Categories of integrable $sl(\infty)$-, $o(\infty)$-, $sp(\infty)$-modules

Ivan Penkov and Vera Serganova

Summary. We investigate several categories of integrable $sl(\infty)$-, $o(\infty)$-, $sp(\infty)$-modules. In particular, we prove that the category of integrable $sl(\infty)$-, $o(\infty)$-, $sp(\infty)$-modules with finite-dimensional weight spaces is semisimple. The most interesting category we study is the category $\widetilde{\text{Tens}}_g$ for $g = sl(\infty)$-, $o(\infty)$-, $sp(\infty)$. Its objects $M$ are defined as integrable $g$-modules of finite Loewy length such that the algebraic dual $M^*$ is also integrable and of finite Loewy length.

We prove that the simple objects of $\widetilde{\text{Tens}}_g$ are precisely the simple tensor modules, i.e. the simple subquotients of the tensor algebra of the direct sum of the natural and conatural representations. We also study injectives in $\widetilde{\text{Tens}}_g$ and compute the $\text{Ext}^1$'s between simple modules.

Finally, we characterize a certain subcategory $\text{Tens}_g$ of $\widetilde{\text{Tens}}_g$ as the unique minimal abelian full subcategory of the category of integrable modules which contains a non-trivial module and is closed under tensor product and algebraic dualization.

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1. Introduction

The category of finite-dimensional representations of a Lie algebra is endowed with a natural contravariant involution

$$M \rightsquigarrow M^*, \tag{1}$$

where * indicates dual space. For categories of infinite-dimensional modules (1) is never an involution as $M \not\cong M^{**}$. This is why one usually looks for a “restricted
dual” or a “continuous dual” which might still yield a contravariant involution on a given category of infinite-dimensional modules. In this paper, we study two categories of infinite-dimensional modules of certain infinite-dimensional Lie algebras and show, in particular, that there exists an interesting category $\tilde{Tens}_g$ of infinite-dimensional representations on which the functor (1) of algebraic dualization is well-defined and preserves the property of a module to be of finite Loewy length.

More precisely, we study representations of locally finite Lie algebras, i.e. of direct limits of finite-dimensional Lie algebras. There are three well-known classical simple locally finite Lie algebras $sl(\infty), o(\infty), sp(\infty)$, each of them being defined by an obvious direct limit. None of these Lie algebras admits non-trivial finite-dimensional representations, and instead one studies integrable representations (the definition see in section 2 below). However, the category of integrable $g$-modules for $g = sl(\infty), o(\infty), sp(\infty)$ is vast (and “wild” in the technical sense), so it is reasonable to look for interesting subcategories.

One subcategory we study is the category of integrable weight modules with finite-dimensional weight spaces, and this is obviously an analog of the category of finite-dimensional representations of a classical finite-dimensional Lie algebra. It is less obvious that for $g = sl(\infty)$ this category contains some rather interesting simple modules, which are not highest weight modules. The first main result of this paper is the proof of the semisimplicity of this category: an extension of Hermann Weyl’s semisimplicity theorem to the classical Lie algebras $sl(\infty), o(\infty), sp(\infty)$.

The above category is clearly not the only reasonable generalization of the category of finite-dimensional representations, as for instance it does not contain the adjoint representation. Indeed, note that the adjoint representation has an infinite-dimensional weight space, the Cartan subalgebra itself. On the other hand, the adjoint representation is naturally a simple tensor module as defined in [PS]. More generally, we define the category $\tilde{Tens}_g$ for $g \cong sl(\infty), o(\infty), sp(\infty)$ simply as the largest category of integrable $g$-modules which is closed under algebraic dualization and such that every object has finite Loewy length. This category is a (non-rigid) tensor category with respect to the usual tensor product.

The second main contribution of the present paper is the study of the category $\tilde{Tens}_g$. In particular, we study injectives in $\tilde{Tens}_g$ and compute the Ext$^1$’s between simple modules. We also give an alternative characterization of $\tilde{Tens}_g$ by proving that an integrable $g$-module is an object of $\tilde{Tens}_g$ if and only if it admits only finitely many non-isomorphic simple subquotients each of which is a submodule of a suitable finite tensor product of natural and conatural modules.

Finally, we describe a certain subcategory $Tens_g$ of $\tilde{Tens}_g$ as the unique minimal abelian full subcategory of the category of integrable modules which contains a non-trivial module and is closed under tensor product and algebraic dualization.

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2. Basic definitions

The ground field is $\mathbb{C}$ and $\otimes$ stands for $\otimes_{\mathbb{C}}$. If $\mathcal{C}$ is a category, $C \in \mathcal{C}$ indicates that $C$ is an object of $\mathcal{C}$. If $P$ is a set, we denote by $2^P$ the power set of $P$. We recall that the cardinal numbers $\beth_n$ are defined inductively: $\beth_0 = \text{card } \mathbb{Z}$, $\beth_1 = \text{card } 2^\mathbb{Z}$, $\beth_n = \text{card } 2^\beth_{n-1}$, where $P_{n-1}$ is a set of cardinality $\beth_{n-1}$.

In this paper $\mathfrak{g}$ stands for a locally semisimple (complex) Lie algebra. By definition, $\mathfrak{g} = \bigcup_{i \in \mathbb{Z}_{>0}} \mathfrak{g}_i$, where

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathfrak{g}_3 \subset \ldots \quad (2)$$

is a sequence of inclusions of semisimple finite-dimensional Lie algebras. We call the sequence (2) an exhaustion of $\mathfrak{g}$, and we will assume that it is fixed. A locally semisimple Lie algebra is locally simple if it admits an exhaustion (2) so that all $\mathfrak{g}_i$ are simple. It is clear that a locally simple Lie algebra is simple. If no restrictions on $\mathfrak{g}$ are clearly stated, in what follows $\mathfrak{g}$ is assumed to be an arbitrary locally semisimple Lie algebra.

A locally simple algebra $\mathfrak{g}$ is diagonal if an exhaustion (2) can be chosen so that all $\mathfrak{g}_i$ are classical simple Lie algebras and the natural representation $V_i$ of $\mathfrak{g}_i$, when restricted to $\mathfrak{g}_{i-1}$, has the form $k_i V_{i-1} \oplus l_i V_{i-1}^* \oplus C^\gamma$ for some $k_i$, $l_i$ and $s_i \in \mathbb{Z}_{>0}$. Here $V_{i-1}$ stands for the natural representation of $\mathfrak{g}_{i-1}$, $C^\gamma$ stands for the trivial module of dimension $s_i$, and $k_i V_{i-1}$ (respectively, $l_i V_{i-1}^*$) denotes the direct sum of $k_i$ (respectively, $l_i$) copies of $V_{i-1}$ (respectively, $V_{i-1}^*$).

The three classical simple Lie algebras $sl(\infty)$, $o(\infty)$ and $sp(\infty)$ (defined respectively as $sl(\infty) = \bigcup_i sl(i)$, $o(\infty) = \bigcup_i o(i)$, $sp(\infty) := \bigcup_i sp(2i)$) via the natural inclusions $sl(i) \subset sl(i+1)$ etc.) are clearly diagonal. Moreover, $sl(\infty)$, $o(\infty)$, $sp(\infty)$ are (up to isomorphism) the only finitary locally simple Lie algebras $\mathfrak{g}$; finitary means by definition that $\mathfrak{g}$ admits a faithful countable-dimensional $\mathfrak{g}$-module with a basis in which each element $g \in \mathfrak{g}$ acts through a finite matrix, [Ba1], [Ba3]. More generally, there exists also a classification of locally simple diagonal Lie algebras up to isomorphism, [BZh]. We do not use this classification in the present paper and present only the simplest example of a diagonal Lie algebra not isomorphic to $sl(\infty)$, $o(\infty)$ or $sp(\infty)$. This is the Lie algebra $sl(2^\infty)$ defined as the direct limit $\lim_{\rightarrow} sl(2^i)$ under the inclusions

$$sl(2^i) \rightarrow sl(2^{i+1}), \ A \rightarrow \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$}

A $\mathfrak{g}$-module $M$ is integrable if $\dim \text{span}\{m, g \cdot m, g^2 \cdot m, \ldots\} < \infty$ for any $m \in M$ and $g \in \mathfrak{g}$. Since $\mathfrak{g}$ is locally semisimple, this is equivalent to the condition that, when restricted to any semisimple finite-dimensional subalgebra $\mathfrak{f}$ of $\mathfrak{g}$, $M$
is isomorphic to a (not necessarily countable) direct sum of finite-dimensional \( L \)-modules. We denote by \( \text{Int}_g \) the category of integrable \( g \)-modules; \( \text{Int}_g \) is a full subcategory of the category of \( g \)-modules \( g\text{-mod} \).

Any countable-dimensional \( g \)-module \( M \in \text{Int}_g \) can be exhausted by finite dimensional \( g \)-modules \( M_i \), i.e. there exists a chain of finite-dimensional \( g \)-submodules \( M_1 \subset M_2 \subset \ldots \) such that \( M = \varprojlim M_i \). We call \( M \) \emph{locally simple} if all \( M_i \) can be chosen to be simple modules. It is clear that a locally simple module is simple. Note also that if \( M \) is locally simple then any two exhaustions \( \{M_i\} \) and \( \{M'_i\} \) coincide from some point on: that follows from the fact that \( M_i \cap M'_i \neq 0 \) for some \( i \) and hence \( M_j = M'_j = M_j \cap M'_j \) for any \( j \geq i \). We say that a locally simple \( g \)-module \( M = \varprojlim M_i \) is a \emph{highest weight module} if there is a chain of nested Borel subalgebras \( b_i \) of \( g \) such that the \( b_i \)-highest weight space of \( M_i \) is mapped into the \( b_{i+1} \)-highest weight space of \( M_{i+1} \) under the inclusion \( M_i \subset M_{i+1} \). The direct limit of highest weight spaces is then the \emph{\( b \)-highest weight space of \( M \)}, where \( b = \varinjlim b_i \).

By

\[ \Gamma^b_g : g\text{-mod} \to \text{Int}_g, \]

\[ M \mapsto \Gamma^b_g(M) := \{ m \in M, \dim \text{span}\{ m, g \cdot m, g \cdot m^2, \ldots \} < \infty \quad \forall g \in g \} \]

we denote the \emph{functor of \( g \)-integrable vectors}. It is an exercise to check that \( \Gamma^b_g(M) \) is indeed a well-defined \( g \)-submodule of \( M \); the fact that \( \Gamma^b_g(M) \) is integrable is obvious. Furthermore, \( \Gamma^b_g \) is a left-exact functor.

If \( g \) is a diagonal (locally simple) Lie algebra, then one can define a \emph{natural module} \( V \) of \( g \). Indeed, the reader will verify that one can choose a subexhaustion of (2) such that the natural \( g_i \)-module \( V_i \) is a \( g_i \)-submodule of \( V_{i+1} \) for any \( i \). Therefore, fixing arbitrary injective homomorphisms \( V_i \to V_{i+1} \) of \( g_i \)-modules, we obtain a direct system and we set \( V := \varinjlim V_i \). Note that \( V \) depends on the choice of the homomorphisms \( V_i \to V_{i+1} \). In the special case when \( g \cong sl(\infty) \), \( o(\infty) \), \( sp(\infty) \), the homomorphisms \( V_i \to V_{i+1} \) are unique up to proportionality, and one can prove that as a result \( V \) is unique up to isomorphism, i.e. in particular does not depend on the fixed exhaustion of \( g \). In these latter cases we speak about the \emph{natural representation}.

By choosing injective homomorphisms of \( g_i \)-modules \( V_i^* \to V_{i+1}^* \), we obtain a direct system defining a \emph{conatural representation} of \( g \). We denote such a representation by \( V \). For \( g \cong sl(\infty) \), \( o(\infty) \), \( sp(\infty) \), \( V \) is unique up to isomorphism. In fact, \( V \cong V \) for \( g \cong o(\infty) \), \( sp(\infty) \).

3. Injective modules in \( \text{Int}_g \) and semisimplicity of the category \( \text{Int}^\text{fin}_{g,b} \)

**Proposition 3.1.** \( \text{Ext}^1_g(X, M^*) = 0 \) for any \( X, M \in \text{Int}_g \).
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**Proof.** We use that

\[
\text{Ext}^1_\mathfrak{g}(X, M^*) = \text{Ext}^1_\mathfrak{g}(C, \text{Hom}_C(X, M^*)) \simeq H^1(\mathfrak{g}, \text{Hom}_C(X, M^*)) = H^1(\mathfrak{g}, X \otimes M^*),
\]

see for instance [W]. Therefore, it suffices to show that $H^1(\mathfrak{g}, R^*) = 0$ for any integrable $\mathfrak{g}$-module $R$.

Consider the standard complex for the cohomology of $\mathfrak{g}$ with coefficients in $R^*$:

\[
0 \to R^* \to (\mathfrak{g} \otimes R)^* \to (\Lambda^2(\mathfrak{g}) \otimes R)^* \to \ldots
\]

(3)

It is dual to the standard homology complex

\[
0 \leftarrow R \leftarrow \mathfrak{g} \otimes R \leftarrow \Lambda^2(\mathfrak{g}) \otimes R \leftarrow \ldots,
\]

which is the direct limit of complexes

\[
0 \leftarrow R \leftarrow \mathfrak{g}_i \otimes R \leftarrow \Lambda^2(\mathfrak{g}_i) \otimes R \leftarrow \ldots.
\]

Since $H_1(\mathfrak{g}_i, R) = 0$ for each $i$, we get $H_1(\mathfrak{g}, R) = 0$. Therefore, the dual complex (3) has trivial first cohomology, i.e. $H^1(\mathfrak{g}, R^*) = 0$. □

**Proposition 3.2.** For any $M \in \text{Int}_\mathfrak{g}$, $\Gamma_\mathfrak{g}(M^*)$ is an injective object of $\text{Int}_\mathfrak{g}$.

**Proof.** Let $X \in \text{Int}_\mathfrak{g}$. The exact sequence of $\mathfrak{g}$-modules

\[
0 \to \Gamma_\mathfrak{g}(M^*) \to M^* \to M^*/\Gamma_\mathfrak{g}(M^*) \to 0
\]

induces an exact sequence of vector spaces

\[
0 \to \text{Hom}_\mathfrak{g}(X, \Gamma_\mathfrak{g}(M^*)) \overset{\psi}{\to} \text{Hom}_\mathfrak{g}(X, M^*) \to \text{Hom}_\mathfrak{g}(X, M^*/\Gamma_\mathfrak{g}(M^*)) \to \text{Ext}^1_\mathfrak{g}(X, \Gamma_\mathfrak{g}(M^*)) \overset{\psi}{\to} \text{Ext}^1_\mathfrak{g}(X, M^*) = 0.
\]

Since $\text{Hom}_\mathfrak{g}(X, M^*/\Gamma_\mathfrak{g}(M^*)) = 0$ (this follows from the facts that a quotient of an integrable $\mathfrak{g}$-module is again an integrable $\mathfrak{g}$-module and that $\text{Int}_\mathfrak{g}$ is closed with respect to extensions) we conclude that $\psi$ is an isomorphism, i.e. that $\text{Ext}^1_\mathfrak{g}(X, \Gamma_\mathfrak{g}(M^*)) = 0$.

□

**Corollary 3.3.** $\text{Int}_\mathfrak{g}$ has enough injectives.

**Proof.** Let $M \in \text{Int}_\mathfrak{g}$. Then $M \subset M^{**}$ and it is easy to check (using the local semisimplicity of $\mathfrak{g}$) that the projection $M^{**} \to \Gamma_\mathfrak{g}(M^*)$ induces an injection $M \to \Gamma_\mathfrak{g}(M^*)$. This in turn induces an injection $M \to \Gamma_\mathfrak{g}(\Gamma_\mathfrak{g}(M^*))$, and $\Gamma_\mathfrak{g}(\Gamma_\mathfrak{g}(M^*))$ is an injective object of $\text{Int}_\mathfrak{g}$ by Proposition 3.2. □

Note that there is a simpler proof of Corollary 3.3 not referring to Proposition 3.2. Indeed, it is enough to notice that the functor $\Gamma_\mathfrak{g} : \mathfrak{g}\text{-mod} \to \text{Int}_\mathfrak{g}$ is right adjoint to the inclusion functor $\text{Int}_\mathfrak{g} \subset \mathfrak{g}\text{-mod}$. Then the equality

\[
\text{Hom}_\mathfrak{g}(M, J_M) = \text{Hom}_\mathfrak{g}(M, \Gamma_\mathfrak{g}(J_M))
\]
allows us to conclude that, if \( i : M \to J_M \) is an injective homomorphism of \( M \in \text{Int}_g \) into an injective \( g \)-module, then \( \Gamma_g(J_M) \) is an injective object of \( \text{Int}_g \) and \( i \) factors through the inclusion \( \Gamma_g(J_M) \subset J_M \). In particular, this argument allows to reduce the existence of injective hulls in \( \text{Int}_g \) to the well-known existence of injective hulls in \( g \)-mod.

With this in mind, we can view Propositions 3.1 and 3.2 as yielding an explicit construction of an injective module \( \Gamma_g(M^\ast) \) associated to any \( M \in \text{Int}_g \).

In the rest of this section we assume that \( g \) admits a splitting Cartan subalgebra \( h \subset g \), i.e. an abelian subalgebra \( h \subset g \) such that \( g \) decomposes as \( h \oplus \bigoplus_{0 \neq \alpha \in h^*} g^\alpha \).

It is well-known that in this case \( g \) is isomorphic to a direct sum of copies of \( \text{sl}(\infty), o(\infty), \text{sp}(\infty) \) and finite-dimensional simple Lie algebras, see [PStr].

We define the category \( \text{Int}_{g,h}^{\text{wt}} \) as the full subcategory of \( \text{Int}_g \) which consists of weight modules \( M \), i.e. objects \( M \in \text{Int}_g \) which admit a decomposition
\[
M = \bigoplus_{\alpha \in h^*} M^\alpha, \tag{4}
\]
where
\[
M^\alpha = \{ m \in M | h \cdot m = \alpha(h)m \text{ for any } h \in h \}.
\]
Note that (4) is automatically a decomposition of \( h \)-modules. It is also clear that there is a left exact functor \( \Gamma_h^{\text{wt}} : \text{Int}_g \rightsquigarrow \text{Int}_{g,h}^{\text{wt}}, \ M \mapsto \bigoplus_{\alpha \in h^*} M^\alpha. \)

By \( \Gamma_h^{\text{wt}} \) we denote the composition \( \Gamma_h^{\text{wt}} \circ \Gamma_g : \text{g-mod} \rightsquigarrow \text{Int}_{g,h}^{\text{wt}}. \)

**Lemma 3.4.** If \( X \) is an injective object of \( \text{Int}_g \), then \( \Gamma_h^{\text{wt}}(X) \) is an injective object of \( \text{Int}_{g,h}^{\text{wt}}. \)

*Proof.* It suffices to note that \( \Gamma_h^{\text{wt}} \) is right adjoint to the inclusion functor \( \text{Int}_{g,h}^{\text{wt}} \subset \text{Int}_g. \)

**Example 3.5.** Let \( g = \text{sl}(\infty) \) and \( M = V \otimes V^\ast \). Consider the \( g \)-module \( M^\ast \). Let’s think of \( M^\ast = (V \otimes V^\ast)^\ast \) as the space of all infinite matrices \( B = (b_{ij}), i,j \in \mathbb{Z}_{>0}, \) and of \( M \) as the space of finitary infinite matrices \( A = (a_{ij}), i,j \in \mathbb{Z}_{>0}, \) where \( B(A) = \sum_{i,j} b_{ij}a_{ij}. \) Then \( g \) is identified with the subspace \( F \subset (V \otimes V^\ast)^\ast \) of finitary matrices with trace zero, and the \( g \)-module structure on \( M^\ast \) is given by \( A \cdot B = [A,B]. \) We fix the Cartan subalgebra \( h \) to be the algebra of finitary diagonal matrices, and we claim that \( \Gamma_h^{\text{wt}}(M^\ast) = F + D \) where \( D \) is the subspace of diagonal matrices. Indeed, clearly \( D \) equals the \( h \)-weight space \( (M^\ast)^0 \) of weight
0. Furthermore, any weight space of non-zero weight is the span of an elementary non-diagonal matrix, hence $\Gamma^\text{wt}(M^*) = F + D$. Note also that we have a non-splitting exact sequence of $\mathfrak{g}$-modules

$$0 \to \mathfrak{g} \to \Gamma^\text{wt}(M^*) \to T \to 0,$$

where $T = D/D \cap F$ is a trivial $\mathfrak{g}$-module of dimension $\mathfrak{n}_1$.

**Corollary 3.6.** For any $M \in \text{Int}_{\mathfrak{g}}$, $\Gamma^\text{wt}_{\mathfrak{g},\mathfrak{h}}(M^*)$ is an injective object of $\text{Int}^\text{wt}_{\mathfrak{g},\mathfrak{h}}$.

Define now $\text{Int}^\text{fin}_{\mathfrak{g},\mathfrak{h}}$ as the full subcategory of $\text{Int}^\text{wt}_{\mathfrak{g},\mathfrak{h}}$ consisting of $\mathfrak{h}$-weight modules $M = \bigoplus_{\alpha \in \mathfrak{h}^*} M^\alpha$ such that $\dim M^\alpha < \infty$ for any $\alpha \in \mathfrak{h}^*$.

**Theorem 3.7.** The category $\text{Int}^\text{fin}_{\mathfrak{g},\mathfrak{h}}$ is semisimple.

**Proof.** Let $M \in \text{Int}^\text{fin}_{\mathfrak{g},\mathfrak{h}}$ be simple. There is an $\mathfrak{h}$-module isomorphism

$$M = \bigoplus_{\alpha \in \mathfrak{h}^*} M^\alpha.$$

Therefore, $M^* = \prod_{\alpha \in \mathfrak{h}^*} (M^\alpha)^*$. A non-difficult computation shows that $\Gamma^\text{wt}_{\mathfrak{g},\mathfrak{h}}(M^*)$ is isomorphic to $\bigoplus_{\alpha \in \mathfrak{h}^*} (M_\alpha)^*$. Moreover, using the fact that $\dim M^\alpha < \infty$ for all $\alpha$, it is easy to check that $M_* := \bigoplus_{\alpha \in \mathfrak{h}^*} (M_\alpha)^*$ is a simple integrable $\mathfrak{g}$-module. Hence $M_* = \Gamma^\text{wt}_{\mathfrak{g},\mathfrak{h}}(M^*)$. Applying $\Gamma^\text{wt}_{\mathfrak{g},\mathfrak{h}}$ again, we see that

$$\Gamma^\text{wt}_{\mathfrak{g},\mathfrak{h}}(\Gamma^\text{wt}_{\mathfrak{g},\mathfrak{h}}(M^*)^*) = M.$$

Therefore, $M$ is injective in $\text{Int}^\text{wt}_{\mathfrak{g},\mathfrak{h}}$, and thus also in $\text{Int}^\text{fin}_{\mathfrak{g},\mathfrak{h}}$, by Corollary 3.6. \qed

**Example 3.8.**

a) Let $\mathfrak{g} = sl(\infty)$. One checks immediately that all tensor powers $V^\otimes \mathfrak{h}$, $V$ being the natural module, are objects of $\text{Int}^\text{fin}_{\mathfrak{g},\mathfrak{h}}$. The same applies to the tensor powers of the conatural module $V_\iota$. However, the category $\text{Int}^\text{fin}_{\mathfrak{g},\mathfrak{h}}$ contains also more interesting modules as the following one: $M = \varinjlim S_i(V_\iota)$, $V_\iota$ being the natural representation of $sl(i)$ . The module $M$ has 1-dimensional weight spaces, but not a highest weight module, see [DP1, Example 3]. Note also that the adjoint representation is not an object of $\text{Int}^\text{fin}_{\mathfrak{g},\mathfrak{h}}$.

b) Let $\mathfrak{g} = o(\infty)$ and let $\mathfrak{g}$ be exhausted by $\mathfrak{g}_i = o(2i), i \geq 3$. Denote by $S_i^1$ and $S_i^2$ the two non-isomorphic spinor $\mathfrak{g}_i$-modules. Then $S_i^1$ and $S_i^2$ are both isomorphic to $S_{i-1}^1 \oplus S_{i-1}^2$ as $\mathfrak{g}_{i-1}$-modules. Therefore, there is an injective homomorphism of $\mathfrak{g}_{i-1}$-modules $\varphi_{i-1}^{k,h} : S_{i-1}^k \to S_i^k$ for $k, s \in \{1, 2\}$, and moreover $\varphi_{i-1}^{k,h}$ is unique up to proportionality. Any sequence $\{t_i\}_{i \geq 3}$ of elements in $\{1, 2\}$ defines a direct system

$$S_3^3 \to S_4^3 \to S_5^3 \to \ldots,$$

and hence a simple $\mathfrak{g}$-module $S(\{t_i\})$. Using the fact that $S(\{t_i\})$ is locally simple, it is easy to see that $S(\{t_i\}) = S(\{t'_i\})$ if and only if the “tails” of the sequence $\{t_i\}$ and $\{t'_i\}$ coincide, i.e. $t_i = t'_i$ for large enough $i$.
The modules \( S(\{t_i\}) \) are weight modules with 1-dimensional spaces for any Cartan subalgebra \( \mathfrak{h} \) of the form \( \mathfrak{h} = \bigcup_i \mathfrak{h}_i \) where \( \mathfrak{h}_3 \subset \mathfrak{h}_4 \subset \ldots \) are nested Cartan subalgebras of \( \mathfrak{g}_3 = o(6) \subset \mathfrak{g}_4 = o(8) \subset \ldots \). In particular, \( S(\{t_i\}) \in \text{Int}_{\mathfrak{g},\mathfrak{b}}. \)

4. On the integrability of \( M^* \) for \( M \in \text{Int}_{\mathfrak{g}} \)

Lemma 4.1. Let \( M \in \text{Int}_{\mathfrak{g}} \). Then \( M^* \in \text{Int}_{\mathfrak{g}} \) if and only if for any \( i > 0 \) \( \text{Hom}_{\mathfrak{g}_i}(N,M) \neq 0 \) only for finitely many non-isomorphic simple \( \mathfrak{g}_i \)-modules \( N \).

Proof. Fix \( i \). Let \( \Lambda_i \) be the set of integral dominant weights of \( \mathfrak{g}_i \) (for some fixed Borel subalgebra \( \mathfrak{b}_i \) of \( \mathfrak{g}_i \) with fixed Cartan subalgebra \( \mathfrak{h}_i \subset \mathfrak{b}_i \)) and \( V^i_\lambda \) be the simple \( \mathfrak{g}_i \)-module with highest weight \( \lambda \). Denote by \( \Lambda_i(M) \) the set of all \( \lambda \in \Lambda_i \) such that \( \text{Hom}_{\mathfrak{g}_i}(V^i_\lambda,M) \neq 0 \). Since \( M \) is a semisimple \( \mathfrak{g}_i \)-module, we can write \( M \) as

\[
M = \bigoplus_{\lambda \in \Lambda_i(M)} M^\lambda \otimes V^i_\lambda,
\]

where \( M^\lambda := \text{Hom}_{\mathfrak{g}_i}(V^i_\lambda,M) \) is a trivial \( \mathfrak{g}_i \)-module. We have

\[
M^* = \prod_{\lambda \in \Lambda_i(M)} (V^i_\lambda)^* \otimes (M^\lambda)^*.
\]

Suppose that \( \Lambda_i(M) \) is finite. Then for any fixed \( g \in \mathfrak{g}_i \) there is a polynomial \( p_\lambda(z) \) such that \( p_\lambda(g) \cdot (V^i_\lambda)^* = 0 \). Set \( p(z) := \prod_{\lambda \in \Lambda_i(M)} p_\lambda(z) \). Then \( p(g) \cdot M^* = 0 \). Hence \( g \) acts integrably on \( M^* \), i.e. \( M^* \) is integrable over \( \mathfrak{g}_i \).

Now let \( \Lambda_i(M) \) be infinite. Let \( v_\lambda \) be a non-zero vector of weight \( -\lambda \) in \( (V^i_\lambda)^* \otimes (M^\lambda)^* \). One can choose \( h \) in the Cartan subalgebra of \( \mathfrak{g}_i \) such that \( \lambda(h) \neq \mu(h) \) for any \( \mu \neq \lambda \in \Lambda_i(M) \). Let \( v := \prod_{\lambda \in \Lambda_i(M)} (v_\lambda) \in \prod_{\lambda \in \Lambda_i(M)} (V^i_\lambda)^* \otimes (M^\lambda)^* \). Then \( \dim(\mathbb{C}[h] \cdot v) = \infty \), and \( M^* \) is not \( \mathfrak{g}_i \)-integrable. \( \Box \)

Corollary 4.2. Let \( M, M' \in \text{Int}_{\mathfrak{g}} \). If \( M^*, (M')^* \in \text{Int}_{\mathfrak{g}} \), then \( (M \otimes M')^* \in \text{Int}_{\mathfrak{g}} \) and \( M^{**} \in \text{Int}_{\mathfrak{g}} \).

Proposition 4.3. Let \( \mathfrak{g} \) be a locally simple Lie algebra. There exists a non-trivial module \( M \in \text{Int}_{\mathfrak{g}} \) such that \( M^* \) is integrable if and only if \( \mathfrak{g} \) is diagonal.

Proof. First of all, if \( \mathfrak{g} \) is diagonal, then any natural module \( V = \lim_{\to} V_n \) satisfies the finiteness condition of Lemma 4.1, hence \( V^* \) is integrable.

Before we prove the other direction, note that, by passing to a subexhaustion, we can always assume that \( \mathfrak{g} \) is exhausted by classical simple Lie algebras \( \mathfrak{g}_i \) of the same type (A, B, C or D). Let now \( M \in \text{Int}_{\mathfrak{g}} \) be non-trivial and \( M^* \) be integrable.

We will show that \( \mathfrak{g} \) is diagonal. Since \( M \) satisfies the finiteness condition of Lemma 4.1, \( \text{End}_{\mathfrak{g}} M \) and its submodules satisfy this condition too. The adjoint module \( \mathfrak{g} \) is a submodule of \( \text{End}_{\mathfrak{g}} M \), hence this implies that for each \( i \) the number of \( \mathfrak{g}_i \)-isotypic components in \( \mathfrak{g}_{i+k} \) is uniformly bounded for all \( k > 0 \). Since the adjoint module of \( \mathfrak{g}_i \) is isomorphic to \( (V_i \otimes V_i^*)/\mathbb{C} \) in the type A case, to \( S^2(V_i) \) in type C, and to \( \Lambda^2(V_i) \) in types B or D, one can easily check that for each \( i \) the number of \( \mathfrak{g}_i \)-isotypic components in \( V_{i+k} \) is also uniformly bounded by for all \( k > 0 \). Our
goal is to show that for all sufficiently large \( i \), \( V_{i+1} \) restricted to \( \mathfrak{g}_i \) is isomorphic to a direct sum of copies of \( V_i \), \( V_i^* \) and \( \mathbb{C} \).

Let us start with the type A case. Pick an \( sl(2) \)-subalgebra in \( \mathfrak{g}_n \) for some \( n \). The set of \( sl(2) \)-weights in \( V \) is finite. Thus we can let \( k \in \mathbb{Z}_{>0} \) be the maximal weight in this set and fix \( i \) such that \( k \) is a weight of \( V_i \). Then \( sl(2) \subset \mathfrak{g}_i \).

Furthermore, we have an isomorphism of \( \mathfrak{g}_i \)-modules

\[
V_{i+1} = T_{\lambda_1}(V_i) \oplus \cdots \oplus T_{\lambda_j}(V_i),
\]

where each \( \lambda_j \) is a Young diagram and \( T_{\lambda_j}(V_i) \) is the image of the corresponding Young projector in the appropriate tensor power of \( V_i \). Since \( V_{i+1} \) does not have any weight greater than \( k \), each diagram \( \lambda_j \) has only one column. Indeed, otherwise we can put a vector of weight \( k \) in each box of the first row and put other weight vectors in all other boxes of \( \lambda_j \) so that the total sum of all weights of vectors is greater than \( k \), which contradicts the fact that \( k \) is the maximal weight. Next we claim that the length of this column equals 0, 1, \( \dim V_i \), or \( \dim V_i - 1 \). Indeed, if we put in the boxes of \( \lambda_1 \) linearly independent vectors of maximal possible sum of weights, the total sum is not greater than \( k \) only in these four cases. Hence each simple \( \mathfrak{g}_i \)-constituent of \( V_{i+1} \) is isomorphic to \( V_i, V_i^* \) or \( \mathbb{C} \) (the numbers 0 and \( \dim V_i \) correspond both to the trivial 1-dimensional \( \mathfrak{g}_i \)-module).

If each \( \mathfrak{g}_i \) is of type B or C, D, let \( \mathfrak{s}_i \subset \mathfrak{g}_i \) be a maximal root subalgebra of type A. Notice that by the previous argument the restriction of \( V_{i+1} \) on \( \mathfrak{s}_i \) is a sum of natural, conatural and trivial modules. That is only possible if the restriction of \( V_{i+1} \) to \( \mathfrak{g}_i \) is a sum of natural and trivial modules. \( \square \)

Proposition 4.3 follows also from Corollary 3.9 in \([Ba2]\).

**Example 4.4.**

a) Let \( \mathfrak{g} = sl(\infty) \), and let \( M = \lim S^i(V_i) \) be as in Example 3.8, a). Then \( \text{Hom}_{\mathfrak{g}_i}(S^k(V_i), S^j(V_i)) \neq 0 \) for all \( i, k \leq j \). Hence \( \text{Hom}_{\mathfrak{g}_i}(S^k(V_i), M) \neq 0 \) for all \( k > 0 \), and by Lemma 4.1 \( M^* \) is not an object of \( \text{Int}_{\mathfrak{g}} \).

b) Consider the case \( \mathfrak{g} = o(\infty) \) and let \( S(\{t_i\}) \) be the \( \mathfrak{g} \)-module defined in Example 3.8, b). Then if \( N \) is a simple \( \mathfrak{g}_i \)-module, \( \text{Hom}_{\mathfrak{g}_i}(N, S(\{t_i\})) \neq 0 \) if \( N \simeq S^1 \) or \( N \simeq S^2 \). Hence \( S(\{t_i\})^* \in \text{Int}_{\mathfrak{g}_i} \) by Lemma 4.1. Moreover, \( S(\{t_i\})^* \) is injective in \( \text{Int}_{\mathfrak{g}_i} \) by Proposition 3.2.

c) Let \( \mathfrak{g} = sp(\infty) \) and let \( M \) be as in Example 3.5. Then \( \text{Hom}_{\mathfrak{g}_i}(N, M) \neq 0 \) if \( N \) is isomorphic to one of the following simple \( \mathfrak{g}_i \)-modules: trivial, natural, conatural, adjoint. Therefore, \( M^* \) is \( \mathfrak{g} \)-integrable and injective in \( \text{Int}_{\mathfrak{g}_i} \). Furthermore, \( M^* \simeq \mathbb{C} \oplus \mathfrak{g}^* \).

**5. On the Loewy length of \( \Gamma_{\mathfrak{g}}(M^*) \) for \( M \in \text{Int}_{\mathfrak{g}} \)**

Recall that the **socle**, \( \text{soc}(M) \), of a \( \mathfrak{g} \)-module \( M \) is the largest semisimple submodule of \( M \). The **socle filtration** of \( M \) is the filtration of \( \mathfrak{g} \)-modules

\[
0 \subset \text{soc}(M) \subset \text{soc}^1(M) \subset \cdots \subset \text{soc}^k(M) \subset \ldots ,
\]
where \( \text{soc}^i(M) = p_i^{-1}(\text{soc}(M/\text{soc}^{i-1}(M)) \) and \( p_i : M \to M/\text{soc}^{i-1}(M) \) is the natural projection. We say that the socle filtration of \( M \) is exhaustive if \( M = \lim \text{soc}^i(M) \). We say that \( M \) has finite Loewy length if the socle filtration of \( M \) is finite and exhaustive. The Loewy length of \( M \) equals \( k + 1 \) where \( k = \min\{r \mid \text{soc}^r(M) = M\} \).

**Proposition 5.1.** Let \( M \in \text{Int}_g \) be a simple \( g \)-module such that \( \Gamma_g(M^*) \) has finite Loewy length. Then there exist \( n \in \mathbb{Z}_{>0} \) and a direct system \( M_i \) of simple finite-dimensional \( g \)-modules such that \( M = \lim M_i \) and \( \dim \text{Hom}_g(M_i, M_j) = 1 \) for all \( j > i > n \).

We first prove several lemmas.

**Lemma 5.2.** Let \( Q = \lim Q_i \in \text{Int}_g \), where \( Q_i \) are finite-dimensional, not necessarily simple, \( g \)-modules. Assume that for all sufficiently large \( i \) there are simple \( g \)-submodules \( X_i \subset Q_i \) such that \( \dim \text{Hom}_g(X_i, X_{i+1}) > 2 \). Then there exists a locally simple module \( X = \lim X_i \in \text{Int}_g \) and a non-trivial extension of \( g \)-modules

\[
0 \to Q \to Z \to X \to 0.
\]

**Proof.** Fix a sequence of injective homomorphisms of \( g \)-modules \( f_i : X_i \to X_{i+1} \) and set \( X = \lim X_i \). Let \( Z_i := X_i \oplus Q_i \) and consider the injective homomorphisms of \( g \)-modules

\[
a_i : Z_i \to Z_{i+1}, \quad a_i((x, q)) := (f_i(x), t_i(x) + e_i(q)),
\]

where \( t_i \) are some injective homomorphisms \( X_i \to Q_{i+1} \), \( e_i : Q_i \to Q_{i+1} \) are the given inclusions, and \( q \in Q_i \), \( x \in X_i \). Put \( Z := \lim Z_i \).

Then, clearly, \( Q \) is a submodule of \( Z \) and the quotient \( Z/Q \) is isomorphic to \( X \). Thus we have constructed an extension of \( X \) by \( Q \). This extension splits if and only if for all sufficiently large \( i \) there exist non-zero homomorphisms \( p_i : X_i \to Q_i \) such that \( t_i = p_{i+1} \circ f_i - e_i \circ p_i \), see the following diagram:

\[
\begin{array}{ccc}
X_{i+1} & \xrightarrow{p_{i+1}} & Q_{i+1} \\
\uparrow f_i & & \uparrow e_i \\
X_i & \xrightarrow{p_i} & Q_i
\end{array}
\]

Assume that for any choice of \( \{t_i\} \) such a splitting exists. If \( n_i := \dim \text{Hom}_g(X_i, Q_i) \), this assumption implies

\[
\dim \text{Hom}_g(X_i, Q_{i+1}) \leq n_i + n_{i+1}.
\]

On the other hand, \( \dim \text{Hom}_g(X_i, Q_{i+1}) \geq k_i n_{i+1} \) where \( k_i := \dim \text{Hom}_g(X_i, X_{i+1}) \). Since \( k_i > 2 \), we have \( n_{i+1} < n_i \). As \( n_i > 0 \) for all \( i \), we obtain a contradiction. \( \Box \)

**Corollary 5.3.** Let \( Q \in \text{Int}_g \) be a simple \( g \)-module satisfying the assumption of Lemma 5.2. Then \( Q \) admits no non-zero homomorphism into an injective object of \( \text{Int}_g \) of finite Loewy length.
Proof. For any $m > 0$ we will now construct an integrable module $Z^{(m)} \supset Q$ whose socle equals $Q$ and whose Loewy length is greater than $m$. For $m = 1$ this was done in Lemma 5.2. Proceeding by induction, we set
\[
Z_i^{(m)} := X_i \oplus Z_i^{(m-1)} = X_i \oplus (X_i \oplus Z_i^{(m-2)})
\] and define $a_i^{(m)} : Z_i^{(m)} \to Z_i^{(m+1)}$ by
\[
a_i^{(m)}(x, x', z) = (f_i(x), r_i^{(m-1)}(x) + f_i(x'), t_i^{(m-2)}(x') + q_i^{(m-2)}(z)),
\] where now $\{t_i^{(m-2)}\}$ is a set of non-zero homomorphisms $t_i^{(m-2)} : X_i \to Z_i^{(m-2)}$ and $\{r_i^{(m-1)}\}$ is a set of non-zero homomorphisms $r_i^{(m-1)} : X_i \to X_{i+1}$. As in the proof of Lemma 5.2 one can choose $\{t_i^{(m-2)}\}$ and $\{r_i^{(m-1)}\}$ so that $Z^{(m)}$ is a non-split extension of $X$ by $Z^{(m-1)}$, and $Z^{(m)}/Z^{(m-2)}$ is a non-split self-extension of $X$. Therefore, the Loewy length of $Z^{(m)}$ is greater than $m$. The statement follows. \qed

Lemma 5.4. Let $Q = \varinjlim Q_i \in \text{Int}_g$ be a simple $g$-module which admits a non-zero homomorphism into an injective object of $\text{Int}_g$ of finite Loewy length. Then there exist $n \in \mathbb{Z}_{>0}$ and a direct system of simple $g_i$-submodules $S_i$ of $Q$ such that $Q = \varinjlim S_i$ and $\dim \text{Hom}_g(S_i, S_j) = 1$ for all $j > i > n$.

Proof. Decompose each $Q_i$ into a direct sum of isotypic components, $Q_i = Q_i^1 \oplus \cdots \oplus Q_i^{l(i)}$. We define a directed graph $\Gamma$ as follows. The set of vertices $V(\Gamma)$ is by definition $\{Q_i^j\}$, and $V(\Gamma) = \bigcup_{i > 0} V(\Gamma)_i$, where $V(\Gamma)_i = \{Q_i^1, \ldots, Q_i^{l(i)}\}$. There is an edge $A \to B$ in $\Gamma$ if $A \in V(\Gamma)_i, B \in V(\Gamma)_{i+1}$ and $\text{Hom}_g(A, B) \neq 0$.

Let $\Gamma_{>i}$ be the full subgraph of $\Gamma$ whose set of vertices equals $\bigcup_{k>i} V(\Gamma)_k$. For any vertex $A$ of $\Gamma$ we denote by $V(A)$ the set of vertices $B$ such that there is a directed path from $A$ to $B$. Let $\Gamma(A)$ be the full subgraph of $\Gamma$ whose set of vertices equals $V(A)$, and $\Gamma(A)_{>i}$ be the full subgraph of $\Gamma(A)$ whose set of vertices equals $\bigcup_{k>i} (V(\Gamma)_k \cap V(A))$. Note that the simplicity of $Q$ implies that $\Gamma_{>i}$ and $\Gamma(A)_{>i}$ are connected (as undirected graphs). In particular, if $\Gamma(A)$ is a tree, then $\Gamma(A)$ is just a string.

We will now prove that there exists a vertex $A$ such that $\Gamma(A)$ is a tree. Indeed, assume the contrary. This implies that one can find an infinite sequence of vertices $A_1 \in V(\Gamma)_i, A_2 \in V(\Gamma)_{i+1}, \ldots$ such that the number of paths from $A_n$ to $A_{n+1}$ is greater than 2 for all $n$. Then $Q = \varinjlim Q_i$. In addition, one can easily see that $Q$ satisfies the assumption of Lemma 5.2 and hence $Q$ admits no non-zero homomorphism into an injective object of $\text{Int}_g$ of finite Loewy length. Contradiction.

Fix now $A \in V(\Gamma)_i$ such that $\Gamma(A)$ is a tree. Then, as we mentioned above, $V(\Gamma)$ is necessarily a string $A_1 = \{A \to A_{i+1} \to A_{i+2} \ldots\}$. Let $S_j$ be a simple submodule of $A_i, j \geq i$. By Lemma 5.2 there exists $n$, such that $\dim \text{Hom}_g(S_j, S_k) = 1$ for any $k > j \geq n$. Fix $s \in S_n$ and set $S_j = U(s_j) \cdot s$ for all $j \geq n$. Then $S_j$ are simple and $Q = \varinjlim S_j$ satisfies the condition in the lemma. \qed
Lemma 5.5. Let $Q = \lim S_i \in \text{Int}_g$, where $S_i$ are simple $g_i$-modules such that, for some $n$, $\dim \text{Hom}_{g_i}(S_i, S_j) = 1$ for all $j > i > n$. Then $Q^*$ has a unique simple submodule $Q_*$, and $Q_* \in \text{Int}_g$.

Proof. The condition on $Q$ implies that $\dim \text{Hom}_{g_i}(S_i, Q) = 1$ for all sufficiently large $i$. Therefore, $\dim \text{Hom}_{g_i}(S_i^*, Q^*) = 1$ for all sufficiently large $i$. Note also that $Q_* = \lim S_i^*$ is uniquely defined (as $\dim \text{Hom}_{g_i}(S_i, S_{i+1}) = 1$) and is a simple integrable submodule of $Q^*$. Let $S$ be some simple submodule of $Q^*$. Since $Q^* = \lim S_i^*$ and $\text{Hom}_{g_i}(S, Q^*) \neq 0$, we have $\text{Hom}_{g_i}(S, S_i^*) \neq 0$ for some $i$. Therefore, $S_i^* \subset S$ as the multiplicity of $S_i^*$ in $Q^*$ is 1. This implies $S = Q_*$.

We are now ready to prove Proposition 5.1.

Proof of Proposition 5.1. Fix $0 \neq m \in M$ and put $M_i := U(g_i) \cdot m$. Then, by the simplicity of $M$, we have $M = \lim M_i$. Since $\Gamma_{g_i}(M^*)$ has finite Loewy length, $M^*$ has a simple submodule $Q$. By Lemma 5.4, $Q$ satisfies the assumption of Lemma 5.5. The composition of the canonical injection $M \to (M^*)^*$ and the dual map $(M^*)^* \to Q^*$ defines an injective homomorphism $M \to Q^*$. By Lemma 5.5 $M \simeq Q_*$ and, since $Q_*$ also satisfies the assumption of Lemma 5.5, we conclude that the claim of Proposition 5.1 holds for $M$.

The following statement is a direct consequence of Proposition 5.1.

Corollary 5.6. Let $M \in \text{Int}_g$ be a simple $g$-module such that $\Gamma_{g_i}(M^*)$ has finite Loewy length. Then for any sufficiently large $i$ there exists a simple $g_i$-module $N$ such that $\dim \text{Hom}_{g_i}(N, M) = 1$.

The next corollary is a direct consequence of Lemma 5.5 and Proposition 5.1.

Corollary 5.7. Let $M \in \text{Int}_g$ be a simple $g$-module such that $\Gamma_{g_i}(M^*)$ has finite Loewy length. Then $M^*$ has a unique simple submodule $M_*$, and $M_* \in \text{Int}_g$.

Theorem 5.8. Let $g$ be a locally simple Lie algebra which has a non-trivial module $M \in \text{Int}_g$ such that $M^*$ is integrable and has finite Loewy length, then $g$ is isomorphic to $sl(\infty)$, $o(\infty)$ or $sp(\infty)$.

Proof. By Proposition 4.3 we know that $g$ is diagonal. Assume that $g$ is not finitary and there exists $M$ satisfying the conditions of the theorem. Also assume that in the restriction of $V_i$ to $g_{i-1}$ there is no costandard module (for types B, C and D it is automatic). Let $g = \lim g_i$. Fix $n$ and let $\varphi_k : g_n \to g_{n+k}$ denote the inclusion defined by our fixed exhaustion of $g$. Since $g$ is diagonal, there exists a root subalgebra $f_k \subset g_{n+k}$ such that $f_k \simeq g_n \oplus \cdots \oplus g_n$ and $\varphi_k(g_n)$ is the diagonal subalgebra in $f_k$. Let $a_k$ be the number of simple direct summands in $f_k$. Since $g$ is not finitary, $a_k \to \infty$.

Note next that our condition on $M$ implies that $M$ admits a simple subquotient whose dual is integrable and of finite Loewy length. Therefore, without loss of generality, we may assume that $M$ is simple. Then, by Corollary 5.6 $M = \lim M_i$ is a direct limit of simple modules and, by possibly increasing $n$, we have $\dim \text{Hom}_{g_n}(M_n, M_{n+k}) = 1$ for all $k$. Choose a set of Borel subalgebras
b_i \subset g_i \text{ such that } \varphi_k(b_i) \subset b_{n+k}. \text{ Let } h \text{ be the highest coroot of } g_n \text{ and let } \lambda \text{ be the highest weight of some simple } t_k\text{-constituent } L \text{ of } M_{n+k}. \text{ Since } M^* \text{ is integrable, Lemma 4.1 implies that } \lambda(\varphi_k(h)) \text{ is bounded by some number } t. \text{ If } h_1, \ldots, h_{a_k} \text{ are the images of } \varphi_k(h) \text{ in the simple direct summands of } t_k \text{ under the natural projections, we have } \lambda(h_i) \neq 0 \text{ for at most } t \text{ direct summands. Therefore, } L \text{ isomorphic to an outer tensor product of at most } t \text{ non-trivial simple } g_n\text{-modules. Since } M_{n+k} \text{ is invariant under permutation of direct summands of } t_k, \text{ we have at least } a_k - t \text{ simple constituents of } M_{n+k} \text{ obtained from } L \text{ by permutation of the simple direct summands of } t_k. \text{ Note that all these simple constituents are isomorphic as } \varphi_k(g_n)\text{-modules. Thus the multiplicity of any simple } \varphi_{n+k}(g_n)\text{-module in } M_{n+k} \text{ is at least } a_k - t. \text{ Since } a_k \rightarrow \infty, \text{ this contradicts Proposition 5.1.}

The case when the restriction of } V_n \text{ to } g_{n-1} \text{ contains a costandard simple constituent can be handled by a similar argument which we leave to the reader. \square

6. The category \( \widetilde{\text{Tens}}_g \) for } g \simeq sl(\infty), o(\infty), sp(\infty)

Define \( \widetilde{\text{Tens}}_g \) as the largest full subcategory of \( \text{Int}_g \) which is closed under algebraic dualization and such that every object in it has finite Loewy length.

It is clear that \( \widetilde{\text{Tens}}_g \) is closed with respect to finite direct sums, however \( \widetilde{\text{Tens}}_g \) is not closed with respect to arbitrary direct sums (see Corollary 6.17 below). Note also that, if } g \text{ is finite-dimensional and semisimple, the objects of } \widetilde{\text{Tens}}_g \text{ are integrable modules which have finitely many isotypic components.}

It follows from Theorem 5.8 that if } g \text{ is locally simple and } \widetilde{\text{Tens}}_g \text{ contains a non-trivial then } g \text{ is fiitary.

In the rest of this section we assume that } g \simeq sl(\infty), o(\infty) \text{ or } sp(\infty). \text{ Set } T^{p,q} := V^{\otimes p} \otimes (V_{\ast}^{\otimes q}), \text{ where } V \text{ and } V_{\ast} \text{ are respectively the natural and conatural } g\text{-modules (} V_{\ast} \simeq V \text{ when } g \simeq o(\infty), sp(\infty)). \text{ The modules } T^{p,q} \text{ have been studied in [PS]; in particular, } T^{p,q} \text{ has finite length and is semisimple only if } pq = 0 \text{ for } g = sl(\infty), \text{ and if } p + q \leq 1 \text{ for } g = o(\infty), sp(\infty). \text{ Moreover, the Loewy length of } T^{p,q} \text{ equals } \min\{p, q\} + 1 \text{ for } g = sl(\infty) \text{ and } \lfloor \frac{p+q}{2} \rfloor + 1 \text{ for } g = o(\infty), sp(\infty). \text{ A simple module } M \text{ is called a simple tensor module if it is a submodule (or, equivalently, a subquotient) of } T^{p,q} \text{ for some } p, q.

It is well known that there is a choice of nested Borel subalgebras } b_i \subset g_i \text{ such that all simple tensor modules are } b\text{-highest weight modules for } b = \lim_{i \to \infty} b_i, \text{ see [PS]. (Moreover, the positive roots of any such } b \text{ are not generated by the simple roots of } b. \text{ However, in the present paper we will make no further reference to this fact.)}

Denote by } \Theta \text{ the set of all highest weights of simple tensor modules. If } \lambda \in \Theta, \text{ by } V_\lambda \text{ we denote the simple tensor module with highest weight } \lambda, \text{ and, as in section 4, by } V_{\lambda}^\ast \text{ we denote the simple } g_{\ast}\text{-highest weight module with highest weight } \lambda \text{ (here } \lambda \text{ is considered as a weight of } g_{\ast}). \text{ It is clear that every } \lambda \in \Theta \text{ can be written in the form } \lambda = \sum a_i \gamma_i \text{ for some finite set } \gamma_1, \ldots, \gamma_s \text{ of linearly independent weights of } V.
and some \(a_i \in \mathbb{Z}\) (see [PS] for an explicit description of \(\Theta\)). We put \(|\lambda| := \sum |a_i|\).

It is not hard to see that for any \(k\) the set of all \(\mu \in \Theta\) with \(|\mu| \leq k\) is finite. Moreover, all simple subquotients of \(T^{p,q}\) are isomorphic to \(V_\mu\) with \(|\mu| \leq p+q\), and it follows from [PS] that if \(V_\lambda\) is a submodule in \(T^{p,q}\) then \(|\lambda| = p+q\).

Note that \((T^{p,q})^*, (T^{p,q})^{**}, \text{etc.},\) are integrable modules. Indeed, it is easy to see (cf. [PS]) that for any fixed \(\lambda\) and any fixed \(i > 0\) the non-vanishing of \(\text{Hom}_{\mathfrak{g}}(N, V_\lambda)\) for a simple \(\mathfrak{g}\)-module \(N\) implies \(N \simeq V_\mu\) for \(|\mu| \leq |\lambda|\). Hence the condition of Lemma 4.1 is satisfied for \(T^{p,q}\) for fixed \(p,q\). This shows that \((T^{p,q})^* \in \text{Int}_g\). By Corollary 4.2, \((T^{p,q})^{**} \in \text{Int}_g, \text{etc.}..\)

**Lemma 6.1.** Fix \(p,q \in \mathbb{Z}_{\geq 0}\).

a) \((T^{p,q})^*\) has finite Loewy length, and all simple subquotients of \((T^{p,q})^*\) are tensor modules of the form \(V_\lambda\) for \(|\lambda| \leq p+q\).

b) The direct product \(\prod_{f \in \mathcal{F}} T_{f}^{p,q}\) of any family \(\{T_{f}^{p,q}\}_{f \in \mathcal{F}}\) of copies of \((T^{p,q})^*\) has finite Loewy length, and all simple subquotients of \(\prod_{f \in \mathcal{F}} T_{f}^{p,q}\) are tensor modules of the form \(V_\lambda\) for \(|\lambda| \leq p+q\).

**Proof.** First we prove b) using induction in \(p+q\). The case \(p+q = 0\) is trivial. If \(p+q > 0\), without loss of generality we can assume that \(p > 0\) (if \(p = 0\) and \(q > 0\) we replace \(V\) by \(V_i\) in the argument below). There is a canonical injective homomorphism \(U \rightarrow \prod_{f \in \mathcal{F}} T_{f}^{p,q}\), where \(U := V \otimes \prod_{f \in \mathcal{F}} T_{f}^{p-1,q}\), so we can consider \(U\) as a submodule of \(\prod_{f \in \mathcal{F}} T_{f}^{p,q}\). By the induction assumption b) holds for \(\prod_{f \in \mathcal{F}} T_{f}^{p-1,q}\).

Since \(T^{r,s}\) has finite length for all \(r,s\), [PS], this implies that \(U\) has finite Loewy length and all simple subquotients of \(U\) are simple tensor modules of the form \(V_\lambda\) for \(|\lambda| \leq p+q\). The quotient \((\prod_{f \in \mathcal{F}} T_{f}^{p,q})/U\) is isomorphic to a submodule of \(R := \prod_{f \in \mathcal{F}} (V' \otimes T_{f}^{p-1,q})\), where \(V'\) is a copy of the vector space \(V\) with trivial \(\mathfrak{g}\)-module structure. Since \(R \simeq \prod_{f \in \mathcal{F}} (\bigoplus_{i \in \mathbb{Z}} T_{i}^{p-1,q})\), by the induction assumption b) holds for \(R\). Therefore, b) holds for \(\prod_{f \in \mathcal{F}} T_{f}^{p,q}\).

a) To prove that \((T^{p,q})^*\) has finite Loewy length, we consider \(U' := V_\lambda \otimes (T^{p-1,q})^*\) as a submodule of \((T^{p,q})^*\). By the induction assumption, \(U'\) has finite Loewy length. The quotient \((T^{p,q})^*/U'\) is a submodule of \(R' = \prod_{i \in \mathbb{Z}} (T_{i}^{p-1,q})^*\).

The latter \(\mathfrak{g}\)-module has finite Loewy length by the induction assumption and b). The statement about the simple subquotients of \((T^{p,q})^*\) follows by an induction argument similar to the one in the proof of b). This proves a) for \((T^{p,q})^*\). \(\square\)
Example 6.2.

a) We start with the simplest example. Let $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$ and $M = V^* = (T^{1,0})^*$. Then $M \in \overline{Tens}_\mathfrak{g}$ by Lemma 6.1. Furthermore, $M$ is an injective object of $\text{Int}_\mathfrak{g}$ by Proposition 3.2. It is easy to see that $\text{soc}(M) = V_*$ and that $M/\text{soc}(M) = V^*/V_* = \text{a trivial module of cardinality } \mathbb{N}_1$. Since $\text{soc}(M)$ is simple, $M$ is an injective hull of $V_*$. 

b) Let $\mathfrak{g}$ be as in a) but let now $M = V^{**} = (T^{1,0})^{**}$. The exact sequence

$$0 \to V_* \to V^* \to V^*/V_* \to 0$$

yields an exact sequence

$$0 \to (V^*/V_*)^* \to M \to (V_*)^* \to 0.$$ (5)

Since $(V^*/V_*)^*$ is a trivial $\mathfrak{g}$-module (cf. a)), it is injective, and hence (5) splits. This yields an isomorphism $M = V^{**} = (V_*)^* \oplus T$, $T$ being a trivial $\mathfrak{g}$-module of cardinality $\mathbb{N}_2$.

c) Here is a more interesting example. We consider the $\mathfrak{g}$-module $M^*$ where $\mathfrak{g} = sl(\infty)$ and $M = V \oplus V_* = T^{1,1}$ as in Example 3.5. Recall the notation introduced in Example 3.5. In addition, let $Sc$ be the one-dimensional space of scalar matrices, and $F_r$ (respectively $F_c$) denote respectively the spaces of matrices with finitely many non-zero rows (resp., columns) ($F$ has codimension 1 in $F_r \cap F_c$).

It is important to notice that $\mathfrak{g} \cdot M^* \subset F_r + F_c$. We first show that $\text{soc}(M^*) = Sc \oplus F = \mathbb{C} \oplus \mathfrak{g}$. It is obvious that $Sc \oplus F \subset \text{soc}(M^*)$ and that $Sc$ is the largest trivial $\mathfrak{g}$-submodule of $M^*$. To see that $Sc \oplus F = \text{soc}(M^*)$, let $X$ be any non-trivial simple submodule of $\text{soc}(M^*)$. Consider $0 \neq x \in X$. Then $\mathfrak{g} \cdot x \subset F_r + F_c$. Furthermore, it is easy to check that for any $0 \neq y \in F_r + F_c$, there exists $A \in \mathfrak{g}$ such that $A \cdot y \in F$ and $A \cdot y \neq 0$. Hence $X = F$, and we have shown that $\text{soc}(M^*) = Sc \oplus F$.

We now compute $\text{soc}(M^*)$. We claim that $F_r + F_c \subset \text{soc}(M^*)$. Since $BA \in F$ for $B \in F_r, A \in F$, the action of $\mathfrak{g}$ on $F_r/F$ is simply left multiplication. Using this it is not difficult to establish an isomorphism of $\mathfrak{g}$-modules $F_r/F \cong \bigoplus_{Q \in \mathbb{Q}} V_q$, where $Q$ is a family of copies of $V$ of cardinality $\mathbb{N}_1$. Similarly, $F_c/F \cong \bigoplus_{Q \in \mathbb{Q}} (V_*)_q$.

(It is convenient to think here of $V_*$ as the space of all row vectors each of which have finitely many non-zero entries.) This implies $F_r + F_c \subset \text{soc}(M^*)$.

Furthermore, $M^*/(F_r + F_c)$ is a trivial $\mathfrak{g}$-module as $\mathfrak{g} \cdot M^* \subset F_r + F_c$. Therefore, in order to compute $\text{soc}(M^*)$ we need to find all $z \in M^*$ such that $\mathfrak{g} \cdot z \subset Sc + F$. A direct computation shows that $\mathfrak{g} \cdot z \subset Sc + F$ if and only $z \in J$, where $J$ denotes the set of matrices each row and each column of which have finitely many non-zero elements. (In fact, $\mathfrak{g} \cdot J \subset F$.) Thus $\text{soc}(M^*) = F_r + F_c + J$, and we obtain the socle filtration of $M^*$:

$$0 \subset Sc \oplus F \subset F_r + F_c + J \subset M^*.$$
Note that $M^*$ is decomposable and is isomorphic to $\mathbb{C} \oplus g^*$. As the socle of $g^*$ is simple (being isomorphic to $g$), $g^*$ is indecomposable. Moreover $g^*$ is an injective hull of $F = g$.

d) We now give an example illustrating statement b) of Lemma 6.1. Let $g = \mathfrak{sl}(\infty), o(\infty), sp(\infty)$ and $M = \prod_{f \in \mathcal{F}} V_f, \{V_f\}_{f \in \mathcal{F}}$ being an infinite family of copies of the natural module $V$. Set $M^{\text{fin}} = \{\psi : \mathcal{F} \to V| \dim(\psi(\mathcal{F})) < \infty\}$. Then $M^{\text{fin}}$ is a $g$-submodule of $M$, and $g.M \subset M^{\text{fin}}$. Hence $M/M^{\text{fin}}$ is a trivial $g$-module. Moreover, $M^{\text{fin}} \simeq \bigoplus_{g \in 2^\mathcal{F}} V_g$, where $2^\mathcal{F}$ is the set of subsets of $\mathcal{F}$. Indeed,

$$M^{\text{fin}} = \lim_{\longrightarrow} \bigoplus_{f \in \mathcal{F}} (V^i)_f = \lim_{\longrightarrow} (\bigotimes_{f \in \mathcal{F}} C_f) \otimes V^i \simeq \lim_{\longrightarrow} \bigoplus_{g \in 2^\mathcal{F}} (C_g \otimes V^i) =$$

$$\lim_{\longrightarrow} (\bigotimes_{g \in 2^\mathcal{F}} (V^i)_g) = \bigoplus_{g \in 2^\mathcal{F}} V_g.$$

This yields an exact sequence

$$0 \to \bigoplus_{g \in 2^\mathcal{F}} V_g \to M \to T \to 0,$$

(6) $T$ being trivial module of dimension $\text{card} 2^\mathcal{F}$. Since $M$ has no non-zero trivial submodules, (6) is in fact the socle filtration of $M$. Consequently the Loewy length of $M$ equals 2.

**Corollary 6.3.** Let $M \in \text{Int}_g$ have finite Loewy length and all simple subquotients of $M$ be isomorphic to $V_\lambda$ where $|\lambda|$ is less or equal than a fixed $k \in \mathbb{Z}_{>0}$. Then

a) for any family $\{M_f\}_{f \in \mathcal{F}}$ of copies of $M$, the $g$-module $\prod_{f \in \mathcal{F}} M_f$ has finite Loewy length and all simple subquotients of $\prod_{f \in \mathcal{F}} M_f$ are isomorphic to $V_\lambda$ with $|\lambda| \leq k$;

b) $M^*$ has finite Loewy length and all simple subquotients of $M^*$ are isomorphic to $V_\lambda$ with $|\lambda| \leq k$;

c) $M \in \text{Tens}_g$.

**Proof.** a) The socle filtration of $M$ induces a finite filtration on $\prod_{f \in \mathcal{F}} M_f$

$$0 \subset \prod_{f \in \mathcal{F}} \text{soc}(M_f) \subset \cdots \subset \prod_{f \in \mathcal{F}} \text{soc}^i(M_f) \subset \cdots \subset \prod_{f \in \mathcal{F}} M_f.$$

Furthermore,

$$\text{soc}^i(M)/\text{soc}^{i-1}(M) \simeq \bigoplus_{|\lambda| \leq k} \bigoplus_{g \in \mathcal{F}_\lambda} (V_\lambda)_g$$

(7) for some families $\{(V_\lambda)_g\}_{g \in \mathcal{F}_\lambda}$ of copies of $V_\lambda$. Hence

$$\prod_{f \in \mathcal{F}} (\text{soc}^i(M_f)/\text{soc}^{i-1}(M_f)) \simeq \bigoplus_{|\lambda| \leq k} \prod_{f \in \mathcal{F}} \left( \bigoplus_{g \in \mathcal{F}_\lambda} (V_\lambda)_g \right)_f.$$


Note that for each \( \lambda \)
\[
\prod_{f \in F} \left( \bigoplus_{g \in F_\lambda} (V_\lambda)_g \right)_f \subset \prod_{(f,g) \in F \times F_\lambda} (V_\lambda)_{(f,g)}.
\]
By Lemma 6.1 b), the \( g \)-module \( \prod_{(f,g) \in F \times F_\lambda} (V_\lambda)_{(f,g)} \) has finite Loewy length and all its simple subquotients are isomorphic to \( V_\mu \) with \( |\mu| \leq |\lambda| \leq k \). The same holds for \( \prod_{f \in F} (\soc^i(M_i)/\soc^{i-1}(M_i)) \). Therefore, a) holds.

b) Since all \( V_\lambda \) with \( |\lambda| \leq k \) satisfy the conditions of Lemma 4.1, \( M \) satisfies the condition of Lemma 4.1 and therefore \( M^* \in \Int_g \).

The socle filtration of \( M \) induces a finite filtration on \( M^* \)
\[
\cdots \subset \soc^i(M)^* \subset \soc^{i-1}(M)^* \subset \cdots.
\]
Using (7) we get
\[
\soc^{i-1}(M)^*/\soc^i(M)^* \simeq \bigoplus_{|\lambda| \leq k} (V_\lambda)_g.
\]
By Lemma 6.1 b) \( V_\lambda^* \) has finite Loewy length and its simple subquotients are isomorphic to \( V_\mu \) with \( |\mu| \leq |\lambda| \), hence by a) the same holds for \( \prod_{g \in F_\lambda} (V_\lambda^*)_g \). This implies that b) holds.

c) Note that if \( M \) satisfies the assumptions of the corollary, then \( M^* \) and all higher duals, \( M^{**} \) etc., satisfy the assumptions of the corollary. Hence \( M \in \Tens_g \).

Remarkably, there is following abstract characterization of simple tensor modules.

**Theorem 6.4.** If \( M \in \Int_g \) is simple and \( \Gamma_g(M^*) \) has finite Loewy length, then \( M \) is a simple tensor module.

**Proof.** By Proposition 5.1, \( M = \lim M_i \) for some \( n \in \mathbb{Z}_+ \) and simple nested \( \mathfrak{g}_i \)-submodules \( M_i \subset M \) with \( \dim \text{Hom}_{\mathfrak{g}_i}(M_i, M) = 1 \) for all \( i \leq n \). If \( \mathfrak{g} = \mathfrak{sl}(\infty) \), it is useful to consider \( M \) as a \( \mathfrak{gl}(\infty) \)-module by extending the \( \mathfrak{sl}(\infty) \)-module structure on \( M_i \) to a \( \mathfrak{gl}(i) \)-module structure in a way compatible with the injections \( M_i \to M_{i+1} \).

It is easy to see that the condition \( \dim \text{Hom}_{\mathfrak{g}_i}(M_i, M) = 1 \) for all \( i \leq n \) ensures the existence of such an extension. Note, furthermore, that \( \dim \text{Hom}_{\mathfrak{gl}(i)}(M_i, M) = 1 \). This allows us to assume that \( \mathfrak{g} = \mathfrak{gl}(\infty) \) and \( \mathfrak{g}_i = \mathfrak{gl}(i) \).

Let now \( \mathfrak{c} \) denote the derived subalgebra of the centralizer of \( \mathfrak{g}_n \) in \( \mathfrak{g} \). Then obviously \( \mathfrak{c} \) is a simple finitary Lie algebra whose action on \( M \) induces a trivial action on \( M_n \). Hence, as a \( \mathfrak{c} \)-module, \( M \) is isomorphic to a quotient of \( U(\mathfrak{g}) \otimes U(\mathfrak{c} \oplus \mathfrak{g}_n) M_n \), or equivalently to a quotient of \( S(\mathfrak{g} / (\mathfrak{c} \oplus \mathfrak{g}_n)) \otimes M_n \). Note that \( \mathfrak{g} / (\mathfrak{c} \oplus \mathfrak{g}_n) \), considered as a \( \mathfrak{c} \)-module has finite length and that its simple subquotients are natural, conatural, and possibly 1-dimensional trivial \( \mathfrak{c} \)-modules. This implies that every simple \( \mathfrak{c} \)-subquotient of \( M \) is a simple tensor \( \mathfrak{c} \)-module. In addition, for \( i \geq n \), the number of non-zero marks of the highest weight of any simple \( \mathfrak{g}_i \)-submodule of \( M \) is not greater than \( n \) plus the multiplicity of the non-trivial simple constituents.
Lemma 6.6. The simple objects of Corollary 6.5. is a subquotient of \( \mathfrak{m} \).

Proof. The first statement follows from Corollary 5.7. The second statement follows immediately from the fact that \( \text{Hom}_\mathfrak{g}(\mathfrak{m}_\lambda^i, (M_\lambda)^*) \neq 0 \) implies \( |\mu| < |\lambda| \).

18 Ivan Penkov and Vera Serganova of the \( \mathfrak{g}_n \)-module \( \mathfrak{g}/(\mathfrak{e} \oplus \mathfrak{g}_n) \). In particular, if \( \lambda_i \) denotes the highest weight of \( M_i \) then \( \lambda_i \) has at most \( 3n \) non-zero marks.

Consider first the case when \( \mathfrak{g} = \mathfrak{gl}(\infty) \). Then every weight \( \lambda_i \) can be written uniquely in the form

\[
a_i^1 \varepsilon_1 + \cdots + a_i^k \varepsilon_k + b_i^1 \varepsilon_{n-k} + \cdots + b_i^k \varepsilon_n
\]

for some fixed \( k \), \( a_i^1 \geq a_i^2 \geq \cdots \geq a_i^k \geq 0 \) and \( 0 \geq b_i^1 \geq \cdots \geq b_i^k \). We claim that for sufficiently large \( i \) the weight stabilizes, i.e. \( a_i^j = a_i^{j+1} = \cdots = a_i^k = \cdots \) and \( b_i^j = b_i^{j+1} = \cdots = b_i^k = \cdots \) for all \( j, 1 \leq j \leq k \). Indeed, assume the contrary. Let \( j \) be the smallest index such that the sequence \( \{ a_i^j \} \) does not stabilize. By the branching rule for \( \mathfrak{gl}(m) \subset \mathfrak{gl}(m+1) \) (see for instance \([GW]\)) the sequence \( \{ a_i^j \} \) is non-decreasing. Hence there is \( p \) such that \( a_i^{p+1} > a_i^p \). Set \( \mu = \lambda_i + \varepsilon_j \). Then the multiplicity of \( M_{p-1} \) in \( V_{p}^\mu \) is not zero and the multiplicity of \( V_{p}^\mu \) in \( M_{p+1} \) is at least 2. Contradiction. Similarly the sequence \( \{ b_i^j \} \) stabilizes. As it is easy to see, this is sufficient to conclude that \( M \cong V_\lambda \) for some \( \lambda \in \Theta \).

Let \( \mathfrak{g} = o(\infty) \) or \( sp(\infty) \). In the first case we assume that \( \mathfrak{g} = o(2i+1) \). Then \( \lambda_i = a_i^1 \varepsilon_1 + \cdots + a_i^k \varepsilon_k \) for some fixed \( k \) and \( a_i^1 \geq a_i^2 \geq \cdots \geq a_i^k \geq 0 \). The sequence \( \{ a_i^j \} \) is non-decreasing for every fixed \( j \) as follows from the branching laws for the respective pairs \( o(2m+1) \subset o(2m+3) \) and \( sp(2n) \subset sp(2m+2) \), see \([GW]\). Then by repeating the argument in the previous paragraph we prove that \( \{ a_i^j \} \) stabilizes, and consequently \( M \cong V_\lambda \) for some \( \lambda \in \Theta \).

Corollary 6.3 and Theorem 6.4 show that a simple module \( M \in \text{Int}_\mathfrak{g} \) is an object of \( \text{Tens}_\mathfrak{g} \) if and only if \( \Gamma_\mathfrak{g}(M^*) \) has finite Loewy length. Below we will use this fact to give an equivalent definition of \( \text{Tens}_\mathfrak{g} \) (Corollary 6.13). Furthermore, it is easy to check (see also \([PS]\)) that for sufficiently large \( i \) the simple \( \mathfrak{g}_n \)-module \( V_i^\lambda \) occurs in \( Y \) with multiplicity 1, and all other simple \( \mathfrak{g}_n \)-constituents have infinite multiplicity and are isomorphic to \( V_i^\mu \) with \( |\mu| < |\lambda| \). In what follows we call this unique \( \mathfrak{g}_n \)-constituent the canonical \( \mathfrak{g}_n \)-constituent of \( V_\lambda \). Note also that by Corollary 5.7 for each simple object \( M \) of \( \text{Tens}_\mathfrak{g} \), \( M_* \) is a well-defined simple object in \( \text{Tens}_\mathfrak{g} \). Hence \( M_* \) is well defined also for any semisimple object \( M \) of \( \text{Tens}_\mathfrak{g} \): if \( M = \bigoplus_{\lambda \in \Theta} M^\lambda \otimes V_\lambda \) (\( M^\lambda \) being trivial \( \mathfrak{g} \)-modules), then \( M_* = \bigoplus_{\lambda \in \Theta} M^\lambda \otimes (V_\lambda)_* \). It is clear that \( M_* \cong M \) for \( \mathfrak{g} \cong o(\infty), \text{sp}(\infty) \).

Corollary 6.5. The simple objects of \( \text{Tens}_\mathfrak{g} \) are precisely the simple tensor modules.

Lemma 6.6. Let \( M \cong V_\lambda \) be a simple tensor module. Then \( \text{soc}(M_*) \cong M \). If \( V_\mu \) is a subquotient of \( (M_\lambda)^* \) and \( \mu \neq \lambda \), then \( |\mu| < |\lambda| \).

Proof. The first statement follows from Corollary 5.7. The second statement follows immediately from the fact that \( \text{Hom}_\mathfrak{g}(V_\mu^i, (M_\lambda)^*) \neq 0 \) implies \( |\mu| < |\lambda| \).
Corollary 6.7. a) For any simple $M \in \widetilde{\text{Tens}}_g$, $(M_\lambda)^*$ is an injective hull of $M$ in $\text{Int}_g$ (and hence also in $\widetilde{\text{Tens}}_g$).

b) Any indecomposable injective object in $\widetilde{\text{Tens}}_g$ is isomorphic to $M^*$ for some simple module $M \in \widetilde{\text{Tens}}_g$. In particular, any indecomposable injective module is isomorphic to a direct summand of $(T\theta^p)^*$ for some $p, q$.

c) For any $M \in \widetilde{\text{Tens}}_g$, any injective hull $I_M$ of $M$ in $\text{Int}_g$ is an object of $\widetilde{\text{Tens}}_g$.

Proof. a) Follows directly from Proposition 3.2 and Lemma 6.6.

b) To derive b) from a) it suffices to note that an injective module in $\widetilde{\text{Tens}}_g$ is indecomposable if and only if it has simple socle.

c) follows from the fact that $I_M$ is isomorphic to a submodule of $\Gamma_g(M^{**})$, see Corollary 3.3. □

In what follows we set $I_\lambda := ((V_\lambda)_*)^*$.

Corollary 6.8. End$_g(I_\lambda) = \mathbb{C}$.

Proof. If $\varphi \in \text{End}_g(I_\lambda)$, then $\varphi|_{V_\lambda} = c \text{Id}$ for $c \in \mathbb{C}$. Therefore, $V_\lambda \subset \text{Ker}(\varphi - c \text{Id})$. Furthermore, any non-zero $g$-submodule of $I_\lambda$ contains $\text{soc}(I_\lambda) = V_\lambda$, hence $V_\lambda \subset \text{Im}(\varphi - c \text{Id})$. This implies $\varphi - c \text{Id} = 0$, as otherwise $V_\lambda$ would be isomorphic to a subquotient of $I_\lambda/V_\lambda$ contrary to Lemma 6.6. □

Lemma 6.9. Let $X, Y, Z, M \in \widetilde{\text{Tens}}_g$. Assume furthermore that $Y$ is simple, $Y = \text{soc}(M)$, and there exists an exact sequence

$$0 \to X \to Z \overset{p}{\to} Y \to 0.$$ 

Then there exists $\tilde{M} \in \text{Int}_g$ such that $Z \subset \tilde{M}$ and $\tilde{M}/X \simeq M$.

Proof. Let $Y_i$ be the canonical $g_i$-constituent of $Y$. Then $Y = \varprojlim Y_i$. Set $Z_i := p^{-1}(Y_i)$ and $Q_i := Z_i \cap X$. Then $Z_i = Y_i \oplus Q_i$ and there are injective homomorphisms $\varphi_i : Z_i \to Z_{i+1}$

$$\varphi_i(y, q) = (e_i(y), t_i(y) + f_i(q)), \ y \in Y_i, q \in Q_i$$

for some non-zero homomorphisms $e_i : Y_i \to Y_{i+1}$, $t_i : Y_i \to Q_{i+1}$ and $f_i : Q_i \to Q_{i+1}$. Clearly, $Z = \varprojlim Z_i$.

On the other hand, $M = \varprojlim M_i$ for some nested finite-dimensional $g_i$-submodules $M_i \subset M$ such that $Y_i \subset M_i$. Moreover, $\dim \text{Hom}_g(Y_i, M_i) = 1$ by Lemma 6.6. Therefore, $M_i$ has a unique $g_i$-module decomposition $M_i = R_i \oplus Y_i$. The inclusions $\psi_i : M_i \to M_{i+1}$ are given by

$$\psi_i(r, y) = (p_i(r), s_i(r) + e_i(y)), \ y \in Y_i, r \in R_i$$

for some non-zero homomorphisms $p_i : R_i \to R_{i+1}$ and $s_i : R_i \to Y_{i+1}$.

Define $\tilde{M}_i := R_i \oplus Y_i \oplus Q_i$ and let $\tilde{\zeta}_i : \tilde{M}_i \to \tilde{M}_{i+1}$ be given by the formula

$$\zeta(r, y, q) = (p_i(r), s_i(r) + e_i(y), t_i(y) + f_i(q)).$$
Set $\tilde{M} := \varinjlim M_i$. It is easy to check that $\tilde{M}$ satisfies the conditions of the lemma.

\begin{proof}
Lemma 6.10. If $\Hom_{\mathfrak{g}}(I_{\lambda}, I_{\mu}) \neq 0$, then $|\mu| \leq |\lambda|$. If $I$ is any injective object of $\Tens_{\mathfrak{g}}$ and $0 \neq \varphi \in \Hom_{\mathfrak{g}}(I, I_{\mu})$, then $\varphi$ is surjective.

Proof. The first statement follows immediately from Lemma 6.6.

To prove the second statement put $X = \Ker \varphi$, $Y = V^i_{\mu}$, $Z = \varphi^{-1}(Y)$ and $M = I_{\mu}$. Construct $\tilde{M}$ as in Lemma 6.9. By the injectivity of $I$, the injective homomorphism $Z \to M$ extends to a homomorphism $\tilde{M} \to I$. The latter induces a homomorphism $\eta : M = I_{\mu} \to I/X$.

Let now $\varphi : I/X \to I_{\mu}$ denote the injective homomorphism induced by $\varphi$. Then it is obvious that $\varphi \circ \eta(y) = y$ for any $y \in Y$. By Corollary 6.8, we have $\varphi \circ \eta = \text{Id}$. Hence $\varphi$ is an isomorphism, i.e. $\varphi$ is surjective.

Proposition 6.11. The Loewy length of $I_{\lambda}$ equals $|\lambda| + 1$.

Proof. By Lemma 6.6 we know that the Loewy length of $I_{\lambda}$ is at most $|\lambda| + 1$. We prove equality by induction in $|\lambda|$. Fix $\mu \in \Theta$ such that $|\mu| = |\lambda| - 1$ and $\Hom_{\mathfrak{g}}(V^i_{\mu}, V^i_{\lambda} + 1) \neq 0$. We claim that $\Ext^1(V^i_{\mu}, V^i_{\lambda}) \neq 0$. Indeed, consider non-zero homomorphisms $\varphi_i \in \Hom_{\mathfrak{g}}(V^i_{\mu}, V^i_{\lambda} + 1)$. Set $X = \varinjlim X_i$, where $X_i = V^i_{\mu} \oplus V^i_{\lambda}$, $q_i : X_i \to X_{i+1}$ is given by $q_i(x, y) = (e_i(x), \varphi_i(x) + f_i(y))$ for $x \in V^i_{\mu}$, $y \in V^i_{\lambda}$, and $e_i : V^i_{\mu} \to V^i_{\mu} + 1$ and $f_i : V^i_{\lambda} \to V^i_{\lambda} + 1$ denote the fixed inclusions. It is easy to see that $X$ is a non-trivial extension of $V^i_{\mu}$ by $V^i_{\lambda}$.

This implies the existence of a non-zero homomorphism $I_{\lambda} \to I_{\mu}$. By Lemma 6.10, this homomorphism is surjective. Hence the Loewy length of $I_{\lambda}$ is greater or equal to the Loewy length of $I_{\mu}$ plus 1. The statement follows.

The following theorem strengthens the claim of Corollary 6.3.

Theorem 6.12. Let $M \in \Int_{\mathfrak{g}}$. Then $M \in \Tens_{\mathfrak{g}}$ if and only if there exists a finite subset $\Theta_M \subset \Theta$ such that any simple subquotient of $M$ is isomorphic to $V^i_{\mu}$ for $\mu \in \Theta_M$.

Proof. Assume that $M \in \Tens_{\mathfrak{g}}$. It is sufficient to prove the existence of $\Theta_M$ for a semisimple $M$ since then the general case follows from Lemma 6.6. Without loss of generality we may assume that $M = \bigoplus_{\lambda \in \Theta} V^i_{\lambda}$, where $V^i_{\lambda}$ are pairwise non-isomorphic. We claim that if $C$ is infinite, then $M^*$ does not have finite Loewy length. Indeed, $M^*$ contains a submodule isomorphic to $\bigoplus_{\lambda \in \Theta} I_{\mu}$, where $V^i_{\mu} = (V^i_{\lambda})_\cdot$. If $C$ is infinite, then $|\mu| = |\lambda|$ is unbounded and the socle filtration of $\bigoplus_{\lambda \in \Theta} I_{\mu}$ is infinite. This contradiction shows that $C$ is finite, i.e. that there exists a finite set $\Theta_M$ as required.

Now assume that $M$ admits a finite set $\Theta_M$ as in the statement of the theorem. We claim first that if $M'$ is a quotient of $M$ and $\Ext^1_{\mathfrak{g}}(M', V^i_{\lambda}) \neq 0$ for some $\lambda \in \Theta$, then $M$ has a subquotient isomorphic to $V^i_{\mu}$ for some $\mu$ with $|\mu| < |\lambda|$. Indeed, by extending the sequence $0 \to V^i_{\lambda} \to I_{\lambda}$ to a minimal injective resolution
Proof. We use induction on the length of the filtration. Assume that

\[ 0 \to V_\lambda \to I_\lambda \to I_\lambda^1 \to \ldots \]

we see that there is a non-zero homomorphism \( M' \to I_\lambda^1 \). Furthermore, by the minimality of the resolution, we have \( \text{soc}(I_\lambda^1) \subset \text{Im} \). Hence by Lemma 6.6 every simple constituent of \( \text{soc}(I_\lambda^1) \) is of the form \( V_\nu \) for \( |\nu| < |\lambda| \). Since \( (\text{Im}) \cap \text{soc}(I_\lambda^1) \neq 0 \), some simple constituent of \( \text{soc}(I_\lambda^1) \) is isomorphic to a subquotient of \( M' \) and thus of \( M \).

We show now that \( M \) has finite Loewy length. Consider a weight \( \lambda \in \Theta_M \) with minimal \( |\lambda| \). The above argument shows that \( \text{Ext}^1(M, V_\lambda) = 0 \) for any quotient \( M' \) of \( M \). This implies that every subquotient of \( M \) isomorphic to \( V_\lambda \) is a quotient of \( M \). Hence \( M \) admits a surjective homomorphism \( \zeta : M \to M_\lambda \), where \( M_\lambda \) is isomorphic to a direct sum of copies of \( V_\lambda \) and \( \Theta_{\ker \zeta} = \Theta_M \setminus \{\lambda\} \). By an induction argument we obtain that \( M \) has finite Loewy length. Therefore, \( M \in \widehat{\text{Tens}}_g \) by Corollary 6.3 c).

**Corollary 6.13.** A \( g \)-module \( M \in \text{Int}_g \) is an object of \( \widehat{\text{Tens}}_g \) if and only if both \( M \) and \( \Gamma_g(M^*) \) have finite Loewy length.

**Proof.** In one direction the statement is trivial. We need to prove that, if \( M \in \text{Int}_g \) satisfies the above two conditions, then \( M^* \in \text{Int}_g \). For a semisimple \( M \) this follows directly from Theorem 6.12 (as we have already pointed out). The argument is completed by induction on the Loewy length. Let \( M \in \text{Int}_g \) have Loewy length \( k \), and \( \Gamma_g(M^*) \) have finite Loewy length. Consider the homomorphism \( \tau : M \to \text{top}(M) \) onto the maximal semisimple quotient \( \text{top}(M) \) of \( M \). Then \( \Gamma_g(\text{top}(M)^*) \subset \Gamma_g(M^*) \), hence \( \text{top}(M) \in \widehat{\text{Tens}}_g \), i.e. in particular \( \text{top}(M)^* \in \text{Int}_g \). Therefore, there is an exact sequence

\[ 0 \to \text{top}(M)^* \to \Gamma_g(M^*) \to \Gamma_g(\text{Ker}(\tau))^* \to 0, \]

implying that \( \Gamma_g(\text{Ker}(\tau)^*) \) has finite Loewy length. Since the Loewy length of \( \text{Ker}(\tau) \) equals \( k - 1 \), the induction assumption allows us to conclude that \( (\text{Ker}(\tau))^* \in \text{Int}_g \). Hence \( \Gamma_g(M^*) = M^* \).

**Corollary 6.14.** \( \widehat{\text{Tens}}_g \) is a tensor category with respect to \( \otimes \).

**Proof.** It suffices to show that \( \widehat{\text{Tens}}_g \) is closed with respect to \( \otimes \). The fact that, if \( M \in \widehat{\text{Tens}}_g \) and \( M' \in \widehat{\text{Tens}}_g \) then \( M \otimes M' \in \widehat{\text{Tens}}_g \), follows immediately from Theorem 6.12.

The following theorem concerns the structure of injective modules in \( \widehat{\text{Tens}}_g \).

**Theorem 6.15.** Any injective module \( I \in \widehat{\text{Tens}}_g \) has a finite filtration \( \{I_j\} \) such that, for each \( j \), \( I_{j+1}/I_j \) is isomorphic to a direct sum of copies of \( I_{\mu_j} \) for some \( \mu_j \in \Theta \).

**Proof.** We use induction on the length of the filtration. Assume that

\[ 0 = I_0 \subset I_1 \subset \ldots \subset I_k \]

is already constructed. Let \( \text{soc}(I/I_k) = \bigoplus_{f \in F} Y_f \) for a family \( \{Y_f\}_{f \in F} \) of simple modules \( Y_f \) (there are only finitely many non-isomorphic modules among
{Y_f}_{f \in \mathcal{F}}$. Denoting by $p$ the projection $I \to I/I_k$, set $X_f := p^{-1}(Y_f)$. By Lemma 6.9, there exists $\bar{Y}_f \in \text{Int}_g$ such that $I_k \subset X_f \subset \bar{Y}_f$ and $\bar{Y}_f/I_k \cong I_{\mu_f}, \mu_f \in \Theta$ being the highest weight of $Y_f$. The inclusion $X_f \subset I$ induces a homomorphism $\psi_f : \bar{Y}_f \to I$. Let $\psi_f : \bar{Y}_f/I_k \to I_{\mu_f} \to I/I_k$ the corresponding homomorphism of quotients. Then $\psi := \bigoplus_{f \in \mathcal{F}} \psi_f : \bigoplus_{f \in \mathcal{F}} I_{\mu_f} \to I$ is injective since its restriction to $\text{soc}(\bigoplus_{f \in \mathcal{F}} I_{\mu_f})$ is an isomorphism. This shows that if $I_{k+1} := p^{-1}(\psi(\bigoplus_{f \in \mathcal{F}} I_{\mu_f}))$, there is an isomorphism $I_{k+1}/I_k \cong \bigoplus_{f \in \mathcal{F}} I_{\mu_f}$.

The filtration $\{I_j\}$ terminates at a finite step as $I$ has finite Loewy length. \qed

Example 6.16.

Let $g = sl(\infty), o(\infty), sp(\infty)$ and let $M$ be a countable direct sum of copies of $V$, i.e. $M = \bigoplus_{f \in \mathcal{F}} V_f$, card$\mathcal{F} = \aleph_0$. Then $(M_*)^*$ can be identified with the set of all infinite matrices $\{b_{ij}\}_{i,j \in \mathbb{Z}_{\geq 0}}$, the action of $g$ being left multiplication. The socle $\text{soc}((M_*)^*)$ is the space of matrices $F_r$ with finitely many non-zero rows and is isomorphic to $\bigoplus_{g \in 2^\mathcal{F}} V_g$. (Note that the module $\bigoplus_{f \in \mathcal{F}} V_f$ considered in Example 6.2 d) is a submodule of $(M_*)^*$ and has the same socle as $(M_*)^*$). We thus obtain the diagram

$$
\bigoplus_{g \in 2^\mathcal{F}} V_g \subset (M_*)^*, \quad\quad M \subset I_M
$$

$I_M$ being an injective hull of $M$. Furthermore, $I_M$ is the largest submodule of $(M_*)^*$ such that $g \cdot I_M = M$. A direct computation shows that $I_M$ coincides with the space of all matrices with finite rows (i.e. each row has finitely many non-zero entries).

Note that $I_M \not\cong \bigoplus_{f \in \mathcal{F}} (I_{\varepsilon_1})_f$ ($\varepsilon_1 \in \Theta$ is the highest weight of $V$). In fact $I_M$ has the following filtration as in Theorem 6.15: $0 \subset \bigoplus_{f \in \mathcal{F}} (I_{\varepsilon_1})_f \subset I_M$. Here $I_M/\bigoplus_{f \in \mathcal{F}} (I_{\varepsilon_1})_f$ is a trivial module of cardinality $2^\mathcal{F}$ which is interpreted as a direct sum of $2^\mathcal{F}$ copies of $I_0$.

For any $k \in \mathbb{Z}_{\geq 0}$ we now define $\text{Tens}_g^{\leq k}$ be the subcategory of modules whose simple quotients are isomorphic to $V_\mu$ with $|\mu| \leq k$. Theorem 6.12 and Corollary 6.3 a) imply the following.

Corollary 6.17. The category $\text{Tens}_g^{\leq k}$ is closed under direct products and direct sums.

Corollary 6.18. a) The category $\text{Tens}_g^{\leq k}$ equals the direct limit $\lim_{\rightarrow} \text{Tens}_g^{\leq k}$.

b) If $\{M_f\}_{f \in \mathcal{F}}$ is an infinite family of objects of $\text{Tens}_g^{\leq k}$, then $\prod_{f \in \mathcal{F}} M_f \in \text{Tens}_g^{\leq k}$ (equivalently, $\bigoplus_{f \in \mathcal{F}} \in \text{Tens}_g^{\leq k}$) if and only if there is $k$ such that $M_f \in \text{Tens}_g^{\leq k}$ for all $f \in \mathcal{F}$.

Proof. a) follows directly from Theorem 6.12.
Consider now $\prod_{j \in \mathcal{E}} M_f$. If $M_f \in \widetilde{\text{Tens}_q}^k$ for some $k$, then $\prod_{j \in \mathcal{E}} M_f \in \widetilde{\text{Tens}_q}^k$ (and thus also $\bigoplus_{j \in \mathcal{E}} M_f \in \widetilde{\text{Tens}_q}$) by Corollary 6.3 a). If no such $k$ exists, then $\bigoplus_{j \in \mathcal{E}} M_f \notin \widetilde{\text{Tens}_q}$ by Theorem 6.12, hence also $\prod_{j \in \mathcal{E}} M_f \notin \widetilde{\text{Tens}_q}$. 

**Corollary 6.19.** Every object in $\widetilde{\text{Tens}_q}$ has a finite injective resolution.

We now introduce the following partial order on $\Theta$: we set $\mu \preceq \lambda$ if for any sufficiently large $i$ there exists $j > i$ such that $\text{Hom}_{\mathfrak{g}}(V_\mu^i, V_\lambda^j) \neq 0$. If $\mu \preceq \lambda$, then $l(\lambda, \mu)$ denotes the length of a maximal chain $\mu \prec \mu_1 \prec \cdots \prec \lambda$ in $\Theta$.

**Lemma 6.20.** $\text{Ext}^1_{\tilde{g}}(V_\mu, V_\lambda) \neq 0$ if and only if $\mu \prec \lambda$. If $\mu \prec \lambda$, then $\text{Ext}^1_{\tilde{g}}(V_\mu, V_\lambda) = \nabla_1$.

**Proof.** Assume that there is a non-trivial extension

$$0 \to V_\lambda \to X \to V_\mu \to 0. \quad (8)$$

We will show that $\mu \prec \lambda$. Let, on the contrary, $\text{Hom}_{\mathfrak{g}}(V_\mu^i, V_\lambda^j) = 0$ for all $j > i$. Then $\text{Hom}_{\mathfrak{g}}(V_\mu^i, V_\lambda) = 0$. Since $\text{dim} \text{Hom}_{\mathfrak{g}}(V_\mu^i, V_\mu) = 1$, we have $\text{dim} \text{Hom}_{\mathfrak{g}}(V_\mu^i, X) = 1$. Let $\varphi : V_\mu^i \to X$ be a non-zero homomorphism. Then $U(\mathfrak{g}) \cdot \varphi(V_\mu^i) \simeq X$. Therefore, $\varphi$ extends to a homomorphism of $\mathfrak{g}$-modules $V_\mu \to X$, and this yields a splitting of the exact sequence (8). Thus, $\text{Ext}^1_{\tilde{g}}(V_\mu, V_\lambda) \neq 0$ implies $\mu \prec \lambda$.

Now let $\mu \prec \lambda$. Then there exists an infinite sequence $i_1, i_2, \ldots$ such that $\text{Hom}_{\mathfrak{g}}(V_{\mu, i}, V_{\lambda, i+1}) \neq 0$ for all $j$. Consider a sequence of non-zero homomorphisms $\varphi_j \in \text{Hom}_{\mathfrak{g}}(V_{\mu, i}, V_{\lambda, i+1})$ and set $Z_j := V_{\mu, i} \oplus V_{\lambda, i+1}$. Denote by $e_j$ (respectively, $f_j$) the inclusion $V_{\mu, i} \to V_{\mu, i+1}$ (resp., $V_{\lambda, i} \to V_{\lambda, i+1}$). Define $\psi_j : Z_j \to Z_{j+1}$ by

$$\psi(x, y) = (e_j(x), \varphi_j(x) + f_j(y)), \ x \in V_{\mu, i}, y \in V_{\lambda, i+1}.$$  

Consider $Z = \lim Z_j$. It is an exercise to check that $Z$ is an extension of $V_\mu$ by $V_\lambda$, and that it does not split if infinitely many $\varphi_j \neq 0$. Hence the dimension of $\text{Ext}^1_{\tilde{g}}(V_\mu, V_\lambda)$ is at least $\nabla_1$. On the other hand, the dimension of $\text{Ext}^1_{\tilde{g}}(V_\mu, V_\lambda)$ is bounded by the multiplicity of $V_\mu$ in $\text{soc}^1(I_\lambda)/\text{soc}(I_\lambda)$. The dimension of $I_\mu = ((V_\mu)_\ast) \ast$ is $\nabla_1$, hence the dimension of $\text{Ext}^1_{\tilde{g}}(V_\mu, V_\lambda)$ is at most $\nabla_1$.

To finish the proof just note that $\text{Ext}^1_{\tilde{g}}(V_\lambda, V_\lambda) = 0$ by Lemma 6.6. 

**Corollary 6.21.** The category $\widetilde{\text{Tens}_q}$ consists of a single block.

**Proof.** According to Lemma 6.20, $\text{Ext}^1_{\tilde{g}}(\mathbb{C}, V_\mu) \neq 0$ for any $\mu \in \Theta$. 

**Proposition 6.22.** For $k \in \mathbb{Z}_{>0}$, set

$$\Theta^k(\lambda) = \{ \mu \prec \lambda | l(\lambda, \mu) \geq k + 1 \}.$$

Then

$$\text{soc}^k(I_\lambda)/\text{soc}^{k-1}(I_\lambda) = \bigoplus_{\mu \in \Theta^k(\lambda)} X^\mu \otimes V_\mu,$$

where each $X^\mu$ is a trivial $\mathfrak{g}$-module of dimension $\nabla_1$. 

Proof. For $k = 1$ the statement follows from Lemma 6.20. Now we proceed by induction on $k$. Note first that if $V_\mu$ is a simple constituent of $\soc^k(I_\lambda)/\soc^{k-1}(I_\lambda)$, then, by Lemma 6.20, $\mu \prec \chi$ for some simple constituent $V_\lambda$ of $\soc^{k-1}(I_\lambda)/\soc^{k-2}(I_\lambda)$. By the induction assumption, $\chi \in \Theta^{k-1}(\lambda)$. In addition, it is clear that $V_\mu$ is a simple constituent of $\soc^k(I_\lambda)/\soc^{k-1}(I_\lambda)$ if and only if there exists a non-zero homomorphism $\varphi : I_\lambda \to I_\mu$, such that $\varphi(\soc^{k-1}(I_\lambda)) = 0$. By Lemma 6.10, $\varphi$ is surjective, so all simple constituents of $\soc^k(I_\mu)/\soc(\mu)$ are also simple constituents of $\soc^k(I_\lambda)/\soc^{k-1}(I_\lambda)$. This implies that $V_\mu$ is a simple constituent of $\soc^k(I_\lambda)/\soc^{k-1}(I_\lambda)$ if and only if there exists $\psi \in \Theta^{k-1}(\lambda)$ such that $\mu \in \Theta^k(\psi)$. Since $\mu \in \Theta^k(\psi)$ if and only if $\mu \in \Theta^k(\lambda)$, the statement follows.

Let $\Tens_{\mathfrak{g}}$ be the full subcategory of $\widetilde{\Tens}_{\mathfrak{g}}$ consisting of modules $M$ whose cardinality $\text{card}M$ is bounded by $\beth_n$ for some $n$ depending on $M$.

**Theorem 6.23.** $\Tens_{\mathfrak{g}}$ is the unique minimal abelian full subcategory of $\Int_{\mathfrak{g}}$ which does not consist of trivial modules only and which is closed under $\otimes$ and $^*$.

Proof. Let $\mathcal{C}$ be a minimal abelian full subcategory of $\Int_\mathfrak{g}$ which contains a non-trivial module $M$ and is closed under $\otimes$ and $^*$. We will show that $V \in \mathcal{C}$. Since $\End_\mathcal{C}M$ is a $\mathfrak{g}$-submodule of $(M^* \otimes M)^*$ (through the map $\varphi(\psi \otimes m) = \psi(\varphi(m))$ for $m \in M$, $\psi \in M^*$, $\varphi \in \End_\mathcal{C}M$), we have $\End_\mathcal{C}M \in \mathcal{C}$. Furthermore, the adjoint module $\mathfrak{g}$ is a submodule of $\End_\mathcal{C}M$. Hence $\mathfrak{g} \in \mathcal{C}$. Recall that $\mathfrak{g}$ is the socle of $V_\ast \otimes V$ for $sl(\infty)$, of $\Lambda^2(V)$ for $o(\infty)$, and of $S^2(V)$ for $sp(\infty)$. In all cases it is easy to see that $\mathfrak{g}^*$ contains a subquotient isomorphic to $V$. Therefore, $V \in \mathcal{C}$. In addition, $V_\ast = \soc(V^*) \in \mathcal{C}$. Therefore, $T^{p,q} \in \mathcal{C}$ for all $p, q$, and $V_\lambda \in \mathcal{C}$ for all $\lambda \in \Theta$. Finally, by Corollary 6.7 a), any $M \in \widetilde{\Tens}_{\mathfrak{g}}$ is a submodule of $(\soc(M)_\ast)^*$, and the statement follows.

We conclude this paper with the remark that the category $\widetilde{\Tens}_{\mathfrak{g}}$, for $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$, is functorial with respect to any homomorphism of locally semisimple Lie algebras $\varphi : \mathfrak{g}^\prime \to \mathfrak{g}$. By this we mean that any $M \in \widetilde{\Tens}_{\mathfrak{g}}$ considered as a $\mathfrak{g}^\prime$-module is an object of $\widetilde{\Tens}_{\mathfrak{g}^\prime}$.

To prove this, recall that the image of $\varphi^\prime$, being a locally semisimple subalgebra of $\mathfrak{g}$, is isomorphic to a direct sum of copies of $sl(\infty), o(\infty), sp(\infty)$ and of finite-dimensional simple Lie algebras, [DP2]. Furthermore, the result of [DP2] implies that as $\mathfrak{g}^\prime$-modules both $V$ and $V_\ast$ have Loewy length at most 2 and that all non-trivial simple constituents of $V$ and $V_\ast$ are isomorphic to the natural and conatural representations $V_\lambda$ and $(V_\lambda)_\ast$, for some simple direct summands $\varphi(\mathfrak{g}^\prime)$ and that all non-trivial constituents occur with finite multiplicity. (The simple trivial representation may occur with up to countable multiplicity in both $\soc(V)$ and $V/\soc(V)$ (respectively, $\soc(V_\ast)$ and $V_\ast/\soc(V_\ast)$.) This allows us to conclude that any single simple object of $\widetilde{\Tens}_{\mathfrak{g}}$ is an object of $\widetilde{\Tens}_{\varphi(\mathfrak{g}^\prime)}$. Hence, by Theorem 6.12, any $M \in \widetilde{\Tens}_{\mathfrak{g}}$ is an object of $\widetilde{\Tens}_{\varphi(\mathfrak{g}^\prime)}$. 
Categories of integrable $sl(\infty)$-, $o(\infty)$-, $sp(\infty)$-modules

References


Ivan Penkov
Jacobs University Bremen, School of Engineering and Science, Campus Ring 1, 28759 Bremen, Germany
e-mail: i.penkov@jacobs-university.de

Vera Serganova
Department of Mathematics, University of California Berkeley, Berkeley CA 94720, USA
e-mail: serganov@math.berkeley.edu