# Dirac Cohomology and classical branching problems

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Joint work with

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$$B < 0 \text{ on } \mathfrak{k}_0,\ B > 0 \text{ on } \mathfrak{p}_0;\ \mathfrak{k} \perp \mathfrak{p}.$$

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$D$ is independent of the choice of basis $b_i$ and $K$-invariant.

$D^2$ is the spin Laplacean (Parthasarathy):

$$D^2 = -(\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + \|\rho_{\mathfrak{g}}\|^2) + (\mathrm{Cas}_{\mathfrak{k}_\Delta} + \|\rho_{\mathfrak{k}}\|^2).$$

Here $\mathrm{Cas}_{\mathfrak{g}}$, $\mathrm{Cas}_{\mathfrak{k}_\Delta}$ are the Casimir elements of $U(\mathfrak{g})$, $U(\mathfrak{k}_\Delta)$.

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$\mathfrak{k}_\Delta$ is the diagonal copy of $\mathfrak{k}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, defined by $\mathfrak{k} \hookrightarrow U(\mathfrak{g})$ and $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p}) \hookrightarrow C(\mathfrak{p})$.

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$H_D(X)$ is a module for the spin double cover $\widetilde{K}$ of $K$.

If $X$ is unitary or finite-dimensional, then

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# Vogan's conjecture

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### Theorem

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*Then $\lambda$ is $\gamma + \rho_\mathfrak{k}$ up to Weyl group $W_\mathfrak{g}$.*

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- Relations to other notions, like $\mathfrak{n}$-cohomology, $(\mathfrak{g}, K)$-cohomology (more details below), characters and branching

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- Can construct reps with $H_D \neq 0$ via "algebraic Dirac induction" (P.-Renard; Prlić)
- There is a translation principle for the Euler characteristic of $H_D$, i.e., the Dirac index (Mehdi-P.-Vogan).

# $\mathfrak{n}$-cohomology

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For a $(\mathfrak{g}, K)$-module $X$,

$$X \otimes S = X \otimes \bigwedge \mathfrak{p}^+ = \operatorname{Hom}(\bigwedge \mathfrak{p}^-, X)$$

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$$D = d + \partial.$$

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Similar result can be proved for $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ such that $\mathfrak{l} \subset \mathfrak{k}$ and $\mathfrak{u} \supset \mathfrak{p}^+$.

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$$H(\mathfrak{g}, K; X) = \mathrm{Hom}_K(H_D(F), H_D(X)).$$

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Dirac index of $X$: the virtual $\widetilde{K}$-module

$$I(X) = H_D(X)^+ - H_D(X)^-$$

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It follows that

$$\operatorname{ch} X(\operatorname{ch} S^+ - \operatorname{ch} S^-) = \operatorname{ch} I(X),$$

where ch denotes the $\widetilde{K}$-character.

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$\sigma'$: the partition obtained by flipping the Young diagram of $\sigma$ over the main diagonal.

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For $\sigma$ with $(\sigma')_1 + (\sigma')_2 \leq k$, $E^\sigma$ is the irreducible representation of $O(k)$ with highest weight obtained from $\sigma$ by adjusting the first column (which encodes the action of the component group).

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$$C^{\sigma}_{\mu} := \sum_{\nu \in P_R} c^{\sigma}_{\mu\nu} \qquad D^{\sigma}_{\mu} := \sum_{\nu \in P_C} c^{\sigma}_{\mu\nu}.$$

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We prove a formula equivalent to that of Enright and Willenbring using Dirac cohomology.

# Further notation

For any $n$-tuple $\sigma$, define

$$\sigma^{\diamond} = \sigma + (\underbrace{\frac{k}{2}, \dots, \frac{k}{2}}_{n}) \qquad \text{and} \qquad \sigma^{-\diamond} = \sigma - (\underbrace{\frac{k}{2}, \dots, \frac{k}{2}}_{n}).$$

# The result

Let $\sigma$ and $\mu$ be partitions with at most $k$ parts. Assume $(\mu')_1 + (\mu')_2 \leq k$.

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# Howe duality for $GL(k, \mathbb{C}) \times \mathfrak{u}(n)$

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Under the natural $GL(k, \mathbb{C}) \times \mathfrak{u}(n)$-action, $\mathcal{P}(M_{k \times n})$ decomposes as

$$\mathcal{P}(M_{k \times n}) \cong \bigoplus_{\sigma} (F^{\sigma})^* \otimes F^{\sigma},$$

with the sum over all partitions $\sigma$ with at most $\min(k, n)$ parts. (As usual, $\sigma$ is extended by adding zeros at the end if necessary.)

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Likewise, $GL(k, \mathbb{C}) \times \mathfrak{u}(k)$ and $Sp(k, \mathbb{C}) \times \mathfrak{so}^*(2k)$ are see-saw dual pairs, and hence the LRF for $Sp(k) \subset GL(k)$ will follow if we can find $K$-type multiplicities for the unitary lowest weight module $L(\mu^\diamond)$ for $(\mathfrak{so}^*(2k, \mathbb{R}), \mathfrak{u}(k))$.

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Let $F^\mu$ be the irreducible finite-dimensional representation of $\mathfrak{k}$ with highest weight $\mu$. Let $\mathfrak{p}^-$ act on $F^\mu$ by zero.

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$$\operatorname{ch} L(\mu) = \sum_\xi \operatorname{ch} N(\xi + \rho_n) - \sum_\eta \operatorname{ch} N(\eta + \rho_n). \qquad (*)$$

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This finishes the proof modulo the character formula (*).

# $K$-character formulas

To prove (*), we first note that the Dirac index of $N(\mu)$ is $F^{\mu-\rho_n}$, and hence

$$\operatorname{ch} N(\mu)(\operatorname{ch} S^+ - \operatorname{ch} S^-) = \operatorname{ch} F^{\mu-\rho_n}.$$

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We conclude that

$$\left(\mathrm{ch}\, L(\mu) - \sum_{\xi} \mathrm{ch}\, N(\xi + \rho_n) + \sum_{\eta} \mathrm{ch}\, N(\eta + \rho_n)\right)(\mathrm{ch}\, S^+ - \mathrm{ch}\, S^-) = 0.$$

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The point is that the $K$-types appearing in the virtual $K$-module

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In other words, when $F^\nu$ runs through the $K$-types occurring in $V$, the numbers $\langle \nu, \rho_n \rangle$ are bounded from below.