Non-rigidity of the tensor representations of $\mathfrak{gl}_\infty$

by

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Abstract

Let V and V_* be the natural and conatural representation of the complex finitary Lie algebra $\mathfrak{gl}_\infty$, respectively. We study the rigidity of the indecomposable direct summands of the tensor representations $V^{\otimes p} \otimes V_*^{\otimes q}$ of $\mathfrak{gl}_\infty$. The main result is that for $p, q \geq 3$ at least one direct summand of $V^{\otimes p} \otimes V_*^{\otimes q}$ is not rigid in the sense that the socle filtration does not coincide with the radical filtration.

1 Introduction

When studying a category of modules, and one is presented with a class of indecomposable modules, it is often essential to understand whether the modules in this class are rigid or not. An important example is Irving's rigidity theorem for Verma modules, see [B], [BB] and [I].

In [PS], I. Penkov and K. Styrkas introduced the tensor representations $V^{\otimes(p,q)}$ of the locally finite countable dimensional Lie algebras $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$ $\mathfrak{so}_\infty$ and $\mathfrak{sp}_\infty$. Moreover, they found the decomposition of $V^{\otimes(p,q)}$ into indecomposable summands and socle filtrations of the indecomposables. Very recently, Penkov, E. Dan-Cohen and V. Serganova in [DPS] proved that those indecomposable tensor representations are injective modules. In addition, it was observed by Serganova (oral communication), that some indecomposable direct summands of $V^{\otimes(p,q)}$ cannot be rigid as $\mathfrak{gl}_\infty$-modules.

In this thesis, I'm building on this observation and generalize it to a sufficient condition for non-rigidity of an indecomposable direct summand of $V^{\otimes(p,q)}$. I also show that this condition is satisfied by at least one direct summand of $V^{\otimes(p,q)}$ if $p, q \geq 3$. This is a starting point for the investigation of the radical filtration of the indecomposable direct summands of $V^{\otimes(p,q)}$.

2 Loewy filtrations

Let M be a module over a ring or a Lie algebra R. M is called **simple** if it does not contain a proper submodule. M is called **semisimple**, if it satisfies any of the following three equivalent definitions:

(1) M is the direct sum of simple submodules.

(2) M is the sum of some of its simple submodules.

(3) For any submodule $N \subset M$, there exists a complement, i.e. a submodule $K \subset M$ with $M = N \oplus K$.

Proof of their equivalence. (Adapted from [Bu]) The implication (1) $\Rightarrow$ (2) is clear. To see the implication (2) $\Rightarrow$ (3), let N be a submodule of M. Consider all those submodules of M that are sum of simple modules and intersect N only in $\{0\}$. The set of such modules is partially ordered by inclusion, and any chain of nested submodules has an upper bound, namely their union. So we can apply Zorn's Lemma to get a maximal submodule K intersecting N only in $\{0\}$. If M is not equal to $N \oplus K$, then M contains a simple submodule S that is not contained in $N \oplus K$. We have $S \cap (N \oplus K) = 0$ by the simplicity of S, hence $S + K$ is another submodule intersecting N trivially and being a sum of simple modules. This violates the maximality of K. Hence $M = N \oplus K$.

To show (3) $\Rightarrow$ (1), first observe that the condition (3) passes down to submodules: if M satisfies (3), and $K \subset N$ are both submodules of M, then K has a complement in N. Indeed, (3) applied to K and M gives a module L with $M = K \oplus L$, which yields the decomposition $N = K \oplus (N \cap L)$. Thus, N also satisfies (3).

Now consider all submodules of M that are a direct sum of simple submodules. Again, those modules are partially ordered by inclusion and any chain contains an upper bound, namely their union. So by Zorn's Lemma, we pick a maximal submodule of this kind and call it K. If $K \subsetneq M$ we can find an $x \in M$ such that $x \notin K$. Pick a maximal submodule P of Ry (using Zorn's Lemma once more), and find its complement S in Ry (using property (3) on Ry). Then S is simple. Otherwise P would not be maximal, then S could be added to K, and we have a contradiction with the maximality of P. Hence, $K = M$, i.e. M satisfies (1). □

A (strict) **finite filtration** is a sequence of nested modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M,$$

where the quotients M_{i+1}/M_i are nonzero. Then k is the **length** of the filtration. In this thesis, we are interested in **semisimple** filtrations, meaning that the quotients M_{i+1}/M_i are semisimple. All filtrations from now on are assumed to have this property. In general, a given module M does not necessarily admit a semisimple filtration, but the modules we will be concerned with will admit such filtrations.

When a finite filtration has maximal length, it is called **composition series**, and the corresponding length is called the **length** of the module M. The Jordan-Hölder-theorem tells us that any two composition series are equivalent, meaning that the quotients are isomorphic up to permutation. On the other hand, if a filtration has minimal length, it is called **Loewy filtration**. Two examples of Loewy filtrations are the socle filtration and the radical filtration, which we will define below.

As the above equivalence proof shows, any module M contains a maximal semisimple submodule. It is called the **socle** of M and is denoted by $\operatorname{soc} M$. Using the socle, one can define the **socle filtration** of M by setting $\operatorname{soc}^0 M := 0$, $\operatorname{soc}^{n+1} M := \pi_n^{-1}(\operatorname{soc}(M/\operatorname{soc}^n M))$, where $\pi_n : M \to M/\operatorname{soc}^n M$ is the quotient map. The quotients of two successive socles are called **layers** and are denoted $\overline{\operatorname{soc}}^n M := \operatorname{soc}^n M/\operatorname{soc}^{n-1} M$. The layers are always semisimple, by definition. Hence, the socle filtration of a module is indeed a filtration:

$$0 = \operatorname{soc}^0 M \subset \operatorname{soc}^1 M \subset \operatorname{soc}^2 M \subset \cdots \subset M.$$

In the socle filtration, the submodules are as large as they can be while yielding semisimple layers. There is also a second construction, which makes the layers as large as possible. Define the **radical** of M as $\operatorname{rad} M := \bigcap_{\Phi: M \to W} \ker \Phi$. Here, Φ runs over all homomorphisms from M to any semisimple module W. In other words, $\operatorname{rad} M$ is the smallest submodule of M with $M/\operatorname{rad} M$ being semisimple.

By applying rad repeatedly, one gets the radical filtration. Define $\operatorname{rad}^0 M := M$, $\operatorname{rad}^{n+1} := \operatorname{rad}(\operatorname{rad}^n M)$. This yields a decreasing sequence of nested submodules, whose quotients are semisimple:

$$0 \subset \cdots \subset \operatorname{rad}^2 M \subset \operatorname{rad}^1 M \subset \operatorname{rad}^0 M = M.$$

The radical filtration is only a filtration in the above sense if it is finite, i.e. $\operatorname{rad}^k M = 0$ for some k. If that is the case and we reverse the indices: $M_i = \operatorname{rad}^{k-i} M$, then (M_i) is a filtration in the above sense. As with the socle filtration, we define the **layers** of a radical filtration via $\overline{\operatorname{rad}}^n M = \operatorname{rad}^{n-1} M/\operatorname{rad}^n M$.

It is natural to ask how the radical and socle filtration are related. The following theorem gives an answer to that:

Theorem 2.1. *Let M be a module and assume that $\operatorname{soc}^n M = M$ or $\operatorname{rad}^n M = 0$ holds for some $n \geq 0$, but neither holds for any smaller n. Then the other equation is also true and the filtrations satisfy $\operatorname{rad}^{n-i} M \subset \operatorname{soc}^i M$ for all $0 \leq i \leq n$. In other words, if one of the filtrations exhausts M after a finite number of steps, then both filtrations are finite and have the same length. This length is called* **Loewy length** *of M.*

Proof. First assume that $\operatorname{soc}^n M = M$. We want to prove $\operatorname{rad}^i M \subset \operatorname{soc}^{n-i} M$ via induction on i for $i \geq 0$. For $i = 0$, the claim amounts to $M \subset M$, which is true. Assuming $\operatorname{rad}^i M \subset \operatorname{soc}^{n-i} M$, we want to show $\operatorname{rad}^{i+1} M \subset \operatorname{soc}^{n-i-1} M$. Let $\pi : \operatorname{soc}^{n-i} M \to \operatorname{soc}^{n-i} M / \operatorname{soc}^{n-i-1} M$ be the projection homomorphism. The kernel of this homomorphism is $\operatorname{soc}^{n-i-1} M$, and the range of the homomorphism is semisimple. So we have $\operatorname{rad}(\operatorname{soc}^{n-i} M) \subset \ker \pi = \operatorname{soc}^{n-i-1} M$. Since $\operatorname{rad}^i M \subset \operatorname{soc}^{n-i} M$, we have $\operatorname{rad}^{i+1} M \subset \operatorname{rad}(\operatorname{soc}^{n-i} M) \subset \operatorname{soc}^{n-i-1} M$ as desired.

Second, assume $\operatorname{rad}^n M = 0$. Then we will show $\operatorname{rad}^i M \subset \operatorname{soc}^{n-i} M$ by induction on i for $i \leq n$. For $i = n$, we simply have $0 \subset 0$. Let $\operatorname{rad}^i M \subset \operatorname{soc}^{n-i} M$, then we need to show that $\operatorname{rad}^{i-1} M \subset \operatorname{soc}^{n-i+1} M$. Let $\pi : M \to M/\operatorname{soc}^{n-i} M$ be the quotient map on $M/\operatorname{soc}^{n-i} M$, then $\operatorname{soc}^{n-i+1} M = \pi^{-1}(\operatorname{soc}(M/soc^{n-i}M))$. By definition of rad, the quotient $\operatorname{rad}^{i-1} M/\operatorname{rad}^i M$ is semisimple. Together with $\operatorname{rad}^i M \subset \operatorname{soc}^{n-i} M$, this implies that $\operatorname{rad}^{i-1} M \subset \pi^{-1}(\operatorname{soc}(M/soc^{n-i}M)) = \operatorname{soc}^{n-i+1}$.

To summarize, $\operatorname{soc}^n M = M$ implies $\operatorname{rad}^i M \subset \operatorname{soc}^{n-i} M$ for all $i \leq n$, especially $\operatorname{rad}^n M \subset \operatorname{soc}^0 M = 0$, i.e. $\operatorname{rad}^n M = 0$. On the other hand, $\operatorname{rad}^n M = 0$ implies the same, hence $M = \operatorname{rad}^0 M \subset \operatorname{soc}^n M$, i.e. $\operatorname{soc}^n M = M$. Therefore, $\operatorname{soc}^n M = M \Leftrightarrow \operatorname{rad}^n M = 0$, which means that both filtrations have the same length. □

So, if their length is finite, then the two filtrations have the same length and the radical filtration "sits inside" the socle filtration. If the socle and the radical filtrations coincide, the module is called **rigid**. The common length of those two filtrations is indeed the minimal length that any filtration can have, so they are Loewy filtrations. Any other Loewy filtration (M_i) "lies in between" those two filtrations, i.e. $\operatorname{rad}^{n-i} M \subset M_i \subset \operatorname{soc}^i M$ for all i, which can be shown similarly to Theorem 2.1. In particular, any rigid module M has a unique Loewy filtration which is given by both the socle and radical filtration.

Example: Let $\mathbb{C}[x]$ act on $\mathbb{C}^n$ as follows: $x \cdot (v_1, v_2, \ldots, v_n)^\intercal := (v_2, v_3, \ldots, v_n, 0)^\intercal$. Then it is easy to check that all submodules of $\mathbb{C}^n$ are of the form $x^k\mathbb{C}^n \cong \mathbb{C}^{n-k}$. The so obtained $\mathbb{C}[x]$-module $\mathbb{C}^n$ is rigid with socle and radical filtration given by $\operatorname{soc}^i \mathbb{C}^n = \operatorname{rad}^{n-i} \mathbb{C}^n = x^{n-i}\mathbb{C}^n$. All the layers are isomorphic to $\mathbb{C}$, which is a simple module over $\mathbb{C}[x]$ with zero action of x. We denote this graphically in the following way: The layers of a filtration are stacked on top of each other, the smallest modules being in the bottom box, and the successive layers being on top of it. To mark a filtration as socle or radical filtration, we put a small soc or rad as index. It is aligned in such a way that a group of boxes from the bottom form a submodule, and a group of boxes from the top form a quotient. For $\mathbb{C}^n$, the socle and radical filtrations look like this:

$$\mathbb{C}^n \sim \begin{array}{|c|}\hline \mathbb{C} \\ \hline \vdots \\ \hline \mathbb{C} \\ \hline \mathbb{C} \\ \hline \end{array}_{\text{soc}}, \mathbb{C}^n \sim \begin{array}{|c|}\hline \mathbb{C} \\ \hline \vdots \\ \hline \mathbb{C} \\ \hline \mathbb{C} \\ \hline \end{array}_{\text{rad}}$$

Socle and radical both behave well with respect to direct sums: The socle of the sum is the sum of the socles, and the radical of the sum is the sum of the radicals. This means that one can find a socle filtration of a direct sum by aligning the indidual filtrations at the lowest level, and the radical filtration by aligning at the highest level. This enables us to find the filtrations for e.g. $\mathbb{C}^4 \oplus \mathbb{C}^2 \oplus \mathbb{C}^1$ easily. We have the following situation:

$$\mathbb{C}^4 \oplus \mathbb{C}^2 \oplus \mathbb{C}^1 \sim \begin{array}{|c|}\hline \mathbb{C}\\\hline \mathbb{C}\\\hline \mathbb{C}\\\hline \mathbb{C}\\\hline\end{array}_{\mathrm{soc}} \oplus \begin{array}{|c|}\hline \mathbb{C}\\\hline \mathbb{C}\\\hline\end{array}_{\mathrm{soc}} \oplus \begin{array}{|c|}\hline \mathbb{C}\\\hline\end{array}_{\mathrm{soc}} \sim \begin{array}{|c|}\hline \mathbb{C}\\\hline \mathbb{C}\\\hline \mathbb{C}\oplus\mathbb{C}\\\hline \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\\\hline\end{array}_{\mathrm{soc}} \sim \begin{array}{|c|}\hline \mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}\\\hline \mathbb{C}\oplus\mathbb{C}\\\hline \mathbb{C}\\\hline \mathbb{C}\\\hline\end{array}_{\mathrm{rad}}$$

Here we can see what we just proved: The submodules of the radical filtration are contained in the corresponding submodules of the socle filtration. All that can happen when we go from the socle to the radical filtration is that simple modules can move to higher layers. One needs to keep in mind that this is just a graphical notation to highlight properties of the modules we study. Also we note that the layers of a socle filtration do not determine the module up to isomorphism.

3 A brief survey on tensor representations of $\mathfrak{gl}_\infty$

Preliminaries

A **partition** λ is a finite, non-strictly decreasing (also called weakly decreasing) sequence of positive integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k).$$

We set $|\lambda| = \lambda_1 + \cdots + \lambda_k$, and $\lambda_i = 0$ for $i > k$. The empty partition is denoted by 0.

Each partition has a **Young diagram** associated to it, formed by k rows with λ_i boxes in the i'th row. A **Young tableau** is a numbering of those boxes by the numbers 1 to $d := |\lambda|$.

One reason for partitions playing an important role in representation theory, is that they parametrize the irreducible representations of the symmetric group $\mathfrak{S}_d$. This works as follows. Given a tableau (e.g. the canonical one, counting row-by-row, left to right) of λ, we define two subgroups of $\mathfrak{S}_d$ by

$$P_\lambda = \{g \in \mathfrak{S}_d | g \text{ preserves each row}\},$$
$$Q_\lambda = \{g \in \mathfrak{S}_d | g \text{ preserves each column}\}.$$

Within the group algebra $\mathbb{C}[\mathfrak{S}_d]$, we define

$$a_\lambda = \sum_{g \in P_\lambda} e_g,$$
$$b_\lambda = \sum_{g \in Q_\lambda} \operatorname{sgn}(g) \cdot e_g.$$

Finally, we define the **Young projector** c_λ corresponding to some partition λ as $c_\lambda = a_\lambda b_\lambda$, and denote

$$H_\lambda := \mathbb{C}[\mathfrak{S}_d] c_\lambda.$$

When $\mathfrak{S}_d$ is acting on $V^{\otimes d}$ by permuting the factors, c_λ projects $V^{\otimes d}$ onto

$$\mathbb{S}_\lambda V := \operatorname{im}(c_\lambda : V^{\otimes d} \to V^{\otimes d}),$$

which is an irreducible representation of $\mathfrak{gl}_d$. $\mathbb{S}_\lambda$ is called the **Schur functor** corresponding to λ. For more details see [FH], p.77.

The **Littlewood-Richardson coefficients** $N^{\nu}_{\lambda,\mu}$ are nonnegative integers, determined for any partitions λ, μ, ν by the relation $S_\lambda S_\mu = \sum_\nu N^{\nu}_{\lambda,\mu} S_\mu$, where S_λ denotes the Schur symmetric polynomial corresponding to the partition λ. More on these coefficients can be found in section 4.

Tensor representations of $\mathfrak{gl}_\infty$

The following definitions and theorems work for any algebraically closed field $\mathbb{k}$ with characteristic 0, but we want to restrict ourselves to the case $\mathbb{k} = \mathbb{C}$ for this thesis. Let V and V_* be countable dimensional $\mathbb{C}$-vector spaces, and let $\langle \cdot, \cdot \rangle : V \otimes V_* \to \mathbb{C}$ be a non-degenerate pairing. Then the Lie algebra $\mathfrak{gl}_\infty$ is defined to be the vector space $V \otimes V_*$, equipped with the Lie bracket $[u \otimes u^*, v \otimes v^*] = \langle u^*, v \rangle\, u \otimes v^* - \langle v^*, u \rangle\, v \otimes u^*$, for $u, v \in V, u^*, v^* \in V_*$. $\mathfrak{sl}_\infty$ is defined as the kernel of the map $\langle \cdot, \cdot \rangle$.

As it was observed by Mackey (see [M]), it is always possible to find dual bases $\{\xi_i\}_{i \in \mathfrak{I}}$ of V and $\{\xi^*_i\}_{i \in \mathfrak{I}}$ of V_* for some countable index set $\mathfrak{I}$, such that we have $\langle \xi_i, \xi^*_j \rangle = \delta_{i,j}$. With these bases, an alternative definition of $\mathfrak{gl}_\infty$ is possible: $\mathfrak{gl}_\infty$ is the Lie algebra with linear basis $\{E_{i,j} = \xi_i \otimes \xi^*_j, i, j \in \mathfrak{I}\}$ and Lie bracket $[E_{i,j}, E_{k,l}] = \delta_{j,k} E_{i,l} - \delta_{i,l} E_{k,j}$.

We call V the **natural representation** of $\mathfrak{gl}_\infty$, and V_* its **restricted dual**. For non-negative integers p, q, the **tensor representation** $V^{\otimes(p,q)}$ is defined as $V^{\otimes p} \otimes V_*^{\otimes q}$, equipped with the structure of a $\mathfrak{gl}_\infty$-module:

$$(u \otimes u^*)\cdot(v_1 \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots \otimes v_q^*) = \sum_{i=1}^{p} \langle u^*, v_i \rangle\, v_1 \otimes \ldots \otimes v_{i-1} \otimes u \otimes v_{i+1} \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots \otimes v_q^* - \sum_{j=1}^{q} \langle v_j^*, u \rangle\, v_1 \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots \otimes v_{i-1}^* \otimes u^* \otimes v_{i+1}^* \otimes \ldots \otimes v_q^*$$

for $u, v_i \in V$ and $u^*, v_j^* \in V_*$. The product of the symmetric groups $\mathfrak{S}_p \times \mathfrak{S}_q$ acts on $V^{\otimes(p,q)}$ by permuting the factors. This action commutes with the $\mathfrak{gl}_\infty$-action: we say that $V^{\otimes(p,q)}$ is a $(\mathfrak{gl}_\infty, \mathfrak{S}_p \times \mathfrak{S}_q)$-module.

The purpose of this section is to present the result of Penkov and Styrkas in [PS] on the structure of $V^{\otimes(p,q)}$, i.e. the decomposition into indecomposable modules, and their socle filtration. The Jordan-Hölder constituents are identified and described as highest weight modules. The definitions and theorems are taken from section 2 of [PS], where their proofs can be found as well.

For any pair of indices $I = (i, j)$ with $i \in \{1, 2, \ldots, p\}$, $j \in \{1, 2, \ldots, q\}$, define the contraction

$$\Phi_I : V^{\otimes(p,q)} \to V^{\otimes(p-1,q-1)}$$

$$v_1 \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots \otimes v_q^* \mapsto \langle v_j^*, v_i \rangle\, v_1 \otimes \ldots \otimes \hat{v_i} \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots \otimes \hat{v_j^*} \otimes \ldots \otimes v_q^*$$

(Here, the hat on $\hat{v}_i$ means to leave out this term.)

Consider the $(\mathfrak{gl}_\infty, \mathfrak{S}_p \times \mathfrak{S}_q)$-submodule $V^{\{p,q\}}$ of $V^{\otimes(p,q)}$,

$$V^{\{p,q\}} := \bigcap_I \ker \left(\Phi_I : V^{\otimes(p,q)} \to V^{\otimes(p-1,q-1)} \right)$$

and furthermore, set $V^{\{p,0\}} := V^{\otimes p}$ and $V_*^{\{0,q\}} := V_*^{\otimes q}$. For any partitions λ, μ with $|\lambda| = p$ and $|\mu| = q$, define the $\mathfrak{gl}_\infty$-submodule $\Gamma_{\lambda,\mu}$ of $V^{\{p,q\}}$ as

$$\Gamma_{\lambda,\mu} := V^{\{p,q\}} \cap (\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V_*).$$

Highest weight modules

As we will see in the next section, all Jordan-Hölder constituents of $V^{\otimes(p,q)}$ have the form $\Gamma_{\lambda,\mu}$ for $|\lambda| \le p$, $|\mu| \le q$. An important property of the $\mathfrak{gl}_\infty$-modules $\Gamma_{\lambda;\mu}$ is that they are highest weight modules for a certain Borel subalgebra of $\mathfrak{gl}_\infty$. The meaning of this statement is explained in this section.

An element of $\mathfrak{gl}_\infty$ is called **semisimple** if it acts semisimply on $\mathfrak{gl}_\infty$ via the adjoint representation. A subalgebra of $\mathfrak{gl}_\infty$ is called **toral** if all of its non-zero elements are semisimple. It is a standard lemma that a toral subalgebra is abelian.

Let M be a $\mathfrak{gl}_\infty$-module, let $\mathfrak{t}$ be a toral subalgebra of $\mathfrak{gl}_\infty$, and let $\mathfrak{t}^*$ be the dual. Then for $\alpha \in \mathfrak{t}^*$, we call

$$M^\alpha := \{m \in M | \forall t \in \mathfrak{t} : tm = \alpha(t)m\}$$

the **$\mathfrak{t}$-weight space** of M of **weight** α. M is called a **$\mathfrak{t}$-weight $\mathfrak{gl}_\infty$-module** if as a $\mathfrak{t}$-module, M decomposes as the direct sum $\bigoplus_{\alpha\in\mathfrak{t}^*} M^\alpha$.

A **splitting Cartan subalgebra** $\mathfrak{h}$ of $\mathfrak{gl}_\infty$ is a maximal toral subalgebra such that $\mathfrak{gl}_\infty$ is an $\mathfrak{h}$-weight module. If $\mathfrak{h}$ is a splitting Cartan subalgebra, then $\mathfrak{gl}_\infty^\alpha$ for $\alpha \neq 0$ are called the **root spaces** of $\mathfrak{gl}_\infty$, $\Delta := \{\alpha \in \mathfrak{h}^* \backslash \{0\} | \mathfrak{gl}_\infty^\alpha \neq 0\}$ is the set of **roots** of $\mathfrak{gl}_\infty$, and the decomposition

$$\mathfrak{gl}_\infty = \mathfrak{h} \oplus \bigoplus_{\alpha\in\mathfrak{h}^*\backslash\{0\}} \mathfrak{gl}_\infty^\alpha$$

is called the **root decomposition** of $\mathfrak{gl}_\infty$.

A decomposition of the set Δ into two disjoint sets Δ^+ and Δ^- is called **triangular decomposition** of Δ if and only if $\alpha, \beta \in \Delta^+, \alpha + \beta \in \Delta$ implies $\alpha + \beta \in \Delta^+$ and $\alpha \in \Delta^\pm$ implies $-\alpha \in \Delta^\mp$.

A Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is called a **Borel subalgebra** of $\mathfrak{g}$ if there is a triangular decomposition such that $\mathfrak{b} = (\bigoplus_{\alpha\in\Delta^+} \mathfrak{gl}_\infty^\alpha) \Subset \mathfrak{h}$, where $\Subset$ denotes semidirect sum.

If M is a $\mathfrak{g}$-module, and $0 \neq v \in M$, then we call v a **highest weight vector** if it generates a one-dimensional $\mathfrak{b}$-submodule. Any such v must satisfy $(\bigoplus_{\alpha\in\Delta^+} \mathfrak{gl}_\infty^\alpha)v = 0$ and $\forall H \in \mathfrak{h} : Hv = \chi(H)v$ for some fixed weight $\chi \in \mathfrak{h}^*$. M is called **highest weight module** if it is generated by a highest weight vector, χ is called the **highest weight** of M. The highest weight vector v of M is determined uniquely up to multiplication by a complex number.

In our case, the modules $\Gamma_{\lambda,\mu}$ are highest weight modules if we pick the triangular decomposition in the right way. For $\mathfrak{gl}_\infty$, we have the root decomposition $\mathfrak{gl}_\infty = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta} \mathbb{C}X_\alpha)$, where:

$$\mathfrak{h} = \bigoplus_{i\in\mathfrak{I}} \mathbb{C}E_{i,i} \qquad \Delta = \{\epsilon_i - \epsilon_j | i, j \in \mathfrak{I}, i \neq j\}, \qquad X_{\epsilon_i-\epsilon_j} = E_{i,j}$$

and $\epsilon_i \in \mathfrak{h}^*$ is determined by $\epsilon_i(E_{j,j}) = \delta_{i,j}$. If we identify the index set $\mathfrak{I}$ with $\mathbb{Z}\backslash\{0\}$, then the triangular decomposition is given by

$$\Delta^+ = \{\epsilon_i - \epsilon_j | 0 < i < j\} \cup \{\epsilon_i - \epsilon_j | i < j < 0\} \cup \{\epsilon_i - \epsilon_j | j < 0 < i\}.$$

With respect to the corresponding $\mathfrak{b}$, $\Gamma_{\lambda,\mu}$ are indeed irreducible highest weight modules with the weight given in Theorem 3.2 below.

One feature of highest weight modules $\Gamma_{\lambda,\mu}$ that will become important later is the following.

Lemma 3.1. *Any automorphism ϕ of $\Gamma_{\lambda,\mu}$ is of the form $\phi(v) = cv$ for some fixed constant $c \in \mathbb{C}\backslash\{0\}$.*

Proof. $\Gamma_{\lambda,\mu}$ is a highest weight module, and is therefore generated by its highest weight vector v. Because v is unique up to multiplication by a constant, any automorphism of $\Gamma_{\lambda,\mu}$ will send v to some cv, $c \in \mathbb{C}$. But since v generates $\Gamma_{\lambda,\mu}$, this is already enough to determine the automorphism uniquely: Every vector in $\Gamma_{\lambda,\mu}$ is multiplied by c. □

Results about the tensor representations

Now we have all definitions that we need to state the following results about the structure of the tensor representations of $\mathfrak{gl}_\infty$.

Theorem 3.2. *For any p,q there is an isomorphism of $(\mathfrak{gl}_\infty, \mathfrak{S}_p \times \mathfrak{S}_q)$-modules*

$$V^{\{p,q\}} \cong \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} \Gamma_{\lambda;\mu} \otimes (H_\lambda \otimes H_\mu) \tag{1}$$

For any partitions λ, μ, the $\mathfrak{gl}_\infty$-module $\Gamma_{\lambda;\mu}$ is an irreducible highest weight module with highest weight $\chi = \sum_{i\in\mathbb{N}} \lambda_i \epsilon_i - \sum_{i\in\mathbb{N}} \mu_i \epsilon_{-i}$.

For $p = 0$ or $q = 0$, we get the following isomorphisms, which are infinite-dimensional versions of the Schur-Weyl duality for $\mathfrak{gl}_\infty$:

$$V^{\otimes p} \cong \bigoplus_{|\lambda|=p} \Gamma_{\lambda;0} \otimes H_\lambda,$$
$$V_*^{\otimes q} \cong \bigoplus_{|\mu|=q} \Gamma_{0;\mu} \otimes H_\mu.$$

By taking the tensor product of those two isomorphisms, we arrive at a decomposition of $V^{\otimes(p,q)}$. The following theorems state that this is indeed a decomposition into indecomposable modules, and describe their socle filtrations.

First, we come to a theorem that describes the socle filtration of $V^{\otimes(p,q)}$ explicitely as intersection of kernels of the contraction map Φ_I and its iterations.

Theorem 3.3. *Let p,q be nonnegative integers, and let $l = \min(p,q)$. Then the Loewy length of the $\mathfrak{gl}_\infty$-module $V^{\otimes(p,q)}$ is $l+1$ and for each r from 1 to l, we have*

$$\operatorname{soc}^r V^{\otimes(p,q)} = \bigcap_{I_1,\ldots,I_r} \ker\left(\Phi_{I_1,\ldots,I_r} : V^{\otimes(p,q)} \to V^{\otimes(p-r,q-r)}\right) \tag{2}$$

This is used in the proof of the following theorem, which gives a precise description about the direct summands of $V^{\otimes(p,q)}$ and its layers:

Theorem 3.4. *For any partitions λ, μ, the $\mathfrak{gl}_\infty$-module $\Gamma_{\lambda;0} \otimes \Gamma_{0;\mu}$ is indecomposable, and*

$$\overline{\operatorname{soc}}^{(r+1)}(\Gamma_{\lambda;0} \otimes \Gamma_{0;\mu}) \cong \bigoplus_{\lambda',\mu'} \left(\sum_{|\gamma|=r} N^{\lambda}_{\lambda',\gamma} N^{\mu}_{\mu',\gamma} \right) \Gamma_{\lambda';\mu'} \tag{3}$$

Theorems 3.2 and 3.4 taken together yield the following result:

Corollary 3.5. *The decomposition of $V^{\otimes(p,q)}$ into indecomposable $\mathfrak{gl}_\infty$ modules is given by*

$$V^{\otimes(p,q)} \cong \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} (\dim H_\lambda \dim H_\mu) \Gamma_{\lambda;0} \otimes \Gamma_{0;\mu} \tag{4}$$

4 Littlewood-Richardson coefficients

In this section, we want to discuss the coefficients in the isomorphism (3), their basic properties and how they can be computed.

The key to compute the coefficients are the following definitions.

Let λ, μ, γ be partitions. A **skew diagram** of shape λ/μ is the set theoretic difference of the Young diagram with shape λ and the one with shape μ. Here it is required that $\mu_i \leq \lambda_i$ for all i, i.e. that the smaller diagram is contained in the bigger one. A **semistandard skew tableau** of shape λ/μ and weight γ is a skew diagram of shape λ/μ, where each box is filled with a positive integer, such that:

(i) the number i appears exactly γ_i times,

(ii) the numbers along each column are strictly increasing, and

(iii) the numbers along each row are weakly increasing.

A semistandard skew tableau is called **Littlewood-Richardson tableau** if it satisfies one more condition:

(iv) Any tableau obtained by removing 0 or more of the leftmost columns is again a semistandard skew tableau. In other words, the reduced tableau has a weight that is again a partition, and therefore weakly decreasing.

These tableaux can be used to characterize the Littlewood-Richardson coefficients, according to the following rule.

Theorem 4.1 (Littlewood-Richardson Rule). *For partitions λ, μ, γ, the Littlewood-Richardson coefficient $N^{\lambda}_{\mu,\gamma}$ is equal to the number of Littlewood-Richardson tableaux of shape λ/μ and of weight γ.*

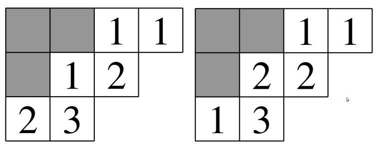

Figure 1: The only two Littlewood-Richardson tableaux of shape $(4,3,2)/(2,1)$ and weight $(3,2,1)$, implying $N^{(4,3,2)}_{(2,1),(3,2,1)} = 2$. The gray boxes are not part of the tableaux.

Here are some basic properties of Littlewood-Richardson tableaux and coefficients that can be easily checked from the definition and will be useful later on.

- Tableaux of shape λ/μ and weigth γ satisfy $|\lambda| = |\mu| + |\gamma|$.
- Every column contains a given number at most once. Together with the fact that γ is decreasing, this implies $\gamma_i \leq \lambda_i$ for all i.
- The rightmost column of any Littlewood-Richardson tableau is filled from top to bottom with consecutive numbers starting from one. This is a consequence of properties (ii) and (iv).
- The first nonempty row of a Littlewood-Richardson tableau is completely filled with ones. This follows from the previous bullet point and from the fact that numbers in the row are weakly increasing.

- The argument in the previous bullet point can be extended to show that the $\lambda_1 - \mu_1$ rightmost columns can be filled in only one way: with number i filling up row i completely.
- Hence, for $\mu = 0$, there is only one tableau of shape $\lambda/0$, and its weight is $\gamma = \lambda$.
- The only tableau of shape λ/λ is the empty one with weight 0, and any tableau with weigth 0 has such a shape, i.e.

$$N^{\lambda}_{\lambda,\gamma} = \begin{cases} 1 & \text{if } \gamma = 0 \\ 0 & \text{if } \gamma \neq 0 \end{cases}, \qquad N^{\lambda}_{\mu,0} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}.$$

- If the k rightmost columns of a diagram and the top m boxes of the $k+1$th column are completely filled with numbers satisfying all conditions, then the $m+1$ box in the $k+1$th column can also be filled according to all rules by adding 1 to number m in that column (or, if $m = 0$, by writing a 1 into that box). Hence any partial filling of a diagram from the right can be completed to a valid tableau in at least one way.

The second to last property has an immediate consequence for our socle filtrations: the isomorphism (3) for r=0 becomes

$$\operatorname{soc}(\Gamma_{\lambda;0} \otimes \Gamma_{0;\mu}) \cong \Gamma_{\lambda;\mu},$$

so the socle of $\Gamma_{\lambda;0} \otimes \Gamma_{0;\mu}$ is always simple.

The property from the last bullet point is useful for effectively computing the coefficients $N^{\lambda}_{\mu,\gamma}$. To compute a certain layer of a socle filtration using formula (3), one needs to find all Littlewood-Richardson tableaux of shape λ/λ' (and μ/μ') for all possible λ', μ' and weights of a given length.

One way to do it is the following: Iterate over all possible partitions λ', for which $\lambda_i \geq \lambda'_i$ for all i, and $|\lambda| - |\lambda'| = r$, where $r+1$ is the number of the layer we want to compute. For each diagram of shape λ/λ', construct all possible Littlewood-Richardson tableaux and count how often each weight appears. To construct the tableaux, start at the rightmost column from the top. For each cell, find out what values can be held in it. The row-condition and the decreasing-weight-condition gives an upper bound, the column-condition gives a lower bound, so there are only finitely many values possible. For each value, proceed recursively with the next cell, top to bottom, column by column.

This recursive algorithm never runs into a dead end (that is exactly what the last bullet point tells us), and all tableaux are constructed in this way since all possible entries are tried out. By keeping track of the used weights, $N^{\lambda}_{\lambda',\gamma}$ can be computed for all γ with one run-through of this algorithm.

I implemented this algorithm in C++ and used it to compute all socle filtrations of our modules for small p and q. The results can be found in the appendix.

5 Results about the radical filtration

This section contains the main result of this paper. We start by looking at some examples of socle filtrations of tensor modules.

The modules $V^{\otimes(p,q)}$ for $p = 2, q = 0$ and $p = 0, q = 2$ are semisimple and decompose into the sum of symmetric and antisymmetric tensors:

$$V \otimes V \cong \Gamma_{(2);0} \oplus \Gamma_{(1,1);0} \cong S^2V \oplus \Lambda^2 V$$
$$V_* \otimes V_* \cong \Gamma_{0;(2)} \oplus \Gamma_{0;(1,1)} \cong S^2V_* \oplus \Lambda^2 V_*$$

The module $V^{\otimes(1,1)}$ (which is isomorphic to $\mathfrak{gl}_\infty$) has socle filtration of length 2, and looks like this:

$$V^{\otimes(1,1)} \sim \begin{array}{|c|}\hline \Gamma_{0;0} \\ \hline \Gamma_{(1);(1)} \\ \hline \end{array}_{\text{soc}} \sim \begin{array}{|c|}\hline \mathbb{C} \\ \hline \mathfrak{sl}_\infty \\ \hline \end{array}_{\text{soc}}$$

It is easy to see that the radical filtration coincides with socle filtration in this case. Indeed, the radical of $V^{\otimes(1,1)}$ is contained in its socle, but the socle is simple. Hence, the bottom layer of the radical filtration is equal to the socle, and, since $V^{\otimes(1,1)}$ has length 2, the socle and radical filtration coincide. This can be generalized to the following statement.

Lemma 5.1. *If all layers except the top layer of a socle filtration are simple, then the module is rigid.*

This can be applied for all modules of type $\Gamma_{(p),0} \otimes \Gamma_{0,(q)}$ or $\Gamma_{(1,\dots,1),0} \otimes \Gamma_{0,(1,\dots,1)}$. For example, the socle filtration of $\Gamma_{(4),0} \otimes \Gamma_{0,(3)}$ is

$$\Gamma_{(4),0} \otimes \Gamma_{0,(3)} \sim \begin{array}{|c|}\hline \Gamma_{(1);(0)} \\ \hline \Gamma_{(2);(1)} \\ \hline \Gamma_{(3);(2)} \\ \hline \Gamma_{(4);(3)} \\ \hline \end{array}_{\text{soc}}$$

Now consider $V^{\otimes(3,3)}$. It decomposes into nine different modules, one of which is $M = \Gamma_{(2,1);0} \otimes \Gamma_{0;(2,1)}$. This is a module of Loewy length 4 and has the following socle filtration.

$$\begin{array}{|c|}\hline \Gamma_{0;0} \\ \hline 2\Gamma_{(1);(1)} \\ \hline \Gamma_{(1,1);(1,1)} \oplus \Gamma_{(1,1);(2)} \oplus \Gamma_{(2);(1,1)} \oplus \Gamma_{(2);(2)} \\ \hline \Gamma_{(2,1);(2,1)} \\ \hline \end{array}_{\text{soc}}$$

This is the smallest example for a socle filtration with a simple constituent of multiplicity greater than one. As it turns out, this multiplicity together with the simple top layer makes it impossible for this module to be rigid.

Before we come to the main results of this thesis, we need a lemma that will be useful later on.

Lemma 5.2. *Let M be a module of Loewy length 2. If M contains a submodule N that does not contain $\operatorname{soc} M$ and satisfies $N + \operatorname{soc} M = M$, then $M = N \oplus A$ for some semisimple module A.*

Proof. By assumption, $B := \operatorname{soc} M \cap N$ is a proper submodule of $\operatorname{soc} M$. By the semisimplicity of the socle, there is a complement to B inside $\operatorname{soc} M$, i.e. a module A with $\operatorname{soc} M = A \oplus B$. We claim that $M = N \oplus A$.

We need to show two things: A and N have zero intersection, and together they span all of M. The first condition is clear from the definition: A lies inside $\operatorname{soc} M$ and is complement to $B = \operatorname{soc} M \cap N$.

The second condition is evident from the assumptions: $\operatorname{soc} M = A \oplus B$ is contained in $N + A$, so $N + A$ also contains $N + \operatorname{soc} M = M$, i.e. N and A generate M. □

Now we are well prepared to prove the following statement:

Proposition 5.3. *If λ and μ are partitions of p and q, respectively, and $l = \min(p, q)$, such that:*

- $\overline{\mathrm{soc}}^{l+1}\Gamma_{\lambda,0}\otimes\Gamma_{0,\mu}$ *is nonzero and simple, and*
- *the decomposition of* $\overline{\mathrm{soc}}^{l}\Gamma_{\lambda,0}\otimes\Gamma_{0,\mu}$ *into simple modules contains at least one module with a multiplicity of at least 2,*

then $\Gamma_{\lambda,0}\otimes\Gamma_{0,\mu}$ *is not rigid.*

Proof. Since there is a natural isomorphism between $V^{\otimes(p,q)}$ and $V^{\otimes(q,p)}$, we can restrict ourselves to the case $p \geq q$.

Assume that on the contrary, $\Gamma_{\lambda,0}\otimes\Gamma_{0,\mu}$ is rigid. Consider the quotient $M = \Gamma_{\lambda,0}\otimes\Gamma_{0,\mu}/\operatorname{soc}^{l-1}(\Gamma_{\lambda,0}\otimes\Gamma_{0,\mu})$. M is by assumption a rigid module of Loewy length 2 with simple top isomorphic to $\Gamma_{\lambda'',0}$, and some module $\Gamma_{\lambda',(1)}$ in the socle with multiplicity $n \geq 2$. Let X be the quotient of M by the complement of $n\Gamma_{\lambda',(1)}$ inside the socle. What remains is a module with the socle filtration

$$X \sim \boxed{\begin{array}{c}\Gamma_{\lambda'',0}\\ \hline n\Gamma_{\lambda',(1)}\end{array}}_{\mathrm{soc}},$$

that is supposed to be rigid. We show below that no such module can be rigid.

It suffices to show that X is decomposable. Indeed, then one of the two summands will be semisimple, otherwise the second layer of X could not be simple. This semisimple summand has to appear in the socle of X, thus it is isomorphic to $k\Gamma_{\lambda',(1)}$ for some $1 \leq k \leq n$. If we succeed, we know that X has the form

$$X \sim \boxed{\begin{array}{c}\Gamma_{\lambda'',0}\\ \hline (n-k)\Gamma_{\lambda',(1)}\end{array}}_{\mathrm{soc}} \oplus \boxed{k\Gamma_{\lambda',(1)}}_{\mathrm{soc}},$$

and hence its radical filtration would be

$$X \sim \boxed{\begin{array}{c}\Gamma_{\lambda'',0}\oplus k\Gamma_{\lambda',(1)}\\ \hline (n-k)\Gamma_{\lambda',(1)}\end{array}}_{\mathrm{rad}}.$$

To show the decomposability of X, we use the fact that all $V^{\otimes(p,q)}$, and hence also their direct summands, are injective modules. This follows from Corollary 4.6 in [DPS].

We now construct an embedding of X into $Y\oplus\cdots\oplus Y$ (n summands), where $Y := \Gamma_{\lambda',0}\otimes\Gamma_{0,(1)}$, as follows. X and nY both have socles isomorphic to $n\Gamma_{\lambda',(1)}$, hence we can embed $\operatorname{soc} X$ into nY. This embedding can be extended to all of X by using the injectivity of $\Gamma_{\lambda',0}\otimes\Gamma_{0,(1)}$ ($\Gamma_{\lambda',0}\otimes\Gamma_{0,(1)}$ is a direct summand of $V^{\otimes(|\lambda'|,1)}$, and therefore injective). Thus, there exists a homomorphism $j : X \to nY$, which becomes an isomorphism when restricted to the socles. The homomorphism j is injective, because its kernel intersects trivially with the socle.

The homomorphism j induces a map of the quotients

$$j^* : X/\operatorname{soc} X \to (nY)/\operatorname{soc}(nY) \cong n(Y/\operatorname{soc} Y).$$

Since j restricts to an isomorphism of $\operatorname{soc} X$ and $\operatorname{soc}(nY)$, the injectivity of j implies the injectivity of j^*. Moreover, $Y/\operatorname{soc}(Y)$ is a semisimple module, whose decomposition into simple modules contains $\Gamma_{\lambda'',0}$ with multiplicity one, since its coefficient is $N^{\lambda'}_{\lambda'',(1)}\cdot N^{(1)}_{0,(1)} = 1$.

Let $Y/\operatorname{soc} Y \cong \bigoplus_{i=0}^{a} A_i$ be a decomposition into simple modules, where $A_0 \cong \Gamma_{\lambda'',0}$ and $A_i \not\cong \Gamma_{\lambda'',0}$ for $i > 0$. This decomposition comes with a family of projections $\pi_{1,i},\ldots,\pi_{n,i}$ mapping from nY into one of the n copies of A_i.

Now we look at the composition $\pi_{k,i}\circ j^*$. Its image is a submodule of A_i, hence by simplicity either 0 or A_i itself. In the latter case, the isomorphism theorem tells us that the quotient of $\Gamma_{\lambda'',0}$ by the kernel is isomorphic to A_i. If $i > 0$, this contradicts the fact that $\Gamma_{\lambda'',0}$ and A_i are simple and not isomorphic. Thus, $\pi_{k,i} \circ j^*$ is the zero map for all $i > 0$.

Let $j_k^* = \pi_{k,0} \circ j^*$, then $j^* = (j_1^*, \dots, j_n^*, 0, \dots, 0)$. As we just saw, the maps j_k^* mapping into $A_0 = \Gamma_{\lambda'',0}$ are either zero maps or surjective. A surjective endomorphism of a simple module must be injective by the first isomorphism theorem, so the maps j_k^* are either zero or automorphisms.

Now is the time to use our knowledge from section 3: Lemma 3.1 tells us that if j_k^* are automorphisms, they are simply multiplication by a nonzero complex number. If they are not automorphisms, they are simply multiplication by 0. Thus, in any case there are n complex numbers $c_1, \dots, c_n \in \mathbb{C}$, such that $j_k^*(v) = c_k v$. Since j^* is injective, c_1 to c_n cannot all be 0 at the same time.

Those numbers can now be used to explicitely write down the submodule of $\operatorname{im} j$, to which we will then apply Lemma 5.2 to show the decomosability of X. Let π denote the projection $\pi : Y \to Y/\operatorname{soc} Y$, and define

$$N := \left\{ (c_1 v, \dots, c_n v) | v \in \pi^{-1}(\Gamma_{\lambda'',0}) \right\}.$$

Then N is a submodule of nY and isomorphic to $\pi^{-1}(\Gamma_{\lambda'',0})$. Hence its socle filtration is

$$N \sim \boxed{\begin{array}{c} \Gamma_{\lambda'',0} \\ \hline \Gamma_{\lambda',(1)} \end{array}}_{\text{soc}} \subset nY \sim \boxed{\begin{array}{c} n\Gamma_{\lambda'',0} \oplus \cdots \\ \hline n\Gamma_{\lambda',(1)} \end{array}}_{\text{soc}}$$

We need to show that N lies in the image of X under j. For any $v \in \pi^{-1}(\Gamma_{\lambda'',0})$, let $(v_1, \dots, v_n) \in \operatorname{im} j$ such that $(\pi(v_1), \dots, \pi(v_n)) = (\pi(c_1 v), \dots, \pi(c_n v))$. Such v_k have to exist because $(c_1\pi(v), \dots, c_n\pi(v))$ lies in the image of j^*. Then

$$(\pi(c_1 v - v_1), \dots, \pi(c_n v - v_n)) = (\pi(c_1 v) - \pi(v_1), \dots, \pi(c_n v) - \pi(v_n)) = 0,$$

thus $(c_1 v - v_1, \dots, c_n v - v_n) \in \operatorname{soc}(nY) \subset \operatorname{im} j$. Since $\operatorname{im} j$ is closed under addition, $(v_1, \dots, v_n) + (c_1 v - v_1, \dots, c_n v - v_n) = (c_1 v, \dots, c_n v)$ is also inside, thus $N \subset \operatorname{im} j$ as desired.

Therefore, we showed that $j^{-1}(N)$ is a submodule of X, has Loewy length 2 and a simple socle. Because the top layer of X is simple, $j^{-1}(N) + \operatorname{soc} X$ has to cover all of X, hence $j^{-1}(N)$ satisfies the conditions of lemma 5.2. We can conclude that X decomposes as described earlier, and is therefore not rigid. □

The proof could have been done by setting $n = 2$ and dividing X by the additional modules in the socle. But in this generality, more was shown:

Lemma 5.4. *If X is a module of Loewy length 2 with top of the form $\Gamma_{\lambda'',0}$ and socle of the form $n\Gamma_{\lambda',(1)}$, then X contains a submodule with top $\Gamma_{\lambda'',0}$ and socle $\Gamma_{\lambda',(1)}$.*

This can be used to prove the following generalization of Proposition 5.3.

Proposition 5.5. *If λ and μ are partitions of p and q, respectively, and $l = \min(p, q)$, such that:*

- *$\overline{\operatorname{soc}}^{l+1}(\Gamma_{\lambda,0} \otimes \Gamma_{0,\mu})$ is nonzero and is the direct sum of $n - 1$ simple modules, and*
- *the decomposition of $\overline{\operatorname{soc}}^{l}(\Gamma_{\lambda,0} \otimes \Gamma_{0,\mu})$ into simple modules contains at least one module with a multiplicity of a least n,*

then $\Gamma_{\lambda,0} \otimes \Gamma_{0,\mu}$ is not rigid.

Proof. As in the proof of Proposition 5.3, define X as appropriate quotient. Let the top layer of the socle filtration of X be $\bigoplus_{i=1}^{n-1} B_i$, where each B_i is of the form Γ_{λ^i,μ^i}, and let the socle be nA, where A is $\Gamma_{\lambda',\mu'}$. Let $\pi : X \to \overline{\mathrm{soc}}^2 X$ be the projection onto the top layer, $\pi_i : X \to B_i$ be the projections onto the simple constituents of $X/\operatorname{soc} X$. Define

$$X_i := \bigcap_{\substack{1\le k\le n-1\\ k\neq i}} \ker \pi_k.$$

Each X_i is a submodule of X with simple top layer isomorphic to B_i and $\operatorname{soc} X_i = \operatorname{soc} X$. Thus, we can apply lemma 5.4, which gives us $n-1$ submodules $Y_i \subset X_i$ with $\pi(Y_i) = B_i$ and $\operatorname{soc} Y_i \cong A$.

The module generated by all those submodules $Y := \sum_{i=1}^{n-1} Y_i$ satisfies $\pi(Y) = \overline{\mathrm{soc}}^2 X$, and its socle is the sum of $n-1$ copies of A. Hence, it does not cover $\operatorname{soc} X$. By lemma 5.2, this gives us a decomposition of X into $Y \oplus kA$ for some k. Hence X is not rigid, and neither is $\Gamma_{\lambda,0} \otimes \Gamma_{0,\mu}$. □

We have shown that $\Gamma_{\lambda,0} \otimes \Gamma_{0,\mu}$ is not rigid if a simple constituent of $\Gamma_{\lambda,0} \otimes \Gamma_{0,\mu}$ has multiplicity in the second top layer of its socle filtration that is larger than the top layer's length. This is in fact not a rare phenomenon, it occurs for any p, q, as long as they are large enough (more precisely, for $p, q \geq 3$).

If $\min(p, q) = 0$, then $V^{\otimes(p,q)}$ decomposes into a direct sum of simple constituents, hence it is rigid for a trivial reason. If $\min(p, q) = 1$, then each direct summand of $V^{\otimes(p,q)}$ has Loewy length 2 with simple bottom layer, hence all direct summands are rigid. For $\min(p, q) = 2$, the direct summands have Loewy length 2 and 3, hence $V^{\otimes(p,q)}$ as a whole cannot be rigid, but that is always the case for $\min(p, q) > 1$. The more interesting question is whether the direct summands by themselves are rigid. For $\min(p, q) = 2$, all multiplicities in the socle filtration are 1, as it can be checked from theorem 3.4 and the Littlewood-Richardson rule. So the theorems above do not yield any answers in that case. But for $\min(p, q) \geq 3$, we find the following, which is the main result of this thesis:

Theorem 5.6. *Let $l = \min(p, q)$ be the Loewy length of $V^{\otimes(p,q)}$. For $l \geq 3$, the decomposition of $V^{\otimes(p,q)}$ into indecomposable modules contains at least one nonrigid direct summand.*

Proof. As before, we restrict ourselves to the case $p \geq q$.

For $l \geq 3$, we give an explicit example for a direct summand of $V^{\otimes(p,q)}$ that is not rigid. Let $\lambda = (2, 1, \ldots, 1)$ and $\mu = (2, 1, \ldots, 1)$ with $|\lambda| = p$ and $|\mu| = q$. Then we claim that $\Gamma_{\lambda,0} \otimes \Gamma_{0,\mu}$ satisfies the condition of Proposition 5.3, and is therefore not rigid.

Assuming $p \geq q$ as before, the top layer of the socle filtration is

$$\begin{aligned}
\overline{\mathrm{soc}}^{(l+1)}(\Gamma_{\lambda;0} \otimes \Gamma_{0;\mu}) &\cong \bigoplus_{\lambda',\mu'} \left(\sum_{|\gamma|=l} N^{\lambda}_{\lambda',\gamma} N^{\mu}_{\mu',\gamma} \right) \Gamma_{\lambda';\mu'} \\
&\cong \bigoplus_{\lambda'} \left(N^{\lambda}_{\lambda',\mu} N^{\mu}_{0,\mu} \right) \Gamma_{\lambda';0} \\
&\cong \bigoplus_{\lambda'} N^{\lambda}_{\lambda',\mu} \Gamma_{\lambda';0}.
\end{aligned}$$

This is because $N^{\mu}_{\mu',\gamma}$, for $|\gamma| = q = |\mu|$, is nonzero only if $\mu' = 0$ and $\gamma = \mu$.

Next, note that $N^{\lambda}_{\lambda',\mu} > 0$ precisely when there are Littlewood-Richardson tableaux of shape $\lambda/\lambda' = (2, 1, \ldots, 1)/\lambda'$ and weight $\mu = (2, 1, \ldots, 1)$. Therefore, the tableau has two columns, and this implies $\lambda'_1 < 2$. But this makes the tableau already unique: λ' has to be a column of length $|\lambda'| = p - q$, (i.e. $\lambda' = (1, ..., 1)$), and the numbers are distributed in the only way: The left column of the diagram contains the numbers from 1 to $q - 1$ from top to bottom, and the single field in the second column contains the second 1. Hence, the top layer of $\Gamma_{\lambda;0} \otimes \Gamma_{0;\mu}$ is simple.

In the second top layer, we need to show the existence of a simple constituent with multiplicity ≥ 2. Such a constituent is $\Gamma_{\lambda',\mu'}$ with $\lambda' = (1, ..., 1)$, $|\lambda| = p - q + 1$, and $\mu' = (1)$, as shown in Figure 2. In general, one needs to set $\gamma = (2, 1, ..., 1)$ and $\gamma' = (1, ..., 1)$, where both partitions have length $q - 1$ (the condition $q \geq 3$ is used here). The multiplicity of $\Gamma_{\lambda',\mu'}$ in the second to top layer in the socle filtration of $\Gamma_{\lambda;0} \otimes \Gamma_{0;\mu}$ is $N^{\lambda}_{\lambda',\gamma} \cdot N^{\mu}_{\mu',\gamma} + N^{\lambda}_{\lambda',\gamma'} \cdot N^{\mu}_{\mu',\gamma'} = 2$.

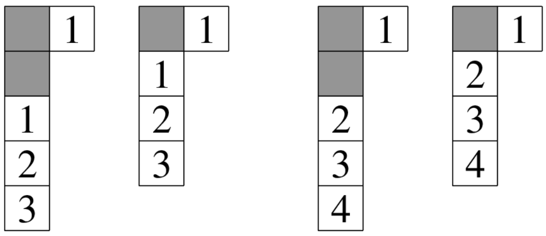

Figure 2: Littlewood-Richardson tableaux for $p = 6, q = 5$, illustrating (from left to right) $N^{\lambda}_{\lambda',\gamma} = 1$, $N^{\mu}_{\mu',\gamma} = 1$, $N^{\lambda}_{\lambda',\gamma'} = 1$, $N^{\mu}_{\mu',\gamma'} = 1$ with $\lambda' = (1, 1), \mu' = (1)$, $\gamma = (2, 1, 1)$ on the left, $\gamma' = (1, 1, 1, 1)$ on the right.

Thus, all conditions of Proposition 5.3 are satisfied, and it can be concluded that $\Gamma_{\lambda;0} \otimes \Gamma_{0;\mu}$ is not rigid. □

6 Conclusion and open questions

We have shown in this thesis that some direct summands of the tensor representations of $\mathfrak{gl}_\infty$ are not rigid, if they have a simple contituent with high enough multiplicity in the second to top layer, and that this actually occures for all $p, q \geq 3$. What is still left to do is the actual computation of the radical filtration of those modules. All we know as of now is that the additional modules in the second top layer of the socle filtration move to the top.

Another open question is: What can be said about the direct summends of the mixed tensor algebra that do not have a high multiplicity in the socle filtration? Abstractly, rigid modules with those socle filtrations can be constructed in most cases, but that does not mean that those constituents of $V^{\otimes(p,q)}$ are actually rigid. This is still unknown territory.

Another question that we will try to tackle in the future are the radical filtrations of the direct summands of the tensor algebra corresponding to other locally finite Lie algebras, such as $\mathfrak{so}(\infty)$ and $\mathfrak{sp}(\infty)$. Penkov and Styrkas found the socle filtrations of these tensor algebras as well, as it can be seen in [PS]. I will try to find out whether the results of this thesis can be extended to those modules as well.

7 Appendix

In the following, we list the socle filtrations of $\Gamma_{\lambda;0} \otimes \Gamma_{0;\mu}$ in the notation described in section 2 (all filtrations here are socle filtrations, so we refrain from marking all of them with soc). We restrict ourselves to the case $|\lambda| \geq |\mu|$, the other case is gained by exchanging the indices in all simple constituents.

For any λ, μ, $\Gamma_{\lambda;0} \otimes \Gamma_{0;\mu}$ is the module with socle $\Gamma_{\lambda;\mu}$.

$V^{\otimes(1,0)}$:

$$\begin{array}{|c|}\hline \Gamma_{(1);(0)} \\ \hline \end{array}$$

$V^{\otimes(2,0)}$:

$$\begin{array}{|c|}\hline \Gamma_{(2);(0)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1);(0)} \\ \hline \end{array}$$

$V^{\otimes(1,1)}$:

$$\begin{array}{|c|}\hline \Gamma_{(0);(0)} \\ \hline \Gamma_{(1);(1)} \\ \hline \end{array}$$

$V^{\otimes(3,0)}$:

$$\begin{array}{|c|}\hline \Gamma_{(3);(0)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(2,1);(0)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1,1);(0)} \\ \hline \end{array}$$

$V^{\otimes(2,1)}$:

$$\begin{array}{|c|}\hline \Gamma_{(1);(0)} \\ \hline \Gamma_{(2);(1)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(1);(0)} \\ \hline \Gamma_{(1,1);(1)} \\ \hline \end{array}$$

$V^{\otimes(4,0)}$:

$$\begin{array}{|c|}\hline \Gamma_{(4);(0)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(3,1);(0)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(2,2);(0)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(2,1,1);(0)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1,1,1);(0)} \\ \hline \end{array}$$

$V^{\otimes(3,1)}$:

$$\begin{array}{|c|}\hline \Gamma_{(2);(0)} \\ \hline \Gamma_{(3);(1)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1);(0)} \oplus \Gamma_{(2);(0)} \\ \hline \Gamma_{(2,1);(1)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1);(0)} \\ \hline \Gamma_{(1,1,1);(1)} \\ \hline \end{array}$$

$V^{\otimes(2,2)}$:

$$\begin{array}{|c|}\hline \Gamma_{(0);(0)} \\ \hline \Gamma_{(1);(1)} \\ \hline \Gamma_{(2);(2)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(1);(1)} \\ \hline \Gamma_{(2);(1,1)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(1);(1)} \\ \hline \Gamma_{(1,1);(2)} \\ \hline \end{array}, \begin{array}{|c|}\hline \Gamma_{(0);(0)} \\ \hline \Gamma_{(1);(1)} \\ \hline \Gamma_{(1,1);(1,1)} \\ \hline \end{array}$$

$V^{\otimes(5,0)}$

$$\begin{array}{|c|}\hline \Gamma_{(5);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(4,1);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(3,2);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(3,1,1);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2,2,1);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2,1,1,1);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1,1,1,1);(0)}\\ \hline\end{array}$$

$V^{\otimes(4,1)}$

$$\begin{array}{|c|}\hline \Gamma_{(3);(0)}\\ \hline \Gamma_{(4);(1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2,1);(0)} \oplus \Gamma_{(3);(0)}\\ \hline \Gamma_{(3,1);(1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2,1);(0)}\\ \hline \Gamma_{(2,2);(1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1,1);(0)} \oplus \Gamma_{(2,1);(0)}\\ \hline \Gamma_{(2,1,1);(1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1,1);(0)}\\ \hline \Gamma_{(1,1,1,1);(1)}\\ \hline\end{array}$$

$V^{\otimes(3,2)}$

$$\begin{array}{|c|}\hline \Gamma_{(1);(0)}\\ \hline \Gamma_{(2);(1)}\\ \hline \Gamma_{(3);(2)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2);(1)}\\ \hline \Gamma_{(3);(1,1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1);(0)}\\ \hline \Gamma_{(1,1);(1)} \oplus \Gamma_{(2);(1)}\\ \hline \Gamma_{(2,1);(2)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1);(0)}\\ \hline \Gamma_{(1,1);(1)} \oplus \Gamma_{(2);(1)}\\ \hline \Gamma_{(2,1);(1,1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1);(1)}\\ \hline \Gamma_{(1,1,1);(2)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1);(0)}\\ \hline \Gamma_{(1,1);(1)}\\ \hline \Gamma_{(1,1,1);(1,1)}\\ \hline\end{array}$$

$V^{\otimes(6,0)}$

$$\begin{array}{|c|}\hline \Gamma_{(6);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(5,1);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(4,2);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(4,1,1);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(3,3);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(3,2,1);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(3,1,1,1);(0)}\\ \hline\end{array}$$

$$, \begin{array}{|c|}\hline \Gamma_{(2,2,2);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2,2,1,1);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2,1,1,1,1);(0)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1,1,1,1,1);(0)}\\ \hline\end{array}$$

$V^{\otimes(5,1)}$

$$\begin{array}{|c|}\hline \Gamma_{(4);(0)}\\ \hline \Gamma_{(5);(1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(3,1);(0)} \oplus \Gamma_{(4);(0)}\\ \hline \Gamma_{(4,1);(1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2,2);(0)} \oplus \Gamma_{(3,1);(0)}\\ \hline \Gamma_{(3,2);(1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2,1,1);(0)} \oplus \Gamma_{(3,1);(0)}\\ \hline \Gamma_{(3,1,1);(1)}\\ \hline\end{array}$$

$$, \begin{array}{|c|}\hline \Gamma_{(2,1,1);(0)} \oplus \Gamma_{(2,2);(0)}\\ \hline \Gamma_{(2,2,1);(1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1,1,1);(0)} \oplus \Gamma_{(2,1,1);(0)}\\ \hline \Gamma_{(2,1,1,1);(1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1,1,1);(0)}\\ \hline \Gamma_{(1,1,1,1,1);(1)}\\ \hline\end{array}$$

$V^{\otimes(4,2)}$

$$\begin{array}{|c|}\hline \Gamma_{(2);(0)}\\ \hline \Gamma_{(3);(1)}\\ \hline \Gamma_{(4);(2)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(3);(1)}\\ \hline \Gamma_{(4);(1,1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1);(0)} \oplus \Gamma_{(2);(0)}\\ \hline \Gamma_{(2,1);(1)} \oplus \Gamma_{(3);(1)}\\ \hline \Gamma_{(3,1);(2)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2);(0)}\\ \hline \Gamma_{(2,1);(1)} \oplus \Gamma_{(3);(1)}\\ \hline \Gamma_{(3,1);(1,1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2);(0)}\\ \hline \Gamma_{(2,1);(1)}\\ \hline \Gamma_{(2,2);(2)}\\ \hline\end{array},$$

$$\begin{array}{|c|}\hline \Gamma_{(1,1);(0)}\\ \hline \Gamma_{(2,1);(1)}\\ \hline \Gamma_{(2,2);(1,1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1);(0)}\\ \hline \Gamma_{(1,1,1);(1)} \oplus \Gamma_{(2,1);(1)}\\ \hline \Gamma_{(2,1,1);(2)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1);(0)} \oplus \Gamma_{(2);(0)}\\ \hline \Gamma_{(1,1,1);(1)} \oplus \Gamma_{(2,1);(1)}\\ \hline \Gamma_{(2,1,1);(1,1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1,1);(1)}\\ \hline \Gamma_{(1,1,1,1);(2)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1);(0)}\\ \hline \Gamma_{(1,1,1);(1)}\\ \hline \Gamma_{(1,1,1,1);(1,1)}\\ \hline\end{array}$$

$V^{\otimes(3,3)}$

$$\begin{array}{|c|}\hline \Gamma_{(0);(0)}\\ \hline \Gamma_{(1);(1)}\\ \hline \Gamma_{(2);(2)}\\ \hline \Gamma_{(3);(3)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1);(1)}\\ \hline \Gamma_{(2);(1,1)} \oplus \Gamma_{(2);(2)}\\ \hline \Gamma_{(3);(2,1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(2);(1,1)}\\ \hline \Gamma_{(3);(1,1,1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1);(1)}\\ \hline \Gamma_{(1,1);(2)} \oplus \Gamma_{(2);(2)}\\ \hline \Gamma_{(2,1);(3)}\\ \hline\end{array},$$

$$\begin{array}{|c|}\hline \Gamma_{(0);(0)}\\ \hline 2\Gamma_{(1);(1)}\\ \hline \Gamma_{(1,1);(1,1)} \oplus \Gamma_{(1,1);(2)} \oplus \Gamma_{(2);(1,1)} \oplus \Gamma_{(2);(2)}\\ \hline \Gamma_{(2,1);(2,1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1);(1)}\\ \hline \Gamma_{(1,1);(1,1)} \oplus \Gamma_{(2);(1,1)}\\ \hline \Gamma_{(2,1);(1,1,1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(1,1);(2)}\\ \hline \Gamma_{(1,1,1);(3)}\\ \hline\end{array},$$

$$\begin{array}{|c|}\hline \Gamma_{(1);(1)}\\ \hline \Gamma_{(1,1);(1,1)} \oplus \Gamma_{(1,1);(2)}\\ \hline \Gamma_{(1,1,1);(2,1)}\\ \hline\end{array}, \begin{array}{|c|}\hline \Gamma_{(0);(0)}\\ \hline \Gamma_{(1);(1)}\\ \hline \Gamma_{(1,1);(1,1)}\\ \hline \Gamma_{(1,1,1);(1,1,1)}\\ \hline\end{array}$$

$V^{\otimes(7,0)}$:

$$\boxed{\Gamma_{(7);(0)}},\ \boxed{\Gamma_{(6,1);(0)}},\ \boxed{\Gamma_{(5,2);(0)}},\ \boxed{\Gamma_{(5,1,1);(0)}},\ \boxed{\Gamma_{(4,3);(0)}},\ \boxed{\Gamma_{(4,2,1);(0)}},$$
$$\boxed{\Gamma_{(4,1,1,1);(0)}},\ \boxed{\Gamma_{(3,3,1);(0)}},\ \boxed{\Gamma_{(3,2,2);(0)}},\ \boxed{\Gamma_{(3,2,1,1);(0)}},\ \boxed{\Gamma_{(3,1,1,1,1);(0)}},\ \boxed{\Gamma_{(2,2,2,1);(0)}},$$
$$\boxed{\Gamma_{(2,2,1,1,1);(0)}},\ \boxed{\Gamma_{(2,1,1,1,1,1);(0)}},\ \boxed{\Gamma_{(1,1,1,1,1,1,1);(0)}}$$

$V^{\otimes(6,1)}$:

$$\begin{array}{|c|}\hline \Gamma_{(5);(0)} \\ \hline \Gamma_{(6);(1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(4,1);(0)} \oplus \Gamma_{(5);(0)} \\ \hline \Gamma_{(5,1);(1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(3,2);(0)} \oplus \Gamma_{(4,1);(0)} \\ \hline \Gamma_{(4,2);(1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(3,1,1);(0)} \oplus \Gamma_{(4,1);(0)} \\ \hline \Gamma_{(4,1,1);(1)} \\ \hline\end{array},$$
$$\begin{array}{|c|}\hline \Gamma_{(3,2);(0)} \\ \hline \Gamma_{(3,3);(1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(2,2,1);(0)} \oplus \Gamma_{(3,1,1);(0)} \oplus \Gamma_{(3,2);(0)} \\ \hline \Gamma_{(3,2,1);(1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(2,1,1,1);(0)} \oplus \Gamma_{(3,1,1);(0)} \\ \hline \Gamma_{(3,1,1,1);(1)} \\ \hline\end{array},$$
$$\begin{array}{|c|}\hline \Gamma_{(2,2,1);(0)} \\ \hline \Gamma_{(2,2,2);(1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(2,1,1,1);(0)} \oplus \Gamma_{(2,2,1);(0)} \\ \hline \Gamma_{(2,2,1,1);(1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1,1,1,1,1);(0)} \oplus \Gamma_{(2,1,1,1);(0)} \\ \hline \Gamma_{(2,1,1,1,1);(1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1,1,1,1,1);(0)} \\ \hline \Gamma_{(1,1,1,1,1,1);(1)} \\ \hline\end{array}$$

$V^{\otimes(5,2)}$:

$$\begin{array}{|c|}\hline \Gamma_{(3);(0)} \\ \hline \Gamma_{(4);(1)} \\ \hline \Gamma_{(5);(2)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(4);(1)} \\ \hline \Gamma_{(5);(1,1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(2,1);(0)} \oplus \Gamma_{(3);(0)} \\ \hline \Gamma_{(3,1);(1)} \oplus \Gamma_{(4);(1)} \\ \hline \Gamma_{(4,1);(2)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(3);(0)} \\ \hline \Gamma_{(3,1);(1)} \oplus \Gamma_{(4);(1)} \\ \hline \Gamma_{(4,1);(1,1)} \\ \hline\end{array},$$
$$\begin{array}{|c|}\hline \Gamma_{(2,1);(0)} \oplus \Gamma_{(3);(0)} \\ \hline \Gamma_{(2,2);(1)} \oplus \Gamma_{(3,1);(1)} \\ \hline \Gamma_{(3,2);(2)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(2,1);(0)} \\ \hline \Gamma_{(2,2);(1)} \oplus \Gamma_{(3,1);(1)} \\ \hline \Gamma_{(3,2);(1,1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1,1,1);(0)} \oplus \Gamma_{(2,1);(0)} \\ \hline \Gamma_{(2,1,1);(1)} \oplus \Gamma_{(3,1);(1)} \\ \hline \Gamma_{(3,1,1);(2)} \\ \hline\end{array},$$
$$\begin{array}{|c|}\hline \Gamma_{(2,1);(0)} \oplus \Gamma_{(3);(0)} \\ \hline \Gamma_{(2,1,1);(1)} \oplus \Gamma_{(3,1);(1)} \\ \hline \Gamma_{(3,1,1);(1,1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(2,1);(0)} \\ \hline \Gamma_{(2,1,1);(1)} \oplus \Gamma_{(2,2);(1)} \\ \hline \Gamma_{(2,2,1);(2)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1,1,1);(0)} \oplus \Gamma_{(2,1);(0)} \\ \hline \Gamma_{(2,1,1);(1)} \oplus \Gamma_{(2,2);(1)} \\ \hline \Gamma_{(2,2,1);(1,1)} \\ \hline\end{array},$$
$$\begin{array}{|c|}\hline \Gamma_{(1,1,1);(0)} \\ \hline \Gamma_{(1,1,1,1);(1)} \oplus \Gamma_{(2,1,1);(1)} \\ \hline \Gamma_{(2,1,1,1);(2)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1,1,1);(0)} \oplus \Gamma_{(2,1);(0)} \\ \hline \Gamma_{(1,1,1,1);(1)} \oplus \Gamma_{(2,1,1);(1)} \\ \hline \Gamma_{(2,1,1,1);(1,1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1,1,1,1);(1)} \\ \hline \Gamma_{(1,1,1,1,1);(2)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1,1,1);(0)} \\ \hline \Gamma_{(1,1,1,1);(1)} \\ \hline \Gamma_{(1,1,1,1,1);(1,1)} \\ \hline\end{array}$$

$V^{\otimes(4,3)}$:

$$\begin{array}{|c|}\hline \Gamma_{(1);(0)} \\ \hline \Gamma_{(2);(1)} \\ \hline \Gamma_{(3);(2)} \\ \hline \Gamma_{(4);(3)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(2);(1)} \\ \hline \Gamma_{(3);(1,1)} \oplus \Gamma_{(3);(2)} \\ \hline \Gamma_{(4);(2,1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(3);(1,1)} \\ \hline \Gamma_{(4);(1,1,1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1);(0)} \\ \hline \Gamma_{(1,1);(1)} \oplus \Gamma_{(2);(1)} \\ \hline \Gamma_{(2,1);(2)} \oplus \Gamma_{(3);(2)} \\ \hline \Gamma_{(3,1);(3)} \\ \hline\end{array},$$
$$\begin{array}{|c|}\hline \Gamma_{(1);(0)} \\ \hline \Gamma_{(1,1);(1)} \oplus 2\Gamma_{(2);(1)} \\ \hline \Gamma_{(2,1);(1,1)} \oplus \Gamma_{(2,1);(2)} \oplus \Gamma_{(3);(1,1)} \oplus \Gamma_{(3);(2)} \\ \hline \Gamma_{(3,1);(2,1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(2);(1)} \\ \hline \Gamma_{(2,1);(1,1)} \oplus \Gamma_{(3);(1,1)} \\ \hline \Gamma_{(3,1);(1,1,1)} \\ \hline\end{array},$$
$$\begin{array}{|c|}\hline \Gamma_{(2);(1)} \\ \hline \Gamma_{(2,1);(2)} \\ \hline \Gamma_{(2,2);(3)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1);(0)} \\ \hline \Gamma_{(1,1);(1)} \oplus \Gamma_{(2);(1)} \\ \hline \Gamma_{(2,1);(1,1)} \oplus \Gamma_{(2,1);(2)} \\ \hline \Gamma_{(2,2);(2,1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1,1);(1)} \\ \hline \Gamma_{(2,1);(1,1)} \\ \hline \Gamma_{(2,2);(1,1,1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1,1);(1)} \\ \hline \Gamma_{(1,1,1);(2)} \oplus \Gamma_{(2,1);(2)} \\ \hline \Gamma_{(2,1,1);(3)} \\ \hline\end{array},$$
$$\begin{array}{|c|}\hline \Gamma_{(1);(0)} \\ \hline 2\Gamma_{(1,1);(1)} \oplus \Gamma_{(2);(1)} \\ \hline \Gamma_{(1,1,1);(1,1)} \oplus \Gamma_{(1,1,1);(2)} \oplus \Gamma_{(2,1);(1,1)} \oplus \Gamma_{(2,1);(2)} \\ \hline \Gamma_{(2,1,1);(2,1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1);(0)} \\ \hline \Gamma_{(1,1);(1)} \oplus \Gamma_{(2);(1)} \\ \hline \Gamma_{(1,1,1);(1,1)} \oplus \Gamma_{(2,1);(1,1)} \\ \hline \Gamma_{(2,1,1);(1,1,1)} \\ \hline\end{array},$$
$$\begin{array}{|c|}\hline \Gamma_{(1,1,1);(2)} \\ \hline \Gamma_{(1,1,1,1);(3)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1,1);(1)} \\ \hline \Gamma_{(1,1,1);(1,1)} \oplus \Gamma_{(1,1,1);(2)} \\ \hline \Gamma_{(1,1,1,1);(2,1)} \\ \hline\end{array},\ \begin{array}{|c|}\hline \Gamma_{(1);(0)} \\ \hline \Gamma_{(1,1);(1)} \\ \hline \Gamma_{(1,1,1);(1,1)} \\ \hline \Gamma_{(1,1,1,1);(1,1,1)} \\ \hline\end{array}$$

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