangle$ induces an injection $W_f \hookrightarrow (V_f)^*$. This forces W_f to be finite-dimensional. Similarly, there is an injection $V_f \hookrightarrow (W_f)^*$. Therefore $\dim V_f = \dim W_f$. □

Given any linear system (U, W), we define two Lie algebras $\mathfrak{gl}_{U,W}$ and $\mathfrak{sl}_{U,W}$. Let $\mathfrak{gl}_{U,W}$ equal the vector space $U \otimes W$, with Lie bracket

$$[u_1 \otimes w_1, u_2 \otimes w_2] = \langle u_2, w_1\rangle u_1 \otimes w_2 - \langle u_1, w_2\rangle u_2 \otimes w_1 \quad \forall u_1, u_2 \in U, w_1, w_2 \in W.$$

We define $\mathfrak{sl}_{U,W}$ to be the kernel of $\langle\cdot,\cdot\rangle$. Note that the pair (V, V_*) is a linear system, and $\mathfrak{sl}_{V,V_*} = \mathfrak{sl}_\infty$ and $\mathfrak{gl}_{V,V_*} = \mathfrak{gl}_\infty$. The next four propositions generalize some simple observations concerning $\mathfrak{sl}_\infty$ and $\mathfrak{gl}_\infty$ to arbitrary $\mathfrak{sl}_{U,W}$ and $\mathfrak{gl}_{U,W}$.

Proposition 2.2. There are isomorphisms of Lie algebras $\mathfrak{sl}_{U,W} \cong \mathfrak{sl}_{W,U}$ and $\mathfrak{gl}_{U,W} \cong \mathfrak{gl}_{W,U}$.

Proof. Consider the linear operators $f\colon \mathfrak{gl}_{U,W} \to \mathfrak{gl}_{W,U}$ and $g\colon \mathfrak{gl}_{W,U} \to \mathfrak{gl}_{U,W}$ such that

$$f(u \otimes w) = -w \otimes u \quad \forall u \otimes w \in \mathfrak{gl}_{U,W},$$

$$g(w \otimes u) = -u \otimes w \quad \forall u \otimes w \in \mathfrak{gl}_{W,U}.$$

They are both Lie algebra homomorphisms and are mutually inverse. Also f restricts to an isomorphism of Lie algebras $\mathfrak{sl}_{U,W} \cong \mathfrak{sl}_{W,U}$. □

Proposition 2.3. The Lie algebra $\mathfrak{sl}_{U,W}$ is simple.

Proof. The set of finite-dimensional subsystems (U_f, W_f) of (U, W) is partially ordered by inclusion, and any two such subsystems have an upper bound. Thus we obtain a direct system of Lie algebras $\{\mathfrak{sl}_{U_f,W_f}\}$ with direct limit $\mathfrak{sl}_{U,W} = \varinjlim \mathfrak{sl}_{U_f,W_f}$. Any nontrivial ideal I of $\mathfrak{sl}_{U,W}$ intersects nontrivially with some $\mathfrak{sl}_{U_f,W_f}$. We conclude that $I \supset \mathfrak{sl}_{U_f,W_f}$ by the simplicity of $\mathfrak{sl}_{U_f,W_f}$. Similarly, $I \supset \mathfrak{sl}_{U'_f,W'_f}$ for any finite-dimensional subsystem (U'_f, W'_f) containing (U_f, W_f) by the simplicity of $\mathfrak{sl}_{U'_f,W'_f}$. This means that $I = \mathfrak{sl}_{U,W}$. □

Proposition 2.4. The Lie algebra $\mathfrak{sl}_{U,W}$ is the commutator subalgebra of $\mathfrak{gl}_{U,W}$.

Proof. It is clear from the definition of the Lie bracket that $[\mathfrak{gl}_{U,W}, \mathfrak{gl}_{U,W}] \subset \mathfrak{sl}_{U,W}$. Conversely, observe that $[\mathfrak{sl}_{U,W}, \mathfrak{sl}_{U,W}]$ is a nontrivial ideal of $\mathfrak{sl}_{U,W}$, therefore equal to $\mathfrak{sl}_{U,W}$ by the simplicity of $\mathfrak{sl}_{U,W}$. To conclude,

$$\mathfrak{sl}_{U,W} = [\mathfrak{sl}_{U,W}, \mathfrak{sl}_{U,W}] = [\mathfrak{gl}_{U,W}, \mathfrak{gl}_{U,W}]. \qquad \square$$

Proposition 2.5.

1. Let A be any finite-dimensional Lie subalgebra of $\mathfrak{gl}_{U,W}$, then $A \subset \mathfrak{gl}_{U_f,W_f}$ for some finite-dimensional subsystem (U_f, W_f).
2. Let A be any finite-dimensional Lie subalgebra of $\mathfrak{sl}_{U,W}$, then $A \subset \mathfrak{sl}_{U_f,W_f}$ for some finite-dimensional subsystem (U_f, W_f).

Proof.

1. Let $\{u_\alpha\}$ be a basis of U and $\{w_\beta\}$ be a basis of W. Then $\{u_\alpha \otimes w_\beta\}$ is a basis of $U \otimes W$. Fix a basis $\{a_1, \cdots, a_n\}$ of A, and let

$$a_i = \sum_{k=1}^{n_i} c_{i,k} \cdot u_{\alpha_{i,k}} \otimes w_{\beta_{i,k}}.$$

   Let (U_f, W_f) be a finite-dimensional subsystem such that $U_f \supset \mathrm{Span}\{u_{\alpha_{i,k}}\}$ and $W_f \supset \mathrm{Span}\{w_{\beta_{i,k}}\}$. Then $A \subset \mathrm{Span}\{u_{\alpha_{i,k}}\} \otimes \mathrm{Span}\{w_{\beta_{i,k}}\} \subset \mathfrak{gl}_{U_f,W_f}$.
2. Given a finite-dimensional subalgebra $A \subset \mathfrak{sl}_{U,W} \subset \mathfrak{gl}_{U,W}$, we have $A \subset \mathfrak{gl}_{U_f,W_f}$ for some finite-dimensional subsystem (U_f, W_f) by statement 1. Then $A \subset \mathfrak{gl}_{U_f,W_f} \cap \ker\langle\cdot,\cdot\rangle = \mathfrak{sl}_{U_f,W_f}$. $\square$

Having defined $\mathfrak{sl}_{U,W}$ and $\mathfrak{gl}_{U,W}$ for a linear system (U, W), it is natural to ask when are $\mathfrak{gl}_{U_1,W_1}$ and $\mathfrak{gl}_{U_2,W_2}$ isomorphic as Lie algebras. A necessary and sufficient condition is given in [PS13].

Definition. Two linear systems (U_1, W_1) and (U_2, W_2) are isomorphic iff one of the following holds:

1. There are vector space isomorphisms $f\colon U_1 \to U_2$ and $g\colon W_1 \to W_2$ such that $\langle f(u), g(w)\rangle = \langle u, w\rangle$ for all $u \in U_1, w \in W_1$.
2. There are vector space isomorphisms $f\colon U_1 \to W_2$ and $g\colon W_1 \to U_2$ such that $\langle g(w), f(u)\rangle = \langle u, w\rangle$ for all $u \in U_1, w \in W_1$.

We write $(U_1, W_1) \cong (U_2, W_2)$.

Theorem 2.6. [PS13, Prop 1.1] The Lie algebras $\mathfrak{sl}_{(U_1,W_1)}$ and $\mathfrak{sl}_{(U_2,W_2)}$ are isomorphic iff the linear systems (U_1, W_1) and (U_2, W_2) are isomorphic.

Corollary 2.7. The Lie algebras $\mathfrak{gl}_{(U_1,W_1)}$ and $\mathfrak{gl}_{(U_2,W_2)}$ are isomorphic iff the linear systems (U_1, W_1) and (U_2, W_2) are isomorphic.

Proof. $\Longleftarrow$ If we have a linear system isomorphism $f\colon U_1 \to U_2$ and $g\colon W_1 \to W_2$, then

$$f \otimes g\colon u \otimes w \mapsto f(u) \otimes g(w) \quad \forall u \otimes w \in \mathfrak{gl}_{U_1,W_1}$$

induces an isomorphism of Lie algebras. If we have isomorphisms $f\colon U_1 \to W_2$ and $g\colon W_1 \to U_2$, then

$$-g \otimes f\colon u \otimes w \mapsto -g(w) \otimes f(u) \quad \forall u \otimes w \in \mathfrak{gl}_{U_1,W_1}$$

induces an isomorphism of Lie algebras.

$\Longrightarrow$ Conversely, if $\mathfrak{gl}_{U_1,W_1} \cong \mathfrak{gl}_{U_2,W_2}$ then the commutator subalgebras $\mathfrak{sl}_{U_1,W_1}$ and $\mathfrak{sl}_{U_2,W_2}$ are isomorphic. By Proposition 2.6 we conclude that $(U_1, W_1) \cong (U_2, W_2)$. □

We call a linear system (U, W) *countable dimensional* if both U and W are countable dimensional. In such cases we can say more about $\mathfrak{gl}_{U,W}$ and $\mathfrak{sl}_{U,W}$ because of the following observation in [Mac43]. The next theorem and corollary tell us that, up to isomorphism, there is only one countable dimensional linear system.

Theorem 2.8. [Mac43] For any countable dimensional linear system (U, W), there exist dual bases $\{\tilde{u}_i\}_{i\in\mathbb{N}}$ of U and $\{\tilde{w}_i\}_{i\in\mathbb{N}}$ of W, where $\langle \tilde{u}_i, \tilde{w}_j\rangle = \delta_{ij}\ \forall i, j$.

Proof. Start with any basis $\{u_i\}_{i\in\mathbb{N}}$ of U and $\{w_i\}_{i\in\mathbb{N}}$ of W and perform the Gram-Schmidt algorithm.

Step 1: If $\langle u_1, w_1\rangle \neq 0$ then go to step 2. If not, then by non-degeneracy of the form $\langle \cdot, \cdot\rangle$ there is a w_j such that $\langle u_1, w_j\rangle \neq 0$. Replace w_1 by $w_1 + w_j$.

Step 2: Set $\tilde{u}_1 = u_1$. Adjust w_1 by a scalar to obtain $\tilde{w}_1$ such that $\langle \tilde{u}_1, \tilde{w}_1\rangle = 1$. Replace u_2 by $\tilde{u}_2 = u_2 - \langle u_2, \tilde{w}_1\rangle \tilde{u}_1$, whence $\langle \tilde{u}_2, \tilde{w}_1\rangle = 0$. Now adjust w_2 to make sure $\langle \tilde{u}_2, w_2\rangle = 1$ and set $\tilde{w}_2 = w_2 - \langle u_1, w_2\rangle \tilde{w}_1$. We have $\langle \tilde{u}_1, \tilde{w}_2\rangle = 0$ and $\langle \tilde{u}_2, \tilde{w}_2\rangle = \langle \tilde{u}_2, w_2\rangle - \langle u_1, w_2\rangle\langle \tilde{u}_2, \tilde{w}_1\rangle = 1$.

Step 3: Having obtained $\tilde{u}_1, \cdots, \tilde{u}_n$ and $\tilde{w}_1, \cdots, \tilde{w}_n$, we set

$$\tilde{u}_{n+1} = u_{n+1} - \sum_{i=1}^{n} \langle u_{n+1}, \tilde{w}_i\rangle \tilde{u}_i,$$

and apply argument in step 1 to adjust w_{n+1} such that $\langle \tilde{u}_{n+1}, w_{n+1}\rangle = 1$. We also set

$$\tilde{w}_{n+1} = w_{n+1} - \sum_{i=1}^{n} \langle \tilde{u}_i, w_{n+1}\rangle \tilde{w}_i.$$

By induction this will give us the dual bases. □

Corollary 2.9. If (U, W) is countable dimensional, then $(U, W) \cong (V, V_*)$. As a consequence $\mathfrak{sl}_{U,W} \cong \mathfrak{sl}_\infty$ and $\mathfrak{gl}_{U,W} \cong \mathfrak{gl}_\infty$.

Proof. Given a pair of dual bases $\{u_n\}_{n\in\mathbb{N}}$ of U and $\{w_n\}_{n\in\mathbb{N}}$ of W, the isomorphisms $f : U \to V$ and $g : W \to V_*$ defined by $f(u_n) = e_n$ and $g(w_n) = e^n$ preserve the form $\langle \cdot, \cdot\rangle$. Therefore the linear systems (U, W) and (V, V_*) are isomorphic. By Theorem 2.6 and Corollary 2.7, we conclude that there are isomorphisms of Lie algebras $\mathfrak{sl}_{U,W} \cong \mathfrak{sl}_\infty$ and $\mathfrak{gl}_{U,W} \cong \mathfrak{gl}_\infty$. □

Theorem 2.8 clearly holds for finite-dimensional linear systems as well, but fails for uncountable dimensional linear systems. Consider $(X := W \oplus W^*, X' := W \oplus W^*)$ where W is a countable dimensional vector space and the form $X \times X' \to \mathbb{C}$ is given by

$$(w + w^*, v + v^*) \mapsto w^*(v) + v^*(w) \quad \forall v, w \in W, v^*, w^* \in W^*.$$

If there were a pair of dual bases, then $W^\perp \subset X'$ would be uncountable dimensional. However, we observe that $W^\perp = W \subset X'$.

3 The Mackey Lie algebras $\mathfrak{gl}^M_{U,W}$

In this section we introduce a canonical Lie algebra $\mathfrak{gl}^M_{U,W}$ corresponding to the linear system (U, W). We discuss the relations between $\mathfrak{sl}_{U,W}, \mathfrak{gl}_{U,W}$ and $\mathfrak{gl}^M_{U,W}$, and realize U and W as non-isomorphic modules of $\mathfrak{gl}^M_{U,W}$.

Definition. We embed $W \hookrightarrow U^*$ and $U \hookrightarrow W^*$ using the pairing $\langle \cdot, \cdot \rangle$. Set

$$\mathfrak{gl}^M_{U,W} := \{\varphi \in \operatorname{End}(U) |\, \varphi^*(W) \subset W\}, \quad \mathfrak{gl}^M_{W,U} := \{\psi \in \operatorname{End}(W) |\, \psi^*(U) \subset U\}.$$

Then $\mathfrak{gl}^M_{U,W}$ and $\mathfrak{gl}^M_{W,U}$ are Lie subalgebras of the Lie algebras $\operatorname{End}(U)$ and $\operatorname{End}(W)$, respectively.

It's easy to see that there is an isomorphism $\mathfrak{gl}^M_{U,W} \cong \mathfrak{gl}^M_{W,U}$ of Lie algebras. The map that sends $\varphi \in \mathfrak{gl}^M_{U,W}$ to $-\varphi^*_{|W}$, is a Lie algebra homomorphism whose inverse homomorphism is given by sending $\psi \in \mathfrak{gl}^M_{W,U}$ to $-\psi^*_{|U}$.

Proposition 3.1 (Penkov, Serganova)**.** The Lie algebra $\mathfrak{gl}_{U,W}$ is isomorphic to the ideal S of $\mathfrak{gl}^M_{U,W}$, where

$$S := \{\varphi \in \mathfrak{gl}^M_{U,W} \colon \dim \varphi(U) < \infty, \dim \varphi^*(W) < \infty\}.$$

Proof. First of all, it is clear that $[h, \varphi] = h\varphi - \varphi h$ has finite-dimensional image in both U and W if one of h, φ does. Thus, S is an ideal. Consider the injection of Lie algebras $\iota \colon \mathfrak{gl}_{U,W} \to \mathfrak{gl}^M_{U,W}$ induced by $u \otimes w \mapsto \langle \cdot, w \rangle u$. The range of ι is indeed in $\mathfrak{gl}^M_{U,W}$ because

$$(\iota(u \otimes w))^*(\langle \cdot, y \rangle) = \langle \langle \cdot, w \rangle u, y \rangle = \langle \cdot, \langle u, y \rangle w \rangle \in W \quad \forall \langle \cdot, y \rangle \in W.$$

Moreover, since elements of $U \otimes W$ are finite linear combinations of pure tensors, under ι they have finite-dimensional images in both U and W, which means that $\operatorname{im} \iota \subset S$.

We now show that $\operatorname{im} \iota \supset S$. Let (U_f, W_f) be a finite-dimensional subsystem of (U, W). Consider the Lie subalgebra

$$S_{U_f, W_f} := \{\varphi \in \operatorname{End}(U) \colon \varphi(U) \subset U_f \text{ and } \varphi^*(W) \subset W_f\} \subset \mathfrak{gl}^M_{U,W}.$$

Observe that $\langle \varphi(v), W \rangle = \langle v, \varphi^*(W) \rangle = 0$ for all $v \in W_f^\perp$. Therefore $U \supset \ker \varphi \supset W_f^\perp$ for any $\varphi \in S_{U_f, W_f}$. In addition, since $\varphi(U) \subset U_f$, the subspace S_{U_f, W_f} can at most have dimension $\dim U_f \cdot \dim W_f$. Since $\mathfrak{gl}_{U_f, W_f}$ injects into S_{U_f, W_f} under ι and does have dimension $\dim U_f \cdot \dim W_f$, we conclude that ι restricts to an isomorphism from $\mathfrak{gl}_{U_f, W_f}$ to S_{U_f, W_f}. Since any $\varphi \in S$ lies in some S_{U_f, W_f}, we have $\operatorname{im} \iota = S$. □

Corollary 3.2. If (U, W) is a finite-dimensional linear system, then $\mathfrak{gl}_{U,W} \cong \mathfrak{gl}^M_{U,W}$.

We identify $\mathfrak{gl}_{U_f,W_f}$ with S_{U_f,W_f}, and $\mathfrak{gl}_{U,W}$ with S inside $\mathfrak{gl}^M_{U,W}$ whenever appropriate.

Proposition 3.3. [PS13, Lemma 6.1] The Lie algebra $\mathfrak{gl}^M_{U,W}$ has a unique simple ideal $\mathfrak{sl}_{U,W}$.

Proposition 3.4. The Lie algebras $\mathfrak{gl}^M_{U_1,W_1}$ and $\mathfrak{gl}^M_{U_2,W_2}$ are isomorphic iff the linear systems (U_1, W_1) and (U_2, W_2) are isomorphic.

Proof. $\Longleftarrow$ This is obvious because the definition of $\mathfrak{gl}^M_{U,W}$ is intrinsic to the linear system (U, W). In the following we construct the isomorphism explicitly. If we have a linear system isomorphism $f : U_1 \to U_2$ and $g : W_1 \to W_2$, then we first show that $f^*_{|W_2} = g^{-1}$ and $(f^{-1})^*_{|W_1} = g$. Indeed

$$f^*(\langle \cdot, w_2 \rangle) = \langle f(\cdot), w_2 \rangle = \langle f^{-1}f(\cdot), g^{-1}(w_2) \rangle = \langle \cdot, g^{-1}(w_2) \rangle \quad \forall w_2 \in W_2,$$

$$(f^{-1})^*(\langle \cdot, w_1 \rangle) = \langle f^{-1}(\cdot), w_1 \rangle = \langle ff^{-1}(\cdot), g(w_1) \rangle = \langle \cdot, g(w_1) \rangle \quad \forall w_1 \in W_1.$$

Consider the map that sends $\varphi \in \mathfrak{gl}^M(U_1, W_1)$ to $f\varphi f^{-1}$. By the above

$$(f\varphi f^{-1})^*_{|W_2} = ((f^{-1})^*\varphi^* f^*)_{|W_2} = g\varphi^* g^{-1},$$

where the right hand side keeps W_2 stable. This map is a homomorphism of Lie algebras between $\mathfrak{gl}^M_{U_1,W_1}$ and $\mathfrak{gl}^M_{U_2,W_2}$, which has an inverse that sends $\psi \in \mathfrak{gl}^M_{U_2,W_2}$ to $f^{-1}\psi f$.

If we have a linear system isomorphism $f : U_1 \to W_2$ and $g : W_1 \to U_2$, then the map that sends φ to $f\varphi f^{-1}$ is an isomorphism of Lie algebras $\mathfrak{gl}^M_{U_1,W_1} \cong \mathfrak{gl}^M_{W_2,U_2}$ by the similar argument as above. But the latter is isomorphic to $\mathfrak{gl}^M_{U_2,W_2}$.

$\Longrightarrow$ Conversely if $\mathfrak{gl}^M_{U_1,W_1} \cong \mathfrak{gl}^M_{U_2,W_2}$, then $\mathfrak{sl}_{U_1,W_1} \cong \mathfrak{sl}_{U_2,W_2}$ because they are both the unique simple ideals of the respective Lie algebras. By Theorem 2.6, the linear systems (U_1, W_1) and (U_2, W_2) are isomorphic. $\square$

Corollary 3.5. If (U, W) is countable, then $\mathfrak{gl}^M_{U,W} \cong \mathfrak{gl}^M_{V,V_*}$.

In addition, we observe the following about $\mathfrak{gl}^M_{U,W}$.

Proposition 3.6.

1. $\mathfrak{gl}^M_{U,W} \subset \mathfrak{gl}^M_{U,U^*} = \mathrm{End}(U)$.
2. $\mathfrak{gl}^M_{U,W}$ has one dimensional center $\mathbb{C}\mathrm{Id}$.
3. $\mathfrak{gl}_{U,W} \oplus \mathbb{C}\mathrm{Id}$ is an ideal of $\mathfrak{gl}^M_{U,W}$.

Proof.

1. Obvious.
2. Assume to the contrary that φ is in the center of $\mathfrak{gl}^M_{U,W}$ and $\varphi \notin \mathbb{C}\mathrm{Id}$. Then there exists $u \in U$ such that $\varphi(u) = u'$ and $u \neq \lambda u'$ for any $\lambda \in \mathbb{C}$, in particular $u \neq 0$. Extend the linearly independent set $\{u, u'\}$ to a basis of U, and define a linear operator ψ such that $\psi(u') = u$ and $\psi(v) = 0$ for any v in the basis that does not equal u'. Clearly $\psi \in \mathfrak{gl}_{U,W} \subset \mathfrak{gl}^M_{U,W}$, but

$$\varphi \circ \psi(u) = 0 \neq u = \psi \circ \varphi(u).$$

   Contradiction.

3. $\mathbb{C}\mathrm{Id} \cap \mathfrak{gl}_{U,W} = 0$. The sum of two ideals is an ideal. $\square$

Define $\mathfrak{gl}_\infty^M := \mathfrak{gl}_{V,V_*}^M$. Then $\mathfrak{gl}_\infty^M$ is isomorphic to the Lie algebra $\mathfrak{gl}_{U,W}^M$ for any countable dimensional linear system (U, W). In fact, $\mathfrak{gl}_\infty^M$ can be conveniently thought of as a certain Lie algebra of matrices.

Define $\mathrm{Mat}_{\mathbb{N}}^M$ to be the vector space consisting of matrices $(A_{ij})_{i,j\in\mathbb{N}}$ such that each row and each column has finitely many nonzero entries. Then $\mathrm{Mat}_{\mathbb{N}}^M$ is a Lie algebra and there is an isomorphism of Lie algebras $\mathfrak{gl}_\infty^M \cong \mathrm{Mat}_{\mathbb{N}}^M$.

We obtain the following commutative diagrams

$$\begin{array}{ccccc}
\mathfrak{sl}_\infty & \hookrightarrow & \mathfrak{gl}_\infty & \hookrightarrow & \mathfrak{gl}_\infty^M \\
\updownarrow & & \updownarrow & & \updownarrow \\
\mathrm{Mat}_{\mathbb{N}}^0 & \hookrightarrow & \mathrm{Mat}_{\mathbb{N}} & \hookrightarrow & \mathrm{Mat}_{\mathbb{N}}^M .
\end{array}$$

We know that V and V^* have dual $\mathfrak{gl}_\infty$-module structures and V_* is a submodule of V^*. Generalizing this, we give U and U^* dual $\mathfrak{gl}_{U,W}^M$-module structures, and let W be a submodule of U^*. By restriction, U and W become $\mathfrak{gl}_{U,W}$ and $\mathfrak{sl}_{U,W}$ modules as well. The actions are explicitly given by

$$\begin{aligned}
&\varphi \cdot u = \varphi(u), \quad \varphi \cdot w = -\varphi^*(w) \quad \text{for } \varphi \in \mathfrak{gl}_{U,W}^M, u \in U, w \in W, \\
&u \otimes w \cdot x = \langle x, w\rangle u, \quad u \otimes w \cdot y = -\langle u, y\rangle w \quad \text{for } u \otimes w \in \mathfrak{gl}_{U,W}, x \in U, y \in W.
\end{aligned}$$

Note that for any nonnegative integers p, q, the tensor product $U^{\otimes p} \otimes W^{\otimes q}$ also becomes a $\mathfrak{gl}_{U,W}^M$-module. Moreover, U and W are not isomorphic as $\mathfrak{gl}_{U,W}^M, \mathfrak{gl}_{U,W}$ or $\mathfrak{sl}_{U,W}$-modules, simply because $U \not\cong W$ as $\mathfrak{sl}_{U_f,W_f}$-modules for finite-dimensional subsystems (U_f, W_f) with $\dim U_f = \dim W_f \geq 3$. Indeed, we know the following decomposition of $\mathfrak{sl}_{U_f,W_f}$-modules

$$U \cong U_f \oplus \left(\bigoplus_{n\in\mathbb{N}} \mathbb{C}\right) \quad \text{and} \quad W \cong W_f \oplus \left(\bigoplus_{n\in\mathbb{N}} \mathbb{C}\right),$$

but $W_f \cong U_f^* \not\cong U_f$ as $\mathfrak{sl}_{U_f,W_f}$-modules if $\mathfrak{sl}_{U_f,W_f} \not\cong \mathfrak{sl}(2)$.

4 The categories $\mathbb{T}_{\mathfrak{sl}_{U,W}}$, $\mathbb{T}_{\mathfrak{gl}_{U,W}^M}$ and the functor $\mathcal{F}^h$

In this section we present results about the categories $\mathbb{T}_{\mathfrak{sl}_{U,W}}$ and $\mathbb{T}_{\mathfrak{gl}_{U,W}^M}$ introduced in [PS13]. In particular, an object in $\mathbb{T}_{\mathfrak{sl}_{U,W}}$ or $\mathbb{T}_{\mathfrak{gl}_{U,W}^M}$ is isomorphic to a finite length submodule of a direct sum of finitely many copies of $T(U \oplus W)$. Moreover, the categories of tensor modules $\mathbb{T}_{\mathfrak{sl}_{U,W}}$ and $\mathbb{T}_{\mathfrak{gl}_{U,W}^M}$ are both equivalent to $\mathbb{T}_{\mathfrak{sl}_\infty}$ as monoidal categories.

Definition. Let M be an $\mathfrak{sl}_{U,W}$-module, we say M is $\mathfrak{sl}_{U,W}$-integrable iff for every $\varphi \in \mathfrak{sl}_{U,W}, m \in M$, we have $\dim\{m, \varphi \cdot m, \varphi^2 \cdot m, \cdots\} < \infty$.

A subalgebra $\mathfrak{k}$ of $\mathfrak{sl}_{U,W}$ has *finite co-rank* iff it contains $\mathfrak{sl}_{W_f^\perp,U_f^\perp}$ for some finite-dimensional subsystem (U_f, W_f). We say that a $\mathfrak{sl}_{U,W}$-module M satisfies the *large annihilator condition* if for every $m \in M$, its annihilator $\mathrm{Ann}(m) := \{g \in \mathfrak{sl}(U,W) : g \cdot m = 0\}$ is a finite co-rank subalgebra of $\mathfrak{sl}_{U,W}$.

Define $\mathbb{T}_{\mathfrak{sl}_{U,W}}$ to be the full subcategory of $\mathfrak{sl}_{U,W}$-mod where the objects are finite length $\mathfrak{sl}_{U,W}$-integrable modules that satisfy the large annihilator condition.

Despite the abstract definition, the category $\mathbb{T}_{\mathfrak{sl}_{U,W}}$ is nothing but the category of tensor modules.

Theorem 4.1. [PS13, Corollay 5.12] The following are equivalent:

1. M is an object of $\mathbb{T}_{\mathfrak{sl}_{U,W}}$.
2. M is isomorphic to a finite length submodule of a direct sum of finitely many copies of $T(U \oplus W)$;
3. M is isomorphic to a finite length subquotient of a direct sum of finitely many copies of $T(U \oplus W)$.

Theorem 4.2. [PS13, Theorem 5.5] For any linear system (U, W), there is an equivalence $\mathbb{T}_{\mathfrak{sl}_{U,W}} \leftrightsquigarrow \mathbb{T}_{\mathfrak{sl}_\infty}$ of monoidal tensor categories.

The category $\mathbb{T}_{\mathfrak{gl}^M_{U,W}}$ is defined analogously, but with slight differences. A subalgebra $\mathfrak{k}$ of $\mathfrak{gl}^M_{U,W}$ is called *finite co-rank* iff $\mathfrak{t} \supset \mathfrak{gl}^M_{W_f^\perp,U_f^\perp}$ for some finite-dimensional subsystem (U_f, W_f). A $\mathfrak{gl}^M_{U,W}$-module M is said to satisfy the *large annihilator condition* if $\mathrm{Ann}(m)$ is a finite co-rank subalgebra of $\mathfrak{gl}^M_{U,W}$ for any $m \in M$.

We define $\mathbb{T}_{\mathfrak{gl}^M_{U,W}}$ to be the full subcategory of $\mathfrak{gl}^M_{U,W}$-mod whose objects are finite length $\mathfrak{sl}_{U,W}$-integrable modules that satisfy the large annihilator condition.

Similar to the $\mathbb{T}_{\mathfrak{sl}_{U,W}}$ case, the category $\mathbb{T}_{\mathfrak{gl}^M_{U,W}}$ is nothing but the category of tensor modules.

Theorem 4.3. [PS13, Theorem 7.9 a)] The following are equivalent:

1. M is an object of $\mathbb{T}_{\mathfrak{gl}^M_{U,W}}$;
2. M is isomorphic to a finite length submodule of a direct sum of finitely many copies of $T(U \oplus W)$;
3. M is isomorphic to a finite length subquotient of a direct sum of finitely many copies of $T(U \oplus W)$.

Theorem 4.4. [PS13, Theorem 7.9 b)] For any object M in the category $\mathbb{T}_{\mathfrak{gl}^M_{U,W}}$, let $\mathrm{Res}(M)$ denote M regarded as a module of $\mathfrak{sl}_{U,W}$. Then Res: $\mathbb{T}_{\mathfrak{gl}^M_{U,W}} \rightsquigarrow \mathbb{T}_{\mathfrak{sl}_{U,W}}$ is a well defined functor that is fully faithful and essentially surjective. Moreover, $\mathrm{Res}(M \otimes N) = \mathrm{Res}(M) \otimes \mathrm{Res}(M)$. Thus $\mathbb{T}_{\mathfrak{gl}^M_{U,W}}$ and $\mathbb{T}_{\mathfrak{sl}_{U,W}}$ are equivalent as monoidal tensor categories.

We introduce the notion of *modules twisted by an automorphism*, or *twisted modules.* Let M be a $\mathfrak{g}$-module, and $h \in \mathrm{Aut}(\mathfrak{g})$ be an automorphism of $\mathfrak{g}$. Then the module twisted by h is the same as M as a vector space but with new action

$$\varphi \cdot_{new} w = h(\varphi) \cdot_{old} w \quad \forall \varphi \in \mathfrak{g}, \forall w \in M.$$

We denote this new $\mathfrak{g}$-module structure by M^h . In the language of representations, the pull-back of the representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(M)$ along $h\colon \mathfrak{g} \to \mathfrak{g}$ gives the representation $\rho \circ h$, which corresponds to the twisted module M^h.

Recall that the *socle* of a module M, denoted by soc M, is the sum of simple submodules of M. If $f : M \to N$ is a $\mathfrak{g}$-module isomorphism, then clearly $f(\text{soc } M) = \text{soc } N$.

We observe the following about twisted modules.

Proposition 4.5. Let U, W be $\mathfrak{g}$-modules and $g, h \in \text{Aut}(\mathfrak{g})$. Then the following holds:

1. $(W^h)^g = W^{g\circ h}$ as $\mathfrak{g}$-modules.
2. If $f : W \to U$ is a $\mathfrak{g}$-module homomorphism, then $f : W^h \to U^h$ is also a $\mathfrak{g}$-module homomorphism. In particular, if there is an isomorphism of $\mathfrak{g}$-modules $W \cong U$, then there is also an isomorphism of $\mathfrak{g}$-modules $W^h \cong U^h$.
3. Any $h \in \text{Aut}(\mathfrak{g})$ induces a covariant functor $\mathcal{F}^h$ that sends W to W^h, and a morphism $f : W \to U$ to $f : W^h \to U^h$. The functor $\mathcal{F}^h$ has an inverse functor $\mathcal{F}^{h^{-1}}$, thus it is an automorphism of the category $\mathfrak{g}$-mod.
4. The functor $\mathcal{F}^h$ commutes with the contravariant functor $(\ \cdot\ )^*$, that is, $(U^*)^h = (U^*)^h$.
5. The functor $\mathcal{F}^h$ preserves the socle of a module, that is, soc $W^h = (\text{soc } W)^h$.

Proof.

1. The $\mathfrak{g}$-modules $(W^h)^g$ and $W^{g\circ h}$ have the same action

$$\varphi \cdot_{new} w = g(h(\varphi)) \cdot_{old} w \quad \forall \varphi \in \mathfrak{g}, w \in W.$$

2. Since $f\colon W \to U$ is a $\mathfrak{g}$-homomorphism, for every $\varphi \in \mathfrak{g}$ we have the following commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & U \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ W & \xrightarrow{f} & U. \end{array}$$

Then for every $\varphi \in \mathfrak{g}$ we also have the following commutative diagram

$$\begin{array}{ccc} W^h & \xrightarrow{f} & U^h \\ {\scriptstyle h(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle h(\varphi)} \\ W^h & \xrightarrow{f} & U^h. \end{array}$$

Thus $f\colon W^h \to U^h$ is a $\mathfrak{g}$-module homomorphism.

3. It is clear from statement 2 that $\mathcal{F}^h$ takes a module W to W^h and morphisms $f\colon W \to U$ to the same morphism $f\colon W^h \to U^h$, and is a well-defined functor from $\mathfrak{g}$-mod to itself. The functors $\mathcal{F}^{h^{-1}}$ and $\mathcal{F}^h$ are mutually inverse by statement 1, thus $\mathcal{F}^h$ is an automorphism of the category $\mathfrak{g}$-mod.
4. Both actions are given by

$$(\varphi \cdot u^*)(u) = -u^*(h(\varphi) \cdot u) \quad \forall \varphi \in \mathfrak{g}, u \in U, u^* \in U^*.$$

5. Since $\mathcal{F}^h$ has an inverse $\mathcal{F}^{h^{-1}}$, it maps simple submodules to simple submodules. Also it is easy to see that for any submodules $W_1, W_2 \subset U$, we have $(W_1 + W_2)^h = W_1^h + W_2^h$. Therefore soc $W^h = (\text{soc } W)^h$. □

One important observation is that U^h and W^h are simple $\mathfrak{sl}_{U,W}$-modules for any $h \in \text{Aut}(\mathfrak{sl}_{U,W})$. Further, we have soc $(U^h)^* = W^h$ and soc $(W^h)^* = U^h$.

Proposition 4.6. As $\mathfrak{sl}_{U,W}, \mathfrak{gl}_{U,W}$ and $\mathfrak{gl}^M_{U,W}$-modules,

1. U and W are simple.
2. soc $U^* = W$ and soc $W^* = U$.
3. soc $(U^h)^* = W^h$ and soc $(W^h)^* = U^h$ for any $h \in \text{Aut}(\mathfrak{gl}^M_\infty)$.

Proof.

1. It suffices to check the simplicity of U and W as $\mathfrak{sl}_{U,W}$-modules. Let $X \subset U$ be a nontrivial submodule of U, then it intersects nontrivially with some finite-dimensional subspace $U_f \subset U$. Since U_f is a simple $\mathfrak{sl}_{U_f,W_f}$-module, we must have $X \supset U_f$. Therefore we must have $X \supset U'_f$ for any finite-dimensional subspace $U'_f \supset U_f$. We conclude that $X = U$, since U is the union of all its finite-dimensional subspaces. The same argument applies to W.
2. Let X be a nontrivial $\mathfrak{sl}_{U,W}$-submodule of U^*. It suffices to show that $X \supset W$. Since W is a simple module, all we need to show is that $X \cap W \neq 0$. Suppose the contrary, then take any nonzero element u^* in X, we must have $u^* \in U^* - W$. There exists some $u \in U$ such that $\langle u, u^* \rangle \neq 0$ by the non-degeneracy of the form $\langle \cdot, \cdot \rangle$. Pick some nonzero $w \in (\mathbb{C}u)^\perp \subset W$, then $u \otimes w \in \mathfrak{sl}_{U,W}$. However $u \otimes w \cdot u^* = \langle u, u^* \rangle w \in W \cap X$. Contradiction. The same argument can be applied to prove that soc $W^* = U$.
3. By Proposition 4.5 statement 4 and 5,

$$\text{soc } (U^h)^* = \text{soc } (U^*)^h = (\text{soc } U^*)^h = W^h,$$

$$\text{soc } (W^h)^* = \text{soc } (W^*)^h = (\text{soc } W^*)^h = U^h.$$ □

5 Automorphisms of $\mathfrak{sl}_\infty$ and $\mathfrak{gl}^M_\infty$

In this section we investigate the automorphism groups of $\mathfrak{sl}_\infty$ and $\mathfrak{gl}^M_\infty$.

We say the linear system (U, W) is *self-dual* if there is an isomorphism of vector spaces $f : U \to W$ such that $\langle f^{-1}(w), f(u) \rangle = \langle u, w \rangle$ for every $u \in U, w \in W$. In this case, according to the proof of Proposition 3.4, we observe that

$$f^*_{|U} = f, \quad (f^{-1})^*_{|W} = f^{-1}.$$

By Proposition 3.4, there is an isomorphism of Lie algebras $\gamma : \mathfrak{gl}^M_{U,W} \to \mathfrak{gl}^M_{W,U}$ defined by $\gamma : \varphi \mapsto f \varphi f^{-1}$. The Lie algebra $\mathfrak{gl}^M_{W,U}$ is again isomorphic to $\mathfrak{gl}^M_{U,W}$ by the map $\sigma : \varphi \mapsto -\varphi^*_{|U}$.

The composition $\tau := \sigma\gamma$ is an automorphism of $\mathfrak{gl}^M_{U,W}$, which is explicitly

$$\tau\colon \varphi \mapsto -f^{-1}\varphi^* f \quad \forall \varphi \in \mathfrak{gl}^M_{U,W}.$$

From the definition of τ we see that τ is an involution. Moreover, for every $\varphi \in \mathfrak{gl}^M_{U,W}$ the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{\ f\ } & W \\ {\scriptstyle \tau(\varphi)}\downarrow & & \downarrow{\scriptstyle -\varphi^*} \\ U & \xrightarrow{\ f\ } & W. \end{array}$$

Therefore we conclude that $f : U^\tau \to W$ is an isomorphism of $\mathfrak{gl}^M_{U,W}$-modules. Consequently $W^\tau \cong (U^\tau)^\tau = U$ as $\mathfrak{gl}^M_{U,W}$-modules. Since the Lie algebra $\mathfrak{gl}^M_{U,W}$ has a unique simple ideal $\mathfrak{sl}_{U,W}$, the automorphism τ restricts to an automorphism of $\mathfrak{sl}_{U,W}$.

If the linear system (U, W) has a pair of dual bases relative to $\langle\cdot,\cdot\rangle$, then clearly (U, W) is self-dual. The converse is false. Again, consider the linear system (X, X') which is defined at the end of section 2. Let $f : X \to X'$ be the isomorphism given by $f(w, w^*) = (w, w^*)$. Then $\langle f^{-1}(x), f(y)\rangle = \langle y, x\rangle$ for every $x \in X', y \in X$. Therefore the linear system (X, X') is self-dual, but it does not have a pair of dual bases.

In the countable dimensional case, the linear system (V, V_*) has a pair of dual bases. Let $\epsilon : V \to V_*$ be the isomorphism of vector spaces induced by $\epsilon(e_n) = e^n\ \forall e_n \in \mathcal{B}$. Then we observe that $\tau : \varphi \mapsto -\epsilon^{-1}\varphi^*\epsilon$ is the involution of the Lie algebra $\mathrm{Mat}^M_{\mathbb{N}}$ given by $A \mapsto -A^t$.

In the following, let $\mathfrak{g} = \mathfrak{sl}_\infty, \mathfrak{gl}^M_\infty$. Define $\tilde{G} := \{g \in \mathrm{Aut}(V)|\ g^*(V_*) = V_*\}$. Conjugation by an element in $\tilde{G}$ induces an automorphism of $\mathfrak{g}$. Therefore we have a group representation $\rho : \tilde{G} \to \mathrm{Aut}(\mathfrak{g})$ with kernel $\mathbb{C}^*$, where $\mathbb{C}^*$ denotes nonzero complex numbers. We set $\tilde{G}_0 := \mathrm{im}\ \rho$.

Theorem 5.1 (Beidar et.al, [BBCM02])**.** The subgroup $\tilde{G}_0$ has index 2 in $\mathrm{Aut}(\mathfrak{sl}_\infty)$, the quotient is represented by the involution τ. In fact, $\mathrm{Aut}(\mathfrak{sl}_\infty) = \tilde{G}_0 \rtimes \{\mathrm{Id}, \tau\}$.

Corollary 5.2. If $h \in \tilde{G}_0$, then $V^h \cong V$ and $V^{\tau\circ h} \cong V_*$ as $\mathfrak{sl}_\infty$-modules.

Proof. If $h \in \tilde{G}_0$, then $h(\varphi) = g^{-1}\varphi g$ for every $\varphi \in \mathfrak{sl}_\infty$ where $g \in \tilde{G}$. It follows that $g : V^h \to V$ is an automorphism of $\mathfrak{sl}_\infty$-modules. Therefore $V^{\tau\circ h} = (V^h)^\tau \cong V^\tau \cong V_*$ as $\mathfrak{sl}_\infty$-modules. □

Let M be a $\mathfrak{gl}^M_\infty$-module. We say that $\mathfrak{sl}_\infty$ *acts densely* on M, if for every $\varphi \in \mathfrak{gl}^M_\infty$ and any choice of finitely many vectors $r_1, \cdots, r_n \in M$, there exists $\psi \in \mathfrak{sl}_\infty$ such that $\psi \cdot r_k = \varphi \cdot r_k$ for $k = 1, \cdots, n$.

Lemma 5.3. [PS13, Lemma 8.2] Let L and L' be $\mathfrak{gl}^M_\infty$-modules which have finite length as $\mathfrak{sl}_\infty$-modules. Then

$$\mathrm{Hom}_{\mathfrak{sl}_\infty}(L, L') = \mathrm{Hom}_{\mathfrak{gl}^M_\infty}(L, L').$$

In particular, if L and L' are isomorphic as $\mathfrak{sl}_\infty$-modules, then they are isomorphic as $\mathfrak{gl}^M_\infty$-modules.

Proposition 5.4. [PS13, Proposition 8.1] Let M be a finite length $\mathfrak{gl}_\infty^M$-module that is $\mathfrak{sl}_\infty$-integrable. Then M is an object of $\mathbb{T}_{\mathfrak{gl}_\infty^M}$ iff $\mathfrak{sl}_\infty$ acts densely on it.

Corollary 5.5. Let $h \in \mathrm{Aut}(\mathfrak{gl}_\infty^M)$. Then there exists an isomorphism of $\mathfrak{gl}_\infty^M$-modules $V^h \cong V$ or $V^h \cong V_*$.

Proof. If $h \in \mathrm{Aut}(\mathfrak{gl}_\infty^M)$, then h restricts to an automorphism of $\mathfrak{sl}_\infty$. We show that $\mathfrak{sl}_\infty$ acts densely on V^h. For any $\varphi \in \mathfrak{gl}_\infty^M$ and $r_1, \cdots, r_n \in V$, we can find an element $\eta \in \mathfrak{sl}_\infty$ such that $\eta(r_k) = h(\varphi)(r_k)$ for $k = 1, \cdots, n$. Therefore $\psi := h^{-1}(\eta)$ is an element of $\mathfrak{sl}_\infty$ such that $\psi \cdot r_k = \varphi \cdot r_k$ for $k = 1, \cdots, n$. By Proposition 5.4, we conclude that $V^h \in \mathbb{T}_{\mathfrak{gl}_\infty^M}$. We know from Corollary 5.2 that either $V^h \cong V$ or $V^h \cong V_*$ as $\mathfrak{sl}_\infty$-modules, therefore we conclude that either $V^h \cong V$ or $V^h \cong V_*$ as $\mathfrak{gl}_\infty^M$-modules by Lemma 5.3. □

Theorem 5.6. Let $h \in \mathrm{Aut}(\mathfrak{gl}_\infty^M)$.

1. If there is an isomorphism $V^h \cong V$ of $\mathfrak{gl}_\infty^M$-modules, then $h \in \tilde{G}_0$.
2. If there is an isomorphism $V^h \cong V_*$ of $\mathfrak{gl}_\infty^M$-modules, then $\tau \circ h \in \tilde{G}_0$.

Proof.

1. If $f \colon V^h \to V$ is an $\mathfrak{gl}_\infty^M$-isomorphism, then $fh(\varphi) = \varphi f$ for every $\varphi \in \mathfrak{gl}_\infty^M$. Since f is an isomorphism, we conclude that $h(\varphi) = f^{-1}\varphi f$. What remains to be shown is that $f \in \tilde{G}$.

   Observe that the dual operator $f^* \colon V^* \to (V^h)^*$ is a $\mathfrak{gl}_\infty^M$-isomorphism. Since $\operatorname{soc} V^* = V_*$ and $\operatorname{soc}\,(V^h)^* = V_*^h$ by Proposition 4.6, we conclude that V_* is f^*-stable. Therefore, $f \in \tilde{G}$ and $h \in \tilde{G}_0$.
2. If $V^h \cong V_*$, then we have the following isomorphisms of $\mathfrak{gl}_\infty^M$-modules

$$V^{\tau \circ h} = (V^h)^\tau \cong V_*^\tau \cong (V^\tau)^\tau = V^{\tau^2} = V.$$

   By statement 1 we conclude that $\tau \circ h \in \tilde{G}_0$. □

Corollary 5.7.

1. $\mathrm{Aut}(\mathfrak{gl}_\infty^M) = \tilde{G}_0 \rtimes \{\mathrm{Id}, \tau\}$.
2. Every automorphism of $\mathfrak{sl}_\infty$ extends uniquely to an automorphism of $\mathfrak{gl}_\infty^M$.

Proof.

1. Follows directly from Corollary 5.5 and Theorem 5.6.
2. Follows directly from statement 1 and Theorem 5.1. □

In conclusion, we computed the automorphism group of $\mathfrak{gl}_\infty^M$ using knowledge about the categories $\mathbb{T}_{\mathfrak{sl}_\infty}$ and $\mathbb{T}_{\mathfrak{gl}_\infty^M}$. As a problem for the future, it would be interesting to determine the automorphism groups of $\mathfrak{sl}_{U,W}$, $\mathfrak{gl}_{U,W}$ and $\mathfrak{gl}_{U,W}^M$ for an arbitrary linear system (U, W).

References

[BBCM02] K.I. Beidar, M. Brësar, M.A. Chebotar, and W.S. Martindale, 3rd, On Herstein's Lie map conjectures, III. *J. Algebra* 249 (2002), 59-94.

[DCPS11] E. Dan-Cohen, I. Penkov, and V. Serganova, A Koszul category of representations of finitary Lie algebras. Preprint (2011), arXiv:1105.3407.

[Mac43] G. Mackey. On infinite dimensional linear spaces. *Trans. AMS* 57 (1945), 155-207.

[PS13] I. Penkov and V. Serganova. Tensor representations of Mackey Lie algebras and their dense subalgebras. *Developments and Retrospectives in Lie Theory: Algebraic Methods*, Springer Verlag (to appear) (2014).