INTEGRABLE $\mathfrak{sl}(\infty)$-MODULES AND CATEGORY $\mathcal{O}$ FOR $\mathfrak{gl}(m|n)$

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ABSTRACT. We introduce and study new categories $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ of integrable $\mathfrak{g} = \mathfrak{sl}(\infty)$-modules which depend on the choice of a certain reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k} \subset \mathfrak{g}$. The simple objects of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ are tensor modules as in the previously studied category $\mathbb{T}_{\mathfrak{g}}$ [DPS]; however, the choice of $\mathfrak{k}$ provides for more flexibility of nonsimple modules in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ compared to $\mathbb{T}_{\mathfrak{g}}$. We then choose $\mathfrak{k}$ to have two infinite-dimensional diagonal blocks, and show that a certain injective object $\mathbf{K}_{m|n}$ in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ realizes a categorical $\mathfrak{sl}(\infty)$-action on the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$, the integral category $\mathcal{O}$ of the Lie superalgebra $\mathfrak{gl}(m|n)$. We show that the socle of $\mathbf{K}_{m|n}$ is generated by the projective modules in $\mathcal{O}^{\mathbb{Z}}_{m|n}$, and compute the socle filtration of $\mathbf{K}_{m|n}$ explicitly. We conjecture that the socle filtration of $\mathbf{K}_{m|n}$ reflects a "degree of atypicality filtration" on the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$. We also conjecture that a natural tensor filtration on $\mathbf{K}_{m|n}$ arises via the Duflo–Serganova functor sending the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ to $\mathcal{O}^{\mathbb{Z}}_{m-1|n-1}$. We prove a weaker version of this latter conjecture for the direct summand of $\mathbf{K}_{m|n}$ corresponding to finite-dimensional $\mathfrak{gl}(m|n)$-modules.

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1. INTRODUCTION

Categorification has set a trend in mathematics in the last two decades and has proved important and useful. The opposite process of studying a given category via a combinatorial or algebraic object such as a single module has also borne ample fruit. An example is Brundan's idea from 2003 to study the category $\mathcal{F}^{\mathbb{Z}}_{m|n}$ of finite-dimensional integral modules over the Lie superalgebra $\mathfrak{gl}(m|n)$ via the weight structure of the $\mathfrak{sl}(\infty)$-module $\Lambda^m\mathbf{V}\otimes\Lambda^n\mathbf{V}_*$, where $\mathbf{V}$ and $\mathbf{V}_*$ are the two nonisomorphic defining (natural) representations of $\mathfrak{sl}(\infty)$. Using this approach Brundan computes decomposition numbers in $\mathcal{F}^{\mathbb{Z}}_{m|n}$ [B]. An extension of Brundan's approach was proposed in the work of Brundan, Losev and Webster in [BLW], where a new proof of the Brundan–Kazhdan–Lusztig conjecture for the category $\mathcal{O}$ over the Lie superalgebra $\mathfrak{gl}(m|n)$ is given. (The first proof of the Brundan–Kazhdan–Lusztig conjecture for the category $\mathcal{O}$ over the Lie superalgebra $\mathfrak{gl}(m|n)$ was given by Cheng, Lam and Wang in [CLW].) The same approach was also used by Brundan and Stroppel in [BS], where the algebra of endomorphisms of a projective generator in $\mathcal{F}^{\mathbb{Z}}_{m|n}$ is described as a certain diagram algebra and the Koszulity of $\mathcal{F}^{\mathbb{Z}}_{m|n}$ is established.

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The representation theory of the Lie algebra $\mathfrak{sl}(\infty)$ is of independent interest and has been developing actively also for about two decades. In particular, several categories of $\mathfrak{sl}(\infty)$-modules have been singled out and studied in detail, see [DP, PStyr, DPS, PS, Nam].

The category $\mathbb{T}_{\mathfrak{sl}(\infty)}$ from [DPS] has been playing a prominent role: its objects are finite-length submodules of a direct sum of several copies of the tensor algebra $T(\mathbf{V}\oplus\mathbf{V}_*)$. In [DPS] it is proved that $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is a self-dual Koszul category, in [SS] it has been shown that $\mathbb{T}_{\mathfrak{sl}(\infty)}$ has a universality property, and in [FPS] $\mathbb{T}_{\mathfrak{sl}(\infty)}$ has been used to categorify the Boson-Fermion Correspondence.

Our goal in the present paper is to find an appropriate category of $\mathfrak{sl}(\infty)$-modules which contains modules relevant to the representation theory of the Lie superalgebras $\mathfrak{gl}(m|n)$. For this purpose, we introduce and study the categories $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$, where $\mathfrak{g}=\mathfrak{sl}(\infty)$ and $\mathfrak{k}$ is a reductive subalgebra of $\mathfrak{g}$ containing the diagonal subalgebra and consisting of finitely many blocks along the diagonal. The Lie algebra $\mathfrak{k}$ is infinite dimensional and is itself isomorphic to the commutator subalgebra of a finite direct sum of copies of $\mathfrak{gl}(n)$ (for varying n) and copies of $\mathfrak{gl}(\infty)$. When $\mathfrak{k}=\mathfrak{g}$, this new category coincides with $\mathbb{T}_{\mathfrak{g}}$. A well-known property of the category $\mathbb{T}_{\mathfrak{g}}$ states that for every $\mathbf{M}\in\mathbb{T}_{\mathfrak{g}}$, any vector $m\in\mathbf{M}$ is annihilated by a "large" subalgebra $\mathfrak{g}'\subset\mathfrak{g}$, i.e. by an algebra which contains the commutator subalgebra of the centralizer of a finite-dimensional subalgebra $\mathfrak{s}\subset\mathfrak{g}$. For a general $\mathfrak{k}$ as above, the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ has the same simple objects as $\mathbb{T}_{\mathfrak{g}}$ but requires the following for a nonsimple module $\mathbf{M}$: the annihilator in $\mathfrak{k}$ of every $m\in\mathbf{M}$ is a large subalgebra of $\mathfrak{k}$. This makes the nonsimple objects of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ more "flexible" than in those of $\mathbb{T}_{\mathfrak{g}}$, the degree of flexibility being governed by $\mathfrak{k}$.

In Section 3, we study the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ in detail, one of our main results being an explicit computation of the socle filtration of an indecomposable injective object $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ (where $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are two Young diagrams), see Theorem 20. An effect which can be observed here is that with a sufficient increase in the number of infinite blocks of $\mathfrak{k}$, the layers of the socle filtration of $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ grow in a "self-similar" manner. This shows that $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ is an intricate extension of the category $\mathbb{T}_{\mathfrak{g}}$ within the category of all integrable $\mathfrak{g}$-modules.

In Section 4, we show that studying the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ achieves our goal of improving the understanding of the integral category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ for the Lie superalgebra $\mathfrak{gl}(m|n)$. More precisely, we choose $\mathfrak{k}$ to have two blocks, both of them infinite. Then we show that the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ is a categorification of an injective object $\mathbf{K}_{m|n}$ in the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$. In order to accomplish this, we exploit the properties of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ as a category, and not just as a collection of modules. The object $\mathbf{K}_{m|n}$ of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ can be defined as the complexified reduced Grothendieck group of the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$, endowed with an $\mathfrak{sl}(\infty)$-module structure (categorical action of $\mathfrak{sl}(\infty)$). For $m,n\geq 1$, $\mathbf{K}_{m|n}$ is an object of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$, but not of $\mathbb{T}_{\mathfrak{g}}$. We prove that the socle of $\mathbf{K}_{m|n}$ as an $\mathfrak{sl}(\infty)$-module is the submodule generated by classes of projective $\mathfrak{gl}(m|n)$-modules in $\mathcal{O}^{\mathbb{Z}}_{m|n}$. Moreover, we conjecture that the socle filtration of $\mathbf{K}_{m|n}$ (which we already know from Section 3) arises from filtering the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ according to the degree of atypicality of $\mathfrak{gl}(m|n)$-modules. We provide some partial evidence toward this conjecture.

We also show that the category $\mathcal{F}^{\mathbb{Z}}_{m|n}$ of finite-dimensional integral $\mathfrak{gl}(m|n)$-modules categorifies a direct summand $\mathbf{J}_{m|n}$ of $\mathbf{K}_{m|n}$ which is nothing but an injective hull in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ of Brundan's module $\Lambda^m\mathbf{V}\otimes\Lambda^n\mathbf{V}_*$, see Corollary 28. (Note that the module $\Lambda^m\mathbf{V}\otimes\Lambda^n\mathbf{V}_*$ is an injective object of $\mathbb{T}_{\mathfrak{g}}$, but is not injective in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ when $\mathfrak{k}$ has two (or more) infinite blocks.)

Finally, we conjecture that a natural filtration on the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ defined via the Duflo–Serganova functor $DS : \mathcal{O}^{\mathbb{Z}}_{m|n} \to \mathcal{O}^{\mathbb{Z}}_{m-1|n-1}$ categorifies the tensor filtration of $\mathbf{K}_{m|n}$, i.e. the coarsest filtration of $\mathbf{K}_{m|n}$ whose successive quotients are objects of $\mathbb{T}_{\mathfrak{g}}$. We have a similar conjecture for the direct summand $\mathbf{J}_{m|n}$ of $\mathbf{K}_{m|n}$, and we provide evidence for this conjecture in Proposition 42.

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3. New categories of integrable $\mathfrak{sl}(\infty)$-modules

3.1. **Preliminaries.** Let $\mathbf{V}$ and $\mathbf{V}_*$ be countable-dimensional vector spaces with fixed bases $\{v_i\}_{i\in\mathbb{Z}}$ and $\{v_j^*\}_{j\in\mathbb{Z}}$, together with a nondegenerate pairing $\langle\cdot,\cdot\rangle : \mathbf{V}\otimes\mathbf{V}_* \to \mathbb{C}$ defined by $\langle v_i, v_j^*\rangle = \delta_{ij}$. Then $\mathfrak{gl}(\infty) := \mathbf{V}\otimes\mathbf{V}_*$ has a Lie algebra structure such that

$$[v_i\otimes v_j^*, v_k\otimes v_l^*] = \langle v_k, v_j^*\rangle v_i\otimes v_l^* - \langle v_i, v_l^*\rangle v_k\otimes v_j^*.$$

We can identify $\mathfrak{gl}(\infty)$ with the space of infinite matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with finitely many nonzero entries, where the vector $v_i\otimes v_j^*$ corresponds to the matrix E_{ij} with 1 in the i,j-position and zeros elsewhere. Then $\langle\cdot,\cdot\rangle$ corresponds to the trace map, and its kernel is the Lie algebra $\mathfrak{sl}(\infty)$, which is generated by $e_i := E_{i,i+1}$, $f_i := E_{i+1,i}$ with $i\in\mathbb{Z}$. One can also realize $\mathfrak{sl}(\infty)$ as a direct limit of finite-dimensional Lie algebras $\mathfrak{sl}(\infty) = \varinjlim \mathfrak{sl}(n)$. In contrast to the finite-dimensional setting, the exact sequence

$$0 \to \mathfrak{sl}(\infty) \to \mathfrak{gl}(\infty) \to \mathbb{C} \to 0$$

does not split, and the center of $\mathfrak{gl}(\infty)$ is trivial.

Let $\mathfrak{g} = \mathfrak{sl}(\infty)$. The representations $\mathbf{V}$ and $\mathbf{V}_*$ are the defining representations of $\mathfrak{g}$. The tensor representations $\mathbf{V}^{\otimes p}\otimes\mathbf{V}_*^{\otimes q}$, $p,q\in\mathbb{Z}_{\geq 0}$ have been studied in [PStyr]. They are not semisimple when $p,q>0$; however, each simple subquotient of $\mathbf{V}^{\otimes p}\otimes\mathbf{V}_*^{\otimes q}$ occurs as a submodule of $\mathbf{V}^{\otimes p'}\otimes\mathbf{V}_*^{\otimes q'}$ for some p',q'. The simple submodules of $\mathbf{V}^{\otimes p}\otimes\mathbf{V}_*^{\otimes q}$ can be parameterized by two Young diagrams $\boldsymbol{\lambda},\boldsymbol{\mu}$, and we denote them $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$.

Recall that the *socle* of a module $\mathbf{M}$, denoted $\operatorname{soc}\mathbf{M}$, is the largest semisimple submodule of $\mathbf{M}$. The *socle filtration* of $\mathbf{M}$ is defined inductively by $\operatorname{soc}^0\mathbf{M} := \operatorname{soc}\mathbf{M}$ and $\operatorname{soc}^i\mathbf{M} := p_i^{-1}(\operatorname{soc}(\mathbf{M}/(\operatorname{soc}^{i-1}\mathbf{M})))$, where $p_i : \mathbf{M}\to\mathbf{M}/(\operatorname{soc}^{i-1}\mathbf{M})$ is the natural projection. We also use the notation $\overline{\operatorname{soc}}^i\mathbf{M} := \operatorname{soc}^i\mathbf{M}/\operatorname{soc}^{i-1}\mathbf{M}$ for the layers of the socle filtration.

Schur-Weyl duality for $\mathfrak{sl}(\infty)$ implies that the module $\mathbf{V}^{\otimes p}\otimes\mathbf{V}_*^{\otimes q}$ decomposes as

$$\mathbf{V}^{\otimes p}\otimes\mathbf{V}_*^{\otimes q} = \bigoplus_{|\boldsymbol{\lambda}|=p,|\boldsymbol{\mu}|=q} (\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))\otimes(Y_{\boldsymbol{\lambda}}\otimes Y_{\boldsymbol{\mu}}), \tag{3.1}$$

where $Y_{\boldsymbol{\lambda}}$ and $Y_{\boldsymbol{\mu}}$ are irreducible S_p- and S_q-modules, and $\mathbb{S}_{\boldsymbol{\lambda}}$ denotes the Schur functor corresponding to the Young diagram (equivalently, partition) $\boldsymbol{\lambda}$. Each module $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ is indecomposable and its socle filtration is described in [PStyr]. Moreover, Theorem 2.3 of [PStyr] claims that

$$\overline{\operatorname{soc}}^k(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)) \cong \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',|\gamma|=k} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\gamma} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\gamma}\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'} \tag{3.2}$$

where $N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}}$ are the standard Littlewood-Richardson coefficients. In particular, $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ has simple socle $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$. It was also shown in [PStyr, Theorem 2.2] that the socle of $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$ equals the intersection of the kernels of all contraction maps

$$\Phi_{ij} : \mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q} \to \mathbf{V}^{\otimes (p-1)} \otimes \mathbf{V}_*^{\otimes (q-1)} \tag{3.3}$$

$$v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^* \mapsto \langle v_j^*, v_i \rangle v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes \widehat{v_j^*} \otimes \cdots \otimes v_q^*$$

A $\mathfrak{g}$-module is called a *tensor module* if it is isomorphic to a submodule of a finite direct sum of $\mathfrak{sl}(\infty)$-modules of the form $\mathbf{V}^{\otimes p_i} \otimes \mathbf{V}_*^{\otimes q_i}$ for $p_i, q_i \in \mathbb{Z}_{\geq 0}$. The category of tensor modules $\mathbb{T}_{\mathfrak{g}}$ is by definition the full subcategory of $\mathfrak{g}$-mod consisting of tensor modules [DPS]. A finite-length $\mathfrak{g}$-module $\mathbf{M}$ lies in $\mathbb{T}_{\mathfrak{g}}$ if and only if $\mathbf{M}$ is integrable and satisfies the large annihilator condition [DPS]. Recall that a $\mathfrak{g}$-module $\mathbf{M}$ is called *integrable* if $\dim\{m, x\cdot m, x^2\cdot m, \ldots\} < \infty$ for any $x \in \mathfrak{g}$, $m \in \mathbf{M}$. A $\mathfrak{g}$-module is said to satisfy the *large annihilator condition* if for each $m \in \mathbf{M}$, the annihilator $\mathrm{Ann}_{\mathfrak{g}} m$ contains the commutator subalgebra of the centralizer of a finite-dimensional subalgebra of $\mathfrak{g}$.

The modules $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$, $p, q \in \mathbb{Z}_{\geq 0}$ are injective in the category $\mathbb{T}_{\mathfrak{g}}$. Moreover, every indecomposable injective object of $\mathbb{T}_{\mathfrak{g}}$ is isomorphic to an indecomposable direct summand of $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$ for some $p, q \in \mathbb{Z}_{\geq 0}$ [DPS]. Consequently, by (3.1), an indecomposable injective in $\mathbb{T}_{\mathfrak{g}}$ is isomorphic to $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ for some $\boldsymbol{\lambda}, \boldsymbol{\mu}$.

The category $\mathbb{T}_{\mathfrak{g}}$ is a subcategory of the category $\widetilde{Tens}_{\mathfrak{g}}$, which was introduced in [PS] as the full subcategory of $\mathfrak{g}$-mod whose objects $\mathbf{M}$ are defined to be the integrable $\mathfrak{g}$-modules of finite Loewy length such that the algebraic dual $\mathbf{M}^* = \mathrm{Hom}_{\mathbb{C}}(\mathbf{M}, \mathbb{C})$ is also integrable and of finite Loewy length. The categories $\mathbb{T}_{\mathfrak{g}}$ and $\widetilde{Tens}_{\mathfrak{g}}$ have the same simple objects $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ [PS, DPS]. The indecomposable injective objects of $\widetilde{Tens}_{\mathfrak{g}}$ are (up to isomorphism) the modules $(\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*$, and $\mathrm{soc}(\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^* \cong \mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ [PS]. A recent result of [CP2] shows that the Grothendieck envelope $\overline{Tens}_{\mathfrak{g}}$ of $\widetilde{Tens}_{\mathfrak{g}}$ is an ordered tensor category, and that any injective object in $\overline{Tens}_{\mathfrak{g}}$ is a direct sum of indecomposable injectives from $\widetilde{Tens}_{\mathfrak{g}}$.

3.2. **The categories $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$.** In this section, we introduce new categories of integrable $\mathfrak{sl}(\infty)$-modules. This is motivated in part by the applications to the representation theory of the Lie superalgebras $\mathfrak{gl}(m|n)$.

Let $\mathfrak{g} = \mathfrak{sl}(\infty)$ with the natural representation denoted $\mathbf{V}$. Consider a decomposition

$$\mathbf{V} = \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_r, \tag{3.4}$$

for some vector subspaces $\mathbf{V}_i$ of $\mathbf{V}$. Let $\mathfrak{l}$ be the Lie subalgebra of $\mathfrak{g}$ preserving this decomposition. Then $\mathfrak{k} := [\mathfrak{l}, \mathfrak{l}]$ is isomorphic to $\mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r$, where each $\mathfrak{k}_i$ is isomorphic to $\mathfrak{sl}(n_i)$ or $\mathfrak{sl}(\infty)$.

Definition 1. Denote by $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ the full subcategory of $\widetilde{Tens}_{\mathfrak{g}}$ consisting of modules $\mathbf{M}$ satisfying the large annihilator condition as a module over $\mathfrak{k}_i$ for all $i = 1, \ldots, r$. By $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ we denote the full subcategory of $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ consisting of finite-length modules.

Both categories $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ and $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ are abelian symmetric monoidal categories with respect to the usual tensor product of $\mathfrak{g}$-modules. Two categories $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ and $\widetilde{\mathbb{T}}_{\mathfrak{g},\bar{\mathfrak{k}}}$ are equal if $\mathfrak{k}$ and $\bar{\mathfrak{k}}$ have finite corank in $\mathfrak{k} + \bar{\mathfrak{k}}$, so we will henceforth assume without loss of generality that each $\mathbf{V}_i$ in decomposition (3.4) is infinite dimensional. Note that $\mathbb{T}_{\mathfrak{g},\mathfrak{g}} = \mathbb{T}_{\mathfrak{g}}$.

We define the functor $\Gamma_{\mathfrak{g},\mathfrak{k}} : \widetilde{Tens}_{\mathfrak{g}} \to \widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ by taking the maximal submodule lying in $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$. Then

$$\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{M}) = \bigcup \mathbf{M}^{\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_r}, \tag{3.5}$$

where the union is taken over all finite corank subalgebras $\mathfrak{s}_1 \subset \mathfrak{k}_1, \dots, \mathfrak{s}_r \subset \mathfrak{k}_r$.

Lemma 2. *Let $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ be as in Definition 1.*

(1) *The simple objects of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ and of $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ are isomorphic to $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$.*
(2) *The functor $\Gamma_{\mathfrak{g},\mathfrak{k}}$ sends injective modules in $\widetilde{Tens}_{\mathfrak{g}}$ to injective modules in $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$.*
(3) *The category $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ has enough injective modules.*
(4) *The indecomposable injective objects of $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ are isomorphic to $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$.*

Proof. (1) The category $\mathbb{T}_{\mathfrak{g}}$ is a full subcategory of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ and of $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$, which are both full subcategories of $\widetilde{Tens}_{\mathfrak{g}}$. Since the categories $\mathbb{T}_{\mathfrak{g}}$ and $\widetilde{Tens}_{\mathfrak{g}}$ have the same simple objects $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$, the claim follows.
(2) This follows from the definition of $\Gamma_{\mathfrak{g},\mathfrak{k}}$, since $\mathrm{Hom}_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(X, \Gamma_{\mathfrak{g},\mathfrak{k}}(Y)) = \mathrm{Hom}_{\widetilde{Tens}_{\mathfrak{g}}}(X,Y)$ for all $X \in \mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ and $Y \in \widetilde{Tens}_{\mathfrak{g}}$.
(3) Every module $\mathbf{M}$ in $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ can be embedded into $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{M}^{**})$, which is injective in $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$, since $\mathbf{M}^{**}$ is injective in $\widetilde{Tens}_{\mathfrak{g}}$ [PS].
(4) This follows from (1) and (2), since $(\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*$ is an indecomposable injective object of $\widetilde{Tens}_{\mathfrak{g}}$, and consequently $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$ is an indecomposable injective object of $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ with $\mathrm{soc}\,\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*) \cong \mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$.

□

Remark 3. It will follow from Corollary 12 that the indecomposable injective objects $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$ are objects of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$. Consequently, $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ and $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ have the same indecomposable injectives.

3.3. **The functor R and Jordan-Hölder multiplicities.** In this section, we calculate the Jordan-Hölder multiplicities of the indecomposable injective objects of the categories $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$. One of the main tools we use for this computation is the functor R, which we will now introduce.

Let

$$\mathbf{V}' = \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_{r-1}, \quad \mathfrak{g}' = \mathfrak{g} \cap \mathfrak{gl}(\mathbf{V}'), \quad \mathfrak{k}' = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_{r-1}. \tag{3.6}$$

Let $(\mathbf{V}_r)_* \subset \mathbf{V}_*$ be the annihilator of $\mathbf{V}' = \mathbf{V}_1 \oplus \cdots \oplus \mathbf{V}_{r-1}$ with respect to the pairing $\langle \cdot, \cdot \rangle$. We have $\mathfrak{g}' \cong \mathfrak{sl}(\infty)$ and $\mathfrak{k}' \subset \mathfrak{g}'$.

Define a functor R from the category $\mathfrak{g}$–mod of all $\mathfrak{g}$-modules to the category $\mathfrak{g}'$–mod by setting

$$\mathrm{R}(\mathbf{M}) = \mathbf{M}^{\mathfrak{k}_r}.$$

It follows from the definition that after restricting to $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ we have a functor $\mathrm{R} : \widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}} \to \widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'}$.

Lemma 4. *The following diagram of functors is commutative:*

$$\begin{array}{ccc} \mathfrak{g}\text{–mod} & \xrightarrow{\mathrm{R}} & \mathfrak{g}'\text{–mod} \\ {\scriptstyle \Gamma_{\mathfrak{g},\mathfrak{k}}}\downarrow & & {\scriptstyle \Gamma_{\mathfrak{g}',\mathfrak{k}'}}\downarrow \\ \widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}} & \xrightarrow{\mathrm{R}} & \widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'} \end{array} .$$

Proof. By (3.5) we have

$$\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{M}) = \bigcup \mathbf{M}^{\mathfrak{s}_1\oplus\cdots\oplus\mathfrak{s}_r}$$

for any $\mathfrak{g}$-module $\mathbf{M}$. Then

$$\mathrm{R}(\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{M})) = (\bigcup \mathbf{M}^{\mathfrak{s}_1\oplus\cdots\oplus\mathfrak{s}_r})^{\mathfrak{k}_r} = \bigcup \mathbf{M}^{\mathfrak{s}_1\oplus\cdots\oplus\mathfrak{s}_{r-1}\oplus\mathfrak{k}_r} = \bigcup (\mathrm{R}(\mathbf{M}))^{\mathfrak{s}_1\oplus\cdots\oplus\mathfrak{s}_{r-1}} = \Gamma_{\mathfrak{g}',\mathfrak{k}'}(\mathrm{R}(\mathbf{M})).$$

□

Lemma 5. *If $\boldsymbol{\lambda},\boldsymbol{\mu}$ are Young diagrams, then*

$$\mathrm{R}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} (\mathbb{S}_{\boldsymbol{\lambda}'}(\mathrm{R}(\mathbf{V}))\otimes\mathbb{S}_{\boldsymbol{\mu}'}(\mathrm{R}(\mathbf{V}_*)))^*.$$

Proof. Since $\mathrm{R}(\mathbf{V}) = \mathbf{V}'$, we have the decompositions

$$\mathbf{V} = \mathrm{R}(\mathbf{V})\oplus\mathbf{V}_r,\quad \mathbf{V}_* = \mathrm{R}(\mathbf{V}_*)\oplus(\mathbf{V}_r)_*.$$

We also have the identity

$$\mathbb{S}_{\boldsymbol{\lambda}}(V\oplus W) = \bigoplus N^{\boldsymbol{\lambda}}_{\boldsymbol{\mu},\boldsymbol{\nu}} \mathbb{S}_{\boldsymbol{\mu}}(V)\otimes\mathbb{S}_{\boldsymbol{\nu}}(W), \tag{3.7}$$

which holds for all vector spaces V and W. These imply

$$\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma},\boldsymbol{\gamma}'} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}'} \mathbb{S}_{\boldsymbol{\lambda}'}(\mathrm{R}(\mathbf{V}))\otimes\mathbb{S}_{\boldsymbol{\gamma}}(\mathbf{V}_r)\otimes\mathbb{S}_{\boldsymbol{\mu}'}(\mathrm{R}(\mathbf{V}_*))\otimes\mathbb{S}_{\boldsymbol{\gamma}'}((\mathbf{V}_r)_*).$$

By definition

$$\mathrm{R}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*) = \mathrm{Hom}_{\mathfrak{g}'}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*),\mathbb{C}),$$

and it follows from (3.2) that

$$\dim\mathrm{Hom}_{\mathfrak{g}'}(\mathbb{S}_{\boldsymbol{\gamma}}(\mathbf{V}_r)\otimes\mathbb{S}_{\boldsymbol{\gamma}'}((\mathbf{V}_r)_*),\mathbb{C}) = \delta_{\boldsymbol{\gamma},\boldsymbol{\gamma}'},$$

$\delta_{\boldsymbol{\gamma},\boldsymbol{\gamma}'}$ being Kronecker's delta. Therefore,

$$\mathrm{Hom}_{\mathfrak{g}'}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*),\mathbb{C}) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} (\mathbb{S}_{\boldsymbol{\lambda}'}(\mathrm{R}(\mathbf{V}))\otimes\mathbb{S}_{\boldsymbol{\mu}'}(\mathrm{R}(\mathbf{V}_*)))^*.$$

□

Lemma 6. *If $0\to A\to B\to C\to 0$ is an exact sequence of modules in $\widetilde{Tens}_{\mathfrak{g}}$, then the dual exact sequence $0\to C^*\to B^*\to A^*\to 0$ splits.*

Proof. This follows from the fact that C^* is injective in $\widetilde{Tens}_{\mathfrak{g}}$. □

Lemma 7. *The functor $\mathrm{R}:\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}\to\widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'}$ sends an indecomposable injective object to an injective object.*

Proof. Let $\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}} = \Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*)$. Then by Lemma 4 we have

$$\mathrm{R}(\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) = \Gamma_{\mathfrak{g}',\mathfrak{k}'}(\mathrm{R}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*)),$$

and hence by Lemma 5

$$\mathrm{R}(\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} \Gamma_{\mathfrak{g}',\mathfrak{k}'}((\mathbb{S}_{\boldsymbol{\lambda}'}(\mathrm{R}(\mathbf{V}))\otimes\mathbb{S}_{\boldsymbol{\mu}'}(\mathrm{R}(\mathbf{V}_*)))^*). \tag{3.8}$$

Therefore, $\mathrm{R}(\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}})$ is injective in $\widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'}$. Every indecomposable injective object in $\widetilde{\mathbb{T}}_{\mathfrak{g},\mathfrak{k}}$ is isomorphic to $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{L}^*)$ for some simple object $\mathbf{L} = \mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$, and by Lemma 6, $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{L}^*)$ is a direct summand of $\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}} = \Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*)$. Since the functor R is left exact, $\mathrm{R}(\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{L}^*))$ is a direct summand of $\mathrm{R}(\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}})$. Hence, $\mathrm{R}(\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{L}^*))$ is injective in $\widetilde{\mathbb{T}}_{\mathfrak{g}',\mathfrak{k}'}$. □

Lemma 8. *Let $\mathbf{V} = V_n \oplus \mathbf{W}$ and $\mathbf{V}_* = V_n^* \oplus \mathbf{W}_*$ be decompositions with $\dim V_n = n$, $\mathbf{W}^\perp = V_n^*$ and $\mathbf{W}_*^\perp = V_n$. Let $\mathfrak{s}$ be the commutator subalgebra of $\mathbf{W} \otimes \mathbf{W}_*$. Let $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}}$ be a module such that all its simple constituents are of the form $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ with $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}| \leq n$. Then the length of $\mathbf{M}^{\mathfrak{s}}$ in the category of $\mathfrak{sl}(n)$-modules equals the length of $\mathbf{M}$ in $\mathbb{T}_{\mathfrak{g}}$.*

Proof. It follows from (3.7) and the fact that $\mathbb{S}_{\boldsymbol{\lambda}}(V_n)$ and $\mathbb{S}_{\boldsymbol{\mu}}(V_n^*)$ are nonzero (since $\dim V_n \geq |\boldsymbol{\lambda}|, |\boldsymbol{\mu}|$) that

$$(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^{\mathfrak{s}} = \mathbb{S}_{\boldsymbol{\lambda}}(V_n) \otimes \mathbb{S}_{\boldsymbol{\mu}}(V_n^*).$$

The description of the layers of the socle filtration of $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ in (3.2) shows that the length of $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ equals the length of $\mathbb{S}_{\boldsymbol{\lambda}}(V_n) \otimes \mathbb{S}_{\boldsymbol{\mu}}(V_n^*)$. Furthermore, since the socle $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ of $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ coincides with the set of vectors annihilated by all contraction maps (see (3.3)), and the set of vectors in $\mathbb{S}_{\boldsymbol{\lambda}}(V_n) \otimes \mathbb{S}_{\boldsymbol{\mu}}(V_n^*)$ annihilated by all contraction maps is the simple $\mathfrak{sl}(n)$-module $V_n^{\boldsymbol{\lambda},\boldsymbol{\mu}}$, we obtain $(\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}})^{\mathfrak{s}} = V_n^{\boldsymbol{\lambda},\boldsymbol{\mu}}$. It then follows from left exactness that the functor $(\cdot)^{\mathfrak{s}}$ does not increase the length.

Let $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}}$, and let $k(\mathbf{M})$ be the maximum of $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$ over all simple constituents $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ of $\mathbf{M}$. We proceed by proving the statement by induction on $k(\mathbf{M})$ with the obvious base case $k(\mathbf{M}) = 0$. Consider an exact sequence

$$0 \to \mathbf{M} \to \mathbf{I} \to \mathbf{N} \to 0,$$

where $\mathbf{I}$ is an injective hull of $\mathbf{M}$ in $\mathbb{T}_{\mathfrak{g}}$. From the description of the socle filtration of an injective module in $\mathbb{T}_{\mathfrak{g}}$ (see (3.2)), we have $k(\mathbf{N}) < k(\mathbf{M})$. Therefore, the length $l(\mathbf{N})$ of $\mathbf{N}$ equals the length $l(\mathbf{N}^{\mathfrak{s}})$ of $\mathbf{N}^{\mathfrak{s}}$ by the induction assumption. On the other hand, since $\mathbf{I}$ is injective and hence isomorphic to a direct sum of $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ with $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}| \leq n$, the length of $\mathbf{I}$ equals the length of $\mathbf{I}^{\mathfrak{s}}$. Now if $l(\mathbf{M}^{\mathfrak{s}}) < l(\mathbf{M})$, then

$$l(\mathbf{N}^{\mathfrak{s}}) \geq l(\mathbf{I}^{\mathfrak{s}}) - l(\mathbf{M}^{\mathfrak{s}}) > l(\mathbf{I}) - l(\mathbf{M}) = l(\mathbf{N}),$$

which is a contradiction. □

Corollary 9. *Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{g}$ as in Lemma 8, and let $\mathbf{M} \in \widetilde{\mathbb{T}}_{\mathfrak{g}}$ be a module such that all its simple constituents are of the form $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ with $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}| \leq n$. Then $\mathbf{M} = U(\mathfrak{g})\mathbf{M}^{\mathfrak{s}}$.*

Proof. Since $\mathbf{M}$ is a direct limit of modules of finite length it suffices to prove the statement for $\mathbf{M} \in \mathbb{T}_{\mathfrak{g}}$. This can be easily done by induction on the length of $\mathbf{M}$. Indeed, consider an exact sequence $0 \to \mathbf{N} \to \mathbf{M} \to \mathbf{L} \to 0$ with simple $\mathbf{L}$. Lemma 8 implies that $0 \to \mathbf{N}^{\mathfrak{s}} \to \mathbf{M}^{\mathfrak{s}} \to \mathbf{L}^{\mathfrak{s}} \to 0$ is also exact, because the functor $(\cdot)^{\mathfrak{s}}$ is left exact and $l(\mathbf{L}^{\mathfrak{s}}) = l(\mathbf{M}^{\mathfrak{s}}) - l(\mathbf{N}^{\mathfrak{s}})$. Now if $U(\mathfrak{g})\mathbf{M}^{\mathfrak{s}} \neq \mathbf{M}$ then, since $U(\mathfrak{g})\mathbf{N}^{\mathfrak{s}} = \mathbf{N}$ by the induction assumption, we obtain $U(\mathfrak{g})\mathbf{M}^{\mathfrak{s}} = \mathbf{N}$. This implies $\mathbf{M}^{\mathfrak{s}} = \mathbf{N}^{\mathfrak{s}}$, and hence $l(\mathbf{L}^{s}) = 0$, which contradicts Lemma 8. □

Lemma 10. *For any $\mathbf{M} \in \mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ we have $U(\mathfrak{g})\mathrm{R}(\mathbf{M}) = \mathbf{M}$.*

Proof. Recall the definition of $k(\mathbf{M})$ from the proof of Lemma 8, and recall the decomposition (3.4). Let $\mathbf{U}$ be a subspace of $\mathbf{V}$, and $\mathbf{U}_*$ be a subspace of $\mathbf{V}_*$ such that $\mathbf{V}_r \subset \mathbf{U}$ and $(\mathbf{V}_r)_* \subset \mathbf{U}_*$, each of codimension $k(\mathbf{M})$.

Denote by $\mathfrak{l} \subset \mathfrak{g}$ the commutator subalgebra of $\mathbf{U} \otimes \mathbf{U}_*$, and by $\mathrm{Res}_{\mathfrak{l}}$ the restriction functor from $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ to $\widetilde{\mathbb{T}}_{\mathfrak{l}}$. The identity (3.7) implies that $k(\mathrm{Res}_{\mathfrak{l}}\,\mathbf{M}) = k(\mathbf{M})$. By Corollary 9 with $\mathfrak{g} = \mathfrak{l}$ and $\mathfrak{s} = \mathfrak{k}_r$, we get $\mathbf{M} = U(\mathfrak{l})\mathrm{R}(\mathbf{M})$. The statement follows. □

Lemma 11. *The functor* $\mathrm{R} : \mathbb{T}_{\mathfrak{g},\mathfrak{k}} \to \mathbb{T}_{\mathfrak{g}',\mathfrak{k}'}$ *is exact and sends a simple module* $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \in \mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ *to the corresponding simple module* $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \in \mathbb{T}_{\mathfrak{g}',\mathfrak{k}'}$*, and hence induces an isomorphism between the Grothendieck groups of* $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ *and* $\mathbb{T}_{\mathfrak{g}',\mathfrak{k}'}$.

Proof. Since $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is in fact an object of $\mathbb{T}_{\mathfrak{g}}$, the statement about simple modules follows by the argument concerning contraction maps from the proof of Lemma 8.

Since R is left exact, we have the inequality

$$l(\mathrm{R}(\mathbf{M})) \leq l(\mathbf{M}). \tag{3.9}$$

Thus, to prove exactness of R it suffices to show that R preserves the length, i.e. $l(\mathbf{M}) = l(\mathrm{R}(\mathbf{M}))$. We prove this by induction on $l(\mathbf{M})$. Consider an exact sequence of $\mathfrak{g}$-modules

$$0 \to \mathbf{N} \to \mathbf{M} \to \mathbf{L} \to 0,$$

such that $\mathbf{L}$ is simple. By the induction hypothesis we have $l(\mathrm{R}(\mathbf{N})) = l(\mathbf{N})$. If we assume that $l(\mathrm{R}(\mathbf{M})) < l(\mathbf{M})$, then $l(\mathrm{R}(\mathbf{M})) = l(\mathbf{N})$ and so $\mathrm{R}(\mathbf{N}) = \mathrm{R}(\mathbf{M})$. But then by Lemma 10, we have $\mathbf{N} = \mathbf{M}$, which is a contradiction. □

Corollary 12. *For any* $\boldsymbol{\lambda}, \boldsymbol{\mu}$, *the module* $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*)$ *has finite length. Hence, the module* $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}} := \Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$ *has finite length and is an object of the category* $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$.

Proof. It was proven in [DPS] that $\Gamma_{\mathfrak{g},\mathfrak{g}}((\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))^*)$ has finite length in $\widetilde{\mathbb{T}}_{\mathfrak{g}}$ (see the proof of Proposition 4.5 in [DPS] and note that the functor $\Gamma_{\mathfrak{g},\mathfrak{g}}$ is denoted by $\mathcal{B}$ in [DPS]). Using (3.8), the first claim follows by induction on the number r of components in the decomposition of $\mathbf{V}$. For the second claim, observe that Lemma 6 implies $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is isomorphic to a direct summand of the module $\Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}_*))^*)$. □

Lemma 13. *Let* $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *denote an injective hull of the simple module* $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *in* $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$*, and let* $\mathbf{J}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *denote an injective hull of* $\mathrm{R}(\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}})$ *in* $\mathbb{T}_{\mathfrak{g}',\mathfrak{k}'}$. *Then*

$$\mathrm{R}(\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} \mathbf{J}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}.$$

Proof. We have $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \cong \Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$ and $\mathbf{J}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \cong \Gamma_{\mathfrak{g}',\mathfrak{k}'}((\mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}})^*)$. Let

$$\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}} = \Gamma_{\mathfrak{g},\mathfrak{k}}((\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}_*))^*), \quad \mathbf{Q}^{\boldsymbol{\lambda},\boldsymbol{\mu}} = \Gamma_{\mathfrak{g}',\mathfrak{k}'}((\mathbb{S}_{\boldsymbol{\mu}}(\mathrm{R}(\mathbf{V})) \otimes \mathbb{S}_{\boldsymbol{\lambda}}(\mathrm{R}(\mathbf{V}_*))^*).$$

Then we have

$$\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \cong \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} \mathbf{I}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}, \quad \mathbf{Q}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \cong \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} \mathbf{J}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}. \tag{3.10}$$

Indeed, using Lemma 6, we can deduce from (3.2) that

$$(\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}) \otimes \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V}_*))^* = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\boldsymbol{\gamma}} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\boldsymbol{\gamma}} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\boldsymbol{\gamma}} (\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'})^*,$$

and then by applying $\Gamma_{\mathfrak{g},\mathfrak{k}}$ to both sides we obtain (3.10).

By (3.8), we have

$$\mathrm{R}(\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) = \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\gamma} N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}',\gamma} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}',\gamma} \mathbf{Q}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}.$$

Let $\mathfrak{I}_{\mathfrak{g},\mathfrak{k}}$ denote the complexified Grothendieck group of the additive subcategory of $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ generated by indecomposable injective modules. Then $\{[\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$ and $\{[\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$ both form a basis for $\mathfrak{I}_{\mathfrak{g},\mathfrak{k}}$. Let $A = (A^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{\boldsymbol{\lambda}',\boldsymbol{\mu}'})$ be the change of basis matrix on $\mathfrak{I}_{\mathfrak{g},\mathfrak{k}}$ given by (3.10) which expresses $\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in terms of $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$. The same matrix A expresses $\mathbf{Q}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in terms of $\mathbf{J}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ by (3.10).

The functor R induces a linear operator from $\mathfrak{I}_{\mathfrak{g},\mathfrak{k}}$ to $\mathfrak{I}_{\mathfrak{g}',\mathfrak{k}'}$ which is represented by the matrix A with respect to both bases $\{[\mathbf{P}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$ and $\{[\mathbf{Q}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$. Hence, the matrix which represents R with respect to the bases $\{[\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$ and $\{[\mathbf{J}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]\}$ is again A as $A = AA(A^{-1})$. □

Corollary 14. *The Jordan-Hölder multiplicities of the indecomposable injective modules* $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ *are given by*

$$[\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}} : \mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}] = \sum_{\boldsymbol{\lambda}',\boldsymbol{\mu}',\gamma_1,\dots,\gamma_r} N^{\boldsymbol{\lambda}}_{\gamma_1,\dots,\gamma_r,\boldsymbol{\lambda}'} N^{\boldsymbol{\mu}}_{\gamma_1,\dots,\gamma_r,\boldsymbol{\mu}'}.$$

Proof. After applying the functor R to the module $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ $(r-1)$ times, we obtain a direct sum of injective modules in the category $\mathbb{T}_{\mathfrak{g}}$. The multiplicity of each indecomposable injective in this sum is thus determined by applying the matrix A^{r-1} to $[\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}]$. The Jordan-Hölder multiplicities of an indecomposable injective module in $\mathbb{T}_{\mathfrak{g}}$ are also given by the matrix A (see 3.2). Therefore,

$$[\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}] = \sum (A^r)^{\boldsymbol{\lambda},\boldsymbol{\mu}}_{\boldsymbol{\lambda}',\boldsymbol{\mu}'} [\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}].$$

□

3.4. **The socle filtration of indecomposable injective objects in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$.** In this section, we describe the socle filtration of the injective objects $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$.

We consider the restriction functor

$$\mathrm{Res}_{\mathfrak{k}} : \mathbb{T}_{\mathfrak{g},\mathfrak{k}} \to \mathbb{T}_{\mathfrak{k}},$$

where $\mathbb{T}_{\mathfrak{k}}$ denotes the category of integrable $\mathfrak{k}$-modules of finite length which satisfy the large annihilator condition for each $\mathfrak{k}_i$ (recall (3.4)). Note that simple objects of $\mathbb{T}_{\mathfrak{k}}$ are outer tensor products of simple objects of the categories $\mathbb{T}_{\mathfrak{k}_i}$ for each $\mathfrak{k}_i$, $i = 1, \dots, r$, (recall that $\mathfrak{k}_i \cong \mathfrak{sl}(\infty)$); we will use the notation

$$\mathbf{V}^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r} := \mathbf{V}_1^{\boldsymbol{\lambda}_1,\boldsymbol{\mu}_1} \boxtimes \cdots \boxtimes \mathbf{V}_r^{\boldsymbol{\lambda}_r,\boldsymbol{\mu}_r}.$$

Injective hulls of simple objects in $\mathbb{T}_{\mathfrak{k}}$ will be denoted by $\mathbf{I}_{\mathfrak{k}}^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}$, and they are also outer tensor products of injective $\mathfrak{k}_i$-modules:

$$\mathbf{I}_{\mathfrak{k}}^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r} := \left(\mathbb{S}_{\boldsymbol{\lambda}_1}(\mathbf{V}_1) \otimes \mathbb{S}_{\boldsymbol{\mu}_1}(\mathbf{V}_1)_*\right) \boxtimes \cdots \boxtimes \left(\mathbb{S}_{\boldsymbol{\lambda}_r}(\mathbf{V}_r) \otimes \mathbb{S}_{\boldsymbol{\mu}_r}(\mathbf{V}_r)_*\right).$$

Recall that for every object $\mathbf{M}$ in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ we denote by $k(\mathbf{M})$ the maximum of $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$ for all simple constituents $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ of $\mathbf{M}$. Similarly for every object $\mathbf{X}$ in $\mathbb{T}_{\mathfrak{k}}$ we denote by $c(\mathbf{X})$ the maximum of $|\boldsymbol{\lambda}_1| + \cdots + |\boldsymbol{\lambda}_r| + |\boldsymbol{\mu}_1| + \cdots + |\boldsymbol{\mu}_r|$ for all simple constituents $\mathbf{V}^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}$ of $\mathbf{X}$. It follows from Corollary 14 that

$$k(\mathbf{M}) = k(\operatorname{soc} \mathbf{M}), \quad c(\mathbf{X}) = c(\operatorname{soc} \mathbf{X}). \tag{3.11}$$

The identities

$$k(\mathbf{M}\otimes\mathbf{N}) = k(\mathbf{M}) + k(\mathbf{N}), \quad c(\mathbf{X}\otimes\mathbf{Y}) = c(\mathbf{X}) + c(\mathbf{Y}). \tag{3.12}$$

follow easily from the Littlewood–Richardson rule, and we leave their proof to the reader.

Lemma 15. *The restriction functor* $\operatorname{Res}_{\mathfrak{k}}$ *maps the category* $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ *to the category* $\mathbb{T}_{\mathfrak{k}}$*, and it maps* $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ *to an injective module. Furthermore, we have the identity*

$$c(\operatorname{Res}_{\mathfrak{k}}\mathbf{M}) = k(\mathbf{M}).$$

Proof. After applying identity (3.7) r-times to $\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$, we get

$$\operatorname{Res}_{\mathfrak{k}}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)) \cong \bigoplus N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r} \mathbf{I}_{\mathfrak{k}}^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}.$$

This implies the first and the second assertions of the lemma. Identity (3.11) implies that it is sufficient to prove the last assertion for $\mathbf{M} = \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$. Hence, this assertion follows from the above computation. □

Conjecture 16. *Suppose* $\operatorname{Ext}^k_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) \neq 0$*. Then* $|\boldsymbol{\lambda}| - |\boldsymbol{\lambda}'| = |\boldsymbol{\mu}| - |\boldsymbol{\mu}'| = k$*.*

Remark 17. For $\mathfrak{k} = \mathfrak{g}$, this was proven in [DPS]. Proving this conjecture would imply that the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ is Koszul. We prove the case $k = 1$.

Proposition 18. *Suppose* $\operatorname{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}) \neq 0$*. Then* $|\boldsymbol{\lambda}| - |\boldsymbol{\lambda}'| = |\boldsymbol{\mu}| - |\boldsymbol{\mu}'| = 1$*.*

Proof. Since $\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}$ is isomorphic to a simple constituent of $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$, we know by Corollary 14 that $|\boldsymbol{\lambda}| - |\boldsymbol{\lambda}'| = |\boldsymbol{\mu}| - |\boldsymbol{\mu}'| = s \geq 1$. It remains to show that $s = 1$. We will do this in two steps.

First, we show that $\operatorname{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)) \neq 0$ implies $s = 1$. Consider a nonsplit short exact sequence in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$

$$0 \to \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*) \to \mathbf{M} \to \mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'} \to 0. \tag{3.13}$$

Let $\varphi : \mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}\otimes\mathfrak{g} \to \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ be a cocyle which defines this extension. By Lemma 15, the module $\operatorname{Res}_{\mathfrak{k}}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))$ is injective in $\mathbb{T}_{\mathfrak{k}}$, and therefore the sequence (3.13) splits over $\mathfrak{k}$. Without loss of generality we may assume that $\varphi(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}\otimes\mathfrak{k}) = 0$. Then the cocycle condition implies that $\varphi : \mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}\otimes(\mathfrak{g}/\mathfrak{k}) \to \mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)$ is a nonzero homomorphism of $\mathfrak{k}$-modules. Consequently, the image of φ contains a simple submodule in the socle of $\operatorname{Res}_{\mathfrak{k}}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))$. By Lemma 15, we have

$$\operatorname{soc}\operatorname{Res}_{\mathfrak{k}}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*)) = \bigoplus N^{\boldsymbol{\lambda}}_{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r} N^{\boldsymbol{\mu}}_{\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r} V^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}.$$

In particular,

$$c(V^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}) = |\boldsymbol{\lambda}_1| + \cdots + |\boldsymbol{\lambda}_r| + |\boldsymbol{\mu}_1| + \cdots + |\boldsymbol{\mu}_r| = |\boldsymbol{\lambda}| + |\boldsymbol{\mu}|$$

for every simple submodule $V^{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_r,\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_r}$ of $\operatorname{soc}\operatorname{Res}_{\mathfrak{k}}(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))$. Therefore,

$$c(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}\otimes(\mathfrak{g}/\mathfrak{k})) \geq |\boldsymbol{\lambda}| + |\boldsymbol{\mu}|,$$

and so (3.12) implies

$$c(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}) + c(\mathfrak{g}/\mathfrak{k}) \geq |\boldsymbol{\lambda}| + |\boldsymbol{\mu}|.$$

Since $\mathfrak{g}/\mathfrak{k} \cong \bigoplus_{i\neq j}(\mathbf{V}_i\otimes(\mathbf{V}_j)_*)$, we have

$$c(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}) = |\boldsymbol{\lambda}'| + |\boldsymbol{\mu}'|, \quad c(\mathfrak{g}/\mathfrak{k}) = 2,$$

and thus $|\boldsymbol{\lambda}|-|\boldsymbol{\lambda}'|+|\boldsymbol{\mu}|-|\boldsymbol{\mu}'|=2s\leq 2$. This yields $s=1$.

Assume now to the contrary that $s\geq 2$. Set

$$\mathbf{X}=(\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))/\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$$

and consider the long exact sequence of Ext

$$\cdots\to\mathrm{Hom}_{\mathfrak{g}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{X})\to\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}})\to\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))\to\dots.$$

Since $s\geq 2$, $\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}$ is not isomorphic to a submodule of $\operatorname{soc}\mathbf{X}$, so $\mathrm{Hom}_{\mathfrak{g}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{X})=0$, and by the already considered case when $s=1$, we have

$$\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbb{S}_{\boldsymbol{\lambda}}(\mathbf{V})\otimes\mathbb{S}_{\boldsymbol{\mu}}(\mathbf{V}_*))=0.$$

Hence, $\mathrm{Ext}^1_{\mathbb{T}_{\mathfrak{g},\mathfrak{k}}}(\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}})=0$, which is a contradiction. □

Corollary 19. *Suppose that $\mathbf{M}\in\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ has a simple socle $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ and the multiplicity of $\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}$ in $\overline{\operatorname{soc}}^k\mathbf{M}$ is nonzero. Then $|\boldsymbol{\lambda}|-|\boldsymbol{\lambda}'|=|\boldsymbol{\mu}|-|\boldsymbol{\mu}'|=k$.*

Proof. This follows by induction on $|\boldsymbol{\lambda}|+|\boldsymbol{\mu}|$. By Proposition 18, the module $\mathbf{M}/\operatorname{soc}\mathbf{M}$ embeds into a direct sum of injective indecomposable modules $\bigoplus\mathbf{I}^{\gamma,\boldsymbol{\nu}}$ with simple socles $\mathbf{V}^{\gamma,\boldsymbol{\nu}}$ satisfying $|\boldsymbol{\lambda}|-|\gamma|=|\boldsymbol{\mu}|-|\boldsymbol{\nu}|=1$, and by induction each $\mathbf{I}^{\gamma,\boldsymbol{\nu}}$ satisfies our claim. If the multiplicity of $\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}$ is nonzero in $\overline{\operatorname{soc}}^k\mathbf{M}=\overline{\operatorname{soc}}^{k-1}(\mathbf{M}/\operatorname{soc}\mathbf{M})\subset\overline{\operatorname{soc}}^{k-1}(\bigoplus\mathbf{I}^{\gamma,\boldsymbol{\nu}})$, then $|\gamma|-|\boldsymbol{\lambda}'|=|\boldsymbol{\nu}|-|\boldsymbol{\mu}'|=k-1$. The result follows. □

Finally, by combining Corollary 14 and Corollary 19 we obtain the following.

Theorem 20. *The layers of the socle filtration of an indecomposable injective $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ satisfy*

$$\overline{\operatorname{soc}}^k\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}\cong\bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}'}\bigoplus_{|\gamma_1|+\dots+|\gamma_r|=k}N^{\boldsymbol{\lambda}}_{\gamma_1,\dots,\gamma_r,\boldsymbol{\lambda}'}N^{\boldsymbol{\mu}}_{\gamma_1,\dots,\gamma_r,\boldsymbol{\mu}'}\mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'},$$

where r is the number of (infinite) blocks in $\mathfrak{k}$ (see (3.4)).

Example 21. Consider an injective hull of the adjoint representation of $\mathfrak{sl}(\infty)$ in the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ in the case that $\mathfrak{k}$ has k (infinite) blocks. Then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ each consist of one box, and $\operatorname{soc}V^{\boldsymbol{\lambda},\boldsymbol{\mu}}=\mathfrak{sl}(\infty)$ and $\overline{\operatorname{soc}}^1V^{\boldsymbol{\lambda},\boldsymbol{\mu}}=\mathbb{C}^k$, the trivial representation of dimension k. The self-similarity effect mentioned in the introduction amounts here to the increase of the dimension of $\overline{\operatorname{soc}}^1$ by 1 when the number of blocks of $\mathfrak{k}$ increases by 1.

Remark 22. Let's observe that the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ is another example of an ordered tensor category as defined in [CP1]. Indeed, the set I in the notation of [CP1] can be chosen as the set of pairs of Young diagrams $(\boldsymbol{\lambda},\boldsymbol{\mu})$, and then the object X_i for $i=(\boldsymbol{\lambda},\boldsymbol{\mu})$ equals $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$.

4. $\mathfrak{sl}(\infty)$-modules arising from category $\mathcal{O}$ for $\mathfrak{gl}(m|n)$

For the remainder of this paper, we let $\mathfrak{k}=\mathfrak{k}_1\oplus\mathfrak{k}_2$ be the commutator subalgebra of the Lie algebra preserving a fixed decomposition $\mathbf{V}=\mathbf{V}_1\oplus\mathbf{V}_2$ such that both $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are isomorphic to $\mathfrak{sl}(\infty)$ ($r=2$ in (3.4)).

4.1. **Category $\mathcal{O}$ for the Lie superalgebra $\mathfrak{gl}(m|n)$.** Let $\mathcal{O}_{m|n}$ denote the category of $\mathbb{Z}_2$-graded modules over $\mathfrak{gl}(m|n)$ which when restricted to $\mathfrak{gl}(m|n)_{\bar{0}}$, belong to the BGG category $\mathcal{O}_{\mathfrak{gl}(m|n)_{\bar{0}}}$ [M, Section 8.2.3]. This category depends only on a choice of simple roots for the Lie algebra $\mathfrak{gl}(m|n)_{\bar{0}}$, and not for all of $\mathfrak{gl}(m|n)$. We denote by $\mathcal{O}^{\mathbb{Z}}_{m|n}$ the Serre subcategory of $\mathcal{O}_{m|n}$ consisting of modules with integral weights. Any simple object in $\mathcal{O}^{\mathbb{Z}}_{m|n}$ is isomorphic to $L(\lambda)$ (the unique simple quotient of the Verma module $M(\lambda)$) for some $\lambda \in \Phi$, where Φ denotes the set of integral weights. Any object in the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$ has finite length.

We denote by $\mathcal{F}^{\mathbb{Z}}_{m|n}$ the Serre subcategory of $\mathcal{O}^{\mathbb{Z}}_{m|n}$ consisting of finite-dimensional modules. Let $\Pi : \mathcal{O}^{\mathbb{Z}}_{m|n} \to \mathcal{O}^{\mathbb{Z}}_{m|n}$ be the parity reversing functor. We define the *reduced Grothendieck group* $K_{m|n}$ (respectively, $J_{m|n}$) to be the quotient of the Grothendieck group of $\mathcal{O}^{\mathbb{Z}}_{m|n}$ (respectively, $\mathcal{F}^{\mathbb{Z}}_{m|n}$) by the relation $[\Pi M] = -[M]$. The elements $[L(\lambda)]$ with $\lambda \in \Phi$ (respectively, $\lambda \in \Phi^+$) form a basis for $K_{m|n}$ (respectively, $J_{m|n}$).

We introduce an action of $\mathfrak{sl}(\infty)$ on $\mathbf{K}_{m|n} := K_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$ following Brundan [B]. Our starting point is to define the translation functors E_i and F_i on the category $\mathcal{O}^{\mathbb{Z}}_{m|n}$. Consider the invariant form $\mathrm{str}(XY)$ on $\mathfrak{gl}(m|n)$ and let X_j, Y_j be a pair of $\mathbb{Z}_2$-homogeneous dual bases of $\mathfrak{gl}(m|n)$ with respect to this form. Then for two $\mathfrak{gl}(m|n)$-modules V and W we define the operator

$$\Omega : V \otimes W \to V \otimes W,$$

$$\Omega(v \otimes w) := \sum_j (-1)^{p(X_j)(p(v)+1)} X_j v \otimes Y_j w,$$

where $p(X_j)$ denotes the parity of the $\mathbb{Z}_2$-homogeneous element X_j. It is easy to check that $\Omega \in \mathrm{End}_{\mathfrak{gl}(m|n)}(V \otimes W)$. Let U and U^* denote the natural and conatural $\mathfrak{gl}(m|n)$-modules. For every $M \in \mathcal{O}^{\mathbb{Z}}_{m|n}$ we let $\mathrm{E}_i(M)$ (respectively, $\mathrm{F}_i(M)$) be the generalized eigenspace of Ω in $M \otimes U^*$ (respectively, $M \otimes U$) with eigenvalue i. Then, as it follows from [BLW], the functor $\cdot \otimes U^*$ (respectively, $\cdot \otimes U$) decomposes into the direct sum of functors $\oplus_{i \in \mathbb{Z}} \mathrm{E}_i(\cdot)$ (respectively, $\oplus_{i \in \mathbb{Z}} \mathrm{F}_i(\cdot)$). Moreover, the functors E_i and F_i are mutually adjoint functors on $\mathcal{O}^{\mathbb{Z}}_{m|n}$. We will denote by e_i and f_i the linear operators which the functors E_i and F_i induce on $\mathbf{K}_{m|n}$.

If we identify e_i and f_i with the Chevalley generators $E_{i,i+1}$ and $F_{i+1,i}$ of $\mathfrak{sl}(\infty)$, then $\mathbf{K}_{m|n}$ inherits the natural structure of a $\mathfrak{sl}(\infty)$-module. This follows from [B, BLW]. Another proof can be obtained by using Theorem 3.11 of [CS] and (4.2) below. Weight spaces with respect to the diagonal subalgebra $\mathfrak{h} \subset \mathfrak{sl}(\infty)$ correspond to the complexified reduced Grothendieck groups of the blocks of $\mathcal{O}^{\mathbb{Z}}_{m|n}$.

Let $\mathbf{J}_{m|n} := J_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$, and let $\mathbf{T}_{m|n} \subset \mathbf{K}_{m|n}$ denote the subspace generated by the classes $[M(\lambda)]$ of all Verma modules $M(\lambda)$ for $\lambda \in \Phi$. Let furthermore $\mathbf{\Lambda}_{m|n} \subset \mathbf{J}_{m|n}$ denote the subspace generated by the classes $[K(\lambda)]$ of all Kac modules $K(\lambda)$ for $\lambda \in \Phi^+$ (for the definition of a Kac module see for example [B]). Then $\mathbf{T}_{m|n}$ is an $\mathfrak{sl}(\infty)$-submodule isomorphic to $\mathbf{V}^{\otimes m} \otimes \mathbf{V}_*^{\otimes n}$ and $\mathbf{\Lambda}_{m|n}$ is a submodule of $\mathbf{T}_{m|n}$ isomorphic to $\Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$ [B]. To see this, let $\{v_i\}_{i \in \mathbb{Z}}$ and $\{w_i\}_{i \in \mathbb{Z}}$ be the standard dual bases in $\mathbf{V}$ and $\mathbf{V}_*$ (i.e. $\mathfrak{h}$-eigenbases in $\mathbf{V}$ and $\mathbf{V}_*$), and let $\bar{\lambda} := \lambda + (m-1, \dots, 1, 0|0, -1, \dots, 1-n)$,

$$m_\lambda := v_{\bar{\lambda}_1} \otimes \cdots \otimes v_{\bar{\lambda}_m} \otimes v^*_{-\bar{\lambda}_{m+1}} \otimes \cdots \otimes v^*_{-\bar{\lambda}_{m+n}}.$$

The map $[M(\lambda)] \mapsto m_\lambda$ establishes an isomorphism $\mathbf{T}_{m|n} \cong \mathbf{V}^{\otimes m} \otimes \mathbf{V}_*^{\otimes n}$, and restricts to an isomorphism

$$\begin{aligned}\mathbf{\Lambda}_{m|n} &\cong \Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_* \\ [K(\lambda)] &\mapsto k_\lambda := v_{\bar{\lambda}_1} \wedge \cdots \wedge v_{\bar{\lambda}_m} \otimes v^*_{-\bar{\lambda}_{m+1}} \wedge \cdots \wedge v^*_{-\bar{\lambda}_{m+n}}.\end{aligned}$$

Lemma 23. *The $\mathfrak{sl}(\infty)$-module $\mathbf{K}_{m|n}$ satisfies the large annihilator condition as a module over $\mathfrak{k}_1$ and $\mathfrak{k}_2$, that is, $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{K}_{m|n}) = \mathbf{K}_{m|n}$.*

Proof. Note that an $\mathfrak{sl}(\infty)$-module $\mathbf{M}$ satisfies the large annihilator condition over $\mathfrak{k}_1$ and $\mathfrak{k}_2$ if and only if for each $x \in \mathbf{M}$, we have $e_i x = f_i x = 0$ for all but finitely many $i \in \mathbb{Z}$. Indeed, if $e_i x = f_i x = 0$ for all but finitely many $i \in \mathbb{Z}$, then the subalgebra generated by the e_i, f_i that annihilate x contains the commutator subalgebra of the centralizer of a finite-dimensional subalgebra. The other direction is also clear.

Since the classes of simple $\mathfrak{gl}(m|n)$-modules $[L(\lambda)]$ form a basis of $\mathbf{K}_{m|n}$, we just need to show that for each $L(\lambda)$ we have $\mathrm{E}_i(L(\lambda)) = \mathrm{F}_i(L(\lambda)) = 0$ for almost all $i \in \mathbb{Z}$. However, since $\mathbf{T}_{m|n}$ satisfies the large annihilator condition, we know that the analogous statement is true for $M(\lambda)$. Therefore, since $L(\lambda)$ is a quotient of $M(\lambda)$, the exactness of the functors E_i and F_i implies the desired statement for $L(\lambda)$. □

If we consider the Cartan involution σ of $\mathfrak{sl}(\infty)$, $\sigma(e_i) = -f_i$, $\sigma(f_i) = -e_i$, we obtain

$$\langle gx, y\rangle = -\langle x, \sigma(g)y\rangle \tag{4.1}$$

for all $g \in \mathfrak{sl}(\infty)$. If $\mathbf{X}$ is a $\mathfrak{sl}(\infty)$-module, we denote by $\mathbf{X}^\vee$ the twist of the algebraic dual $\mathbf{X}^*$ by σ. Note that $(\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}})^\vee = \mathbf{V}^{\boldsymbol{\mu},\boldsymbol{\lambda}}$. Hence, if $\mathbf{X}$ is a semisimple object of finite length in $\widetilde{Tens}_\mathfrak{g}$, then $\mathbf{X}^\vee$ is an injective hull of $\mathbf{X}$ in $\widetilde{Tens}_\mathfrak{g}$.

Let $\mathcal{P}_{m|n}$ denote the semisimple subcategory of $\mathcal{O}^{\mathbb{Z}}_{m|n}$ which consists of projective $\mathfrak{gl}(m|n)$-modules, and let $P_{m|n}$ denote the reduced Grothendieck group of $\mathcal{P}_{m|n}$. The $\mathfrak{sl}(\infty)$-module $\mathbf{P}_{m,n} := P_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$ is the socle of $\mathbf{T}_{m,n}$ [CS, Theorem 3.11]. Note that for any projective module $P \in \mathcal{P}_{m|n}$ the functor $\mathrm{Hom}_{\mathfrak{gl}(m|n)}(P, \cdot)$ on $\mathcal{O}^{\mathbb{Z}}_{m|n}$ is exact, and for any module $M \in \mathcal{F}_{m|n}$ the functor $\mathrm{Hom}_{\mathfrak{gl}(m|n)}(\cdot, M)$ on $\mathcal{P}_{m|n}$ is exact. Moreover, we have the dual bases in $\mathbf{K}_{m|n}$ and $\mathbf{P}_{m|n}$ given by the classes of irreducible modules and indecomposable projective modules, respectively.

Consider the pairing $\mathbf{K}_{m|n} \times \mathbf{P}_{m|n} \to \mathbb{C}$ defined by

$$\langle [M], [P]\rangle := \dim \mathrm{Hom}_{\mathfrak{gl}(m|n)}(P, M).$$

Since the functors E_i and F_i are adjoint, we have

$$\langle e_i x, y\rangle = \langle x, f_i y\rangle$$

and

$$\langle f_i x, y\rangle = \langle x, e_i y\rangle,$$

for all $i \in \mathbb{Z}$, $x \in \mathbf{K}_{m|n}$, $y \in \mathbf{P}_{m|n}$. Thus, there is an embedding of $\mathfrak{sl}(\infty)$-modules

$$\Psi : \mathbf{K}_{m|n} \hookrightarrow \mathbf{P}^\vee_{m|n} \tag{4.2}$$

given by $[M] \mapsto \langle [M], \cdot\rangle$.

Theorem 24. *The $\mathfrak{sl}(\infty)$-module $\mathbf{K}_{m|n}$ is an injective hull in the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$ of the semisimple module $\mathbf{P}_{m|n}$. Furthermore, there is an isomorphism*

$$\mathbf{K}_{m|n} \cong \bigoplus_{|\boldsymbol{\lambda}|=m,|\boldsymbol{\mu}|=n} \mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \otimes (Y_{\boldsymbol{\lambda}} \otimes Y_{\boldsymbol{\mu}})$$

where $Y_{\boldsymbol{\lambda}}$, $Y_{\boldsymbol{\mu}}$ are irreducible modules over S_m and S_n respectively, and $\mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is an injective hull of the simple module $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ in $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$. Consequently, the layers of the socle filtration of $\mathbf{K}_{m|n}$ are given by

$$\overline{\operatorname{soc}}^k \mathbf{K}_{m|n} \cong \bigoplus_{|\boldsymbol{\lambda}|=m,|\boldsymbol{\mu}|=n} (\overline{\operatorname{soc}}^k \mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}})^{\oplus(\dim Y_{\boldsymbol{\lambda}} \dim Y_{\boldsymbol{\mu}})}$$

where

$$\overline{\operatorname{soc}}^k \mathbf{I}^{\boldsymbol{\lambda},\boldsymbol{\mu}} \cong \bigoplus_{\boldsymbol{\lambda}',\boldsymbol{\mu}'} \bigoplus_{|\gamma_1|+|\gamma_2|=k} N^{\boldsymbol{\lambda}}_{\gamma_1,\gamma_2,\boldsymbol{\lambda}'} N^{\boldsymbol{\mu}}_{\gamma_1,\gamma_2,\boldsymbol{\mu}'} \mathbf{V}^{\boldsymbol{\lambda}',\boldsymbol{\mu}'}.$$

Proof. The module $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^\vee_{m|n})$ is an injective hull of the semisimple module $\mathbf{P}_{m|n}$ in the category $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$, so it suffices to show that the image of $\mathbf{K}_{m|n}$ under the embedding (4.2) equals $\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^\vee_{m|n})$. The fact that $\Psi(\mathbf{K}_{m|n}) \subset \Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^\vee_{m|n})$ follows from Lemma 23. Herein, we will identify $\mathbf{K}_{m|n}$ with its image $\Psi(\mathbf{K}_{m|n}) = \operatorname{span}\{\langle l_\lambda, \cdot\rangle \mid \lambda \in \Phi\}$, where $l_\lambda := [L(\lambda)]$.

Now $\operatorname{soc}(\Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^\vee_{m|n})) = \mathbf{P}_{m|n}$, since $\mathbf{P}_{m|n}$ is semisimple, and $\operatorname{soc}\mathbf{T}_{m|n} = \mathbf{P}_{m|n}$ by [CS, Theorem 3.11]. Therefore, since $\mathbf{T}_{m|n} \subset \mathbf{K}_{m|n} \subset \Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^\vee_{m|n})$, we have $\operatorname{soc}\mathbf{K}_{m|n} = \mathbf{P}_{m|n}$.

We will show that $\mathbf{K}_{m|n} = \Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^\vee_{m|n})$. To accomplish this, we use the existence of the dual bases $p_\lambda := [P(\lambda)] \in \mathbf{P}_{m|n}$ and $l_\lambda \in \mathbf{K}_{m|n}$, where $L(\lambda)$ denotes the irreducible $\mathfrak{gl}(m|n)$-module with highest weight $\lambda \in \Phi$ and $P(\lambda)$ is a projective cover of $L(\lambda)$.

Fix $\omega \in \Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^\vee_{m|n})$. To prove that $\omega \in \mathbf{K}_{m|n} = \operatorname{span}\{\langle l_\lambda, \cdot\rangle \mid \lambda \in \Phi\}$, it suffices to show that $\omega(p_\lambda) = 0$ for almost all $\lambda \in \Phi$. For each $q, r \in \mathbb{Z}$, with $q < r$, we let $\mathfrak{g}_{q,r} := \mathfrak{g}^-_q \oplus \mathfrak{g}^+_r$, where $\mathfrak{g}^-_q$ is the subalgebra of $\mathfrak{g}$ generated by e_i, f_i for $i < q$ and $\mathfrak{g}^+_r$ is the subalgebra of $\mathfrak{g}$ generated by e_i, f_i for $i > r$. By the annihilator condition, ω is $\mathfrak{g}_{q,r}$-invariant for suitable q and r. Fix such q and r. Then since ω is $\mathfrak{g}_{q,r}$-invariant, it suffices to show that $p_\lambda \in \mathfrak{g}_{q,r}\mathbf{P}_{m|n}$ for almost all $\lambda \in \Phi$.

If $p_\lambda \in \mathbf{P}_{m|n} \cap (\mathfrak{g}_{q,r}\mathbf{T}_{m|n})$, then $p_\lambda \in \mathfrak{g}_{q,r}\mathbf{P}_{m|n}$. Indeed, for any $\mathfrak{g}_{q,r}$-module $\mathbf{M}$ we have

$$\mathfrak{g}_{q,r}\mathbf{M} = \bigcap_{\varphi \in \operatorname{Hom}_{\mathfrak{g}_{q,r}}(\mathbf{M},\mathbb{C})} \ker\ \varphi.$$

Now any $\mathfrak{g}_{q,r}$-module homomorphism $\varphi : \mathbf{P}_{m|n} \to \mathbb{C}$ lifts to a $\mathfrak{g}_{q,r}$-module homomorphism $\varphi : \mathbf{K}_{m|n} \to \mathbb{C}$, since the trivial module $\mathbb{C}$ is injective in the full subcategory of $\mathfrak{g}_{q,r}$-mod consisting of integrable finite-length $\mathfrak{g}_{q,r}$-modules satisfying the large annihilator condition [DPS]. Hence, the claim follows.

For each $\lambda \in \Phi$ we define $\operatorname{supp}(\bar\lambda)$ to be the multiset $\{\bar\lambda_1, \ldots, \bar\lambda_m, -\bar\lambda_{m+1}, \ldots, -\bar\lambda_{m+n}\}$, where

$$\bar\lambda := \lambda + (m-1, \ldots, 1, 0|0, -1, \ldots, 1-n).$$

The set of $\lambda \in \Phi$ such that $\operatorname{supp}(\bar\lambda) \cap (\mathbb{Z}_{<(q-m-n)} \cup \mathbb{Z}_{>(r+m+n)}) = \emptyset$ is finite. Hence, to finish the proof of the theorem, it suffices to show the following.

Lemma 25. *If $\operatorname{supp}(\bar\lambda) \cap \mathbb{Z}_{<(q-m-n)} \neq \emptyset$, then $p_\lambda \in \mathfrak{g}^-_q\mathbf{T}_{m|n}$. Similarly, if $\operatorname{supp}(\bar\lambda) \cap \mathbb{Z}_{>(r+m+n)} \neq \emptyset$, then $p_\lambda \in \mathfrak{g}^+_r\mathbf{T}_{m|n}$.*

Proof. We will prove the first statement; the proof of the second statement is similar. We can write $p_\lambda = \sum_\nu c_\nu m_\nu$, where each $c_\nu \in \mathbb{Z}_{>0}$ and $m_\nu = [M(\nu)]$ is the class of the Verma module $M(\nu)$ over $\mathfrak{gl}(m|n)$ of highest weight $\nu \in \Phi$.

We claim that $\mathrm{supp}(\bar{\nu}) \cap \mathbb{Z}_{<q} \neq \emptyset$ for every m_ν which occurs in the decomposition of p_λ. Indeed, recall that $P(\lambda)$ is a direct summand in the induced module $\mathrm{Ind}_{\mathfrak{gl}(m|n)_{\bar{0}}}^{\mathfrak{gl}(m|n)} P^0(\lambda)$, where $P^0(\lambda)$ is a projective cover of the simple $\mathfrak{gl}(m|n)_{\bar{0}}$-module with highest weight λ. Now

$$[P^0(\lambda)] = \sum_{w \in \mathcal{W}} b_{w\cdot\lambda}[M^0(w \cdot \lambda)], \tag{4.3}$$

where $M^0(\mu)$ denotes the Verma module over $\mathfrak{gl}(m|n)_{\bar{0}}$ with highest weight μ, $\mathcal{W}$ denotes the Weyl group of $\mathfrak{gl}(m|n)_{\bar{0}}$ and $w \cdot \lambda$ denotes the $\rho_{\bar{0}}$-shifted action of $\mathcal{W}$. The isomorphism of $\mathfrak{gl}(m|n)$-modules

$$M(\mu) \cong \mathrm{Ind}_{\mathfrak{gl}(m|n)_{\bar{0}} \oplus \mathfrak{gl}(m|n)_1}^{\mathfrak{gl}(m|n)} M^0(\mu)$$

implies that

$$\mathrm{Ind}_{\mathfrak{gl}(m|n)_{\bar{0}}}^{\mathfrak{gl}(m|n)} M^0(\mu) \cong \mathrm{Ind}_{\mathfrak{gl}(m|n)_{\bar{0}} \oplus \mathfrak{gl}(m|n)_1}^{\mathfrak{gl}(m|n)} (M^0(\mu) \otimes U(\mathfrak{gl}(m|n)_1).$$

Therefore, $\mathrm{Ind}_{\mathfrak{gl}(m|n)_{\bar{0}}}^{\mathfrak{gl}(m|n)} M^0(\mu)$ admits a filtration by Verma modules $M(\mu+\gamma)$ where γ runs over the set of weights of $U(\mathfrak{gl}(m|n)_1)$. Since $\mathrm{supp}(\gamma) \subset \{-m-n, \dots, m+n\}$ for every γ, we have

$$|(\overline{\mu+\gamma})_i - \bar{\mu}_i| \leq m+n.$$

Combining this with (4.3) we obtain that for each $i \leq m+n$, $|\bar{\nu}_i - \bar{\lambda}_{w(i)}| < m+n$, for some $w \in \mathcal{W}$. The claim follows.

Following the notations of Lemma 47 from the appendix, we set

$$\mathbf{W}_1 = \mathrm{span}\{v_i, \mid i < q\}, \quad \mathbf{W}_2 = \mathrm{span}\{v_j, \mid j \geq q\}.$$

Then $\mathfrak{g}_q^- = \mathfrak{sl}(\mathbf{W}_1) = \mathfrak{s}$. By above, every m_ν occurring in the decomposition of p_λ is contained in $\mathbf{Y}_{m|n}$. Hence $p_\lambda \in \mathbf{Y}_{m|n}$. Since we also have $p_\lambda \in \mathrm{soc}\,\mathbf{T}_{m|n}$, Lemma 47 implies that $p_\lambda \in \mathfrak{g}_q^- \mathbf{T}_{m|n}$. □

Hence, $\mathbf{K}_{m|n} = \Gamma_{\mathfrak{g},\mathfrak{k}}(\mathbf{P}^\vee_{m|n})$, and the description of the socle filtration now follows from Theorem 20. □

4.2. **The symmetric group action on $\mathbf{K}_{m|n}$.** Recall that we have a natural action of the product of symmetric groups $S_m \times S_n$ on $\mathbf{T}_{m|n}$, which commutes with the $\mathfrak{sl}(\infty)$-module structure on $\mathbf{T}_{m|n}$. Moreover, it follows from [DPS, Sect. 6] that

$$\mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{T}_{m|n}) = \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{P}_{m|n}) = \mathbb{C}[S_m \times S_n]. \tag{4.4}$$

A similar result is true for $\mathbf{K}_{m|n}$:

Proposition 26.

$$\mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n}) = \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{P}_{m|n}) = \mathbb{C}[S_m \times S_n].$$

Proof. Recall that $\mathbf{P}_{m|n}$ is the socle of $\mathbf{K}_{m|n}$ by Theorem 24. Every $\varphi \in \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n})$ maps the socle to the socle, hence we have a homomorphism

$$\mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n}) \to \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{P}_{m|n}). \tag{4.5}$$

Let $\mathbf{K}'_{m|n} = \mathbf{K}_{m|n}/\mathbf{P}_{m|n}$. By Theorem 20, for every simple module $\mathbf{V}^{\lambda,\mu}$ we have

$$[\mathbf{K}'_{m|n} : \mathbf{V}^{\lambda,\mu}][\mathbf{P}_{m|n} : \mathbf{V}^{\lambda,\mu}] = 0.$$

Therefore, every $\varphi \in \mathrm{End}_{\mathfrak{sl}(\infty)}(\mathbf{K}_{m|n})$ such that $\varphi(\mathbf{P}_{m|n}) = 0$ is identically zero, since for such φ the socle of $\operatorname{im}\varphi$ is zero. In other words, homomorphism (4.5) is injective. The surjectivity follows from the fact that every $\varphi : \mathbf{P}_{m|n} \to \mathbf{P}_{m|n} \hookrightarrow \mathbf{K}_{m|n}$ extends to $\tilde{\varphi} : \mathbf{K}_{m|n} \to \mathbf{K}_{m|n}$ by the injectivity of $\mathbf{K}_{m|n}$. □

4.3. **The Zuckerman functor $\Gamma_{\mathfrak{gl}(m|n)}$ and the category $\mathcal{F}^{\mathbb{Z}}_{m|n}$.** Let us recall the definition of the derived Zuckerman functor. A systematic treatment of the Zuckerman functor for Lie superalgebras can be found in [S]. Assume that M is a finitely generated $\mathfrak{gl}(m|n)$-module which is semisimple over the Cartan subalgebra of $\mathfrak{gl}(m|n)$. Let $\Gamma_{\mathfrak{gl}(m|n)}(M)$ denote the subspace of $\mathfrak{gl}(m|n)_0$-finite vectors. Then $\Gamma_{\mathfrak{gl}(m|n)}(M)$ is a finite-dimensional $\mathfrak{gl}(m|n)$-module, and hence $\Gamma_{\mathfrak{gl}(m|n)}$ is a left exact functor from the category of finitely generated $\mathfrak{gl}(m|n)$-modules, semisimple over the Cartan subalgebra, to the category $\mathcal{F}_{m|n}$ of finite-dimensional modules. The corresponding right derived functor $\Gamma^i_{\mathfrak{gl}(m|n)}$ is called the *i-th derived Zuckerman functor*. Note that $\Gamma^i_{\mathfrak{gl}(m|n)}(X) = 0$ for $i > \dim \mathfrak{gl}(m|n)_0 - (m+n)$. We are interested in the restriction of this functor

$$\Gamma^i_{\mathfrak{gl}(m|n)} : \mathcal{O}^{\mathbb{Z}}_{m|n} \to \mathcal{F}^{\mathbb{Z}}_{m|n}.$$

Let us consider the linear operator $\gamma : \mathbf{K}_{m|n} \to \mathbf{J}_{m|n}$ given by

$$\gamma([M]) = \sum_i (-1)^i [\Gamma^i_{\mathfrak{gl}(m|n)} M].$$

This operator is well defined as for any short exact sequence of $\mathfrak{gl}(m|n)$-modules

$$0 \to N \to M \to L \to 0,$$

we have the Euler characteristic identity

$$\gamma([M]) = \gamma([N]) + \gamma([L]).$$

It is well known that $\Gamma^i_{\mathfrak{gl}(m|n)}$ commutes with the functors $\cdot \otimes U$ and $\cdot \otimes U^*$, and with the projection to the block $(\mathcal{O}^{\mathbb{Z}}_{m|n})_\chi$ with a fixed central character χ. Therefore, γ is a homomorphism of $\mathfrak{sl}(\infty)$-modules.

Proposition 27. *The homomorphism γ is given by the formula*

$$\gamma = \sum_{s \in S_m \times S_n} \mathrm{sgn}(s) s, \tag{4.6}$$

where the action of s on $\mathbf{K}_{m|n}$ is defined in Proposition 26.

Proof. By Proposition 26, it suffices to check the equality (4.6) on vectors in $\mathbf{T}_{m|n}$, which amounts to checking that for all Verma modules $M(\lambda)$

$$\gamma([M(\lambda)]) = \sum_{s \in S_m \times S_n} \mathrm{sgn}(s)[M(s \cdot \lambda)], \tag{4.7}$$

where $s \cdot \lambda = s(\lambda + \rho) - \rho$ and $\rho = (m-1, \dots, 0|0, -1, \dots, 1-n)$.

Consider the functor Res_0 of restriction to $\mathfrak{gl}(m|n)_0$. This is an exact functor from the category of finitely generated $\mathfrak{gl}(m|n)$-modules, semisimple over the Cartan subalgebra, to the similar category of $\mathfrak{gl}(m|n)_0$-modules. It is clear from the definition of $\Gamma^i_{\mathfrak{gl}(m|n)}$ that

$$\mathrm{Res}_0\,\Gamma^i_{\mathfrak{gl}(m|n)} = \Gamma^i_{\mathfrak{gl}(m|n)_0}\,\mathrm{Res}_0\,. \tag{4.8}$$

Recall that every Verma module $M(\lambda)$ over $\mathfrak{gl}(m|n)$ has a finite filtration with successive quotients isomorphic to Verma modules $M^0(\mu)$ over $\mathfrak{gl}(m|n)_0$. Hence by (4.8) it suffices to check the analogue of (4.7) for even Verma modules:

$$\gamma^0([M^0(\lambda)]) = \sum_{s\in S_m\times S_n} \mathrm{sgn}(s)[M^0(s\cdot\lambda)], \tag{4.9}$$

where γ^0 is the obvious analogue of γ. To prove (4.9) we observe that $[M^0(\lambda)] = [M^0(\lambda)^\vee]$ where $X^\vee$ stands for the contragredient dual of X.

It is easy to compute $\Gamma^i_{\mathfrak{gl}(m|n)_0} M^0(\lambda)^\vee$. Let $\mathfrak{t}$ denote the Cartan subalgebra of $\mathfrak{gl}(m|n)$, and let $\mathfrak{n}_0^+$, $\mathfrak{n}_0^-$ be the maximal nilpotent ideals of the Borel and opposite Borel subalgebras of $\mathfrak{gl}(m|n)_0$, respectively. From the definition of the derived Zuckerman functor, the following holds for any $\mu\in\Phi^+$

$$\mathrm{Hom}_{\mathfrak{gl}(m|n)_0}(L^0(\mu), \Gamma^i_{\mathfrak{gl}(m|n)_0} M) \simeq \mathrm{Ext}^i(L^0(\mu), M),$$

where the extension is taken in the category of modules semisimple over $\mathfrak{t}$. If $M = M^0(\lambda)^\vee$, then M is cofree over $U(\mathfrak{n}_0^+)$ and therefore

$$\mathrm{Ext}^i(L^0(\mu), M^0(\lambda)^\vee) \simeq \mathrm{Hom}_{\mathfrak{t}}(H_i(\mathfrak{n}_0^-, L^0(\mu)), \mathbb{C}_\lambda).$$

Now we apply Kostant's theorem to conclude that

$$\Gamma^i_{\mathfrak{gl}(m|n)_0} M^0(\lambda)^\vee = \begin{cases} L^0(\mu) & \text{if } \mu = s\cdot\lambda \text{ for } s\in S_m\times S_n,\ l(s)=i,\\ 0 & \text{otherwise.}\end{cases}$$

Here μ is the only dominant weight in $(S_m\times S_n)\cdot\lambda$ and hence s is unique. Moreover, if $\lambda+\rho$ is a singular weight then $\Gamma^i_{\mathfrak{gl}(m|n)_0} M^0(\lambda)^\vee = 0$ for all i. Combining this with the Weyl character formula

$$[L^0(\mu)] = \sum_{s\in S_m\times S_n} \mathrm{sgn}(s)[M^0(s\cdot\mu)]$$

we obtain (4.9), and hence the proposition. □

Corollary 28. *We have* $\mathbf{J}_{m|n} = \gamma(\mathbf{K}_{m|n})$ *and* $\mathbf{K}_{m|n} = \mathbf{J}_{m|n}\oplus\ker\gamma$. *In particular,* $\mathbf{J}_{m|n}$ *is an injective hull of* $\mathbf{\Lambda}_{m|n}\cong\Lambda^m\mathbf{V}\otimes\Lambda^n\mathbf{V}_*$.

Recall that $\mathbf{\Lambda}_{m|n}\subset\mathbf{J}_{m|n}$ denotes the subspace generated by the classes of all Kac modules. Let $\mathcal{Q}_{m|n}$ denote the additive subcategory of $\mathcal{F}^{\mathbb{Z}}_{m|n}$ which consists of projective finite-dimensional $\mathfrak{gl}(m|n)$-modules, and let $Q_{m|n}$ denote the reduced Grothendieck group of $\mathcal{Q}_{m|n}$. It was proven in [CS, Theorem 3.11] that $\mathbf{Q}_{m|n} := Q_{m|n}\otimes_{\mathbb{Z}}\mathbb{C}$ is the socle of the module $\mathbf{\Lambda}_{m|n}$, implying that $\mathbf{Q}_{m|n}\cong\mathbf{V}^{(m)^\perp,(n)^\perp}$, where $\perp$ indicates the conjugate partition. Corollary 28 implies the following.

Corollary 29. $\mathbf{J}_{m|n}$ *is an injective hull of* $\mathbf{Q}_{m|n}$, *and the socle filtration of* $\mathbf{J}_{m|n}$ *is*

$$\overline{\mathrm{soc}}^i\mathbf{J}_{m|n}\cong\left(\mathbf{V}^{(m-i)^\perp(n-i)^\perp}\right)^{\oplus(i+1)}.$$

4.4. **The Duflo–Serganova functor and the tensor filtration.** In this section, we discuss the relationship between the Duflo–Serganova functor and submodules of the $\mathfrak{sl}(\infty)$-modules $\mathbf{K}_{m|n}$ and $\mathbf{J}_{m|n}$.

Let $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$ be a finite-dimensional contragredient Lie superalgebra. For any odd element $x \in \mathfrak{a}_{\bar{1}}$ which satisfies $[x, x] = 0$, the *Duflo–Serganova functor* DS_x is defined by

$$DS_x : \ \mathfrak{a} - \mathrm{mod} \to \mathfrak{a}_x - \mathrm{mod}$$
$$M \mapsto \ker_M x/xM,$$

where $\ker_M x/xM$ is a module over the Lie superalgebra $\mathfrak{a}_x := \mathfrak{a}^x/[x, \mathfrak{a}]$ (here $\mathfrak{a}^x$ denotes the centralizer of x in $\mathfrak{a}$) [DS]. In what follows we set

$$M_x := DS_x(M).$$

The Duflo–Serganova functor DS_x is a symmetric monoidal functor, [DS], see also Proposition 5 in [Ser].

It is known that the functor DS is not exact, nevertheless it induces a homomorphism ds_x between the reduced Grothendieck groups of the categories $\mathfrak{a}$-mod and $\mathfrak{a}_x$-mod defined by $ds_x([M]) = [M_x]$. (Recall that "reduced" indicates passage to the quotient by the relation $[\Pi M] = -[M]$, where Π is the parity reversing functor.) This follows from the following statement, see Section 1.1 in [GS].

Lemma 30. *For every exact sequence of $\mathfrak{a}$-modules*

$$0 \to M_1 \xrightarrow{\psi} M_2 \xrightarrow{\varphi} M_3 \to 0$$

there exists an exact sequence of $\mathfrak{a}_x$-modules

$$0 \to E \to DS_x(M_1) \xrightarrow{DS_x(\psi)} DS_x(M_2) \xrightarrow{DS_x(\varphi)} DS_x(M_3) \to \Pi E \to 0,$$

for an appropriate $\mathfrak{a}_x$-module E.

Proof. Set $E := \mathrm{Ker}(DS_x(\psi))$, $E' := \mathrm{Coker}(DS_x(\varphi))$, and consider the exact sequence

$$0 \to E \to DS_x(M_1) \to DS_x(M_2) \to DS_x(M_3) \to E' \to 0.$$

The odd morphism $\psi^{-1}x\varphi^{-1} : DS_x(M_3) \to DS_x(M_1)$ induces an isomorphism $E' \to \Pi E$. □

In [HR] the existence of the homomorphism ds_x was proven for finite-dimensional modules.

Remark 31. If $0 \to C_1 \to \cdots \to C_k \to 0$ is a complex of $\mathfrak{a}$-modules with odd differentials, the Euler characteristic of this complex is defined as the element $\sum_{i=1}^k [C_i]$ in the reduced Grothendieck group. If H_i denotes the i-th cohomology group, then

$$\sum_{i=1}^k [C_i] = \sum_{i=1}^k [H_i].$$

The absence of the usual sign follows from the relation $[\Pi M] = -[M]$ and the fact that the differentials are odd. For example, for an acyclic complex $0 \to X \to \Pi X \to 0$ the Euler characteristic is zero.

Let $\mathfrak{a} = \mathfrak{gl}(m|n)$ and suppose $\operatorname{rank} x = k$. Then $\mathfrak{a}_x \cong \mathfrak{gl}(m-k|n-k)$. Let $\mathcal{O}^{ind}_{m|n}$ be the category whose objects are direct limits of objects in $\mathcal{O}_{m|n}$. Then by Lemma 5.2 in [CS] the restriction of DS_x to $\mathcal{O}_{m|n}$ is a well-defined functor

$$DS_x : \mathcal{O}_{m|n} \to \mathcal{O}^{ind}_{m-k|n-k}.$$

Lemma 32. *The functor $DS_x : \mathcal{O}^{\mathbb{Z}}_{m|n} \to (\mathcal{O}^{\mathbb{Z}}_{m-k|n-k})^{ind}$ commutes with translation functors.*

Proof. Recall that U is the natural $\mathfrak{gl}(m|n)$-module. Since DS is a monoidal functor, we have a canonical isomorphism

$$(M \otimes U)_x \simeq M_x \otimes U_x.$$

Moreover, a direct computation shows that U_x is isomorphic to the natural $\mathfrak{gl}(m-k|n-k)$-module. We will use these observations to show that there is a canonical isomorphism

$$\mathrm{E}_i(M_x) \simeq (\mathrm{E}_i(M))_x. \tag{4.10}$$

Recall the notations of Section 3.1. Define the homomorphism of $\mathfrak{gl}(m|n)$-modules

$$\omega_{m|n} : \mathbb{C} \to \mathfrak{gl}(m|n) \otimes \mathfrak{gl}(m|n), \quad 1 \mapsto \sum (-1)^{p(X_j)} X_j \otimes Y_j.$$

We have $DS_x(\omega_{m|n}) = \omega_{m-k|n-k}$. Consider the composition

$$\Omega : M \otimes U \xrightarrow{1\otimes\omega_{m|n}\otimes 1} M \otimes \mathfrak{gl}(m|n) \otimes \mathfrak{gl}(m|n) \otimes U \xrightarrow{r_M \otimes l_U} M \otimes U,$$

where $r_M : M \otimes \mathfrak{gl}(m|n) \to M$ is the morphism of right action, and $l_U : \mathfrak{gl}(m|n) \otimes U \to U$ is the morphism of left action. The morphism $DS_x(\Omega) : M_x \otimes U_x \to M_x \otimes U_x$ is defined in a similar manner in the category of $\mathfrak{gl}(m-k|n-k)$-modules. Recall that

$$\mathrm{E}_i(M) = \{v \in M \otimes U \mid (\Omega - i)^N v = 0 \quad \text{for some } N > 0\};$$

similarly

$$\mathrm{E}_{\mathrm{i}}(M_x) = \{v \in M_x \otimes U_x \mid (DS_x(\Omega) - i)^N v = 0 \quad \text{for some } N > 0\}.$$

This implies the existence of the isomorphism (4.10) as desired.

The proof for F_i is similar. □

We are going to strengthen the result of [CS] by proving the following proposition.

Proposition 33. *The restriction of DS_x to $\mathcal{O}_{m|n}$ is a well-defined functor*

$$DS_x : \mathcal{O}_{m|n} \to \mathcal{O}_{m-k|n-k}.$$

To prove the proposition we first consider the case when $k = 1$.

Lemma 34. *If $k = 1$, then the restriction of DS_x to $\mathcal{O}_{m|n}$ is a well-defined functor*

$$DS_x : \mathcal{O}_{m|n} \to \mathcal{O}_{m-1|n-1}.$$

Proof. By Theorem 5.1 in [CS] we may assume without loss of generality that x is a generator of the root space $\mathfrak{gl}(m|n)_\alpha$ for some $\alpha = \pm(\varepsilon_i - \delta_j)$. Moreover, we can choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{gl}(m|n)$ so that α is a simple root. Let M be an object in the category $\mathcal{O}_{m|n}$ and M^μ denote the weight space of weight μ. The set of all weights of M is denoted by $\operatorname{supp} M$. Let $x_\mu : M^\mu \to M^{\mu+\alpha}$ be the restriction of x as an operator on M. Then

$$M_x = \oplus_{\mu \in \operatorname{supp} M} M^\mu_x \quad \text{where} \quad M^\mu_x = \ker x_\mu / x_{\mu-\alpha}(M^{\mu-\alpha}).$$

Let us first check that all weight multiplicities of M_x are finite with respect to the Cartan subalgebra $\mathfrak{h}_x := \ker \varepsilon_i \cap \ker \delta_j$ of $\mathfrak{g}_x$. We have to show that for any $\nu \in \mathfrak{h}_x^*$

$$\sum_{\mu \in \operatorname{supp} M, \mu|_{\mathfrak{h}_x} = \nu} \dim M_x^\mu < \infty. \tag{4.11}$$

Note that $\dim M_x^\mu \neq 0$ implies $(\mu, \alpha) = 0$, by $\mathfrak{sl}(1|1)$-representation theory. If $(\mu', \alpha') = 0$ and $\mu|_{\mathfrak{h}_x} = \mu'|_{\mathfrak{h}_x}$, then $\mu - \mu' \in \mathbb{C}\alpha$. Denote by Δ_s the set of simple roots of $\mathfrak{b}$. Since M is an object of $\mathcal{O}_{m|n}$, M has a finite filtration by highest weight modules. Therefore it suffices to consider the case when M is a highest weight module. Let λ be the highest weight of M. Then every $\mu \in \operatorname{supp} M$ has the form $\lambda - \sum_{\beta \in \Delta_s} k_\beta \beta$ for some $k_\beta \in \mathbb{Z}_{\geq 0}$ satisfying $k_\alpha \leq 1 + \sum_{\beta \in \Delta_S \setminus \alpha} k_\beta$. Therefore, for any $\mu \in \operatorname{supp} M$ the set $(\mu + \mathbb{C}\alpha) \cap \operatorname{supp} M$ is finite. Hence, for any $\nu \in \mathfrak{h}_x^*$ the set of $\mu \in \operatorname{supp} M$ such that $\mu|_{\mathfrak{h}_x} = \nu$ and $(\mu, \alpha) = 0$ is finite. Since all weight spaces of M are finite dimensional, this implies (4.11).

To finish the proof we observe that Lemma 32 implies $\mathrm{E}_i(M_x) = \mathrm{F}_i(M_x) = 0$ for almost all $i \in \mathbb{Z}$. Now for each $i \in \operatorname{supp}(\bar{\lambda})$, at least one of the $\mathrm{E}_i, \mathrm{E}_{i+1}, \mathrm{F}_i, \mathrm{F}_{i+1}$ does not annihilate $L_{\mathfrak{g}_x}(\lambda)$. Together this implies that the set S_M of all weights λ satisfying $[M_x : L_{\mathfrak{g}_x}(\lambda)] \neq 0$ is a finite set. On the other hand, since M_x has finite weight multiplicities, every simple constituent occurs in M_x with finite multiplicity. Hence M_x has finite length. □

Proof. Now we prove Proposition 33 by induction on $\operatorname{rank}(x) = k$. By Theorem 5.1 in [CS], x is B_0-conjugate to $x_1 + \cdots + x_k$, where $x_i \in \mathfrak{gl}(m|n)_{\alpha_i}$ for some linearly independent set of mutually orthogonal odd roots $\beta_1, \ldots, \beta_k$. So without loss of generality we may suppose that $x = x_1 + \cdots + x_k$. Let $y = x_1 + \cdots + x_{k-1}$. Choose $h_y \in \mathfrak{h}_{x_k}$ and $h_{x_k} \in \mathfrak{h}_y$ such that $\alpha(h_y), \alpha(h_{x_k}) \in \mathbb{Z}$ for all roots α of $\mathfrak{gl}(m|n)$, $[h_y, y] = y$ and $[h_{x_k}, x_k] = x_k$. Assume that $M \in \mathcal{O}_{m|n}$ and $\operatorname{supp} M \in \lambda + Q$, where Q is the root lattice. Then $\operatorname{ad} h_y - \lambda(h_y)$ and $\operatorname{ad} h_{x_k} - \lambda(h_{x_k})$ define a $\mathbb{Z} \times \mathbb{Z}$-grading on M and the differentials y and x_k form a bicomplex. Moreover, M_x is nothing but the cohomology $\bigoplus_r H^r(y + x_k, M)$ of the total complex.

Consider the second term

$$E_2^{p,q}(M) = H^p(x_k, H^q(y, M))$$

of the spectral sequence of this bicomplex. By the induction assumption $M_y \in \mathcal{O}_{m-k+1|n-k+1}$, and in particular, $H^q(y, M) \neq 0$ for finitely many q. The induction assumption implies that $H^p(x_k, H^q(y, M)) \in \mathcal{O}_{m-k|n-k}$ does not vanish for finitely many p. This yields $\bigoplus_{p,q} E_2^{p,q}(M) \in \mathcal{O}_{m-k|n-k}$. Since $\bigoplus_r H^r(y + x_k, M)$ is a subquotient of $\bigoplus_{p,q} E_2^{p,q}(M)$, we obtain

$$M_x = \bigoplus_r H^r(y + x_k, M) \in \mathcal{O}_{m-k|n-k}.$$

□

Next note that the restriction of DS_x to $\mathcal{O}^{\mathbb{Z}}_{m|n}$ is a well-defined functor

$$\mathcal{O}^{\mathbb{Z}}_{m|n} \to \mathcal{O}^{\mathbb{Z}}_{m-k|n-k}.$$

Since DS_x is a well-defined functor from $\mathcal{O}^{\mathbb{Z}}_{m|n}$ to $\mathcal{O}^{\mathbb{Z}}_{m-k|n-k}$ we see that $ds_x : K_{m|n} \to K_{m-k|n-k}$ is a well-defined group homomorphism.

Lemma 35. *If $x = x_1 + \cdots + x_k$ with commuting $x_1, \ldots, x_k$ of rank 1, then on $K_{m|n}$ we have the identity*

$$ds_x = ds_{x_k} \circ \cdots \circ ds_{x_1}.$$

Proof. We retain the notation of the proof of Proposition 33. Clearly, it suffices to check that

$$ds_x = ds_{x_k} \circ ds_y,$$

where $y = x_1 + \cdots + x_{k-1}$. The Euler characteristic of the E_s-terms of the spectral sequence from the proof of Proposition 33 remains unchanged for $s \geq 2$:

$$[\bigoplus_{p,q} E_2^{p,q}(M)] = [\bigoplus_{p,q} E_s^{p,q}(M)].$$

As the spectral sequence converges to $[M_x]$, we obtain

$$ds_{x_k} \circ ds_y([M]) = [\bigoplus_{p,q} E_2^{p,q}(M)] = [M_x] = ds_x([M]).$$

□

For the category of finite-dimensional modules the above statement is proven in [HR].

Proposition 36. *The complexification $ds_x : \mathbf{K}_{m|n} \to \mathbf{K}_{m-k|n-k}$ is a homomorphism of $\mathfrak{sl}(\infty)$-modules, as is its restriction $ds_x : \mathbf{J}_{m|n} \to \mathbf{J}_{m-k|n-k}$ to the $\mathfrak{sl}(\infty)$-submodule $\mathbf{J}_{m|n} := J_{m|n} \otimes_{\mathbb{Z}} \mathbb{C}$.*

Proof. This follows from the fact that the Duflo–Serganova functor commutes with translation functors, see Lemma 32. □

Remark 37. Note that in [HR] the ring $J_{m|n}$ is denoted by $\mathcal{J}_G$ where $G = GL(m|n)$.

Let $X_{\mathfrak{a}} = \{x \in \mathfrak{a}_{\bar{1}} : [x,x] = 0\}$, and let

$$\mathcal{B}_{\mathfrak{a}} = \{B \subset \Delta_{iso} \mid B = \{\beta_1, \ldots, \beta_k \mid (\beta_i, \beta_j) = 0,\ \beta_i \neq \pm\beta_j\}\} \tag{4.12}$$

be the set of subsets of linearly independent mutually orthogonal isotropic roots of $\mathfrak{a}$. Then the orbits of the action of the adjoint group $G_{\bar{0}}$ of $\mathfrak{a}_{\bar{0}}$ on $X_{\mathfrak{a}}$ are in one-to-one correspondence with the orbits of the Weyl group $\mathcal{W}$ of $\mathfrak{a}_{\bar{0}}$ on $\mathcal{B}_{\mathfrak{a}}$ via the correspondence

$$B = \{\beta_1, ..., \beta_k\} \mapsto x = x_{\beta_1} + \cdots + x_{\beta_k} \in X_{\mathfrak{a}}, \tag{4.13}$$

where each $x_{\beta_i} \in \mathfrak{a}_{\beta_i}$ is chosen to be nonzero [DS, Theorem 4.2].

Lemma 38. *Let $\mathfrak{a} = \mathfrak{gl}(m|n)$. Fix $x \in X_{\mathfrak{a}}$ and set $k = |B_x|$, where $B_x \in \mathcal{B}_{\mathfrak{a}}$ corresponds to x. The homomorphism $ds_x : J_{m|n} \to J_{m-k|n-k}$ depends only k, and not on x.*

Proof. This follows from the description of ds_x given in [HR, Theorem 10], using the fact that supercharacters of finite-dimensional modules are invariant under the Weyl group $\mathcal{W} = S_m \times S_n$ of $\mathfrak{gl}(m|n)$. If $B_1, B_2 \in \mathcal{B}$ with $|B_1| = |B_2|$ then there exists $w \in \mathcal{W}$ satisfying: $\pm\beta \in w(B_1)$ if and only if $\pm\beta \in B_2$. So if $f \in J_{m|n}$ we have that

$$ds_{x_1}(f) = f|_{\beta_1^1,\ldots,\beta_k^1=0} = w(f)|_{w(\beta_1^1),\ldots,w(\beta_k^1)=0} = w(f)|_{\beta_1^2,\ldots,\beta_k^2=0} = f|_{\beta_1^2,\ldots,\beta_k^2=0} = ds_{x_2}(f).$$

□

Note that Lemma 38 does not hold if we replace $J_{m|n}$ with $K_{m|n}$.

Remark 39. Since the homomorphism $ds_x : \mathbf{J}_{m|n} \to \mathbf{J}_{m-k|n-k}$ does not depend on x, we denote it by ds^k, where $|B_x| = k$, and we let $ds := ds^1$.

Now we introduce a filtration of an $\mathfrak{sl}(\infty)$-module $\mathbf{M}$, whose layers are tensor modules.

Definition 40. The *tensor filtration* of an $\mathfrak{sl}(\infty)$-module $\mathbf{M}$ is defined inductively by

$$\operatorname{tens}^0 \mathbf{M} := \operatorname{tens} \mathbf{M} := \Gamma_{\mathfrak{g},\mathfrak{g}}(\mathbf{M}), \qquad \operatorname{tens}^i \mathbf{M} := p_i^{-1}(\operatorname{tens}(\mathbf{M}/(\operatorname{tens}^{i-1} \mathbf{M}))),$$

where $p_i : \mathbf{M} \to \mathbf{M}/(\operatorname{tens}^{i-1} \mathbf{M})$ is the natural projection.

We also use the notation $\overline{\operatorname{tens}}^i \mathbf{M} = \operatorname{tens}^i \mathbf{M}/\operatorname{tens}^{i-1} \mathbf{M}$.

Note that $\operatorname{tens} \mathbf{M}$ is the maximal tensor submodule of $\mathbf{M}$.

Example 41. The socle of $\mathbf{J}_{1|1}$ is isomorphic to the adjoint module of $\mathfrak{sl}(\infty)$, and $\overline{\operatorname{soc}}^1 \mathbf{J}_{1|1} = \mathbb{C} \oplus \mathbb{C}$. Note that this is a special case of Example 21 in the case that $\mathfrak{k}$ has two infinite blocks.

Consider now the tensor filtration of $\mathbf{J}_{1|1}$. This filtration also has length 2, $\operatorname{tens} \mathbf{J}_{1|1} = \mathbf{\Lambda}_{1|1} \cong \mathbf{V} \otimes \mathbf{V}_*$ and $\overline{\operatorname{tens}}^1 \mathbf{J}_{1|1} \cong \mathbb{C}$. The module $\mathbf{J}_{1|1}$ admits a nice matrix realization. Indeed, we can identify the $\mathfrak{sl}(\infty)$-module $\mathbf{\Lambda}_{1|1}$ with the matrix realization of $\mathfrak{gl}(\infty)$ (see Section 3.1), and then extend it by the diagonal matrix D which has entries $D_{ii} = 1$ for $i \geq 1$ and 0 elsewhere. The action of $\mathfrak{sl}(\infty)$ in this realization of $\mathbf{J}_{1|1}$ is the adjoint action.

Proposition 42. *For each k, let $ds^k : \mathbf{J}_{m|n} \to \mathbf{J}_{m-k|n-k}$ be the homomorphism induced by the Duflo–Serganova functor (see Remark 39). Set $t := 1 + \min\{m, n\}$ and let $\mathbf{M}_k^t := \ker ds^k$. Consider the filtration of $\mathfrak{sl}(\infty)$-modules*

$$\mathbf{M}_1^t \subset \mathbf{M}_2^t \subset \cdots \subset \mathbf{M}_t^t = \mathbf{J}_{m|n}.$$

Then $\mathbf{M}_1^t = \mathbf{\Lambda}_{m|n}$ and $\mathbf{M}_{k+1}^t/\mathbf{M}_k^t \cong \Lambda^{m-k}\mathbf{V} \otimes \Lambda^{n-k}\mathbf{V}_*$. This filtration is the tensor filtration of $\mathbf{J}_{m|n}$, that is, $\operatorname{tens}^{k-1} \mathbf{J}_{m|n} = \ker ds^k$.

Proof. In the proof we let m and n vary. It follows from [HR, Theorems 17 and 20] that for every $m, n \in \mathbb{Z}_{>0}$ the map $ds : \mathbf{J}_{m|n} \to \mathbf{J}_{m-1|n-1}$ is surjective and the kernel is spanned by the classes of Kac modules. So we have an exact sequence of $\mathfrak{sl}(\infty)$-modules

$$0 \to \mathbf{\Lambda}_{m|n} \to \mathbf{J}_{m|n} \overset{ds}{\to} \mathbf{J}_{m-1|n-1} \to 0.$$

Thus, we obtain the following diagram of $\mathfrak{sl}(\infty)$-modules for each $l = |m - n|$, in which the horizontal arrows represent the map ds.

$$\begin{array}{ccccccccccc}
\twoheadrightarrow & \mathbf{M}_5^5 & \twoheadrightarrow & \mathbf{M}_4^4 & \twoheadrightarrow & \mathbf{M}_3^3 & \twoheadrightarrow & \mathbf{M}_2^2 & \twoheadrightarrow & \mathbf{M}_1^1 \\
 & \cup & & \cup & & \cup & & \cup & & \cup \\
\twoheadrightarrow & \mathbf{M}_4^5 & \twoheadrightarrow & \mathbf{M}_3^4 & \twoheadrightarrow & \mathbf{M}_2^3 & \twoheadrightarrow & \mathbf{M}_1^2 & \twoheadrightarrow & 0 \\
 & \cup & & \cup & & \cup & & \cup & & \\
\twoheadrightarrow & \mathbf{M}_3^5 & \twoheadrightarrow & \mathbf{M}_2^4 & \twoheadrightarrow & \mathbf{M}_1^3 & \twoheadrightarrow & 0 & & \\
 & \cup & & \cup & & \cup & & & & \\
\twoheadrightarrow & \mathbf{M}_2^5 & \twoheadrightarrow & \mathbf{M}_1^4 & \twoheadrightarrow & 0 & & & & \\
 & \cup & & \cup & & & & & & \\
\twoheadrightarrow & \mathbf{M}_1^5 & \twoheadrightarrow & 0 & & & & & &
\end{array}$$

By induction we get $\mathbf{M}_{k+1}^t/\mathbf{M}_k^t \cong \mathbf{M}_1^{t-k} = \mathbf{\Lambda}_{m-k|n-k}$, and by [B], $\mathbf{\Lambda}_{m-k|n-k} \cong \Lambda^{m-k}\mathbf{V} \otimes \Lambda^{n-k}\mathbf{V}_*$. Hence, the first claim follows.

For the second claim, suppose for sake of contradiction that for some k, the module $\mathbf{M}_{k+1}^t/\mathbf{M}_k^t$ is not the maximal tensor submodule of $\mathbf{J}_{m|n}/\mathbf{M}_k^t$. By projecting to $\mathbf{J}_{m-k|n-k}$, we obtain that $\mathbf{M}_1^t$ is not the maximal tensor submodule of $\mathbf{J}_{m|n}$, for some m, n. Since

$\mathbf{M}_1^t = \mathbf{\Lambda}_{m|n} \cong \Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$ is injective in the category $\mathbb{T}_{\mathfrak{g}}$ [DPS], this implies that $\operatorname{soc} \mathbf{J}_{m|n}$ is larger than $\operatorname{soc} \mathbf{M}_1^t$, which is a contradiction since $\operatorname{soc} \mathbf{J}_{m|n} = \operatorname{soc} \mathbf{\Lambda}_{m|n} = \mathbf{P}_{m|n}$. □

In the rest of this subsection, we fix x to be a generator of the root space corresponding to $\delta_j - \varepsilon_i$. We denote by $ds_{ij} : \mathbf{K}_{m|n} \to \mathbf{K}_{m-1|n-1}$ the $\mathfrak{sl}(\infty)$-module homomorphism ds_x.

Proposition 43. *We have*

$$\bigcap_{i,j} \ker ds_{ij} = \mathbf{T}_{m|n}.$$

Proof. It follows from [HR] that $ds_{ij}[M] = 0$ if and only if $e^{\varepsilon_i} - e^{\delta_j}$ divides the supercharacter $\operatorname{sch} M$ of M. Hence, $[M]$ lies in the intersection of kernels of all ds_{ij} if and only if $\prod_{i,j}(e^{\varepsilon_i} - e^{\delta_j})$ divides $\operatorname{sch} M$. This means that $\operatorname{sch} M$ is a linear combination of supercharacters induced from the parabolic subalgebra $\mathfrak{gl}(m|n)_{\bar{0}} \oplus \mathfrak{gl}(m|n)_1$. Therefore, $\operatorname{sch} M$ is a linear combination of supercharacters of Verma modules. □

Proposition 44. *We have* $\operatorname{tens} \mathbf{K}_{m|n} = \mathbf{T}_{m|n}$. *Moreover,* $\mathbf{K}_{m|n}$ *has an exhausting tensor filtration of length* $\min(m, n) + 1$.

Proof. Obviously $\operatorname{tens} \mathbf{K}_{m|n} \supset \mathbf{T}_{m|n}$. Assume that $\operatorname{tens} \mathbf{K}_{m|n} \neq \mathbf{T}_{m|n}$. Then since $\mathbf{T}_{m|n}$ is injective in $\mathbb{T}_{\mathfrak{g}}$ the socle of $\operatorname{tens} \mathbf{K}_{m|n}$ is larger than the socle of $\mathbf{T}_{m|n}$, but this is a contradiction since $\operatorname{soc} \mathbf{T}_{m|n} = \operatorname{soc} \mathbf{K}_{m|n}$. The second claim can be proven by induction on $\min(m, n)$, since $\mathbf{K}_{m|n}/\mathbf{T}_{m|n}$ is isomorphic to a submodule of $\mathbf{K}_{m-1|n-1}^{\oplus mn}$ via the map $\oplus_{ij} ds_{ij}$. □

4.5. **Meaning of the socle filtration.** Now we will define a filtration on the category $\mathcal{O}_{m|n}^{\mathbb{Z}}$. For a $\mathfrak{gl}(m|n)$-module M, let

$$X_M = \{x \in X_{\mathfrak{gl}(m|n)} \mid DS_x(M) \neq 0\},$$

and let $X_{\mathfrak{gl}(m|n)}^k$ be the subset of all elements in $X_{\mathfrak{gl}(m|n)}$ of rank less than or equal to k. We define $[\mathcal{O}_{m|n}^{\mathbb{Z}}]^k$ to be the full subcategory of $\mathcal{O}_{m|n}^{\mathbb{Z}}$ consisting of all modules M such that $X_M \subset X_{\mathfrak{gl}(m|n)}^k$. Note that $[\mathcal{O}_{m|n}^{\mathbb{Z}}]^k$ is not an abelian category. Furthermore, we define $[\mathcal{O}_{m|n}^{\mathbb{Z}}]_-^k$ to be the full subcategory of $\mathcal{O}_{m|n}^{\mathbb{Z}}$ consisting of all modules M such that

$$X_M \cap \mathfrak{gl}(m|n)_{-1} \subset X_{\mathfrak{gl}(m|n)}^k.$$

Let $\mathbf{K}_{m|n}^k$ denote the complexification of the subgroup in $\mathbf{K}_{m|n}$ generated by the classes of modules lying in $[\mathcal{O}_{m|n}^{\mathbb{Z}}]^k$, and let $(\mathbf{K}_{m|n}^k)_-$ be defined similarly for the category $[\mathcal{O}_{m|n}^{\mathbb{Z}}]_-^k$. Since both categories are invariant under the functors E_i and F_i, both $\mathbf{K}_{m|n}^k$ and $(\mathbf{K}_{m|n}^k)_-$ are $\mathfrak{sl}(\infty)$-submodules of $\mathbf{K}_{m|n}$.

Conjecture 45. $\mathbf{K}_{m|n}^k = \operatorname{soc}^{k+1} \mathbf{K}_{m|n}$ *and* $(\mathbf{K}_{m|n}^k)_- = \operatorname{tens}^{k+1} \mathbf{K}_{m|n}$.

Here we prove a weaker statement. Recall that $\mathcal{O}_{m|n}^{\mathbb{Z}}$ has block decomposition:

$$\mathcal{O}_{m|n}^{\mathbb{Z}} = \bigoplus (\mathcal{O}_{m|n}^{\mathbb{Z}})_\chi,$$

where $(\mathcal{O}_{m|n}^{\mathbb{Z}})_\chi$ is the subcategory of modules admitting generalized central character χ. The complexified reduced Grothendieck group of $(\mathcal{O}_{m|n}^{\mathbb{Z}})_\chi$ coincides with the weight subspace $(\mathbf{K}_{m|n})_\chi$. The degree of atypicality of χ is defined in [DS]. In [CS] it is proven that $(\mathcal{O}_{m|n}^{\mathbb{Z}})_\chi \subset [\mathcal{O}_{m|n}^{\mathbb{Z}}]^k$ if the degree of atypicality of χ is not greater than k. Note that the degree of atypicality of the highest weight χ of the irreducible $\mathfrak{sl}_\infty$-module $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is equal to $m-|\boldsymbol{\lambda}| = n-|\boldsymbol{\mu}|$ and the degree of atypicality of any weight of $\mathbf{V}^{\boldsymbol{\lambda},\boldsymbol{\mu}}$ is not less than the degree of atypicality of the highest weight. Combining this observation with the description of the socle filtration of $\mathbf{K}_{m|n}$ we obtain the following.

Proposition 46. $\operatorname{soc}^{k+1}\mathbf{K}_{m|n}$ *is the submodule in* $\mathbf{K}_{m|n}$ *generated by weight vectors of weights with degree of atypicality less or equal to k. Therefore we have* $\operatorname{soc}^{k+1}\mathbf{K}_{m|n}\subset\mathbf{K}^k_{m|n}$.

5. Appendix

In this section, we prove the technical lemma used in Lemma 25, which in turn is needed for the proof of Theorem 24.

Consider decompositions $\mathbf{V}=\mathbf{W}_1\oplus\mathbf{W}_2$ and $(\mathbf{V})_*=(\mathbf{W}_1)_*\oplus(\mathbf{W}_2)_*$ such that $\mathbf{W}_1^\perp=(\mathbf{W}_2)_*$ and $\mathbf{W}_2^\perp=(\mathbf{W}_1)_*$. Denote by $\mathfrak{s}$ the subalgebra $\mathfrak{sl}(\mathbf{W}_1)$ of $\mathfrak{g}$. Let $\mathbf{T}_{m|n}=\mathbf{V}^{\otimes m}\otimes\mathbf{V}_*^{\otimes n}$, and let $\mathbf{Y}_{m|n}$ be the intersection with $\mathbf{T}_{m|n}$ of the ideal generated by $\mathbf{W}_1\oplus(\mathbf{W}_1)_*$ in the tensor algebra $T(\mathbf{V}\oplus\mathbf{V}_*)$. Then $\mathbf{T}_{m|n}$ considered as an $\mathfrak{s}$-module admits the decomposition

$$\operatorname{Res}_{\mathfrak{s}}\mathbf{T}_{m|n}=(\mathbf{W}_2^{\otimes m}\otimes(\mathbf{W}_2)_*^{\otimes n})\oplus\mathbf{Y}_{m|n}.$$

Lemma 47. *We have*

$$(\operatorname{soc}\mathbf{T}_{m|n})\cap\mathbf{Y}_{m|n}\subset\mathfrak{s}\mathbf{Y}_{m|n}.$$

Proof. Note that $\mathbf{Y}_{m|n}$ is an object of $\widetilde{\mathbb{T}}_{\mathfrak{s}}$ and

$$\mathfrak{s}\mathbf{Y}_{m|n}=\bigcap_{\varphi\in\operatorname{Hom}_{\mathfrak{s}}(\mathbf{Y}_{m|n},\mathbb{C})}\ker\ \varphi. \tag{5.1}$$

Let τ denote a map from $\{1,\dots,m+n\}$ to $\{1,2\}$. Denote by $\mathbf{T}^\tau_{m|n}$ the subspace of $\mathbf{T}_{m|n}$ spanned by $v_1\otimes\cdots\otimes v_m\otimes u_{m+1}\otimes\cdots\otimes u_{m+n}$ with $v_i\in\mathbf{W}_{\tau(i)}$ and $u_j\in(\mathbf{W}_{\tau(j)})_*$. Clearly,

$$\operatorname{Res}_{\mathfrak{s}}\mathbf{T}_{m|n}=\bigoplus_\tau\mathbf{T}^\tau_{m|n},$$

and we have an $\mathfrak{s}$-module isomorphism

$$\mathbf{T}^\tau_{m|n}\cong\mathbf{W}_1^{\otimes p(\tau)}\otimes\mathbf{W}_2^{\otimes(m-p(\tau))}\otimes(\mathbf{W}_1)_*^{\otimes q(\tau)}\otimes(\mathbf{W}_2)_*^{\otimes(n-q(\tau))},$$

where

$$p(\tau):=|\tau^{-1}(1)\cap\{1,\dots,m\}|,\quad q(\tau):=|\tau^{-1}(1)\cap\{m+1,\dots,m+n\}|.$$

Furthermore,

$$\mathbf{Y}_{m|n}=\bigoplus_{p(\tau)+q(\tau)>0}\mathbf{T}^\tau_{m|n}.$$

Recall from [PStyr, Theorem 2.1] that

$$\operatorname{soc}\mathbf{T}_{m|n}=\bigcap_{1\le i\le m,m<j\le m+n}\ker\Phi_{ij},$$

where Φ_{ij} is defined in (3.3). For $r=1,2$, let $\Phi_{ij}^{\mathbf{W}_r}:\mathbf{T}_{m|n}\to\mathbf{T}_{m-1|n-1}$ be defined by

$$v_1\otimes\cdots\otimes v_m\otimes u_{m+1}\otimes\cdots\otimes u_{m+n}\mapsto\langle u_j,v_i\rangle^{\mathbf{W}_r}v_1\otimes\cdots\otimes\widehat{v_i}\otimes\cdots\otimes v_m\otimes u_{m+1}\otimes\cdots\otimes\widehat{u_j}\otimes\cdots\otimes u_{m+n},$$

where $\langle\cdot,\cdot\rangle^{\mathbf{W}_r}$ is defined on homogeneous elements by

$$\langle u_j, v_i\rangle^{\mathbf{W}_r} := \begin{cases} \langle u_j, v_i\rangle \text{ if } u_j, v_i \in \mathbf{W}_r \\ 0 \text{ otherwise.} \end{cases}$$

Next, recall from [DPS] that $\mathrm{Hom}_{\mathfrak{s}}(\mathbf{W}_1^{\otimes p} \otimes (\mathbf{W}_1)_*^{\otimes q}, \mathbb{C}) = 0$ if $p \neq q$, and if $p = q$, is spanned by compositions of contractions $\Phi^{\mathbf{W}_1}_{1,j_1} \dots \Phi^{\mathbf{W}_1}_{p,j_p}$ for all possible permutations $j_1, \dots, j_p$. Using (5.1) we can conclude that $\mathfrak{s}\mathbf{Y}^{\tau}_{m|n} = \mathbf{Y}^{\tau}_{m|n}$ if $p(\tau) \neq q(\tau)$, whereas if $p = p(\tau) = q(\tau)$ we have

$$\mathfrak{s}\mathbf{Y}^{\tau}_{m|n} = \bigcap_{i_1,\dots,i_p,j_1,\dots,j_p \in \tau^{-1}(1)} \ker \Phi^{\mathbf{W}_1}_{i_1,j_1} \dots \Phi^{\mathbf{W}_1}_{i_p,j_p}.$$

Observe that

$$\Phi_{ij} = \Phi^{\mathbf{W}_1}_{ij} + \Phi^{\mathbf{W}_2}_{ij}. \tag{5.2}$$

We claim that if $y = \sum_\tau y_\tau \in \mathbf{Y}_{m|n}$ and $\Phi_{ij}(y) = 0$ for all i, j, then $y_\tau \in \mathfrak{s}\mathbf{T}^{\tau}_{m|n}$ for all τ. The statement is trivial for every τ such that $p(\tau) \neq q(\tau)$. Now we proceed to prove the claim in the case $p(\tau) = q(\tau) = p$ by induction on p.

Let $p = 1$ and consider τ' with $p(\tau') = 1 = q(\tau')$. Let $i \leq m$ and $j > m$ be such that $\tau'(i) = \tau'(j) = 1$. Note that $\Phi_{i,j}(y'_\tau) \in (\mathbf{W}_2^{\otimes m-1} \otimes (\mathbf{W}_2)_*^{\otimes n-1})$ and for $\tau \neq \tau'$ we have $\Phi_{i,j}(y_\tau) \in Y_{m-1|n-1}$. Therefore, $\Phi_{i,j}(y_{\tau'}) = \Phi^{\mathbf{W}_1}_{i,j}(y_{\tau'}) = 0$ and hence $y_{\tau'} \in \mathfrak{s}\mathbf{T}^{\tau'}_{m|n}$.

Now consider $y_{\tau'}$ such that $p(\tau') = p = q(\tau')$. Let $i_1, \dots, i_p \leq m$ and $j_1, \dots j_p > m$ such that $\tau'(i) = \tau'(j) = 1$. We would like to show that

$$\Phi^{\mathbf{W}_1}_{i_1,j_1} \dots \Phi^{\mathbf{W}_1}_{i_p,j_p}(y_{\tau'}) = \Phi_{i_1,j_1} \dots \Phi_{i_p,j_p}(y_{\tau'}) = 0. \tag{5.3}$$

Note that τ' has the property

$$\Phi_{i_1,j_1} \dots \Phi_{i_p,j_p}(y_{\tau'}) \;\in\; \mathbf{W}_2^{\otimes m-p} \otimes (\mathbf{W}_2)_*^{\otimes n-p}. \tag{5.4}$$

Suppose that τ'' also has property (5.4). Then $(\tau'')^{-1}(1) \subset (\tau')^{-1}(1)$, and if $\Phi_{i_1,j_1} \dots \Phi_{i_p,j_p}(y_{\tau''}) \neq 0$, then $\tau''(i_r) = \tau''(j_r)$ for all $r = 1, \dots, p$. For every such $\tau'' \neq \tau'$ we have $p(\tau'') = q(\tau'') := l < p$. Let $\{i_{r_1}, \dots, i_{r_l}, j_{r_1}, \dots, j_{r_l}\} = (\tau'')^{-1}(1)$. Then by induction assumption $y_{\tau''} \in \mathfrak{s}\mathbf{T}^{\tau''}_{m|n}$ and hence

$$\Phi^{\mathbf{W}_1}_{i_{r_1},j_{r_1}} \dots \Phi^{\mathbf{W}_1}_{i_{r_l},j_{r_l}}(y_{\tau''}) = \Phi_{i_{r_1},j_{r_1}} \dots \Phi_{i_{r_l},j_{r_l}}(y_{\tau''}) = 0.$$

But then

$$\Phi_{i_1,j_1} \dots \Phi_{i_p,j_p}(y_{\tau''}) = 0,$$

which implies

$$\Phi_{i_1,j_1} \dots \Phi_{i_p,j_p}(y_{\tau'}) = 0.$$

Now (5.3) follows, and this implies $y_{\tau'} \in \mathfrak{s}\mathbf{T}^{\tau}_{m|n}$. □

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