Categories of integrable $sl(\infty)$-, $o(\infty)$-, $sp(\infty)$-modules

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Summary. We investigate several categories of integrable $\mathrm{sl}(\infty)$-, $o(\infty)$-, $sp(\infty)$-modules. In particular, we prove that the category of integrable $\mathrm{sl}(\infty)$-, $o(\infty)$-, $sp(\infty)$-modules with finite-dimensional weight spaces is semisimple. The most interesting category we study is the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ for $\mathfrak{g} = \mathrm{sl}(\infty)$-, $o(\infty)$-,
$sp(\infty)$. Its objects M are defined as integrable $\mathfrak{g}$-modules of finite Loewy length such that the algebraic dual M^* is also integrable and of finite Loewy length.

We prove that the simple objects of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are precisely the simple tensor modules, i.e. the simple subquotients of the tensor algebra of the direct sum of the natural and conatural representations. We also study injectives in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ and compute the Ext^1's between simple modules.

Finally, we characterize a certain subcategory $\mathrm{Tens}_{\mathfrak{g}}$ of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ as the unique minimal abelian full subcategory of the category of integrable modules which contains a non-trivial module and is closed under tensor product and algebraic dualization.

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1. Introduction

The category of finite-dimensional representations of a Lie algebra is endowed with a natural contravariant involution

$$M \rightsquigarrow M^*, \tag{1}$$

where * indicates dual space. For categories of infinite-dimensional modules (1) is never an involution as $M \not\simeq M^{**}$. This is why one usually looks for a "restricted dual" or a "continuous dual" which might still yield a contravariant involution on a given category of infinite-dimensional modules. In this paper, we study two categories of infinite-dimensional modules of certain infinite-dimensional Lie algebras and show, in particular, that there exists an interesting category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ of infinite-dimensional representations on which the functor (1) of algebraic dualization is well-defined and preserves the property of a module to be of finite Loewy length.

More precisely, we study representations of locally finite Lie algebras, i.e. of direct limits of finite-dimensional Lie algebras. There are three well-known classical simple locally finite Lie algebras $sl(\infty), o(\infty), sp(\infty)$, each of them being defined by an obvious direct limit. None of these Lie algebras admits non-trivial finite-dimensional representations, and instead one studies integrable representations (the definition see in section 2 below). However, the category of integrable $\mathfrak{g}$-modules for $\mathfrak{g} = \mathrm{sl}(\infty)$-, $o(\infty)$-, $sp(\infty)$ is vast (and "wild" in the technical sense), so it is reasonable to look for interesting subcategories.

One subcategory we study is the category of integrable weight modules with finite-dimensional weight spaces, and this is obviously an analog of the category of finite-dimensional representations of a classical finite-dimensional Lie algebra. It is less obvious that for $\mathfrak{g} = sl(\infty)$ this category contains some rather interesting simple modules, which are not highest weight modules. The first main result of this paper is the proof of the semisimplicity of this category: an extension of Hermann Weyl's semisimplicity theorem to the classical Lie algebras $sl(\infty), o(\infty), sp(\infty)$.

The above category is clearly not the only reasonable generalization of the category of finite-dimensional representations, as for instance it does not contain the adjoint representation. Indeed, note that the adjoint representation has an infinite-dimensional weight space, the Cartan subalgebra itself. On the other hand, the adjoint representation is naturally a simple tensor module as defined in [PS]. More generally, we define the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ for $\mathfrak{g} \cong sl(\infty)$, $o(\infty)$, $sp(\infty)$ simply as the largest category of integrable $\mathfrak{g}$-modules which is closed under algebraic dualization and such that every object has finite Loewy length. This category is a (non-rigid) tensor category with respect to the usual tensor product.

The second main contribution of the present paper is the study of the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. In particular, we study injectives in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ and compute the Ext^1's between simple modules. We also give an alternative characterization of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ by proving that an integrable $\mathfrak{g}$-module is an object of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ if and only if it admits only finitely many non-isomorphic simple subquotients each of which is a submodule of a suitable finite tensor product of natural and conatural modules.

Finally, we describe a certain subcategory $\mathrm{Tens}_{\mathfrak{g}}$ of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ as the unique minimal abelian full subcategory of the category of integrable modules which contains a non-trivial module and is closed under tensor product and algebraic dualization.

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2. Basic definitions

The ground field is $\mathbb{C}$ and $\otimes$ stands for $\otimes_{\mathbb{C}}$. If $\mathcal{C}$ is a category, $C \in \mathcal{C}$ indicates that C is an object of $\mathcal{C}$. If P is a set, we denote by 2^P the power set of P. We recall that the cardinal numbers $\beth_n$ are defined inductively: $\beth_0 = \operatorname{card}\mathbb{Z}$, $\beth_1 = \operatorname{card} 2^{\mathbb{Z}}$, $\beth_n = \operatorname{card} 2^{P_{n-1}}$, where P_{n-1} is a set of cardinality $\beth_{n-1}$.

In this paper $\mathfrak{g}$ stands for a *locally semisimple* (complex) Lie algebra. By definition, $\mathfrak{g} = \bigcup_{i\in\mathbb{Z}_{>0}} \mathfrak{g}_i$ where

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathfrak{g}_3 \subset \dots \tag{2}$$

is a sequence of inclusions of semisimple finite-dimensional Lie algebras. We call the sequence (2) an *exhaustion* of $\mathfrak{g}$, and we will assume that it is fixed. A locally semisimple Lie algebra is *locally simple* if it admits an exhaustion (2) so that all $\mathfrak{g}_i$ are simple. It is clear that a locally simple Lie algebra is simple. If no restrictions on $\mathfrak{g}$ are clearly stated, in what follows $\mathfrak{g}$ is assumed to be an arbitrary locally semisimple Lie algebra.

A locally simple algebra $\mathfrak{g}$ is *diagonal* if an exhaustion (2) can be chosen so that all $\mathfrak{g}_i$ are classical simple Lie algebras and the natural representation V_i of $\mathfrak{g}_i$, when restricted to $\mathfrak{g}_{i-1}$, has the form $k_iV_{i-1} \oplus l_iV_{i-1}^* \oplus \mathbb{C}^{s_i}$ for some k_i, l_i and $s_i \in \mathbb{Z}_{\geq 0}$. Here V_{i-1} stands for the natural representation of $\mathfrak{g}_{i-1}$, $\mathbb{C}^{s_i}$ stands for the trivial module of dimension s_i, and k_iV_{i-1} (respectively, $l_iV_{i-1}^*$) denotes the direct sum of k_i(respectively, l_i) copies of V_{i-1} (respectively, V_{i-1}^*).

The three classical simple Lie algebras $sl(\infty), o(\infty)$ and $sp(\infty)$ (defined respectively as $sl(\infty) = \cup_i sl(i)$, $o(\infty) = \cup_i o(i)$, $sp(\infty) := \cup_i sp(2i)$ via the natural inclusions $sl(i) \subset sl(i+1)$) etc.) are clearly diagonal. Moreover, $sl(\infty), o(\infty), sp(\infty)$ are (up to isomorphism) the only finitary locally simple Lie algebras $\mathfrak{g}$; *finitary* means by definition that $\mathfrak{g}$ admits a faithful countable-dimensional $\mathfrak{g}$-module with a basis in which each element $g \in \mathfrak{g}$ acts through a finite matrix, [Ba1], [Ba3]. More generally, there exists also a classification of locally simple diagonal Lie algebras up to isomorphism, [BZh]. We do not use this classification in the present paper and present only the simplest example of a diagonal Lie algebra not isomorphic to $sl(\infty), o(\infty)$ or $sp(\infty)$. This is the Lie algebra $sl(2^\infty)$ defined as the direct limit $\varinjlim sl(2^i)$ under the inclusions

$$sl(2^i) \to sl(2^{i+1}), A \to \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

A $\mathfrak{g}$-module M is *integrable* if $\dim \operatorname{span}\{m, g\cdot m, g^2\cdot m, \cdots\} < \infty$ for any $m \in M$ and $g \in \mathfrak{g}$. Since $\mathfrak{g}$ is locally semisimple, this is equivalent to the condition that, when restricted to any semisimple finite-dimensional subalgebra $\mathfrak{f}$ of $\mathfrak{g}$, M

is isomorphic to a (not necessarily countable) direct sum of finite-dimensional $\mathfrak{k}$-modules. We denote by $\mathrm{Int}_{\mathfrak{g}}$ the category of integrable $\mathfrak{g}$-modules; $\mathrm{Int}_{\mathfrak{g}}$ is a full subcategory of the category of $\mathfrak{g}$-modules $\mathfrak{g}$-mod.

Any countable-dimensional $\mathfrak{g}$-module $M \in \mathrm{Int}_{\mathfrak{g}}$ can be exhausted by finite dimensional $\mathfrak{g}_i$-modules M_i, i. e. there exists a chain of finite-dimensional $\mathfrak{g}_i$-submodules $M_1 \subset M_2 \subset \dots$ such that $M = \varinjlim M_i$. We call M *locally simple* if all M_i can be chosen to be simple modules. It is clear that a locally simple module is simple. Note also that if M is locally simple then any two exhaustions $\{M_i\}$ and $\{M_i'\}$ coincide from some point on: that follows from the fact that $M_i \cap M_i' \neq 0$ for some i and hence $M_j = M_j' = M_j \cap M_j'$ for any $j \geq i$. We say that a locally simple $\mathfrak{g}$-module $M = \varinjlim M_i$ is a *highest weight module* if there is a chain of nested Borel subalgebras $\mathfrak{b}_i$ of $\mathfrak{g}_i$ such that the $\mathfrak{b}_i$-highest weight space of M_i is mapped into the $\mathfrak{b}_{i+1}$-highest weight space of M_{i+1} under the inclusion $M_i \subset M_{i+1}$. The direct limit of highest weight spaces is then the $\mathfrak{b}$-*highest weight space of* M, where $\mathfrak{b} = \varinjlim \mathfrak{b}_i$.

By

$$\Gamma_{\mathfrak{g}} : \mathfrak{g} - \mathrm{mod} \rightsquigarrow \mathrm{Int}_{\mathfrak{g}},$$

$$M \mapsto \Gamma_{\mathfrak{g}}(M) := \{m \in M, \dim \mathrm{span}\{m, g \cdot m, g \cdot m^2, \cdots\} < \infty \quad \forall g \in \mathfrak{g}\}$$

we denote the *functor of* $\mathfrak{g}$*-integrable vectors*. It is an exercise to check that $\Gamma_{\mathfrak{g}}(M)$ is indeed a well-defined $\mathfrak{g}$-submodule of M; the fact that $\Gamma_{\mathfrak{g}}(M)$ is integrable is obvious. Furthermore, $\Gamma_{\mathfrak{g}}$ is a left-exact functor.

If $\mathfrak{g}$ is a diagonal (locally simple) Lie algebra, then one can define a *natural module* V of $\mathfrak{g}$. Indeed, the reader will verify that one can choose a subexhaustion of (2) such that the natural $\mathfrak{g}_i$-module V_i is a $\mathfrak{g}_i$-submodule of V_{i+1} for any i. Therefore, fixing arbitrary injective homomorphisms $V_i \to V_{i+1}$ of $\mathfrak{g}_i$-modules, we obtain a direct system and we set $V := \varinjlim V_i$. Note that V depends on the choice of the homomorphisms $V_i \to V_{i+1}$. In the special case when $\mathfrak{g} \cong sl(\infty)$, $o(\infty)$, $sp(\infty)$, the homomorphisms $V_i \to V_{i+1}$ are unique up to proportionality, and one can prove that as a result V is unique up to isomorphism, i.e. in particular does not depend on the fixed exhaustion of $\mathfrak{g}$. In these latter cases we speak about *the natural representation.*

By choosing injective homomorphisms of $\mathfrak{g}_i$-modules $V_i^* \to V_{i+1}^*$, we obtain a direct system defining a *conatural representation* of $\mathfrak{g}$. We denote such a representation by V_*. For $\mathfrak{g} \cong sl(\infty)$, $o(\infty)$, $sp(\infty)$ V_* is unique up to isomorphism. In fact, $V \simeq V_*$ for $\mathfrak{g} \cong o(\infty)$, $sp(\infty)$.

3. Injective modules in $\mathrm{Int}_{\mathfrak{g}}$ and semisimplicity of the category $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$

Proposition 3.1. $\mathrm{Ext}^1_{\mathfrak{g}}(X, M^*) = 0$ *for any* $X, M \in \mathrm{Int}_{\mathfrak{g}}$.

Proof. We use that

$$\mathrm{Ext}^1_{\mathfrak{g}}(X, M^*) = \mathrm{Ext}^1_{\mathfrak{g}}(\mathbb{C}, \mathrm{Hom}_{\mathbb{C}}(X, M^*)) \simeq H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{C}}(X, M^*)) = H^1(\mathfrak{g}, (X \otimes M)^*),$$

see for instance [W]. Therefore, it suffices to show that $H^1(\mathfrak{g}, R^*) = 0$ for any integrable $\mathfrak{g}$-module R.

Consider the standard complex for the cohomology of $\mathfrak{g}$ with coefficients in R^*:

$$0 \to R^* \to (\mathfrak{g} \otimes R)^* \to (\Lambda^2(\mathfrak{g}) \otimes R)^* \to \dots \tag{3}$$

It is dual to the standard homology complex

$$0 \leftarrow R \leftarrow \mathfrak{g} \otimes R \leftarrow \Lambda^2(\mathfrak{g}) \otimes R \leftarrow \dots,$$

which is the direct limit of complexes

$$0 \leftarrow R \leftarrow \mathfrak{g}_i \otimes R \leftarrow \Lambda^2(\mathfrak{g}_i) \otimes R \leftarrow \dots.$$

Since $H_1(\mathfrak{g}_i, R) = 0$ for each i, we get $H_1(\mathfrak{g}, R) = 0$. Therefore, the dual complex (3) has trivial first cohomology, i.e. $H^1(\mathfrak{g}, R^*) = 0$. □

Proposition 3.2. *For any $M \in \mathrm{Int}_{\mathfrak{g}}$, $\Gamma_{\mathfrak{g}}(M^*)$ is an injective object of $\mathrm{Int}_{\mathfrak{g}}$.*

Proof. Let $X \in \mathrm{Int}_{\mathfrak{g}}$. The exact sequence of $\mathfrak{g}$-modules

$$0 \to \Gamma_{\mathfrak{g}}(M^*) \to M^* \to M^*/\Gamma_{\mathfrak{g}}(M^*) \to 0$$

induces an exact sequence of vector spaces

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(X, \Gamma_{\mathfrak{g}}(M^*)) \overset{\varphi}{\to} \mathrm{Hom}_{\mathfrak{g}}(X, M^*) \to \mathrm{Hom}_{\mathfrak{g}}(X, M^*/\Gamma_{\mathfrak{g}}(M^*)) \to$$

$$\to \mathrm{Ext}^1_{\mathfrak{g}}(X, \Gamma_{\mathfrak{g}}(M^*)) \overset{\psi}{\to} \mathrm{Ext}^1_{\mathfrak{g}}(X, M^*) = 0.$$

Since $\mathrm{Hom}_{\mathfrak{g}}(X, M^*/\Gamma_{\mathfrak{g}}(M^*)) = 0$ (this follows from the facts that a quotient of an integrable $\mathfrak{g}$-module is again an integrable $\mathfrak{g}$-module and that $\mathrm{Int}_{\mathfrak{g}}$ is closed with respect to extensions) we conclude that ψ is an isomorphism, i.e. that

$$\mathrm{Ext}^1_{\mathfrak{g}}(X, \Gamma_{\mathfrak{g}}(M^*)) = 0.$$

□

Corollary 3.3. $\mathrm{Int}_{\mathfrak{g}}$ *has enough injectives.*

Proof. Let $M \in \mathrm{Int}_{\mathfrak{g}}$. Then $M \subset M^{**}$ and it is easy to check (using the local semisimplicity of $\mathfrak{g}$) that the projection $M^{**} \to \Gamma_{\mathfrak{g}}(M^*)^*$ induces an injection $M \to \Gamma_{\mathfrak{g}}(M^*)^*$. This in turn induces an injection $M \to \Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(M^*)^*)$, and $\Gamma_{\mathfrak{g}}(\Gamma_{\mathfrak{g}}(M^*)^*)$ is an injective object of $\mathrm{Int}_{\mathfrak{g}}$ by Proposition 3.2. □

Note that there is a simpler proof of Corollary 3.3 not referring to Proposition 3.2. Indeed, it is enough to notice that the functor $\Gamma_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \rightsquigarrow \mathrm{Int}_{\mathfrak{g}}$ is right adjoint to the inclusion functor $\mathrm{Int}_{\mathfrak{g}} \subset \mathfrak{g}$-mod. Then the equality

$$\mathrm{Hom}_{\mathfrak{g}}(M, J_M) = \mathrm{Hom}_{\mathfrak{g}}(M, \Gamma_{\mathfrak{g}}(J_M))$$

allows us to conclude that, if $i : M \to J_M$ is an injective homomorphism of $M \in \mathrm{Int}_{\mathfrak{g}}$ into an injective $\mathfrak{g}$-module, then $\Gamma_{\mathfrak{g}}(J_M)$ is an injective object of $\mathrm{Int}_{\mathfrak{g}}$ and i factors through the inclusion $\Gamma_{\mathfrak{g}}(J_M) \subset J_M$. In particular, this argument allows to reduce the existence of injective hulls in $\mathrm{Int}_{\mathfrak{g}}$ to the well-known existence of injective hulls in $\mathfrak{g}$-mod.

With this in mind, we can view Propositions 3.1 and 3.2 as yielding an explicit construction of an injective module $\Gamma_{\mathfrak{g}}(M^*)$ associated to any $M \in \mathrm{Int}_{\mathfrak{g}}$.

In the rest of this section we assume that $\mathfrak{g}$ admits a splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, i.e. an abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g}$ decomposes as

$$\mathfrak{h} \oplus \bigoplus_{0 \neq \alpha \in \mathfrak{h}^*} \mathfrak{g}^{\alpha},$$

where

$$\mathfrak{g}^{\alpha} = \{g \in \mathfrak{g} | [h, g] = \alpha(h) g \,\text{for any}\, h \in \mathfrak{h}\}.$$

It is well-known that in this case $\mathfrak{g}$ is isomorphic to a direct sum of copies of $sl(\infty), o(\infty), sp(\infty)$ and finite-dimensional simple Lie algebras, see [PStr].

We define the category $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ as the full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ which consists of *weight modules* M, i.e. objects $M \in \mathrm{Int}_{\mathfrak{g}}$ which admit a decomposition

$$M = \bigoplus_{\alpha \in \mathfrak{h}^*} M^{\alpha}, \tag{4}$$

where

$$M^{\alpha} = \{m \in M | h \cdot m = \alpha(h) m \,\text{for any}\, h \in \mathfrak{h}\}.$$

Note that (4) is automatically a decomposition of $\mathfrak{h}$-modules. It is also clear that there is a left exact functor

$$\Gamma^{\mathrm{wt}}_{\mathfrak{h}} : \mathrm{Int}_{\mathfrak{g}} \rightsquigarrow \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}, \ M \mapsto \bigoplus_{\alpha \in \mathfrak{h}^*} M^{\alpha}.$$

By $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ we denote the composition

$$\Gamma^{\mathrm{wt}}_{\mathfrak{h}} \circ \Gamma_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \rightsquigarrow \mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}.$$

Lemma 3.4. *If X is an injective object of $\mathrm{Int}_{\mathfrak{g}}$, then $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(X)$ is an injective object of $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$.*

Proof. It suffices to note that $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}$ is right adjoint to the inclusion functor $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}} \subset \mathrm{Int}_{\mathfrak{g}}$. □

Example 3.5. Let $\mathfrak{g} = sl(\infty)$ and $M = V \otimes V_*$. Consider the $\mathfrak{g}$-module M^*. Let's think of $M^* = (V \otimes V_*)^*$ as the space of all infinite matrices $B = (b_{ij}), i, j \in \mathbb{Z}_{>0}$, and of M as the space of finitary infinite matrices $A = (a_{ij}), i, j \in \mathbb{Z}_{>0}$, where $B(A) = \sum_{i,j} b_{ij} a_{ji}$. Then $\mathfrak{g}$ is identified with the subspace $F \subset (V \otimes V_*)^*$ of finitary matrices with trace zero, and the $\mathfrak{g}$-module structure on M^* is given by $A \cdot B = [A, B]$. We fix the Cartan subalgebra $\mathfrak{h}$ to be the algebra of finitary diagonal matrices, and we claim that $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M^*) = F + D$ where D is the subspace of diagonal matrices. Indeed, clearly D equals the $\mathfrak{h}$-weight space $(M^*)^0$ of weight

0. Furthermore, any weight space of non-zero weight is the span of an elementary non-diagonal matrix, hence $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M^*) = F + D$. Note also that we have a non-splitting exact sequence of $\mathfrak{g}$-modules

$$0 \to \mathfrak{g} \to \Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M^*) \to T \to 0,$$

where $T = D/D \cap F$ is a trivial $\mathfrak{g}$-module of dimension $\beth_1$.

Corollary 3.6. *For any* $M \in \mathrm{Int}_{\mathfrak{g}}$*,* $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M^*)$ *is an injective object of* $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$.

Define now $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ as the full subcategory of $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ consisting of $\mathfrak{h}$-weight modules $M = \bigoplus_{\alpha\in\mathfrak{h}^*} M^\alpha$ such that $\dim M^\alpha < \infty$ for any $\alpha \in \mathfrak{h}^*$.

Theorem 3.7. *The category* $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ *is semisimple.*

Proof. Let $M \in \mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ be simple. There is an $\mathfrak{h}$-module isomorphism

$$M = \oplus_{\alpha\in\mathfrak{h}^*} M^\alpha.$$

Therefore, $M^* = \prod_{\alpha\in\mathfrak{h}^*}(M^\alpha)^*$. A non-difficult computation shows that $\Gamma^{\mathrm{wt}}_{\mathfrak{h}}(M^*)$ is isomorphic to $\oplus_{\alpha\in\mathfrak{h}^*}(M_\alpha)^*$. Moreover, using the fact that $\dim M^\alpha < \infty$ for all α, it is easy to check that $M_* := \oplus_{\alpha\in\mathfrak{h}^*}(M_\alpha)^*$ is a simple integrable $\mathfrak{g}$-module. Hence $M_* = \Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M^*)$. Applying $\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$ again, we see that

$$\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(\Gamma^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}(M^*)^*) = M.$$

Therefore, M is injective in $\mathrm{Int}^{\mathrm{wt}}_{\mathfrak{g},\mathfrak{h}}$, and thus also in $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$, by Corollary 3.6. $\square$

Example 3.8.

a) Let $\mathfrak{g} = sl(\infty)$. One checks immediately that all tensor powers $V^{\otimes k}$, V being the natural module, are objects of $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$. The same applies to the tensor powers of the conatural module V_*. However, the category $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$ contains also more interesting modules as the following one: $M = \varinjlim S^i(V_i)$, V_i being the natural representation of $sl(i)$. The module M has 1-dimensional weight spaces, but is not a highest weight module, see [DP1, Example 3]. Note also that the adjoint representation is not an object of $\mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$.

b) Let $\mathfrak{g} = o(\infty)$ and let $\mathfrak{g}$ be exhausted by $\mathfrak{g}_i = o(2i)$, $i \geq 3$. Denote by S^1_i and S^2_i the two non-isomorphic spinor $\mathfrak{g}_i$-modules. Then S^1_i and S^2_i are both isomorphic to $S^1_{i-1} \oplus S^2_{i-1}$ as $\mathfrak{g}_{i-1}$-modules. Therefore, there is an injective homomorphism of $\mathfrak{g}_{i-1}$-modules $\varphi^{ks}_{i-1} : S^k_{i-1} \to S^s_i$ for $k, s \in \{1, 2\}$, and moreover φ^{ks}_{i-1} is unique up to proportionality. Any sequence $\{t_i\}_{i\geq 3}$ of elements in $\{1, 2\}$ defines a direct system

$$S^{t_3}_3 \xrightarrow{\varphi^{t_3,t_4}_3} S^{t_4}_4 \xrightarrow{\varphi^{t_4,t_5}_4} S^{t_5}_5 \xrightarrow{\varphi^{t_5,t_6}_5} \dots,$$

and hence a simple $\mathfrak{g}$-module $S(\{t_i\})$. Using the fact that $S(\{t_i\})$ is locally simple, it is easy to see that $S(\{t_i\}) = S(\{t'_i\})$ if and only if the "tails" of the sequence $\{t_i\}$ and $\{t'_i\}$ coincide, i.e. $t_i = t'_i$ for large enough i.

The modules $S(\{t_i\})$ are weight modules with 1-dimensional spaces for any Cartan subalgebra $\mathfrak{h}$ of the form $\mathfrak{h} = \cup_i \mathfrak{h}_i$ where $\mathfrak{h}_3 \subset \mathfrak{h}_4 \subset \dots$ are nested Cartan subalgebras of $\mathfrak{g}_3 = o(6) \subset \mathfrak{g}_4 = o(8) \subset \dots$. In particular, $S(\{t_i\}) \in \mathrm{Int}^{\mathrm{fin}}_{\mathfrak{g},\mathfrak{h}}$.

4. On the integrability of M^* for $M \in \mathrm{Int}_\mathfrak{g}$

Lemma 4.1. *Let $M \in \mathrm{Int}_\mathfrak{g}$. Then $M^* \in \mathrm{Int}_\mathfrak{g}$ if and only if for any $i > 0$ $\mathrm{Hom}_{\mathfrak{g}_i}(N, M) \neq 0$ only for finitely many non-isomorphic simple $\mathfrak{g}_i$-modules N.*

Proof. Fix i. Let Λ_i be the set of integral dominant weights of $\mathfrak{g}_i$ (for some fixed Borel subalgebra $\mathfrak{b}_i$ of $\mathfrak{g}_i$ with fixed Cartan subalgebra $\mathfrak{h}_i \subset \mathfrak{b}_i$) and V^i_λ be the simple $\mathfrak{g}_i$-module with highest weight λ. Denote by $\Lambda_i(M)$ the set of all $\lambda \in \Lambda_i$ such that $\mathrm{Hom}_{\mathfrak{g}_i}(V^i_\lambda, M) \neq 0$. Since M is a semisimple $\mathfrak{g}_i$-module, we can write M as

$$M = \oplus_{\lambda \in \Lambda_i(M)} M^\lambda \otimes V^i_\lambda,$$

where $M^\lambda := \mathrm{Hom}_{\mathfrak{g}_i}(V^i_\lambda, M)$ is a trivial $\mathfrak{g}_i$-module. We have

$$M^* = \prod_{\lambda \in \Lambda_i(M)} (V^i_\lambda)^* \otimes (M^\lambda)^*.$$

Suppose that $\Lambda_i(M)$ is finite. Then for any fixed $g \in \mathfrak{g}_i$ there is a polynomial $p_\lambda(z)$ such that $p_\lambda(g) \cdot (V^i_\lambda)^* = 0$. Set $p(z) := \prod_{\lambda \in \Lambda_i(M)} p_\lambda(z)$. Then $p(g) \cdot M^* = 0$. Hence g acts integrably on M^*, i.e. M^* is integrable over $\mathfrak{g}_i$.

Now let $\Lambda_i(M)$ be infinite. Let v_λ be a non-zero vector of weight $-\lambda$ in $(V^i_\lambda)^* \otimes (M^\lambda)^*$. One can choose h in the Cartan subalgebra of $\mathfrak{g}_i$ such that $\lambda(h) \neq \mu(h)$ for any $\mu \neq \lambda \in \Lambda_i(M)$. Let $v := \prod_{\lambda \in \Lambda_i(M)} (v_\lambda) \in \prod_{\lambda \in \Lambda_i(M)} (V^i_\lambda)^* \otimes (M^\lambda)^*$. Then $\dim(\mathbb{C}[h] \cdot v) = \infty$, and M^* is not $\mathfrak{g}_i$-integrable. $\square$

Corollary 4.2. *Let $M, M' \in \mathrm{Int}_\mathfrak{g}$. If $M^*, (M')^* \in \mathrm{Int}_\mathfrak{g}$, then $(M \otimes M')^* \in \mathrm{Int}_\mathfrak{g}$ and $M^{**} \in \mathrm{Int}_\mathfrak{g}$.*

Proposition 4.3. *Let $\mathfrak{g}$ be a locally simple Lie algebra. There exists a non-trivial module $M \in \mathrm{Int}_\mathfrak{g}$ such that M^* is integrable if and only if $\mathfrak{g}$ is diagonal.*

Proof. First of all, if $\mathfrak{g}$ is diagonal, then any natural module $V = \varinjlim V_n$ satisfies the finiteness condition of Lemma 4.1, hence V^* is integrable.

Before we prove the other direction, note that, by passing to a subexhaustion, we can always assume that $\mathfrak{g}$ is exhausted by classical simple Lie algebras $\mathfrak{g}_i$ of the same type (A, B, C or D). Let now $M \in \mathrm{Int}_\mathfrak{g}$ be non-trivial and M^* be integrable. We will show that $\mathfrak{g}$ is diagonal. Since M satisfies the finiteness condition of Lemma 4.1, $\mathrm{End}_\mathbb{C} M$ and its submodules satisfy this condition too. The adjoint module $\mathfrak{g}$ is a submodule of $\mathrm{End}_\mathbb{C} M$, hence this implies that for each i the number of $\mathfrak{g}_i$-isotypic components in $\mathfrak{g}_{i+k}$ is uniformly bounded for all $k > 0$. Since the adjoint module of $\mathfrak{g}_i$ is isomorphic to $(V_i \otimes V_i^*)/\mathbb{C}$ in the type A case, to $S^2(V_i)$ in type C, and to $\Lambda^2(V_i)$ in types B or D, one can easily check that for each i the number of $\mathfrak{g}_i$-isotypic components in V_{i+k} is also uniformly bounded by for all $k > 0$. Our goal is to show that for all sufficiently large i, V_{i+1} restricted to $\mathfrak{g}_i$ is isomorphic to a direct sum of copies of V_i, V_i^* and $\mathbb{C}$.

Let us start with the type A case. Pick an $sl(2)$-subalgebra in $\mathfrak{g}_n$ for some n. The set of $sl(2)$-weights in V is finite. Thus we can let $k \in \mathbb{Z}_{>0}$ be the maximal weight in this set and fix i such that k is a weight of V_i. Then $sl(2) \subset \mathfrak{g}_i$. Furthermore, we have an isomorphism of $\mathfrak{g}_i$-modules

$$V_{i+1} = T_{\lambda_1}(V_i) \oplus \cdots \oplus T_{\lambda_s}(V_i),$$

where each λ_j is a Young diagram and $T_{\lambda_j}(V_i)$ is the image of the corresponding Young projector in the appropriate tensor power of V_i. Since V_{i+1} does not have any weight greater than k, each diagram λ_j has only one column. Indeed, otherwise we can put a vector of weight k in each box of the first row and put other weight vectors in all other boxes of λ_j so that the total sum of all weights of vectors is greater than k, which contradicts the fact that k is the maximal weight. Next we claim that the length of this column equals $0, 1, \dim V_i$, or $\dim V_i - 1$. Indeed, if we put in the boxes of λ_i linearly independent vectors of maximal possible sum of weights, the total sum is not greater than k only in these four cases. Hence each simple $\mathfrak{g}_i$-constituent of V_{i+1} is isomorphic to V_i, V_i^* or $\mathbb{C}$ (the numbers 0 and $\dim V_i$ correspond both to the trivial 1-dimensional $\mathfrak{g}_i$-module).

If each $\mathfrak{g}_i$ is of type B or C, D, let $\mathfrak{s}_i \subset \mathfrak{g}_i$ be a maximal root subalgebra of type A. Notice that by the previous argument the restriction of V_{i+1} on $\mathfrak{s}_i$ is a sum of natural, conatural and trivial modules. That is only possible if the restriction of V_{i+1} to $\mathfrak{g}_i$ is a sum of natural and trivial modules. $\square$

Proposition 4.3 follows also from Corollary 3.9 in [Ba2].

Example 4.4.

a) Let $\mathfrak{g} = sl(\infty)$, and let $M = \varinjlim S^i(V_i)$ be as in Example 3.8, a). Then $\mathrm{Hom}_{\mathfrak{g}_i}(S^k(V_i), S^j(V_j)) \neq 0$ for all i, $k \leq j$. Hence $\mathrm{Hom}_{\mathfrak{g}_i}(S^k(V_i), M) \neq 0$ for all $k > 0$, and by Lemma 4.1 M^* is not an object of $\mathrm{Int}_{\mathfrak{g}}$.

b) Consider the case $\mathfrak{g} = o(\infty)$ and let $S(\{t_i\})$ be the $\mathfrak{g}$-module defined in Example 3.8, b). Then if N is a simple $\mathfrak{g}_i$-module, $\mathrm{Hom}_{\mathfrak{g}_i}(N, S(\{t_i\})) \neq 0$ iff $N \simeq S_i^1$ or $N \simeq S_i^2$. Hence $S(\{t_i\})^* \in \mathrm{Int}_{\mathfrak{g}}$ by Lemma 4.1. Moreover, $S(\{t_i\})^*$ is injective in $\mathrm{Int}_{\mathfrak{g}}$ by Proposition 3.2.

c) Let $\mathfrak{g} = sl(\infty)$ and let M be as in Example 3.5. Then $\mathrm{Hom}_{\mathfrak{g}_i}(N, M) \neq 0$ if N is isomorphic to one of the following simple $\mathfrak{g}_i$-modules: trivial, natural, conatural, adjoint. Therefore, M^* is $\mathfrak{g}$-integrable and injective in $\mathrm{Int}_{\mathfrak{g}}$. Furthermore, $M^* \cong \mathbb{C} \oplus \mathfrak{g}^*$.

5. On the Loewy length of $\Gamma_{\mathfrak{g}}(M^*)$ for $M \in \mathrm{Int}_{\mathfrak{g}}$

Recall that the *socle*, $\mathrm{soc}(M)$, of a $\mathfrak{g}$-module M is the largest semisimple submodule of M. The *socle filtration* of M is the filtration of $\mathfrak{g}$-modules

$$0 \subset \mathrm{soc}(M) \subset \mathrm{soc}^1(M) \subset \cdots \subset \mathrm{soc}^i(M) \subset \ldots,$$

where $\mathrm{soc}^i(M) = p_i^{-1}(\mathrm{soc}(M/\mathrm{soc}^{i-1}(M))$ and $p_i : M \to M/\mathrm{soc}^{i-1}(M)$ is the natural projection. We say that the socle filtration of M is *exhaustive* if $M = \varinjlim(\mathrm{soc}^i(M))$. We say that M has *finite Loewy length* if the socle filtration of M is finite and exhaustive. The *Loewy length* of M equals $k+1$ where $k = \min\{r \mid \mathrm{soc}^r(M) = M\}$.

Proposition 5.1. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$ be a simple $\mathfrak{g}$-module such that $\Gamma_{\mathfrak{g}}(M^*)$ has finite Loewy length. Then there exist $n \in \mathbb{Z}_{>0}$ and a direct system M_i of simple finite-dimensional $\mathfrak{g}_i$-modules such that $M = \varinjlim M_i$ and $\dim \mathrm{Hom}_{\mathfrak{g}_i}(M_i, M_j) = 1$ for all $j > i > n$.*

We first prove several lemmas.

Lemma 5.2. *Let $Q = \varinjlim Q_i \in \mathrm{Int}_{\mathfrak{g}}$, where Q_i are finite-dimensional, not necessarily simple, $\mathfrak{g}_i$-modules. Assume that for all sufficiently large i there are simple $\mathfrak{g}_i$-submodules $X_i \subset Q_i$ such that $\dim \mathrm{Hom}_{\mathfrak{g}_i}(X_i, X_{i+1}) > 2$. Then there exists a locally simple module $X = \varinjlim X_i \in \mathrm{Int}_{\mathfrak{g}}$ and a non-trivial extension of $\mathfrak{g}$-modules*

$$0 \to Q \to Z \to X \to 0.$$

Proof. Fix a sequence of injective homomorphisms of $\mathfrak{g}_i$-modules $f_i : X_i \to X_{i+1}$ and set $X = \varinjlim X_i$. Let $Z_i := X_i \oplus Q_i$ and consider the injective homomorphisms of $\mathfrak{g}_i$-modules

$$a_i : Z_i \to Z_{i+1},\ a_i((x,q)) := (f_i(x), t_i(x) + e_i(q)),$$

where t_i are some injective homomorphisms $X_i \to Q_{i+1}$, $e_i : Q_i \to Q_{i+1}$ are the given inclusions, and $q \in Q_i$, $x \in X_i$. Put $Z := \varinjlim Z_i$.

Then, clearly, Q is a submodule of Z and the quotient Z/Q is isomorphic to X. Thus we have constructed an extension of X by Q. This extension splits if and only if for all sufficiently large i there exist non-zero homomorphisms $p_i : X_i \to Q_i$ such that $t_i = p_{i+1} \circ f_i - e_i \circ p_i$, see the following diagram:

$$\begin{array}{ccc} X_{i+1} & \stackrel{p_{i+1}}{\to} & Q_{i+1} \\ \uparrow f_i & t_i \nearrow & \uparrow e_i \\ X_i & \stackrel{p_i}{\to} & Q_i. \end{array}$$

Assume that for any choice of $\{t_i\}$ such a splitting exists. If $n_i := \dim \mathrm{Hom}_{\mathfrak{g}_i}(X_i, Q_i)$, this assumption implies

$$\dim \mathrm{Hom}_{\mathfrak{g}_i}(X_i, Q_{i+1}) \leq n_i + n_{i+1}.$$

On the other hand, $\dim \mathrm{Hom}_{\mathfrak{g}_i}(X_i, Q_{i+1}) \geq k_i n_{i+1}$ where $k_i := \dim \mathrm{Hom}_{\mathfrak{g}_i}(X_i, X_{i+1})$. Since $k_i > 2$, we have $n_{i+1} < n_i$. As $n_i > 0$ for all i, we obtain a contradiction. □

Corollary 5.3. *Let $Q \in \mathrm{Int}_{\mathfrak{g}}$ be a simple $\mathfrak{g}$-module satisfying the assumption of Lemma 5.2. Then Q admits no non-zero homomorphism into an injective object of $\mathrm{Int}_{\mathfrak{g}}$ of finite Loewy length.*

Proof. For any $m > 0$ we will now construct an integrable module $Z^{(m)} \supset Q$ whose socle equals Q and whose Loewy length is greater than m. For $m = 1$ this was done in Lemma 5.2. Proceeding by induction, we set

$$Z_i^{(m)} := X_i \oplus Z_i^{(m-1)} = X_i \oplus (X_i \oplus Z_i^{(m-2)})$$

and define $a_i^{(m)} : Z_i^{(m)} \to Z_{i+1}^{(m)}$ by

$$a_i^{(m)}(x, x', z) = (f_i(x), r_i^{(m-1)}(x) + f_i(x'), t_i^{(m-2)}(x') + q_i^{(m-2)}(z)),$$

where now $\{t_i^{(m-2)}\}$ is a set of non-zero homomorphisms $t_i^{(m-2)} : X_i \to Z_{i+1}^{(m-2)}$ and $\{r_i^{(m-1)}\}$ is a set of non-zero homomorphisms $r_i^{(m-1)} : X_i \to X_{i+1}$. As in the proof of Lemma 5.2 one can choose $\{t_i^{(m-2)}\}$ and $\{r_i^{(m-1)}\}$ so that $Z^{(m)}$ is a non-split extension of X by $Z^{(m-1)}$, and $Z^{(m)}/Z^{(m-2)}$ is a non-split self-extension of X. Therefore, the Loewy length of $Z^{(m)}$ is greater than m. The statement follows. □

Lemma 5.4. *Let $Q = \varinjlim Q_i \in \mathrm{Int}_{\mathfrak{g}}$ be a simple $\mathfrak{g}$-module which admits a non-zero homomorphism into an injective object of $\mathrm{Int}_{\mathfrak{g}}$ of finite Loewy length. Then there exist $n \in \mathbb{Z}_{>0}$ and a direct system of simple $\mathfrak{g}_i$-submodules S_i of Q such that $Q = \varinjlim S_i$ and $\dim \mathrm{Hom}_{\mathfrak{g}_i}(S_i, S_j) = 1$ for all $j > i > n$.*

Proof. Decompose each Q_i into a direct sum of isotypic components, $Q_i = Q_i^1 \oplus \cdots \oplus Q_i^{l(i)}$. We define a directed graph Γ as follows. The set of vertices $V(\Gamma)$ is by definition $\{Q_i^j\}$, and $V(\Gamma) = \cup_{i>0} V(\Gamma)_i$, where $V(\Gamma)_i = \{Q_i^1, \dots, Q_i^{l(i)}\}$. There is an edge $A \to B$ in Γ if $A \in V(\Gamma_i)$, $B \in V(\Gamma_{i+1})$ and $\mathrm{Hom}_{\mathfrak{g}_i}(A, B) \neq 0$.

Let $\Gamma_{>i}$ be the full subgraph of Γ whose set of vertices equals $\cup_{k>i} V(\Gamma)_k$. For any vertex A of Γ we denote by $V(A)$ the set of vertices B such that there is a directed path from A to B. Let $\Gamma(A)$ be the full subgraph of Γ whose set of vertices equals $V(A)$, and $\Gamma(A)_{>i}$ be the full subgraph of $\Gamma(A)$ whose set of vertices equals $\cup_{k>i}(V(\Gamma)_k \cap V(A))$. Note that the simplicity of Q implies that $\Gamma_{>i}$ and $\Gamma(A)_{>i}$ are connected (as undirected graphs). In particular, if $\Gamma(A)$ is a tree, then $\Gamma(A)$ is just a string.

We will now prove that there exists a vertex A such that $\Gamma(A)$ is a tree. Indeed, assume the contrary. This implies that one can find an infinite sequence of vertices $A_1 \in V(\Gamma)_{i_1}, A_2 \in V(\Gamma)_{i_2}, \dots$ such that the number of paths from A_n to A_{n+1} is greater than 2 for all n. Then $Q = \varinjlim Q_{i_k}$. In addition, one can easily see that Q satisfies the assumption of Lemma 5.2 and hence Q admits no non-zero homomorphism into an injective object of $\mathrm{Int}_{\mathfrak{g}}$ of finite Loewy length. Contradiction.

Fix now $A \in V(\Gamma)_i$ such that $\Gamma(A)$ is a tree. Then, as we mentioned above, $V(\Gamma)$ is necessarily a string $A_i = \{A \to A_{i+1} \to A_{i+2} \dots\}$. Let S_j be a simple submodule of A_j, $j \geq i$. By Lemma 5.2 there exists n, such that $\dim \mathrm{Hom}_{\mathfrak{g}_j}(S_j, S_k) = 1$ for any $k > j \geq n$. Fix $s \in S_n$ and set $S_j = U(\mathfrak{g}_j) \cdot s$ for all $j \geq n$. Then S_j are simple and $Q = \varinjlim S_j$ satisfies the condition in the lemma. □

Lemma 5.5. *Let $Q = \varinjlim S_i \in \mathrm{Int}_{\mathfrak{g}}$, where S_i are simple $\mathfrak{g}_i$-modules such that, for some n, $\dim \mathrm{Hom}_{\mathfrak{g}_i}(S_i, S_j) = 1$ for all $j > i > n$. Then Q^* has a unique simple submodule Q_*, and $Q_* \in \mathrm{Int}_{\mathfrak{g}}$.*

Proof. The condition on Q implies that $\dim \mathrm{Hom}_{\mathfrak{g}_i}(S_i, Q) = 1$ for all sufficiently large i. Therefore, $\dim \mathrm{Hom}_{\mathfrak{g}_i}(S_i^*, Q^*) = 1$ for all sufficiently large i. Note also that $Q_* = \varinjlim S_i^*$ is uniquely defined (as $\dim \mathrm{Hom}_{\mathfrak{g}_i}(S_i, S_{i+1}) = 1$) and is a simple integrable submodule of Q^*. Let S be some simple submodule of Q^*. Since $Q^* = \varprojlim S_i^*$ and $\mathrm{Hom}_{\mathfrak{g}}(S, Q^*) \neq 0$, we have $\mathrm{Hom}_{\mathfrak{g}_i}(S, S_i^*) \neq 0$ for some i. Therefore, $S_i^* \subset S$ as the multiplicity of S_i^* in Q^* is 1. This implies $S = Q_*$. □

We are now ready to prove Proposition 5.1.

Proof of Proposition 5.1. Fix $0 \neq m \in M$ and put $M_i := U(\mathfrak{g}_i) \cdot m$. Then, by the simplicity of M, we have $M = \varinjlim M_i$. Since $\Gamma_{\mathfrak{g}}(M^*)$ has finite Loewy length, M^* has a simple submodule Q. By Lemma 5.4, Q satisfies the assumption of Lemma 5.5. The composition of the canonical injection $M \to (M^*)^*$ and the dual map $(M^*)^* \to Q^*$ defines an injective homomorphism $M \to Q^*$. By Lemma 5.5 $M \simeq Q_*$ and, since Q_* also satisfies the assumption of Lemma 5.5, we conclude that the claim of Proposition 5.1 holds for M.□

The following statement is a direct consequence of Proposition 5.1.

Corollary 5.6. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$ be a simple $\mathfrak{g}$-module such that $\Gamma_{\mathfrak{g}}(M^*)$ has finite Loewy length. Then for any sufficiently large i there exists a simple $\mathfrak{g}_i$-module N such that $\dim \mathrm{Hom}_{\mathfrak{g}_i}(N, M) = 1$.*

The next corollary is a direct consequence of Lemma 5.5 and Proposition 5.1.

Corollary 5.7. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$ be a simple $\mathfrak{g}$-module such that $\Gamma_{\mathfrak{g}}(M^*)$ has finite Loewy length. Then M^* has a unique simple submodule M_*, and $M_* \in \mathrm{Int}_{\mathfrak{g}}$.*

Theorem 5.8. *Let $\mathfrak{g}$ be a locally simple Lie algebra which has a non-trivial module $M \in \mathrm{Int}_{\mathfrak{g}}$ such that M^* is integrable and has finite Loewy length, then $\mathfrak{g}$ is isomorphic to $sl(\infty)$, $o(\infty)$ or $sp(\infty)$.*

Proof. By Proposition 4.3 we know that $\mathfrak{g}$ is diagonal. Assume that $\mathfrak{g}$ is not finitary and there exists M satisfying the conditions of the theorem. Also assume that in the restriction of V_i to $\mathfrak{g}_{i-1}$ there is no costandard module (for types B, C and D it is automatic). Let $\mathfrak{g} = \varinjlim \mathfrak{g}_i$. Fix n and let $\varphi_k : \mathfrak{g}_n \to \mathfrak{g}_{n+k}$ denote the inclusion defined by our fixed exhaustion of $\mathfrak{g}$. Since $\mathfrak{g}$ is diagonal, there exists a root subalgebra $\mathfrak{l}_k \subset \mathfrak{g}_{n+k}$ such that $\mathfrak{l}_k \simeq \mathfrak{g}_n \oplus \cdots \oplus \mathfrak{g}_n$ and $\varphi_k(\mathfrak{g}_n)$ is the diagonal subalgebra in $\mathfrak{l}_k$. Let a_k be the number of simple direct summands in $\mathfrak{l}_k$. Since $\mathfrak{g}$ is not finitary, $a_k \to \infty$.

Note next that our condition on M implies that M admits a simple subquotient whose dual is integrable and of finite Loewy length. Therefore, without loss of generality, we may assume that M is simple. Then, by Corollary 5.6 $M = \varinjlim M_i$ is a direct limit of simple modules and, by possibly increasing n, we have $\dim \mathrm{Hom}_{\mathfrak{g}_n}(M_n, M_{n+k}) = 1$ for all k. Choose a set of Borel subalgebras

$\mathfrak{b}_i \subset \mathfrak{g}_i$ such that $\varphi_k(\mathfrak{b}_n) \subset \mathfrak{b}_{n+k}$. Let h be the highest coroot of $\mathfrak{g}_n$ and let λ be the highest weight of some simple $\mathfrak{l}_k$-constituent L of M_{n+k}. Since M^* is integrable, Lemma 4.1 implies that $\lambda(\varphi_k(h))$ is bounded by some number t. If $h_1, \dots, h_{a_k}$ are the images of $\varphi_k(h)$ in the simple direct summands of $\mathfrak{l}_k$ under the natural projections, we have $\lambda(h_j) \neq 0$ for at most t direct summands. Therefore, L isomorphic to an outer tensor product of at most t non-trivial simple $\mathfrak{g}_n$-modules. Since M_{n+k} is invariant under permutation of direct summands of $\mathfrak{l}_k$, we have at least $a_k - t$ simple constituents of M_{n+k} obtained from L by permutation of the simple direct summands of $\mathfrak{l}_k$. Note that all these simple constituents are isomorphic as $\varphi_k(\mathfrak{g}_n)$-modules. Thus the multiplicity of any simple $\varphi_{n+k}(\mathfrak{g}_n)$-module in M_{n+k} is at least $a_k - t$. Since $a_k \to \infty$, this contradicts Proposition 5.1.

The case when the restriction of V_n to $\mathfrak{g}_{n-1}$ contains a costandard simple constituent can be handled by a similar argument which we leave to the reader. $\square$

6. The category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ for $\mathfrak{g} \simeq sl(\infty), o(\infty), sp(\infty)$

Define $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ as the largest full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ which is closed under algebraic dualization and such that every object in it has finite Loewy length.

It is clear that $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is closed with respect to finite direct sums, however $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is not closed with respect to arbitrary direct sums (see Corollary 6.17 below). Note also that, if $\mathfrak{g}$ is finite-dimensional and semisimple, the objects of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are integrable modules which have finitely many isotypic components.

It follows from Theorem 5.8 that if $\mathfrak{g}$ is locally simple and $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ contains a non-trivial module, then $\mathfrak{g}$ is finitary.

In the rest of this section we assume that $\mathfrak{g} \simeq sl(\infty)$, $o(\infty)$ or $sp(\infty)$. Set $T^{p,q} := V^{\otimes p} \otimes (V_*)^{\otimes q}$, where V and V_* are respectively the natural and conatural $\mathfrak{g}$-modules ($V_* \simeq V$ when $\mathfrak{g} \simeq o(\infty), sp(\infty)$). The modules $T^{p,q}$ have been studied in [PS]; in particular, $T^{p,q}$ has finite length and is semisimple only if $pq = 0$ for $\mathfrak{g} = sl(\infty)$, and if $p + q \leq 1$ for $\mathfrak{g} = o(\infty), sp(\infty)$. Moreover, the Loewy length of $T^{p,q}$ equals $\min\{p, q\}+1$ for $\mathfrak{g} = sl(\infty)$ and $[\frac{p+q}{2}]+1$ for $\mathfrak{g} = o(\infty), sp(\infty)$. A simple module M is called a *simple tensor module* if it is a submodule (or, equivalently, a subquotient) of $T^{p,q}$ for some p, q.

It is well known that there is a choice of nested Borel subalgebras $\mathfrak{b}_i \subset \mathfrak{g}_i$ such that all simple tensor modules are $\mathfrak{b}$-highest weight modules for $\mathfrak{b} = \varinjlim \mathfrak{b}_i$, see [PS]. (Moreover, the positive roots of any such $\mathfrak{b}$ are not generated by the simple roots of $\mathfrak{b}$. However, in the present paper we will make no further reference to this fact.)

Denote by Θ the set of all highest weights of simple tensor modules. If $\lambda \in \Theta$, by V_λ we denote the simple tensor module with highest weight λ, and, as in section 4, by V^i_λ we denote the simple $\mathfrak{g}_i$-highest weight module with highest weight λ (here λ is considered as a weight of $\mathfrak{g}_i$). It is clear that every $\lambda \in \Theta$ can be written in the form $\lambda = \sum a_i \gamma_i$ for some finite set $\gamma_1, ..., \gamma_s$ of linearly independent weights of V

and some $a_i \in \mathbb{Z}$ (see [PS] for an explicit description of Θ). We put $|\lambda| := \sum |a_i|$. It is not hard to see that for any k the set of all $\mu \in \Theta$ with $|\mu| \leq k$ is finite. Moreover, all simple subquotients of $T^{p,q}$ are isomorphic to V_μ with $|\mu| \leq p+q$, and it follows from [PS] that if V_λ is a submodule in $T^{p,q}$ then $|\lambda| = p+q$.

Note that $(T^{p,q})^*, (T^{p,q})^{**}$, etc., are integrable modules. Indeed, it is easy to see (cf. [PS]) that for any fixed λ and any fixed $i > 0$ the non-vanishing of $\mathrm{Hom}_{\mathfrak{g}_i}(N, V_\lambda)$ for a simple $\mathfrak{g}_i$-module N implies $N \simeq V^i_\mu$ for $|\mu| \leq |\lambda|$. Hence the condition of Lemma 4.1 is satisfied for $T^{p,q}$ for fixed p, q. This shows that $(T^{p,q})^* \in \mathrm{Int}_{\mathfrak{g}}$. By Corollary 4.2, $(T^{p,q})^{**} \in \mathrm{Int}_{\mathfrak{g}}$, etc..

Lemma 6.1. *Fix $p, q \in \mathbb{Z}_{\geq 0}$.*

a) $(T^{p,q})^*$ has finite Loewy length, and all simple subquotients of $(T^{p,q})^*$ are tensor modules of the form V_λ for $|\lambda| \leq p+q$.

b) The direct product $\prod_{f\in\mathcal{F}} T^{p,q}_f$ of any family $\{T^{p,q}_f\}_{f\in\mathcal{F}}$ of copies of $T^{p,q}$ has finite Loewy length, and all simple subquotients of $\prod_{f\in\mathcal{F}} T^{p,q}_f$ are tensor modules of the form V_λ for $|\lambda| \leq p+q$.

Proof. First we prove b) using induction in $p+q$. The case $p+q=0$ is trivial. If $p+q>0$, without loss of generality we can assume that $p>0$ (if $p=0$ and $q>0$ we replace V by V_* in the argument below). There is a canonical injective homomorphism $U \to \prod_{f\in\mathcal{F}} T^{p,q}_f$, where $U := V \otimes \prod_{f\in\mathcal{F}} T^{p-1,q}_f$, so we can consider U as a submodule of $\prod_{f\in\mathcal{F}} T^{p,q}_f$. By the induction assumption b) holds for $\prod_{f\in\mathcal{F}} T^{p-1,q}_f$. Since $T^{r,s}$ has finite length for all r, s, [PS], this implies that U has finite Loewy length and all simple subquotients of U are simple tensor modules of the form V_λ for $|\lambda| \leq p+q$. The quotient $(\prod_{f\in\mathcal{F}} T^{p,q}_f)/U$ is isomorphic to a submodule of $R := \prod_{f\in\mathcal{F}} (V' \otimes T^{p-1,q}_f)$, where V' is a copy of the vector space V with trivial $\mathfrak{g}$-module structure. Since $R \simeq \prod_{f\in\mathcal{F}} (\bigoplus_{i\in\mathbb{Z}} T^{p-1,q}_{f,i})$, by the induction assumption b) holds for R. Therefore, b) holds for $\prod_{f\in\mathcal{F}} T^{p,q}_f$.

a) To prove that $(T^{p,q})^*$ has finite Loewy length, we consider $U' := V_* \otimes (T^{p-1,q})^*$ as a submodule of $(T^{p,q})^*$. By the induction assumption, U' has finite Loewy length. The quotient $(T^{p,q})^*/U'$ is a submodule of $R' = \prod_{i\in\mathbb{Z}} (T^{p-1,q}_i)^*$. The latter $\mathfrak{g}$-module has finite Loewy length by the induction assumption and b). The statement about the simple subquotients of $(T^{p,q})^*$ follows by an induction argument similar to the one in the proof of b). This proves a) for $(T^{p,q})^*$. $\square$

Example 6.2.

a) We start with the simplest example. Let $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$ and $M = V^* = (T^{1,0})^*$. Then $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ by Lemma 6.1. Furthermore, M is an injective object of $\mathrm{Int}_{\mathfrak{g}}$ by Proposition 3.2. It is easy to see that $\mathrm{soc}(M) = V_*$ and that $M/\mathrm{soc}(M) = V^*/V_*$ is a trivial module of cardinality $\beth_1$. Since $\mathrm{soc}(M)$ is simple, M is an injective hull of V_*.

b) Let $\mathfrak{g}$ be as in a) but let now $M = V^{**} = (T^{1,0})^{**}$. The exact sequence

$$0 \to V_* \to V^* \to V^*/V_* \to 0$$

yields an exact sequence

$$0 \to (V^*/V_*)^* \to M \to (V_*)^* \to 0. \tag{5}$$

Since $(V^*/V_*)^*$ is a trivial $\mathfrak{g}$-module (cf. a)), it is injective, and hence (5) splits. This yields an isomorphism $M = V^{**} = (V_*)^* \oplus T$, T being a trivial $\mathfrak{g}$-module of cardinality $\beth_2$.

c) Here is a more interesting example. We consider the $\mathfrak{g}$-module M^* where $\mathfrak{g} = \mathrm{sl}(\infty)$ and $M = V \otimes V_* = T^{1,1}$ as in Example 3.5. Recall the notation introduced in Example 3.5. In addition, let Sc be the one-dimensional space of scalar matrices, and F_r (respectively F_c) denote respectively the spaces of matrices with finitely many non-zero rows (resp., columns) (F has codimension 1 in $F_r \cap F_c$). It is important to notice that $\mathfrak{g} \cdot M^* \subset F_r + F_c$.

We first show that $\mathrm{soc}(M^*) = Sc \oplus F = \mathbb{C} \oplus \mathfrak{g}$. It is obvious that $Sc \oplus F \subset \mathrm{soc}(M^*)$ and that Sc is the largest trivial $\mathfrak{g}$-submodule of M^*. To see that $Sc \oplus F = \mathrm{soc}(M^*)$, let X be any non-trivial simple submodule of $\mathrm{soc}(M^*)$. Consider $0 \neq x \in X$. Then $\mathfrak{g} \cdot x \subset F_r + F_c$. Furthermore, it is easy to check that for any $0 \neq y \in F_r + F_c$, there exists $A \in \mathfrak{g}$ such that $A \cdot y \in F$ and $A \cdot y \neq 0$. Hence $X = F$, and we have shown that $\mathrm{soc}(M^*) = Sc \oplus F$.

We now compute $\mathrm{soc}^1(M^*)$. We claim that $F_r + F_c \subset \mathrm{soc}^1(M^*)$. Since $BA \in V \otimes V_*$ for $B \in F_r$, $A \in F$, the action of $\mathfrak{g}$ on F_r/F is simply left multiplication. Using this it is not difficult to establish an isomorphism of $\mathfrak{g}$-modules $F_r/F \simeq \bigoplus_{q \in Q} V_q$, where Q is a family of copies of V of cardinality $\beth_1$. Similarly, $F_c/F \simeq \bigoplus_{q \in Q} (V_*)_q$. (It is convenient to think here of V_* as the space of all row vectors each of which have finitely many non-zero entries.) This implies $F_r + F_c \subset \mathrm{soc}^1(M^*)$.

Furthermore, $M^*/(F_r + F_c)$ is a trivial $\mathfrak{g}$-module as $\mathfrak{g} \cdot M^* \subset F_r + F_c$. Therefore, in order to compute $\mathrm{soc}^1(M^*)$ we need to find all $z \in M^*$ such that $\mathfrak{g} \cdot z \subset Sc + F$. A direct computation shows that $\mathfrak{g} \cdot z \in Sc + F$ if and only $z \in J$, where J denotes the set of matrices each row and each column of which have finitely many non-zero elements. (In fact, $\mathfrak{g} \cdot J \subset F$). Thus $\mathrm{soc}^1(M^*) = F_r + F_c + J$, and we obtain the socle filtration of M^*:

$$0 \subset Sc \oplus F \subset F_r + F_c + J \subset M^*.$$

In particular, the Loewy length of M^* equals 3, the irreducible subquotients of M^* up to isomorphism are $\mathbb{C}, V, V_*, \mathfrak{g}$, and all of them occur with multiplicity $\beth_1$, except $\mathfrak{g}$ which occurs with multiplicity 1.

Note that M^* is decomposable and is isomorphic to $\mathbb{C} \oplus \mathfrak{g}^*$. As the socle of $\mathfrak{g}^*$ is simple (being isomorphic to $\mathfrak{g}$), $\mathfrak{g}^*$ is indecomposable. Moreover $\mathfrak{g}^*$ is an injective hull of $F = \mathfrak{g}$.

d) We now give an example illustrating statement b) of Lemma 6.1. Let $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$ and $M = \prod_{f\in\mathcal{F}} V_f$, $\{V_f\}_{f\in\mathcal{F}}$ being an infinite family of copies of the natural module V. Set $M^{\mathrm{fin}} = \{\psi : \mathcal{F} \to V \,|\, \dim(\psi(\mathcal{F})) < \infty\}$. Then M^{fin} is a $\mathfrak{g}$-submodule of M, and $\mathfrak{g}\cdot M \subset M^{\mathrm{fin}}$. Hence M/M^{fin} is a trivial $\mathfrak{g}$-module. Moreover, $M^{\mathrm{fin}} \simeq \bigoplus_{g\in 2^{\mathcal{F}}} V_g$, where $2^{\mathcal{F}}$ is the set of subsets of $\mathcal{F}$. Indeed,

$$M^{\mathrm{fin}} = \varinjlim\Big(\prod_{f\in\mathcal{F}} (V^i)_f\Big) = \varinjlim\Big(\Big(\prod_{f\in\mathcal{F}} \mathbb{C}_f\Big) \otimes V^i\Big) \cong \varinjlim \bigoplus_{g\in 2^{\mathcal{F}}} (\mathbb{C}_g \otimes V^i) =$$

$$\varinjlim\Big(\bigotimes_{g\in 2^{\mathcal{F}}} (V^i)_g\Big) = \bigoplus_{g\in 2^{\mathcal{F}}} V_g.$$

This yields an exact sequence

$$0 \to \bigoplus_{g\in 2^{\mathcal{F}}} V_g \to M \to T \to 0, \tag{6}$$

T being trivial module of dimension $\mathrm{card}\, 2^{\mathcal{F}}$. Since M has no non-zero trivial submodules, (6) is in fact the socle filtration of M. Consequently the Loewy length of M equals 2.

Corollary 6.3. *Let $M \in \mathrm{Int}_{\mathfrak{g}}$ have finite Loewy length and all simple subquotients of M be isomorphic to V_λ where $|\lambda|$ is less or equal than a fixed $k \in \mathbb{Z}_{>0}$. Then*

a) for any family $\{M_f\}_{f\in\mathcal{F}}$ of copies of M, the $\mathfrak{g}$-module $\prod_{f\in\mathcal{F}} M_f$ has finite Loewy length and all simple subquotients of $\prod_{f\in\mathcal{F}} M_f$ are isomorphic to V_λ with $|\lambda| \leq k$;

b) M^* has finite Loewy length and all simple subquotients of M^* are isomorphic to V_λ with $|\lambda| \leq k$;

c) $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$.

Proof. a) The socle filtration of M induces a finite filtration on $\prod_{f\in\mathcal{F}} M_f$

$$0 \subset \prod_{f\in\mathcal{F}} \mathrm{soc}(M_f) \subset \cdots \subset \prod_{f\in\mathcal{F}} \mathrm{soc}^i(M_f) \subset \cdots \subset \prod_{f\in\mathcal{F}} M_f.$$

Furthermore,

$$\mathrm{soc}^i(M)/\mathrm{soc}^{i-1}(M) \simeq \bigoplus_{|\lambda|\leq k} \bigoplus_{g\in\mathcal{F}_\lambda} (V_\lambda)_g \tag{7}$$

for some families $\{(V_\lambda)_g\}_{g\in\mathcal{F}_\lambda}$ of copies of V_λ. Hence

$$\prod_{f\in\mathcal{F}} (\mathrm{soc}^i(M_f)/\mathrm{soc}^{i-1}(M_f)) \simeq \bigoplus_{|\lambda|\leq k} \prod_{f\in\mathcal{F}} \Big(\bigoplus_{g\in\mathcal{F}_\lambda} (V_\lambda)_g\Big)_f.$$

Note that for each λ

$$\prod_{f\in\mathcal{F}}(\bigoplus_{g\in\mathcal{F}_\lambda}(V_\lambda)_g)_f \subset \prod_{(f,g)\in\mathcal{F}\times\mathcal{F}_\lambda}(V_\lambda)_{(f,g)}.$$

By Lemma 6.1 b), the $\mathfrak{g}$-module $\prod_{(f,g)\in\mathcal{F}\times\mathcal{F}_\lambda}(V_\lambda)_{(f,g)}$ has finite Loewy length and all its simple subquotients are isomorphic ot V_μ with $|\mu| \leq |\lambda| \leq k$. The same holds for $\prod_{f\in\mathcal{F}}(\mathrm{soc}^i(M_f)/\mathrm{soc}^{i-1}(M_f))$. Therefore, a) holds.

b) Since all V_λ with $|\lambda| \leq k$ satisfy the conditions of Lemma 4.1, M satisfies the condition of Lemma 4.1 and therefore $M^* \in \mathrm{Int}_\mathfrak{g}$.

The socle filtration of M induces a finite filtration on M^*

$$\cdots \subset \mathrm{soc}^i(M)^* \subset \mathrm{soc}^{i-1}(M)^* \subset \cdots.$$

Using (7) we get

$$\mathrm{soc}^{i-1}(M)^*/\mathrm{soc}^i(M)^* \simeq \bigoplus_{|\lambda|\leq k}\prod_{g\in\mathcal{F}_\lambda}(V_\lambda^*)_g.$$

By Lemma 6.1 b) V_λ^* has finite Loewy length and its simple subquotients are isomorphic to V_μ with $|\mu| \leq |\lambda|$, hence by a) the same holds for $\prod_{g\in\mathcal{F}_\lambda}(V_\lambda^*)_g$. This implies that b) holds.

c) Note that if M satisfies the assumptions of the corollary, then M^* and all higher duals, M^{**} etc., satisfy the assumptions of the corollary. Hence $M \in \widetilde{\mathrm{Tens}}_\mathfrak{g}$. □

Remarkably, there is following abstract characterization of simple tensor modules.

Theorem 6.4. *If $M \in \mathrm{Int}_\mathfrak{g}$ is simple and $\Gamma_\mathfrak{g}(M^*)$ has finite Loewy length, then M is a simple tensor module.*

Proof. By Proposition 5.1, $M = \varinjlim M_i$ for some $n \in \mathbb{Z}_+$ and simple nested $\mathfrak{g}_i$-submodules $M_i \subset M$ with $\dim \mathrm{Hom}_{\mathfrak{g}_i}(M_i, M) = 1$ for all $i \geq n$. If $\mathfrak{g} = sl(\infty)$, it is useful to consider M as a $gl(\infty)$-module by extending the $sl(i)$-module structure on M_i to a $gl(i)$-module structure in a way compatible with the injections $M_i \to M_{i+1}$. It is easy to see that the condition $\dim \mathrm{Hom}_{\mathfrak{g}_i}(M_i, M) = 1$ for all $i \geq n$ ensures the existence of such an extension. Note, furthermore, that $\dim \mathrm{Hom}_{gl(i)}(M_i, M) = 1$. This allows us to assume that $\mathfrak{g} = gl(\infty)$ and $\mathfrak{g}_i = gl(i)$.

Let now $\mathfrak{c}$ denote the derived subalgebra of the centralizer of $\mathfrak{g}_n$ in $\mathfrak{g}$. Then obviously $\mathfrak{c}$ is a simple finitary Lie algebra whose action on M induces a trivial action on M_n. Hence, as a $\mathfrak{c}$-module, M is isomorphic to a quotient of $U(\mathfrak{g})\otimes_{U(\mathfrak{c}\oplus\mathfrak{g}_n)} M_n$, or equivalently to a quotient of $S^\cdot(\mathfrak{g}/(\mathfrak{c}\oplus\mathfrak{g}_n)) \otimes M_n$. Note that $\mathfrak{g}/(\mathfrak{c}\oplus\mathfrak{g}_n)$, considered as a $\mathfrak{c}$-module has finite length and that its simple subquotients are natural, conatural, and possibly 1-dimensional trivial $\mathfrak{c}$-modules. This implies that every simple $\mathfrak{c}$-subquotient of M is a simple tensor $\mathfrak{c}$-module. In addition, for $i \geq n$, the number of non-zero marks of the highest weight of any simple $\mathfrak{g}_i$-submodule of M is not greater than n plus the multiplicity of the non-trivial simple constituents of the $\mathfrak{g}_n$-module $\mathfrak{g}/(\mathfrak{c} \oplus \mathfrak{g}_n)$. In particular, if λ_i denotes the highest weight of M_i then λ_i has at most $3n$ non-zero marks.

Consider first the case when $\mathfrak{g} = gl(\infty)$. Then every weight λ_i can be written uniquely in the form

$$a_1^i \varepsilon_1 + \cdots + a_k^i \varepsilon_k + b_1^i \varepsilon_{n-k} + \cdots + b_k^i \varepsilon_n$$

for some fixed k, $a_1^i \geq a_2^i \geq \cdots \geq a_k^i \geq 0$ and $0 \geq b_1^i \geq \cdots \geq b_k^i$. We claim that for sufficiently large i the weight stabilizes, i.e. $a_j^i = a_j^{i+1} = \cdots = a_j^p = \ldots$ and $b_j^i = b_j^{i+1} = \cdots = b_j^p = \ldots$ for all j, $1 \leq j \leq k$. Indeed, assume the contrary. Let j be the smallest index such that the sequence $\{a_j^i\}$ does not stabilize. By the branching rule for $gl(m) \subset gl(m+1)$ (see for instance [GW]) the sequence $\{a_j^i\}$ is non-decreasing. Hence there is p such that $a_j^{p+1} > a_j^p$. Set $\mu = \lambda_p + \varepsilon_j$. Then the multiplicity of M_{p-1} in V_μ^p is not zero and the multiplicity of V_μ^p in M_{p+1} is not zero. Since $V_\mu^p \neq M_p$, this shows that the multiplicity of M_{p-1} in M_{p+1} is at least 2. Contradiction. Similarly the sequence $\{b_j^i\}$ stabilizes. As it is easy to see, this is sufficient to conclude that $M \simeq V_\lambda$ for some $\lambda \in \Theta$.

Let $\mathfrak{g} = o(\infty)$ or $sp(\infty)$. In the first case we assume that $\mathfrak{g}_i = o(2i+1)$. Then $\lambda_i = a_1^i \varepsilon_1 + \cdots + a_k^i \varepsilon_k$ for some fixed k and $a_1^i \geq a_2^i \geq \cdots \geq a_k^i \geq 0$. The sequence $\{a_j^i\}$ is non-decreasing for every fixed j as follows from the branching laws for the respective pairs $o(2m+1) \subset o(2m+3)$ and $sp(2n) \subset sp(2m+2)$, see [GW]. Then by repeating the argument in the previous paragraph we prove that $\{a_j^i\}$ stabilizes, and consequently $M \simeq V_\lambda$ for some $\lambda \in \Theta$. □

Corollary 6.3 and Theorem 6.4 show that a simple module $M \in \mathrm{Int}_{\mathfrak{g}}$ is an object of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ if and only if $\Gamma_{\mathfrak{g}}(M^*)$ has finite Loewy length. Below we will use this fact to give an equivalent definition of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ (Corollary 6.13). Furthermore, it is easy to check (see also [PS]) that for sufficiently large i the simple $\mathfrak{g}_i$-module V_λ^i occurs in Y with multiplicity 1, and all other simple $\mathfrak{g}_i$-constituents have infinite multiplicity and are isomorphic to V_μ^i with $|\mu| < |\lambda|$. In what follows we call this unique $\mathfrak{g}_i$-constituent the *canonical $\mathfrak{g}_i$-constituent of* V_λ. Note also that by Corollary 5.7 for each simple object M of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$, M_* is a well-defined simple object in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. Hence M_* is well defined also for any semisimple object M of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$: if $M = \bigoplus_{\lambda \in \Theta} M^\lambda \otimes V_\lambda$ (M^λ being trivial $\mathfrak{g}$-modules), then $M_* = \bigoplus_{\lambda \in \Theta} M^\lambda \otimes (V_\lambda)_*$. It is clear that $M_* \cong M$ for $\mathfrak{g} \cong o(\infty)$, $sp(\infty)$.

Corollary 6.5. *The simple objects of* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *are precisely the simple tensor modules.*

Lemma 6.6. *Let $M \cong V_\lambda$ be a simple tensor module. Then* $\mathrm{soc}((M_*)^*) \simeq M$*. If V_μ is a subquotient of $(M_*)^*$ and $\mu \neq \lambda$, then $|\mu| < |\lambda|$.*

Proof. The first statement follows from Corollary 5.7.

The second statement follows immediately from the fact that $\mathrm{Hom}_{\mathfrak{g}_i}(V_\mu^i, (M_*)^*) \neq 0$ implies $|\mu| < |\lambda|$. □

Corollary 6.7. *a) For any simple* $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$, $(M_*)^*$ *is an injective hull of* M *in* $\mathrm{Int}_{\mathfrak{g}}$ *(and hence also in* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$*).*

b) Any indecomposable injective object in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *is isomorphic to* M^* *for some simple module* $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$*. In particular, any indecomposable injective module is isomorphic to a direct summand of* $(T^{p,q})^*$ *for some* p, q*.*

c) For any $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$*, any injective hull* I_M *of* M *in* $\mathrm{Int}_{\mathfrak{g}}$ *is an object of* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$*.*

Proof. a) Follows directly from Proposition 3.2 and Lemma 6.6.

b) To derive b) from a) it suffices to note that an injective module in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is indecomposable if and only if it has simple socle.

c) follows from the fact that I_M is isomorphic to a submodule of $\Gamma_{\mathfrak{g}}(M^{**})$, see Corollary 3.3. □

In what follows we set $I_\lambda := ((V_\lambda)_*)^*$.

Corollary 6.8. $\mathrm{End}_{\mathfrak{g}}(I_\lambda) = \mathbb{C}$.

Proof. If $\varphi \in \mathrm{End}_{\mathfrak{g}}(I_\lambda)$, then $\varphi|_{V_\lambda} = c\,\mathrm{Id}$ for $c \in \mathbb{C}$. Therefore, $V_\lambda \subset \mathrm{Ker}(\varphi - c\,\mathrm{Id})$. Furthermore, any non-zero $\mathfrak{g}$-submodule of I_λ contains $\mathrm{soc}(I_\lambda) = V_\lambda$, hence $V_\lambda \subset \mathrm{Im}(\varphi - c\,\mathrm{Id})$. This implies $\varphi - c\,\mathrm{Id} = 0$, as otherwise V_λ would be isomorphic to a subquotient of I_λ/V_λ contrary to Lemma 6.6. □

Lemma 6.9. *Let* $X, Y, Z, M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$*. Assume furthermore that* Y *is simple,* $Y = \mathrm{soc}(M)$*, and there exists an exact sequence*

$$0 \to X \to Z \xrightarrow{p} Y \to 0.$$

Then there exists $\tilde{M} \in \mathrm{Int}_{\mathfrak{g}}$ *such that* $Z \subset \tilde{M}$ *and* $\tilde{M}/X \simeq M$*.*

Proof. Let Y_i be the canonical $\mathfrak{g}_i$-constituent of Y. Then $Y = \varinjlim Y_i$. Set $Z_i := p^{-1}(Y_i)$ and $Q_i := Z_i \cap X$. Then $Z_i = Y_i \oplus Q_i$ and there are injective homomorphisms $\varphi_i : Z_i \to Z_{i+1}$

$$\varphi_i(y, q) = (e_i(y), t_i(y) + f_i(q)),\ \ y \in Y_i, q \in Q_i$$

for some non-zero homomorphisms $e_i : Y_i \to Y_{i+1}$, $t_i : Y_i \to Q_{i+1}$ and $f_i : Q_i \to Q_{i+1}$. Clearly, $Z = \varinjlim Z_i$.

On the other hand, $M = \varinjlim M_i$ for some nested finite-dimensional $\mathfrak{g}_i$-submodules $M_i \subset M$ such that $Y_i \subset M_i$. Moreover, $\dim \mathrm{Hom}_{\mathfrak{g}_i}(Y_i, M_i) = 1$ by Lemma 6.6. Therefore, M_i has a unique $\mathfrak{g}_i$-module decomposition $M_i = R_i \oplus Y_i$. The inclusions $\psi_i : M_i \to M_{i+1}$ are given by

$$\psi_i(r, y) = (p_i(r), s_i(r) + e_i(y)),\ \ y \in Y_i, r \in R_i$$

for some non-zero homomorphisms $p_i : R_i \to R_{i+1}$ and $s_i : R_i \to Y_{i+1}$.

Define $\tilde{M}_i := R_i \oplus Y_i \oplus Q_i$ and let $\zeta_i : \tilde{M}_i \to \tilde{M}_{i+1}$ be given by the formula

$$\zeta(r, y, q) = (p_i(r), s_i(r) + e_i(y), t_i(y) + f_i(q)).$$

Set $\tilde{M} := \varinjlim \tilde{M}_i$. It is easy to check that $\tilde{M}$ satisfies the conditions of the lemma. $\square$

Lemma 6.10. *If* $\mathrm{Hom}_{\mathfrak{g}}(I_\lambda, I_\mu) \neq 0$*, then* $|\mu| \leq |\lambda|$*. If* I *is any injective object of* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *and* $0 \neq \varphi \in \mathrm{Hom}_{\mathfrak{g}}(I, I_\mu)$*, then* φ *is surjective.*

Proof. The first statement follows immediately from Lemma 6.6.

To prove the second statement put $X = \mathrm{Ker}\varphi$, $Y = V_\mu$, $Z = \varphi^{-1}(Y)$ and $M = I_\mu$. Construct $\tilde{M}$ as in Lemma 6.9. By the injectivity of I, the injective homomorphism $Z \to \tilde{M}$ extends to a homomorphism $\tilde{M} \to I$. The latter induces a homomorphism $\eta : M = I_\mu \to I/X$.

Let now $\bar{\varphi} : I/X \to I_\mu$ denote the injective homomorphism induced by φ. Then it is obvious that $\bar{\varphi} \circ \eta(y) = y$ for any $y \in Y$. By Corollary 6.8, we have $\bar{\varphi} \circ \eta = \mathrm{Id}$. Hence $\bar{\varphi}$ is an isomorphism, i.e. φ is surjective. $\square$

Proposition 6.11. *The Loewy length of* I_λ *equals* $|\lambda| + 1$*.*

Proof. By Lemma 6.6 we know that the Loewy length of I_λ is at most $|\lambda| + 1$. We prove equality by induction in $|\lambda|$. Fix $\mu \in \Theta$ such that $|\mu| = |\lambda| - 1$ and $\mathrm{Hom}_{\mathfrak{g}_i}(V^i_\mu, V^{i+1}_\lambda) \neq 0$. We claim that $\mathrm{Ext}^1(V_\mu, V_\lambda) \neq 0$. Indeed, consider non-zero homomorphisms $\varphi_i \in \mathrm{Hom}_{\mathfrak{g}_i}(V^i_\mu, V^{i+1}_\lambda)$. Set $X = \varinjlim X_i$, where $X_i = V^i_\mu \oplus V^i_\lambda$, $q_i : X_i \to X_{i+1}$ is given by $q_i(x, y) = (e_i(x), \varphi_i(x) + f_i(y))$ for $x \in V_\mu$, $y \in V_\lambda$, and $e_i : V^i_\mu \to V^{i+1}_\mu$ and $f_i : V^i_\lambda \to V^{i+1}_\lambda$ denote the fixed inclusions. It is easy to see that X is a non-trivial extension of V_μ by V_λ.

This implies the existence of a non-zero homomorphism $I_\lambda \to I_\mu$. By Lemma 6.10, this homomorphism is surjective. Hence the Loewy length of I_λ is greater or equal to the Loewy length of I_μ plus 1. The statement follows. $\square$

The following theorem strengthens the claim of Corollary 6.3.

Theorem 6.12. *Let* $M \in \mathrm{Int}_{\mathfrak{g}}$*. Then* $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *if and only if there exists a finite subset* $\Theta_M \subset \Theta$ *such that any simple subquotient of* M *is isomorphic to* V_μ *for* $\mu \in \Theta_M$*.*

Proof. Assume that $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. It is sufficient to prove the existence of Θ_M for a semisimple M since then the general case follows from Lemma 6.6. Without loss of generality we may assume that $M = \bigoplus_{j \in C} V_{\lambda_j}$, where V_{λ_j} are pairwise non-isomorphic. We claim that if C is infinite, then M^* does not have finite Loewy length. Indeed, M^* contains a submodule isomorphic to $\bigoplus_{j \in C} I_{\mu_j}$, where $V_{\mu_j} = (V_{\lambda_j})_*$. If C is infinite, then $|\mu_j| = |\lambda_j|$ is unbounded and the socle filtration of $\bigoplus_{j \in C} I_{\mu_j}$ is infinite. This contradiction shows that C is finite, i.e. that there exists a finite set Θ_M as required.

Now assume that M admits a finite set Θ_M as in the statement of the theorem. We claim first that if M' is a quotient of M and $\mathrm{Ext}^1_{\mathfrak{g}}(M', V_\lambda) \neq 0$ for some $\lambda \in \Theta$, then M has a subquotient isomorphic to V_μ for some μ with $|\mu| < |\lambda|$. Indeed, by extending the sequence $0 \to V_\lambda \to I_\lambda$ to a minimal injective resolution

$0 \to V_\lambda \to I_\lambda \xrightarrow{i} I^1_\lambda \to ...$, we see that there is a non-zero homomorphism $M' \xrightarrow{p} I^1_\lambda$. Furthermore, by the minimality of the resolution, we have $\mathrm{soc}(I^1_\lambda) \subset \mathrm{Im} i$. Hence by Lemma 6.6 every simple constituent of $\mathrm{soc}(I^1_\lambda)$ is of the form V_ν for $|\nu| < |\lambda|$. Since $(\mathrm{Im} p) \cap \mathrm{soc}(I^1_\lambda) \neq 0$, some simple constituent of $\mathrm{soc}(I'_\lambda)$ is isomorphic to a subquotient of M' and thus of M.

We show now that M has finite Loewy length. Consider a weight $\lambda \in \Theta_M$ with minimal $|\lambda|$. The above argument shows that $\mathrm{Ext}^1_{\mathfrak{g}}(M', V_\lambda) = 0$ for any quotient M' of M. This implies that every subquotient of M isomorphic to V_λ is a quotient of M. Hence M admits a surjective homomorphism $\zeta : M \to M_\lambda$, where M_λ is isomorphic to a direct sum of copies of V_λ and $\Theta_{\ker \zeta} = \Theta_M \setminus \{\lambda\}$. By an induction argument we obtain that M has finite Loewy length. Therefore, $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ by Corollary 6.3 c). $\square$

Corollary 6.13. *A $\mathfrak{g}$-module $M \in \mathrm{Int}_{\mathfrak{g}}$ is an object of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ if and only if both M and $\Gamma_{\mathfrak{g}}(M^*)$ have finite Loewy length.*

Proof. In one direction the statement is trivial. We need to prove that, if $M \in \mathrm{Int}_{\mathfrak{g}}$ satisfies the above two conditions, then $M^* \in \mathrm{Int}_{\mathfrak{g}}$. For a semisimple M this follows directly from Theorem 6.12 (as we have already pointed out). The argument is completed by induction on the Loewy length. Let $M \in \mathrm{Int}_{\mathfrak{g}}$ have Loewy length k, and $\Gamma_{\mathfrak{g}}(M^*)$ have finite Loewy length. Consider the homomorphism $\pi : M \to \mathrm{top}(M)$ onto the maximal semisimple quotient $\mathrm{top}(M)$ of M. Then $\Gamma_{\mathfrak{g}}(\mathrm{top}(M)^*) \subset \Gamma_{\mathfrak{g}}(M^*)$, hence $\mathrm{top}(M) \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$, i.e. in particular $\mathrm{top}(M)^* \in \mathrm{Int}_{\mathfrak{g}}$. Therefore, there is an exact sequence

$$0 \to \mathrm{top}(M)^* \to \Gamma_{\mathfrak{g}}(M^*) \to \Gamma_{\mathfrak{g}}((\mathrm{Ker}\pi)^*) \to 0,$$

implying that $\Gamma_{\mathfrak{g}}((\mathrm{Ker}\pi)^*)$ has finite Loewy length. Since the Loewy length of $\mathrm{Ker}\pi$ equals $k-1$, the induction assumption allows us to conclude that $(\mathrm{Ker}\pi)^* \in \mathrm{Int}_{\mathfrak{g}}$. Hence $\Gamma_{\mathfrak{g}}(M^*) = M^*$. $\square$

Corollary 6.14. *$\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is a tensor category with respect to $\otimes$.*

Proof. It suffices to show that $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ is closed with respect to $\otimes$. The fact that, if $M \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ and $M' \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ then $M \otimes M' \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$, follows immediately from Theorem 6.12. $\square$

The following theorem concerns the structure of injective modules in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$.

Theorem 6.15. *Any injective module $I \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ has a finite filtration $\{I_j\}$ such that, for each j, I_{j+1}/I_j is isomorphic to a direct sum of copies of I_{μ_j} for some $\mu_j \in \Theta$.*

Proof. We use induction on the length of the filtration. Assume that

$$0 = I_0 \subset I_1 \subset ... \subset I_k$$

is already constructed. Let $\mathrm{soc}(I/I_k) = \bigoplus_{f \in \mathcal{F}} Y_f$ for a family $\{Y_f\}_{f \in \mathcal{F}}$ of simple modules Y_f (there are only finitely many non-isomorphic modules among

$\{Y_f\}_{f\in\mathcal{F}}$). Denoting by p the projection $I \to I/I_k$, set $X_f := p^{-1}(Y_f)$. By Lemma 6.9, there exists $\tilde{Y}_f \in \mathrm{Int}_{\mathfrak{g}}$ such that $I_k \subset X_f \subset \tilde{Y}_f$ and $\tilde{Y}_f/I_k \simeq I_{\mu_f}$, $\mu_f \in \Theta$ being the highest weight of Y_f. The inclusion $X_f \subset I$ induces a homomorphism $\psi_f : \tilde{Y}_f \to I$. Let $\psi_f : \tilde{Y}_f/I_k \tilde{\to} I_{\mu_f} \to I/I_k$ the corresponding homomorphism of quotients. Then $\bar{\psi} := \bigoplus_{f\in\mathcal{F}} \bar{\psi}_f : \bigoplus_{f\in\mathcal{F}} I_{\mu_f} \to I$ is injective since its restriction to $\mathrm{soc}(\bigoplus_{f\in\mathcal{F}} I_{\mu_f})$ is an isomorphism. This shows that if $I_{k+1} := p^{-1}(\bar{\psi}(\bigoplus_{f\in\mathcal{F}} I_{\mu_f}))$, there is an isomorphism $I_{k+1}/I_k \simeq \bigoplus_{f\in\mathcal{F}} I_{\mu_f}$.

The filtration $\{I_j\}$ terminates at a finite step as I has finite Loewy length. □

Example 6.16.

Let $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$ and let M be a countable direct sum of copies of V, i.e. $M = \bigoplus_{f\in\mathcal{F}} V_f$, $\mathrm{card}\mathcal{F} = \beth_0$. Then $(M_*)^*$ can be identified with the set of all infinite matrices $\{b_{ij}\}_{i,j\in\mathbb{Z}_{>0}}$, the action of $\mathfrak{g}$ being left multiplication. The socle $\mathrm{soc}((M_*)^*)$ is the space of matrices F_r with finitely many non-zero rows and is isomorphic to $\bigoplus_{g\in 2^{\mathcal{F}}} V_g$. (Note that the module $\prod_{f\in\mathcal{F}} V_f$ considered in Example 6.2 d) is a submodule of $(M_*)^*$ and has the same socle as $(M_*)^*$). We thus obtain the diagram

$$\begin{array}{ccc} \bigoplus_{g\in 2^{\mathcal{F}}} V_g & \subset & (M_*)^* \\ \cup & & \cup \\ M & \subset & I_M \end{array},$$

I_M being an injective hull of M. Furthermore, I_M is the largest submodule of $(M_*)^*$ such that $\mathfrak{g}\cdot I_M = M$. A direct computation shows that I_M coincides with the space of all matrices with finite rows (i.e. each row has finitely many non-zero entries).

Note that $I_M \not\simeq \bigoplus_{f\in\mathcal{F}}(I_{\varepsilon_1})_f$ ($\varepsilon_1 \in \Theta$ is the highest weight of V). In fact I_M has the following filtration as in Theorem 6.15: $0 \subset \bigoplus_{f\in\mathcal{F}}(I_{\varepsilon_1})_f \subset I_M$. Here $I_M/\bigoplus_{f\in\mathcal{F}}(I_{\varepsilon_1})_f$ is a trivial module of cardinality $2^{\mathcal{F}}$ which is interpreted as a direct sum of $2^{\mathcal{F}}$ copies of I_0.

For any $k \in \mathbb{Z}_{>0}$ we now define $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$ be the subcategory of modules whose simple quotients are isomorphic to V_μ with $|\mu| \le k$. Theorem 6.12 and Corollary 6.3 a) imply the following.

Corollary 6.17. *The category* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$ *is closed under direct products and direct sums.*

Corollary 6.18. *a) The category* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *equals the direct limit* $\varinjlim \widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$.

b) If $\{M_f\}_{f\in\mathcal{F}}$ *is an infinite family of objects of* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$*, then* $\prod_{f\in\mathcal{F}} M_f \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *(equivalently,* $\bigoplus_{f\in\mathcal{F}} \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$*) if and only if there is* k *such that* $M_f \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$ *for all* $f \in \mathcal{F}$.

Proof. a) follows directly from Theorem 6.12.

Consider now $\prod_{f\in\mathcal{F}} M_f$. If $M_f \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$ for some k, then $\prod_{f\in\mathcal{F}} M_f \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$ (and thus also $\bigoplus_{f\in\mathcal{F}} M_f \in \widetilde{\mathrm{Tens}}_{\mathfrak{g}}^k$) by Corollary 6.3 a). If no such k exists, then $\bigoplus_{f\in\mathcal{F}} M_f \notin \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ by Theorem 6.12, hence also $\prod_{f\in\mathcal{F}} M_f \notin \widetilde{\mathrm{Tens}}_{\mathfrak{g}}$. □

Corollary 6.19. *Every object in* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *has a finite injective resolution.*

We now introduce the following partial order on Θ: we set $\mu \preceq \lambda$ if for any sufficiently large i there exists $j > i$ such that $\mathrm{Hom}_{\mathfrak{g}_i}(V_\mu^i, V_\lambda^j) \neq 0$. If $\mu \preceq \lambda$, then $l(\lambda,\mu)$ denotes the length of a maximal chain $\mu \prec \mu_1 \prec \cdots \prec \lambda$ in Θ.

Lemma 6.20. $\mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda) \neq 0$ *if and only if* $\mu \prec \lambda$. *If* $\mu \prec \lambda$, *then* $\dim \mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda) = \beth_1$.

Proof. Assume that there is a non-trivial extension

$$0 \to V_\lambda \to X \to V_\mu \to 0. \tag{8}$$

We will show that $\mu \prec \lambda$. Let, on the contrary, $\mathrm{Hom}_{\mathfrak{g}_i}(V_\mu^i, V_\lambda^j) = 0$ for all $j > i$. Then $\mathrm{Hom}_{\mathfrak{g}_i}(V_\mu^i, V_\lambda) = 0$. Since $\dim \mathrm{Hom}_{\mathfrak{g}_i}(V_\mu^i, V_\mu) = 1$, we have $\dim \mathrm{Hom}_{\mathfrak{g}_i}(V_\mu^i, X) = 1$. Let $\varphi : V_\mu^i \to X$ be a non-zero homomorphism. Then $U(\mathfrak{g}) \cdot \varphi(V_\mu^i) \simeq X$. Therefore, φ extends to a homomorphism of $\mathfrak{g}$-modules $V_\mu \to X$, and this yields a splitting of the exact sequence (8). Thus, $\mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda) \neq 0$ implies $\mu \prec \lambda$.

Now let $\mu \prec \lambda$. Then there exists an infinite sequence $i_1, i_2, \dots$ such that $\mathrm{Hom}_{\mathfrak{g}_{i_j}}(V_\mu^{i_j}, V_\lambda^{i_{j+1}}) \neq 0$ for all j. Consider a sequence of non-zero homomorphisms $\varphi_j \in \mathrm{Hom}_{\mathfrak{g}_{i_j}}(V_\mu^{i_j}, V_\lambda^{i_{j+1}})$ and set $Z_j := V_\mu^{i_j} \oplus V_\lambda^{i_j}$. Denote by e_j (respectively, f_j) the inclusion $V_\mu^{i_j} \to V_\mu^{i_{j+1}}$ (resp., $V_\lambda^{i_j} \to V_\lambda^{i_{j+1}}$). Define $\psi_j : Z_j \to Z_{j+1}$ by

$$\psi(x,y) = (e_j(x), \varphi_j(x) + f_j(y)),\ x \in V_\mu^{i_j}, y \in V_\lambda^{i_j}.$$

Consider $Z = \varinjlim Z_j$. It is an exercise to check that Z is an extension of V_μ by V_λ, and that it does not split if infinitely many $\varphi_j \neq 0$. Hence the dimension of $\mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda)$ is at least $\beth_1$. On the other hand, the dimension of $\mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda)$ is bounded by the multiplicity of V_μ in $\mathrm{soc}^1(I_\lambda)/\mathrm{soc}(I_\lambda)$. The dimension of $I_\mu = ((V_\mu)_*)^*$ is $\beth_1$, hence the dimension of $\mathrm{Ext}^1_{\mathfrak{g}}(V_\mu, V_\lambda)$ is at most $\beth_1$.

To finish the proof just note that $\mathrm{Ext}^1_{\mathfrak{g}}(V_\lambda, V_\lambda) = 0$ by Lemma 6.6. □

Corollary 6.21. *The category* $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ *consists of a single block.*

Proof. According to Lemma 6.20, $\mathrm{Ext}^1_{\mathfrak{g}}(\mathbb{C}, V_\mu) \neq 0$ for any $\mu \in \Theta$. □

Proposition 6.22. *For* $k \in \mathbb{Z}_{>0}$, *set*

$$\Theta^k(\lambda) = \{\mu \prec \lambda | l(\lambda,\mu) \geq k+1\}.$$

Then

$$\mathrm{soc}^k(I_\lambda)/\mathrm{soc}^{k-1}(I_\lambda) = \bigoplus_{\mu \in \Theta^k(\lambda)} X^\mu \otimes V_\mu,$$

where each X^μ *is a trivial* $\mathfrak{g}$*-module of dimension* $\beth_1$.

Proof. For $k = 1$ the statement follows from Lemma 6.20. Now we proceed by induction on k. Note first that if V_μ is a simple constituent of $\mathrm{soc}^k(I_\lambda)/\mathrm{soc}^{k-1}(I_\lambda)$, then, by Lemma 6.20, $\mu \prec \chi$ for some simple constituent V_χ of $\mathrm{soc}^{k-1}(I_\lambda)/\mathrm{soc}^{k-2}(I_\lambda)$. By the induction assumption, $\chi \in \Theta^{k-1}(\lambda)$. In addition, it is clear that V_μ is a simple constituent of $\mathrm{soc}^k(I_\lambda)/\mathrm{soc}^{k-1}(I_\lambda)$ if and only if there exists a non-zero homomorphism $\varphi : I_\lambda \to I_\mu$, such that $\varphi(\mathrm{soc}^{k-1}(I_\lambda)) = 0$. By Lemma 6.10, φ is surjective, so all simple constituents of $\mathrm{soc}^1(I_\mu)/\mathrm{soc}(I_\mu)$ are also simple constituents of $\mathrm{soc}^k(I_\lambda)/\mathrm{soc}^{k-1}(I_\lambda)$. This implies that V_μ is a simple constituent of $\mathrm{soc}^k(I_\lambda)/\mathrm{soc}^{k-1}(I_\lambda)$ if and only if there exists $\psi \in \Theta^{k-1}(\lambda)$ such that $\mu \in \Theta^1(\psi)$. Since $\mu \in \Theta^1(\psi)$ if and only if $\mu \in \Theta^k(\lambda)$, the statement follows. ☐

Let $\mathrm{Tens}_\mathfrak{g}$ be the full subcategory of $\widetilde{\mathrm{Tens}}_\mathfrak{g}$ consisting of modules M whose cardinality $\mathrm{card}M$ is bounded by $\beth_n$ for some n depending on M.

Theorem 6.23. $\mathrm{Tens}_\mathfrak{g}$ *is the unique minimal abelian full subcategory of* $\mathrm{Int}_\mathfrak{g}$ *which does not consist of trivial modules only and which is closed under* $\otimes$ *and* *.

Proof. Let $\mathcal{C}$ be a minimal abelian full subcategory of $\mathrm{Int}_\mathfrak{g}$ which contains a non-trivial module M and is closed under $\otimes$ and *. We will show that $V \in \mathcal{C}$. Since $\mathrm{End}_\mathbb{C}M$ is a $\mathfrak{g}$-submodule of $(M^* \otimes M)^*$ (through the map $\varphi(\psi \otimes m) = \psi(\varphi(m))$ for $m \in M$, $\psi \in M^*$, $\varphi \in \mathrm{End}_\mathbb{C}M$), we have $\mathrm{End}_\mathbb{C}M \in \mathcal{C}$. Furthermore, the adjoint module $\mathfrak{g}$ is a submodule of $\mathrm{End}_\mathbb{C}M$. Hence $\mathfrak{g} \in \mathcal{C}$. Recall that $\mathfrak{g}$ is the socle of $V_* \otimes V$ for $sl(\infty)$, of $\Lambda^2(V)$ for $o(\infty)$, and of $S^2(V)$ for $sp(\infty)$. In all cases it is easy to see that $\mathfrak{g}^*$ contains a subquotient isomorphic to V. Therefore, $V \in \mathcal{C}$. In addition, $V_* = \mathrm{soc}(V^*) \in \mathcal{C}$. Therefore, $T^{p,q} \in \mathcal{C}$ for all p, q, and $V_\lambda \in \mathcal{C}$ for all $\lambda \in \Theta$. Finally, by Corollary 6.7 a), any $M \in \widetilde{\mathrm{Tens}}_\mathfrak{g}$ is a submodule of $(\mathrm{soc}(M)_*)^*$, and the statement follows. ☐

We conclude this paper with the remark that the category $\widetilde{\mathrm{Tens}}_\mathfrak{g}$, for $\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$, is functorial with respect to any homomorphism of locally semisimple Lie algebras $\varphi : \mathfrak{g}' \to \mathfrak{g}$. By this we mean that any $M \in \widetilde{\mathrm{Tens}}_\mathfrak{g}$ considered as a $\mathfrak{g}'$-module is an object of $\widetilde{\mathrm{Tens}}_\mathfrak{g}$.

To prove this, recall that the image of φ', being a locally semisimple subalgebra of $\mathfrak{g}$, is isomorphic to a direct sum of copies of $sl(\infty), o(\infty), sp(\infty)$ and of finite-dimensional simple Lie algebras, [DP2]. Furthermore, the result of [DP2] implies that as $\mathfrak{g}'$-modules both V and V_* have Loewy length at most 2 and that all non-trivial simple constituents of V and V_* are isomorphic to the natural and conatural representations $V_\mathfrak{s}$ and $(V_\mathfrak{s})_*$ for some simple direct summands $\mathfrak{s}$ of $\varphi(\mathfrak{g}')$ and that all non-trivial constituents occur with finite multiplicity. (The simple trivial representation may occur with up to countable multiplicity in both $\mathrm{soc}(V)$ and $V/\mathrm{soc}(V)$ (respectively, $\mathrm{soc}(V_*)$ and $V_*/\mathrm{soc}(V_*)$.) This allows us to conclude that any single simple object of $\widetilde{\mathrm{Tens}}_\mathfrak{g}$ is an object of $\widetilde{\mathrm{Tens}}_{\varphi(\mathfrak{g}')}$. Hence, by Theorem 6.12, any $M \in \widetilde{\mathrm{Tens}}_\mathfrak{g}$ is an object of $\widetilde{\mathrm{Tens}}_{\varphi(\mathfrak{g}')}$.

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