An Oka principle for equivariant isomorphisms

Gerald W. Schwarz

Brandeis University

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Stein spaces and Oka principle

- With F. Kutzschebauch and F. Lárusson.
- Let X be a complex manifold. Then X is Stein iff X is biholomorphic to a closed complex submanifold of some $\mathbb{C}^n$.
- Holomorphic analogue of smooth complex affine variety.
- Can also define when a complex space is Stein. Analogue of complex affine variety.

Oka Principle

On reduced Stein spaces, there are only topological obstructions to solving holomorphic problems that can be formulated cohomologically.

Grauert's Theorem

- Let G be a complex Lie group and X a reduced Stein space.

Theorem (Grauert)

Inclusion induces an isomorphism between isomorphism classes of holomorphic principal G-bundles on X and topological principal G-bundles on X.

- Note that isomorphism classes of principal G-bundles are given by a certain cohomology set $\mathrm{H}^1(X, \mathcal{G})$ where $\mathcal{G}$ is maps of open sets of X to G.
- Theorem of Grauert is an Oka principle.
- Equivariant version due to Heinzner and Kutzschebauch.

Quotients

- Want an Oka principle for equivariant maps.
- Let X be a connected Stein manifold with holomorphic action of the complex reductive Lie group G.
- We have the quotient space $Z = X/\!\!/G$, a reduced Stein space.
- The space Z has points corresponding to the closed G-orbits in X and the pull-back of the structure sheaf on Z is the sheaf of G-invariant holomorphic functions on X.
- $\pi_X \colon X \to Z$ dual to the inclusion $\mathcal{H}(X)^G \subset \mathcal{H}(X)$.
- Let $x \in X$ such that Gx is closed. Then G_x is reductive and the representation of G_x on $T_x(X)/T_x(Gx)$ is called the slice representation at x.
- Z has a stratification $Z_{(H)}$ where the points in $Z_{(H)}$ correspond to the closed orbits with isotropy group conjugate to the reductive subgroup H of G.

Stratification

- The stratification $Z_{(H)}$ is a locally finite stratification of Z by locally closed smooth subvarieties of Z.
- Example. Let $G = \mathbb{C}^*$ and $V = \mathbb{C}^2$ where $t(a, b) = (ta, t^{-1}b)$, $t \in \mathbb{C}^*$, $(a, b) \in V$. Let x and y be the coordinate functions. Then $\mathcal{O}(V)^G$ is generated by xy.
- Let $\pi = xy \colon V \to Z = \mathbb{C}$. Then $\pi^*\mathcal{H}(Z) = \mathcal{H}(V)^G$.
- Nonzero closed orbits Gx have $G_x = \{e\}$. The origin has isotropy group G. Then the strata of $Z = \mathbb{C}$ are $\mathbb{C} \setminus \{0\}$ and $\{0\}$.

Cohomology class

- Let Y be another Stein G-manifold with quotient mapping $\pi_Y\colon Y \to Z$. Same quotient space as X.
- We say that X and Y are locally isomorphic over Z if there are G-biholomorphisms $\psi_i\colon \pi_X^{-1}(U_i) \simeq \pi_Y^{-1}(U_i)$ which induce the identity on U_i for an open cover $\{U_i\}$ of Z.
- Hoped for Oka principle: X and Y are G-biholomorphic (over $\mathrm{Id}\colon Z \to Z$) iff a topological condition is satisfied.
- For $U \subset Z$ let $\mathcal{F}(U)$ denote the G-equivariant biholomorphisms of $\pi_X^{-1}(U)$ inducing $\mathrm{Id}\colon U \to U$. Sheaf of groups.
- Then $\psi_{ij} := \psi_i^{-1} \circ \psi_j$ is in $\mathcal{F}(U_i \cap U_j)$ and $\{\psi_{ij}\} \in \mathrm{H}^1(Z, \mathcal{F})$.

There is an equivariant biholomorphism $\varphi\colon X \to Y$ over the identity of Z iff $\{\psi_{ij}\}$ is a coboundary.

An example

- Example. Let X and Y be holomorphic principal G-bundles over the Stein manifold Z
- Then $X/\!\!/G = Y/\!\!/G = Z$ and X and Y, as Stein G-manfolds, are locally isomorphic over Z.
- Then X is G-biholomorphic to Y over Z if and only if the two holomorphic principal bundles are isomorphic if and only if the principal bundles are G-homeomorphic (Grauert) if and only if X is G-homeomorphic to Y over Z.

Generic actions

- There is a unique open stratum $Z_{\text{pr}} \subset Z$, called the principal stratum. Let $X_{\text{pr}} = \pi_X^{-1}(Z_{\text{pr}})$.
- We say that X is generic if X_{pr} consists of closed orbits with trivial isotropy group and $\operatorname{codim} X \setminus X_{\text{pr}} \geq 2$.
- X is generic iff every slice representation is generic.
- $X_{\text{pr}} \to Z_{\text{pr}}$ is a principal G-bundle.
- For a fixed simple group H and H-modules W with $W^H = (0)$, up to isomorphism, only finitely many W are not generic!
- Similar statement for H semisimple. Thus "almost any" X is generic.

Special automorphisms

- Let $\psi\colon X \to X$ be holomorphic, equivariant, induce identity on Z. Say ψ is special if there is a holomorphic map $\gamma\colon X \to G$ such that $\psi(x) = \gamma(x) \cdot x$.

Lemma

If X is generic, then every holomorphic ψ is special. Moreover, we have that $\gamma(gx) = g\gamma(x)g^{-1}$.

- Let $\mathcal{G}$ be the sheaf on Z corresponding to equivariant holomorphic $\gamma\colon \pi_X^{-1}(U) \to G$, U open in Z.
- If X is generic, then $\mathcal{F} \simeq \mathcal{G}$, by the Lemma.

Theorem of Heinzner and Kutzschebauch

- Let $\mathcal{G}_c$ be the sheaf of groups corresponding to continuous equivariant maps to G.

Theorem (HK)

The natural map $\mathrm{H}^1(Z, \mathcal{G}) \to \mathrm{H}^1(Z, \mathcal{G}_c)$ is an isomorphism.

Corollary

$X \simeq Y$ over Z, equivariantly, iff a topological condition is satisfied.

G-finite functions

- Now we see what a topological condition should be.
- G acts on $\mathcal{H}(X)$, $f \mapsto g \cdot f$ where $(g \cdot f)(x) = f(g^{-1}x)$, $x \in X$.

Definition

$f \in \mathcal{H}(X)$ is G-finite if $\{g \cdot f \mid g \in G\}$ spans a finite-dimensional G-module.

- The G-finite functions are an $\mathcal{H}(X)^G$-module.
- Let V_i be a finite-dimensional G-module and let $\mathcal{H}(X)_{V_i}$ denote the sum of the subspaces of G-finite functions that transform by V_i. Covariants.
- Assume $\exists$ collection of irreducible representations V_i such that the $\mathcal{H}(X)_{V_i}$ generate the algebra of G-finite functions on X and that the $\mathcal{H}(X)_{V_i}$ are finitely generated $\mathcal{H}(X)^G$-modules. (True locally over Z).

Strongly continuous maps

- Let $\psi\colon X \to X$ be equivariant biholomorphic over Z. Let $f_1, \dots, f_n$ generate the $\mathcal{H}(X)_{V_i}$. Then

$\psi^* f_i = \sum a_{ij}(z) f_j$ where the $a_{ij}(z) \in \mathcal{H}(Z)$.

- ψ is determined by the a_{ij}.
- Let $\varphi\colon X \to X$ be a G-equivariant homeomorphism.

Definition

We say that φ is strongly continuous if $\varphi^* f_i = \sum a_{ij}(z) f_j$ where the $a_{ij}(z)$ are continuous.

Strongly continuous maps

$\varphi^* f_i = \sum_{ij} a_{ij}(z) f_j$.

- The fibers of π are affine G-varieties and the $\mathcal{H}(X)_{V_i}$ generate $\mathcal{O}(\pi_X^{-1}(z))$.
- Hence φ induces a G-automorphism of $\pi_X^{-1}(z)$. So φ is a continuous family of G-isomorphisms of the fibers of π_X.
- Strongly continuous maps are the natural kinds of topological maps one should consider.
- Suppose that $\mathcal{F}$ is represented by a group scheme $\tilde{\mathcal{F}}$ over Z, i.e., the fibers of $\tilde{\mathcal{F}} \to Z$ are groups and $\mathcal{F}(U) \simeq \Gamma(U, \tilde{\mathcal{F}})$. Then the continuous sections of $\tilde{\mathcal{F}}$ are the strongly continuous homeomorphisms.

Main Theorem

- Let $\varphi\colon X \to Y$ be a G-homeomorphism over Z. Then φ is strongly continuous if $\psi_i^{-1} \circ \varphi\colon \pi_X^{-1}(U_i) \to \pi_X^{-1}(U_i)$ is strongly continuous for all i. Recall $\psi_i\colon \pi_X^{-1}(U_i) \simeq \pi_Y^{-1}(U_i)$ over U_i.

Main Theorem

Let $\varphi\colon X \to Y$ be strongly continuous where X and Y are generic. Then there is an equivariant biholomorphism $\varphi'\colon X \to Y$.

Proof of Theorem

- Let $x \in X$, Gx closed and let (W, H) be the slice representation. (So $H = G_x$.) There is an H-saturated open set $0 \in B \subset W$ such that $\sigma_X \colon \pi_X^{-1}(U) \simeq G \times^H B$ where U is a neighborhood of $z = \pi_X(x)$. Slice theorem.
- We similarly have a $\sigma_Y \colon \pi_Y^{-1}(U) \simeq G \times^H B$.
- Then $\varphi_U := \sigma_Y \circ \varphi \circ \sigma_X^{-1} : G \times^H B \to G \times^H B$.
- For $t \in \mathbb{C}^*$ let $t \cdot [g, w] = [g, tw]$ for $[g, w] \in G \times^H B$. We can assume that $G \times^H B$ is stable under this action for $|t| \leq 1$.

Lemma

Let $\varphi_t([g, w]) = t^{-1}\varphi([g, tw])$. Then $\varphi_0 := \lim_{t \to 0} \varphi_t$ exists and is special, where the associated map γ is continuous.

- Using induction and a partition of unity argument, one can show that there is a homotopy φ_t with $\varphi_1 = \varphi$ and φ_0 special.
- Now $\{\psi_{ij}\} \in \mathrm{H}^1(Z, \mathcal{F}) = \mathrm{H}^1(Z, \mathcal{G}) \simeq \mathrm{H}^1(Z, \mathcal{G}_c)$ where the existence of φ_0 shows that the class in $\mathrm{H}^1(Z, \mathcal{G}_c)$ is trivial. QED.

Latest Theorem

- The proof does not actually show that φ is homotopic to a G-biholomorphism of X and Y over Z.
- What about actions that are not generic?

Latest Theorem

Suppose that $\varphi\colon X \to Y$ is strongly continuous. Then there is a homotopy φ_t with $\varphi_1 = \varphi$ and φ_0 a G-biholomorphism of X and Y over Z.

- Can't reduce to HK. Go through Cartan's version of Grauert's original theorem and modify everything to fit our situation.

Latest Theorem

- Preliminary step. Let $\varphi \colon X \to X$ be strongly continuous. Then, after a homotopy, we can arrange the following.

Let $z \in Z$. Then there is a neighborhood U_z of z and $\psi_z \in \mathcal{F}(U_z)$ which agrees with φ on $\pi_X^{-1}(z)$. Moreover, the family $\Psi(x, x') = \psi_{\pi_X(x)}(x')$ is smooth in x and x'.

- We can apply the Grauert proof to the φ which admit an extension Ψ. (They form a sheaf of groups.)

Linearization Problem

- How can G act on $\mathbb{C}^n$? Can we holomorphically change coordinates such that the action of G is linear? Say G-action is linearizable.
- Derksen-Kutzschebauch: For every $G \neq \{e\}$ there is a d and a nonlinearizable action of G on $\mathbb{C}^n$ for $n \geq d$. The quotients $\mathbb{C}^n /\!\!/ G$ are rather horrible.

Theorem

Suppose that V is a G-module and that X and V are locally isomorphic over a common quotient. Then X is equivariantly biholomorphic to V.

Linearization Problem

Theorem

Suppose that V is not too "small" and suppose that $X /\!\!/ G$ and $V /\!\!/ G$ are biholomorphic by a mapping which preserves the Luna strata. Then X is G-biholomorphic to V.

- For any simple Lie group G, only finitely many V with $V^G = (0)$ are too small. Similarly for G-semisimple.
- The Luna stratification is finer than the stratification by conjugacy class of the isotropy group. On each irreducible component of $Z_{(H)}$, the slice representation (W, H) is constant. The Luna stratification is by the isomorphism class of the slice representation.