An Oka principle for equivariant isomorphisms

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• With F. Kutzschebauch and F. Lárusson.
• Let $X$ be a complex manifold. Then $X$ is Stein iff $X$ is biholomorphic to a closed complex submanifold of some $\mathbb{C}^n$.
• Holomorphic analogue of smooth complex affine variety.
• Can also define when a complex space is Stein. Analogue of complex affine variety.

**Oka Principle**

On reduced Stein spaces, there are only topological obstructions to solving holomorphic problems that can be formulated cohomologically.
Grauert’s Theorem

- Let $G$ be a complex Lie group and $X$ a reduced Stein space.

**Theorem (Grauert)**

Inclusion induces an isomorphism between isomorphism classes of holomorphic principal $G$-bundles on $X$ and topological principal $G$-bundles on $X$.

- Note that isomorphism classes of principal $G$-bundles are given by a certain cohomology set $H^1(X, G)$ where $G$ is maps of open sets of $X$ to $G$.
- Theorem of Grauert is an Oka principle.
- Equivariant version due to Heinzner and Kutzschebauch.
Want an Oka principle for equivariant maps.

Let $X$ be a connected Stein manifold with holomorphic action of the complex reductive Lie group $G$.

We have the quotient space $Z = X \sslash G$, a reduced Stein space.

The space $Z$ has points corresponding to the closed $G$-orbits in $X$ and the pull-back of the structure sheaf on $Z$ is the sheaf of $G$-invariant holomorphic functions on $X$.

$\pi_X : X \to Z$ dual to the inclusion $\mathcal{H}(X)^G \subset \mathcal{H}(X)$.

Let $x \in X$ such that $G_x$ is closed. Then $G_x$ is reductive and the representation of $G_x$ on $T_x(X)/T_x(G_x)$ is called the slice representation at $x$.

$Z$ has a stratification $Z_{(H)}$ where the points in $Z_{(H)}$ correspond to the closed orbits with isotropy group conjugate to the reductive subgroup $H$ of $G$. 
• The stratification $Z_{(H)}$ is a locally finite stratification of $Z$ by locally closed smooth subvarieties of $Z$.
• Example. Let $G = \mathbb{C}^*$ and $V = \mathbb{C}^2$ where $t(a, b) = (ta, t^{-1}b)$, $t \in \mathbb{C}^*$, $(a, b) \in V$. Let $x$ and $y$ be the coordinate functions. Then $\mathcal{O}(V)^G$ is generated by $xy$.
• Let $\pi = xy : V \to Z = \mathbb{C}$. Then $\pi^*\mathcal{H}(Z) = \mathcal{H}(V)^G$.
• Nonzero closed orbits $Gx$ have $G_x = \{e\}$. The origin has isotropy group $G$. Then the strata of $Z = \mathbb{C}$ are $\mathbb{C} \setminus \{0\}$ and $\{0\}$.
Let $Y$ be another Stein $G$-manifold with quotient mapping $\pi_Y: Y \to Z$. Same quotient space as $X$.

We say that $X$ and $Y$ are \textit{locally isomorphic over $Z$} if there are $G$-biholomorphisms $\psi_i: \pi_X^{-1}(U_i) \simeq \pi_Y^{-1}(U_i)$ which induce the identity on $U_i$ for an open cover $\{U_i\}$ of $Z$.

Hoped for Oka principle: $X$ and $Y$ are $G$-biholomorphic (over $\text{Id}: Z \to Z$) iff a topological condition is satisfied.

For $U \subset Z$ let $\mathcal{F}(U)$ denote the $G$-equivariant biholomorphisms of $\pi_X^{-1}(U)$ inducing $\text{Id}: U \to U$. Sheaf of groups.

Then $\psi_{ij} := \psi_i^{-1} \circ \psi_j$ is in $\mathcal{F}(U_i \cap U_j)$ and $\{\psi_{ij}\} \in H^1(Z, \mathcal{F})$.

There is an equivariant biholomorphism $\varphi: X \to Y$ over the identity of $Z$ iff $\{\psi_{ij}\}$ is a coboundary.
• Example. Let $X$ and $Y$ be holomorphic principal $G$-bundles over the Stein manifold $Z$
• Then $X//G = Y//G = Z$ and $X$ and $Y$, as Stein $G$-manifolds, are locally isomorphic over $Z$.
• Then $X$ is $G$-biholomorphic to $Y$ over $Z$ if and only if the two holomorphic principal bundles are isomorphic if and only if the principal bundles are $G$-homeomorphic (Grauert) if and only if $X$ is $G$-homeomorphic to $Y$ over $Z$. 
There is a unique open stratum $Z_{pr} \subset Z$, called the principal stratum. Let $X_{pr} = \pi_X^{-1}(Z_{pr})$.

We say that $X$ is generic if $X_{pr}$ consists of closed orbits with trivial isotropy group and $\text{codim } X \setminus X_{pr} \geq 2$.

$f_X$ is generic iff every slice representation is generic.

$X_{pr} \to Z_{pr}$ is a principal $G$-bundle.

For a fixed simple group $H$ and $H$-modules $W$ with $W^H = (0)$, up to isomorphism, only finitely many $W$ are not generic!

Similar statement for $H$ semisimple. Thus “almost any” $X$ is generic.
Let \( \psi : X \to X \) be holomorphic, equivariant, induce identity on \( Z \). Say \( \psi \) is **special** if there is a holomorphic map \( \gamma : X \to G \) such that \( \psi(x) = \gamma(x) \cdot x \).

**Lemma**

If \( X \) is generic, then every holomorphic \( \psi \) is special. Moreover, we have that \( \gamma(gx) = g\gamma(x)g^{-1} \).

Let \( \mathcal{G} \) be the sheaf on \( Z \) corresponding to equivariant holomorphic \( \gamma : \pi_X^{-1}(U) \to G \), \( U \) open in \( Z \).

If \( X \) is generic, then \( \mathcal{F} \simeq \mathcal{G} \), by the Lemma.
Theorem of Heinzner and Kutzschebauch

Let $G_c$ be the sheaf of groups corresponding to continuous equivariant maps to $G$.

**Theorem (HK)**
The natural map $H^1(Z, \mathcal{G}) \to H^1(Z, G_c)$ is an isomorphism.

**Corollary**
$X \simeq Y$ over $Z$, equivariantly, iff a topological condition is satisfied.
Now we see what a topological condition should be.

$G$ acts on $\mathcal{H}(X)$, $f \mapsto g \cdot f$ where $(g \cdot f)(x) = f(g^{-1}x)$, $x \in X$.

**Definition**

$f \in \mathcal{H}(X)$ is $G$-finite if $\{g \cdot f \mid g \in G\}$ spans a finite-dimensional $G$-module.

- The $G$-finite functions are an $\mathcal{H}(X)^G$-module.
- Let $V_i$ be a finite-dimensional $G$-module and let $\mathcal{H}(X)_{V_i}$ denote the sum of the subspaces of $G$-finite functions that transform by $V_i$. Covariants.
- Assume $\exists$ collection of irreducible representations $V_i$ such that the $\mathcal{H}(X)_{V_i}$ generate the algebra of $G$-finite functions on $X$ and that the $\mathcal{H}(X)_{V_i}$ are finitely generated $\mathcal{H}(X)^G$-modules. (True locally over $Z$).
Strongly continuous maps

- Let $\psi : X \to X$ be equivariant biholomorphic over $Z$. Let $f_1, \ldots, f_n$ generate the $\mathcal{H}(X)_V$. Then

$$\psi^* f_i = \sum a_{ij}(z) f_j$$ where the $a_{ij}(z) \in \mathcal{H}(Z)$.

- $\psi$ is determined by the $a_{ij}$.
- Let $\varphi : X \to X$ be a $G$-equivariant homeomorphism.

**Definition**

We say that $\varphi$ is strongly continuous if $\varphi^* f_i = \sum a_{ij}(z) f_j$ where the $a_{ij}(z)$ are continuous.
\( \varphi^* f_i = \sum_{ij} a_{ij}(z) f_j. \)

- The fibers of \( \pi \) are affine \( G \)-varieties and the \( \mathcal{H}(X)_{V_i} \) generate \( \mathcal{O}(\pi_X^{-1}(z)) \).
- Hence \( \varphi \) induces a \( G \)-automorphism of \( \pi_X^{-1}(z) \). So \( \varphi \) is a continuous family of \( G \)-isomorphisms of the fibers of \( \pi_X \).
- Strongly continuous maps are the natural kinds of topological maps one should consider.
- Suppose that \( F \) is represented by a group scheme \( \tilde{F} \) over \( Z \), i.e., the fibers of \( \tilde{F} \to Z \) are groups and \( F(U) \cong \Gamma(U, \tilde{F}) \). Then the continuous sections of \( \tilde{F} \) are the strongly continuous homeomorphisms.
Let $\varphi: X \to Y$ be a $G$-homeomorphism over $Z$. Then $\varphi$ is strongly continuous if $\psi_i^{-1} \circ \varphi: \pi_X^{-1}(U_i) \to \pi_X^{-1}(U_i)$ is strongly continuous for all $i$. Recall $\psi_i: \pi_X^{-1}(U_i) \simeq \pi_Y^{-1}(U_i)$ over $U_i$.

Let $\varphi: X \to Y$ be strongly continuous where $X$ and $Y$ are generic. Then there is an equivariant biholomorphism $\varphi': X \to Y$.
Proof of Theorem

- Let $x \in X$, $G_x$ closed and let $(W, H)$ be the slice representation. (So $H = G_x$.) There is an $H$-saturated open set $0 \in B \subset W$ such that $\sigma_X : \pi_X^{-1}(U) \simeq G \times^H B$ where $U$ is a neighborhood of $z = \pi_X(x)$. Slice theorem.
- We similarly have a $\sigma_Y : \pi_Y^{-1}(U) \simeq G \times^H B$.
- Then $\varphi_U := \sigma_Y \circ \varphi \circ \sigma_X^{-1} : G \times^H B \to G \times^H B$.
- For $t \in \mathbb{C}^*$ let $t \cdot [g, w] = [g, tw]$ for $[g, w] \in G \times^H B$. We can assume that $G \times^H B$ is stable under this action for $|t| \leq 1$.

Lemma

Let $\varphi_t([g, w]) = t^{-1}\varphi([g, tw])$. Then $\varphi_0 := \lim_{t \to 0} \varphi_t$ exists and is special, where the associated map $\gamma$ is continuous.

- Using induction and a partition of unity argument, one can show that there is a homotopy $\varphi_t$ with $\varphi_1 = \varphi$ and $\varphi_0$ special.
- Now $\{\psi_{ij}\} \in H^1(Z, F) = H^1(Z, G) \simeq H^1(Z, G_c)$ where the existence of $\varphi_0$ shows that the class in $H^1(Z, G_c)$ is trivial. QED.
• The proof does not actually show that \( \varphi \) is homotopic to a \( G \)-biholomorphism of \( X \) and \( Y \) over \( Z \).
• What about actions that are not generic?

Latest Theorem

Suppose that \( \varphi : X \rightarrow Y \) is strongly continuous. Then there is a homotopy \( \varphi_t \) with \( \varphi_1 = \varphi \) and \( \varphi_0 \) a \( G \)-biholomorphism of \( X \) and \( Y \) over \( Z \).

• Can’t reduce to HK. Go through Cartan’s version of Grauert’s original theorem and modify everything to fit our situation.
• Preliminary step. Let \( \varphi : X \to X \) be strongly continuous. Then, after a homotopy, we can arrange the following.

Let \( z \in Z \). Then there is a neighborhood \( U_z \) of \( z \) and \( \psi_z \in \mathcal{F}(U_z) \) which agrees with \( \varphi \) on \( \pi^{-1}_X(z) \). Moreover, the family \( \Psi(x, x') = \psi_{\pi_X(x)}(x') \) is smooth in \( x \) and \( x' \).

• We can apply the Grauert proof to the \( \varphi \) which admit an extension \( \Psi \). (They form a sheaf of groups.)
Linearization Problem

• How can $G$ act on $\mathbb{C}^n$? Can we holomorphically change coordinates such that the action of $G$ is linear? Say $G$-action is linearizable.

• Derksen-Kutzschebauch: For every $G \neq \{e\}$ there is a $d$ and a nonlinearizable action of $G$ on $\mathbb{C}^n$ for $n \geq d$. The quotients $\mathbb{C}^n / \! \!/ G$ are rather horrible.

**Theorem**

Suppose that $V$ is a $G$-module and that $X$ and $V$ are locally isomorphic over a common quotient. Then $X$ is equivariantly biholomorphic to $V$. 

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Oka principle for isomorphisms
Suppose that $V$ is not too “small” and suppose that $X \parallel G$ and $V \parallel G$ are biholomorphic by a mapping which preserves the Luna strata. Then $X$ is $G$-biholomorphic to $V$.

- For any simple Lie group $G$, only finitely many $V$ with $V^G = (0)$ are too small. Similarly for $G$-semisimple.
- The Luna stratification is finer than the stratification by conjugacy class of the isotropy group. On each irreducible component of $Z(H)$, the slice representation $(W, H)$ is constant. The Luna stratification is by the isomorphism class of the slice representation.