Jacobs University Bremen

Thesis for Bachelor of Science in Mathematics

On the automorphism groups of certain flag ind-varieties

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Abstract

Let F be a generalized flag as defined in [2]. We wish to study the automorphism group $\operatorname{Aut}\mathcal{F}\ell(F,E)$ of the ind-variety of generalized flags $\mathcal{F}\ell(F,E)$ of which F is a point. In this thesis we describe this automorphism group in the cases when F consists only of subspaces of finite dimension or finite codimension.

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1 Introduction

The study of flag varieties is a classical topic in complex geometry. In this thesis we recall the automorphism groups of classical complex flag varieties and then we study the automorphism groups of certain flag ind-varieties.

The simplest type of a flag variety, the Grassmannian, has been previously studied by Chow in [1].

The automorphism group of a finite-dimensional flag variety has been determined in [6]. This automorphism group is $\mathrm{PGL}(V)$ (where V is the complex vector space spanned by the flags in the flag variety) in the general case, and is a group containing $\mathrm{PGL}(V)$ as a normal subgroup of index 2 when the flag is 'symmetric'. We shall reprove this result in section 4.

In section 5 we generalize these results by replacing the assumption of the finite dimension of V by countable dimension. In that case we define the notion of a generalized flag and consider the respective ind-varieties following [2]. We then determine a flag ind-variety's automorphism group when all of the flag's components are either finite dimensional or cofinite dimensional.

In section 3 we briefly consider two specific ind-flag varieties whose automorphism groups have been determined in [4], and prove that they are not isomorphic as groups.

2 Preliminaries

In this section we introduce the necessary definitions and constructions following [3]. We assume that some general facts of algebraic geometry are known.

The base field we will be working over is the field $\mathbb{C}$ of complex numbers. Unless otherwise stated, V is a countable-dimensional complex vector space, $E := \{e_1, e_2, \ldots, e_n, \ldots\}$ is a fixed basis of V, $V_n := \mathrm{Span}\{e_1, e_2, \ldots, e_n\}$, $E_* := \{e_1^*, e_2^*, \ldots, e_n^*, \ldots\}$ where $e_i^*(e_j) := \delta_{i,j}$ ($\delta_{i,j}$ is Kronecker's delta), and $V_* := \mathrm{Span}\{e_1^*, e_2^*, \ldots, e_n^*, \ldots\} \subset V^*$. Furthermore, in what follows we will use the identification $V_{**} = V$ given by $(e_i^*)^* = e_i$.

Definition 2.1. Let $X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n \hookrightarrow \cdots$ be a chain of closed embeddings of algebraic varieties. We call the direct limit $X = \varinjlim X_n$ an *ind-variety*. A morphism of ind-varieties $\phi : X \to Y$ is a set of morphisms $\phi_n : X_n \to Y_{M(n)}$ (for some $M(n)$) that commute with the embeddings. An isomorphism of ind-varieties is a morphism that admits an inverse. We say that $U \subset X$ is open if $U \cap X_n$ is open in X_n for each $n \geq 1$. Furthermore, we define the structure sheaf to be $\mathcal{O}_X = \varprojlim \mathcal{O}_{X_n}$ where $\cdots \to \mathcal{O}_{X_n} \to \cdots \to$

$\mathcal{O}_{X_2} \to \mathcal{O}_{X_1}$ is the projective system induced by the chain of embeddings $X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n \hookrightarrow \ldots$.

Example 2.2. Consider the chain

$$\mathrm{GL}(V_1) \subset \mathrm{GL}(V_2) \subset \cdots \subset \mathrm{GL}(V_n) \subset$$

where the inclusion $\iota_n : \mathrm{GL}(V_n) \subset \mathrm{GL}(V_{n+1})$ is given by $\iota_n(g)(v) = g(v)$ for $v \in V_n$ and $\iota_n(g)(e_{n+1}) = e_{n+1}$. Then the ind-variety $\mathrm{GL}(E,V) = \varinjlim \mathrm{GL}(V_n)$ is well-defined. Note that the elements of $\mathrm{GL}(E,V)$ are invertible linear operators on V that act as identity on almost all elements of E.

Next we define a generalized flag and its corresponding ind-variety.

Let $F = \{C_\alpha\}_{\alpha \in I}$ be a chain of pairwise distinct subspaces of V ordered by inclusion. Denote by F' (respectively F'') the set of elements of F that have an immediate successor (respectively predecessor). Also denote by $F^\dagger$ the set of pairs (C', C'') such that C'' is the immediate successor of C' in F.

Definition 2.3. A *generalized flag* is a chain of subspaces F such that $F = F' \cup F''$ and

$$V \setminus \{0\} = \bigcup_{(C',C'') \in F^\dagger} C'' \setminus C'.$$

Note that if we let V be finite dimensional then the notion of generalized flag coincides with that of usual flag.

Definition 2.4. Let F be a generalized flag as above. We say that F is *E-compatible* if for every $i \in I$ we have $C_i = \operatorname{Span} E_i$ for some $E_i \subset E$.

A basic result in [3] proves that any generalized flag F admits a basis E such that F is E-compatible.

In the classical setting where $\dim V < \infty$, we can define a flag variety as $\mathcal{F}\ell(F,V) = \{(U_1, U_2, \ldots, U_{|I|}) \in \prod_{i \in I} \mathrm{Gr}(\dim C_i, V) \mid U_i \subset U_{i+1}\}$. The automorphism groups of such classical flag varieties will be discussed in section 4.

Now we describe the analogous construction of an ind-variety corresponding to a generalized flag.

Let F be a generalized flag compatible with the basis E and let $F_n = \{C_\alpha \cap V_n \mid C_\alpha \in F\}$.

Consider the closed embeddings $\iota_n : \mathcal{F}\ell(F_n, V_n) \hookrightarrow \mathcal{F}\ell(F_{n+1}, V_{n+1})$ given by

$$\{U_\alpha\} \mapsto \{U'_\alpha\} \text{ where } U'_\alpha = \begin{cases} U_\alpha \oplus \mathbb{C}e_{n+1} & \text{if } e_{n+1} \in C_\alpha \\ U_\alpha & \text{otherwise} \end{cases}.$$

By definition, the ind-variety $\mathcal{F}\ell(F,E)$ is then the inductive limit $\varinjlim \mathcal{F}\ell(F_n,V_n)$. Note that as a set $\mathcal{F}\ell(F,E)$ coincides with the orbit of F under the natural action of $\mathrm{PGL}(E,V)$ on the generalized flags on V. We shall use this identification throughout the thesis.

Example 2.5. Consider the generalized flag $F=\{0\subset U\subset V\}$ for a fixed subspace U spanned by some elements of E. We call the generalized flag ind-variety $\mathcal{F}\ell(F,E)$ an *ind-grassmannian* and denote it by $\mathrm{Gr}(U,E)$.

Although ind-grassmannians are defined by a single subspace U, it can be shown that they are classified up to isomorphism according to $\min(\dim U,\operatorname{codim} U)$:

- if $\dim U<\infty$ then the orbit of U under the action of $\mathrm{GL}(E,V)$ is the set of all subspaces U' with $\dim U'=\dim U$, thus $\mathrm{Gr}(U,E)$ depends only on $\dim U$ and is denoted by $\mathrm{Gr}(\dim U)$.
- if $\operatorname{codim} U<\infty$ then we have the isomorphisms $\mathrm{Gr}(U,E)\cong\mathrm{Gr}(U^\perp,E_*)\cong\mathrm{Gr}(\operatorname{codim} U,V_*)$ given by $W\mapsto W^\perp=\{g\in V_*\mid g(W)=0\}$ and thus $\mathrm{Gr}(U,E)\cong\mathrm{Gr}(\operatorname{codim} U)$.
- if $\min(\dim U,\operatorname{codim} U)=\infty$ then it is proved in Lemma 4.3 of [5] that $\mathrm{Gr}(U,E)$ does not depend up to isomorphism on the choice of U. We will thus denote it by $\mathrm{Gr}(\infty)$.

3 Two special cases

The main goal of this thesis is to describe the groups of automorphisms $\operatorname{Aut}\mathcal{F}\ell(F,E)$ in the special case when the generalized flag F consists only of finite-dimensional and finite-codimensional subspaces. Such groups have previously been described for ind-grassmannians of the form $\mathrm{Gr}(\dim U)$ and $\mathrm{Gr}(\operatorname{codim} U)$ for $\dim U<\infty$ or $\operatorname{codim} U<\infty$, and when F is a maximal increasing generalized flag with $\dim F_\alpha<\infty$ for all α. In [4] it was proven that

- for $F=\{0\subset U\subset V\}$ with $\min(\dim U,\operatorname{codim} U)<\infty$ we have $\operatorname{Aut}\mathcal{F}\ell(F,E)\cong\mathrm{PGL}(V)$;
- for $F=\{0\subset V_1\subset V_2\subset\cdots\subset V_n\subset\dots\}$ with $\dim V_i=i$ we have $\operatorname{Aut}\mathcal{F}\ell(F,E)\cong\mathrm{P}\left(\mathrm{GL}(E,V)\cdot B_E\right)$, where $B_E\subset\mathrm{GL}(E,V)$ is the stabilizer of F under the action of $\mathrm{GL}(E,V)$.

Using the following lemmas, we shall give a brief proof of why these automorphism groups are not isomorphic as abstract groups.

Lemma 3.1. *Every nonabelian simple subgroup of* $\mathrm{P}(\mathrm{GL}(E,V)\cdot B_E)$ *generates the same normal subgroup.*

Proof. Set $G := \mathrm{P}(\mathrm{GL}(E,V)\cdot B_E)$ and let H be the normal subgroup of G generated by the simple subgroup $\mathrm{PGL}(E,V)$. Note that H corresponds to linear maps that act as scalar multiplication on the respective cofinite subspaces, furthermore $H = \bigcup_{g\in G}(g^{-1}\,\mathrm{PGL}(E,V)g)$. Also $H \neq G$, the linear operator f with by $f(e_n) = \sum_i^n e_i$ does not fix any cofinite subspace, and hence the image of f in G does not belong to H. Consider the subgroup $G_n := \langle f \in B_E \mid (e_j^* \cdot f)(e_i) = 0 \text{ for any } 1 \leq i-j \leq n \text{ and } (e_i^* f)(e_i) = 1\rangle$ and its image $\widetilde{G_n}$ in ${}^{G}\!/_{H}$. Then $\widetilde{G_n} \lhd {}^{G}\!/_{H}$ and we obtain a filtration

$$ {}^{G}\!/_{H} \rhd \widetilde{G_0} \rhd \widetilde{G_1} \rhd \cdots \rhd \widetilde{G_n} \rhd \cdots \tag{$\star$}$$

Furthermore, a straightforward computation shows that the quotients of this descending normal series are abelian. Consider now a simple subgroup N of G that is not contained in H. Due to the simplicity of N we obtain that $N\cap H = \langle e\rangle$ and that the image of N in ${}^{G}\!/_{H}$ is isomorphic to ${}^{N}\!/_{N\cap H} \cong N$. Intersecting the filtration ($\star$) with the image of N gives

$$N \rhd (N\cap \widetilde{G_0}) \rhd (N\cap \widetilde{G_1}) \rhd \cdots \rhd (N\cap \widetilde{G_n}) \rhd \cdots$$

Since each $(N\cap \widetilde{G_n})$ is normal in N, it must be either trivial or equal to N. If $N\cap \widetilde{G_n} = N$ for all n then $N \subset \bigcap_n \widetilde{G_n} = \langle e\rangle$ thus N is trivial. Otherwise there exists a minimal n such that $N\cap \widetilde{G_n} = \langle e\rangle$. Considering the abelian quotient at that term, we obtain that it is isomorphic to N, which implies that N is trivial.

Thus any simple subgroup N of G is necessarily contained in H. Denote by N' the normal subgroup of G generated by N. Since $H = \bigcup_{g\in G}(g^{-1}\,\mathrm{PGL}(E,V)g)$, N' must intersect non-trivially some $g^{-1}\,\mathrm{PGL}(E,V)g$ for some g. Fix such a g. Then using the fact that $g^{-1}\,\mathrm{PGL}(E,V)g$ is simple, we must necessarily have $g^{-1}\,\mathrm{PGL}(E,V)g \subset N'$. Therefore

$$h^{-1}\,\mathrm{PGL}(E,V)h = (g^{-1}h)^{-1}g^{-1}\,\mathrm{PGL}(E,V)g(g^{-1}h) \subset (g^{-1}h)^{-1}N'(g^{-1}h) = N',$$

and by varying h over G we obtain $H = N'$. □

Lemma 3.2. *There exist two nonabelian simple subgroups of* $\mathrm{PGL}(V)$ *that generate distinct normal subgroups.*

Proof. Consider the normal subgroup H in $\mathrm{PGL}(V)$ generated by $\mathrm{PGL}(E,V)$. It consists of the images of linear maps that act as the identity on some cofinite subspace of V.

Now consider the embedding $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{SL}(V_2) \hookrightarrow \mathrm{GL}(V)$ which maps a linear map $M : V_2 \to V_2$ to the linear map $M' : V \to V$ satisfying

$$M'(ae_{2n+1} + be_{2n}) = e_1^*(M(ae_1 + be_2))e_{2n+1} + e_2^*(M(ae_1 + be_2))e_{2n}$$

for any $n \geq 1$. Denote the image of the embedding in $\mathrm{GL}(V)$ by G. The nontrivial elements of G do not fix any cofinite subspace of V, and thus the image in $\mathrm{PGL}(V)$ of G intersects H trivially. Therefore this image generates a normal subgroup of $\mathrm{PGL}(V)$ not equal to H. $\square$

Corollary 3.3.

$$\mathrm{P}\left(\mathrm{GL}(E,V) \cdot B_E\right) \ncong \mathrm{PGL}(V).$$

4 Finite Version

We first consider the classical version when V is finite dimensional.

Consider the embedding $\mathcal{F}\ell(F,V) \subset \prod \mathrm{Gr}(d_i,V)$ where $d_i := \dim C_i$. It induces the projection morphisms $p_i : \mathcal{F}\ell(F,V) \to \mathrm{Gr}(d_i,V)$. A classical result states that $\mathrm{Pic}\,\mathrm{Gr}(d_i,V) \cong \mathbb{Z}$ and $\mathrm{Pic}\,\mathcal{F}\ell(F,V) = \bigoplus_i \mathbb{Z}[L_i]$ where $L_i := p_i^*(\mathcal{O}_{\mathrm{Gr}(d_i,V)}(1))$. We call the set $\{L_i\}$ the **preferred set of generators** of $\mathrm{Pic}\,\mathcal{F}\ell(F,V)$.

We assume the following theorem proved by Chow in [1]

Theorem 4.1 (Chow '49)**.** *Let $0 \subsetneq C \subsetneq V$ be a subspace, then the following holds*

- *If $2\dim C = \dim V$ then $\mathrm{Aut}\,\mathrm{Gr}(\dim U, V)$ is the semidirect product of $\mathrm{PGL}(V)$ and the 'flip' morphism*

$$fl : C \mapsto C^{\perp} = \{v \in V_* \mid v(C) = 0\} \subset V_* \cong V.$$

- *If $2\dim C \neq \dim V$, then $\mathrm{Aut}\,\mathrm{Gr}(\dim U, V) \cong \mathrm{PGL}(V)$.*

We will make use of the following lemma.

Lemma 4.2. *Let $0 \subsetneq U \subset U' \subsetneq V$ be a flag in V and $f, f' : V \to V$ be invertible linear maps, such that for any flag $0 \subsetneq W \subset W' \subsetneq V$ with $\dim W = \dim U$ and $\dim W' = \dim U'$ we have $f(W) \subset f'(W')$. Then $f = cf'$ for some $c \in \mathbb{C} \setminus \{0\}$.*

Proof. Assume otherwise. Let $v \in V$ be such that $Z := \mathrm{Span}\{f(v), f'(v)\}$ has dimension 2. Extend v to a basis $E = \{v = v_0, v_1, \ldots, v_n\}$ of V. Then $Z \subset f'(\mathrm{Span}\, X)$ for any $v \in X \subset E$ with $|X| = \dim U'$. In particular,

$$\begin{aligned} Z \subset \bigcap_{v\in X\subset E, |X|=\dim U'} f'(\mathrm{Span}\, X) = \\ f'(\bigcap_{v\in X\subset E, |X|=\dim U'} \mathrm{Span}\, X) = \\ f'(\mathrm{Span} \bigcap_{v\in X\subset E, |X|=\dim U'} X) = \\ \mathrm{Span}\, f'(\bigcap_{v\in X\subset E, |X|=\dim U'} X) = \mathbb{C}f'(v). \end{aligned}$$

This contradicts the assumption that $\dim Z > 1$. □

We can now state our main theorem in the finite case.

Theorem 4.3. *Let V be a finite-dimensional vector space and*

$$F = (C_1 \subset C_2 \subset \cdots \subset C_m)$$

be a flag. Then the following holds:

- *If $\dim C_i = \operatorname{codim} C_{m+1-i}$ for every i, the group $\operatorname{Aut} \mathcal{F}\ell(F, V)$ is the semi-direct product of $\mathrm{PGL}(V)$ and the 'flip' morphism*

$$fl : U_i \mapsto U_i^{\perp} = \{v \in V^* \mid v(U_i) = 0\} \subset V^* \cong V.$$

- *If $\dim C_i \neq \operatorname{codim} C_{m+1-i}$ for some i, then $\operatorname{Aut} \mathcal{F}\ell(F, V) \cong \mathrm{PGL}(V)$.*

In the proof of the main theorem we shall use the following immediate corollary of [Prop. 2.3 from [7]]:

Corollary 4.4. *Let $E = (V_1 \subset \cdots \subset V_k)$ and $E' = (W_1 \subset \cdots \subset W_l)$ be flags on finite-dimensional vector spaces V, W, and let $\phi : \mathcal{F}\ell(E, V) \hookrightarrow \mathcal{F}\ell(E', W)$ be a closed embedding. If $1 \leq r \leq k$ and $1 \leq s \leq l$ are such that $\phi^*(L_s^2) \cong L_r^1$ where $L_s^1 \in \operatorname{Pic} \mathcal{F}\ell(E, V)$ and $L_r^2 \in \operatorname{Pic} \mathcal{F}\ell(E', W)$ are* ***preferred generators****, then there exists a morphism $\psi : \mathrm{Gr}(\dim V_r, V) \to \mathrm{Gr}(\dim W_s, W)$ such that the diagram*

$$\begin{tikzcd} \mathcal{F}\ell(E, V) \arrow[r, hook, "\phi"] \arrow[d, "p_r"] & \mathcal{F}\ell(E', W) \arrow[d, "p_s"] \\ \mathrm{Gr}(\dim V_r, V) \arrow[r, "\psi"] & \mathrm{Gr}(\dim W_s, W) \end{tikzcd}$$

is commutative.

Proof of theorem 4.3. We assume that F is not a Grassmannian. The case of a Grassmannian was considered in theorem 4.1. An automorphism $\phi \in \operatorname{Aut}\mathcal{F}\ell(F,V)$ induces an automorphism $\phi^* \in \operatorname{Aut}\operatorname{Pic}\mathcal{F}\ell(F,V)$. Consider $\phi^*(L_i) = \sum_{j=1}^m \alpha_{i,j}L_j$. Since L_i is generated by its global sections, the same must hold for $\phi^*(L_i)$. Thus $\alpha_{i,j} \geq 0$ for all $i,j \in \{1,\ldots,m\}$, and ϕ^* acts on $\operatorname{Pic}\mathcal{F}\ell(F,V)$ as an invertible matrix whose coefficients are nonnegative integers. Since the same holds for $(\phi^*)^{-1}$, it follows that the automorphism ϕ must act by a permutation on $\{L_i\}$.

For each $i \in \{1,\ldots,m\}$ let $a_i \in \{1,\ldots,m\}$ be such that $\phi^*(L_i) = L_{a_i}$.

By corollary 4.4, the automorphism ϕ induces morphisms $\theta_i : \operatorname{Gr}(\dim C_{a_i}, V) \to \operatorname{Gr}(\dim C_i, V)$. Similarly, by applying the corollary to the morphism ϕ^{-1} we get morphisms $\psi_i : \operatorname{Gr}(\dim C_i, V) \to \operatorname{Gr}(\dim C_{a_i}, V)$. Since the projection maps $p_i : \mathcal{F}\ell(E,V) \to \operatorname{Gr}(\dim C_i, V)$ are surjective, we must have $\theta_i \circ \psi_i = \mathrm{id}_{\operatorname{Gr}(\dim C_i, V)}$. Hence ψ_i are isomorphisms, and either $a_i = i$ or $\dim C_i = \operatorname{codim} C_{a_i}$.

Fix an $i \in \{1,\ldots,m\}$. There are three possibilities:

- $a_i = i$, $2\dim C_i \neq \dim V$, and $\psi_i \in \operatorname{Aut}(\operatorname{Gr}(\dim C_i, V))$ is induced by the action of a $\overline{\psi_i} \in \operatorname{PGL}(V)$;
- $a_i = i$, $2\dim C_i = \dim V$, and $\psi_i \in \operatorname{Aut}(\operatorname{Gr}(\dim C_i, V))$ is induced by the action of a $\overline{\psi_i} \in \operatorname{PGL}(V) \sqcup fl \circ \operatorname{PGL}(V)$;
- $a_i \neq i$,$\dim C_i = \operatorname{codim} C_{a_i}$ (and thus $2\dim C_i \neq \dim V$), $a_{a_i} = i$, ψ_i is induced by the action of a $\overline{\psi_i} \in fl \circ \operatorname{PGL}(V)$; also $\overline{\psi_{a_i}} = \overline{\psi_i}^{-1}$.

The next goal is to determine when do elements $\{\overline{\psi_i}\} \subset \operatorname{PGL}(V) \sqcup fl \circ \operatorname{PGL}(V)$ induce well-defined automorphisms ϕ. We shall use the fact that $\{\overline{\psi_i}\}$ must preserve the incidence relation $\overline{\psi_i}(C_i) \subset \overline{\psi_j}(C_j)$.

If $\overline{\psi_i} \in \operatorname{PGL}(V)$ for every $i \in \{1,\ldots,m\}$, then by lemma 4.2 all $\overline{\psi_i}$ are equal and induce an automorphism $\phi \in \operatorname{PGL}(V)$.

Similarly, if $\overline{\psi_i} \in fl \circ \operatorname{PGL}(V)$ for every $i \in \{1,\ldots,m\}$ then $\dim C_i = \operatorname{codim} C_{m+1-i}$ for every $i \in \{1,\ldots,m\}$. By the above argument, the elements $fl \circ \overline{\psi_i} \in \operatorname{PGL}(V)$ induce an automorphism $fl \circ \phi \in \operatorname{PGL}(V)$ so that $\phi \in fl \circ \operatorname{PGL}(V)$.

If $\overline{\psi_i} \in \operatorname{PGL}(V)$, $\overline{\psi_j} \in fl \circ \operatorname{PGL}(V)$ and $2\dim C_j \neq \dim V$ for some $i,j \in \{1,\ldots,m\}$, then we have 5 cases:

- If $C_i \subset C_j \subset C_{a_j}$, by the definition of ϕ we have

$$\begin{aligned} C_i \subset C_j \subset C_{a_j} \mapsto \psi_i(C_i) \subset (\psi_j)^{-1}(C_{a_j}) \subset \psi_j(C_j) = \\ \overline{\psi_i}(C_i) \subset (\overline{\psi_j}^{-1})(C_{a_j}) \subset (\overline{\psi_j})(C_j). \end{aligned}$$

Then considering the action of $\mathrm{GL}(V)$ on the flag $C_i \subset C_{a_j}$, we obtain

$$\overline{\psi_i}(C_{a_j}) = \sum_{g \in \mathrm{GL}(V), g(C_{a_j}) = C_{a_j}} \overline{\psi_i}(g(C_i)) \subset (\overline{\psi_j}^{-1})(C_{a_j}),$$

which gives a contradiction since $\dim \mathrm{LHS} = \dim C_{a_j} > \dim \mathrm{RHS} = \operatorname{codim} C_{a_j} = \dim C_j$.

- If $C_j \subset C_i \subset C_{a_j}$ with $\dim C_i < \operatorname{codim} C_i$, by the definition of ϕ we have

$$\begin{aligned} C_j \subset C_i \subset C_{a_j} \mapsto (\psi_j)^{-1}(C_{a_j}) \subset \psi_i(C_i) \subset \psi_j(C_j) = \\ (\overline{\psi_j}^{-1})(C_{a_j}) \subset \overline{\psi_i}(C_i) \subset (\overline{\psi_j})(C_j). \end{aligned}$$

Then considering the action of $\mathrm{GL}(V)$ on the flag $C_i \subset C_{a_j}$, we obtain

$$(\overline{\psi_j}^{-1})(C_i) = \sum_{g \in \mathrm{GL}(V), g(C_i) = C_i} (\overline{\psi_j}^{-1})(g(C_{a_j})) \subset \overline{\psi_i}(C_i),$$

which gives a contradiction since $\dim \mathrm{LHS} = \operatorname{codim} C_i > \dim \mathrm{RHS} = \dim C_i$.

- If $C_j \subset C_i \subset C_{a_j}$ with $\dim C_i > \operatorname{codim} C_i$, then by the definition of ϕ we have

$$\begin{aligned} C_j \subset C_i \subset C_{a_j} \mapsto (\psi_j)^{-1}(C_{a_j}) \subset \psi_i(C_i) \subset \psi_j(C_j) = \\ (\overline{\psi_j}^{-1})(C_{a_i}) \subset \overline{\psi_i}(C_i) \subset (\overline{\psi_j})(C_j). \end{aligned}$$

Then considering the action of $\mathrm{GL}(V)$ on the flag $C_j \subset C_i$, we obtain

$$\overline{\psi_i}(C_i) \subset \bigcap_{g \in \mathrm{GL}(V), g(C_i) = C_i} (\overline{\psi_j})(g(C_j)) = (\overline{\psi_j})(C_i),$$

which gives a contradiction since $\dim \mathrm{LHS} = \dim C_i > \dim \mathrm{RHS} = \operatorname{codim} C_i$.

- If $C_j \subset C_i \subset C_{a_j}$ with $\dim C_i = \operatorname{codim} C_i$, then we combine the identities in the two cases above to get

$$\overline{\psi_j}^{-1}(C_i) = \overline{\psi_i}(C_i) = \overline{\psi_j}(C_i),$$

so that $\overline{\psi_j} = \overline{\psi_i}$ as elements of $\mathrm{Aut}(\mathrm{Gr}(\dim C_i, V))$. This is a contradiction since $\overline{\psi_i} \in \mathrm{PGL}(V)$ but $\overline{\psi_j} \in fl \circ \mathrm{PGL}(V)$.

- If $C_j \subset C_{a_j} \subset C_i$, then by the definition of ϕ we have

$$C_j \subset C_{a_j} \subset C_i \mapsto (\psi_j)^{-1}(C_{a_j}) \subset \psi_j(C_j) \subset \psi_i(C_i) = \\ (\overline{\psi_j}^{-1})(C_{a_j}) \subset (\overline{\psi_j})(C_j) \subset \overline{\psi_i}(C_i).$$

  Then considering the action of $\mathrm{GL}(V)$ on the flag $C_j \subset C_i$, we obtain

$$(\overline{\psi_j})(C_j) \subset \bigcap_{g \in \mathrm{GL}(V), g(C_j)=C_j} \overline{\psi_i}(g(C_i)) = \overline{\psi_i}(C_j)$$

  which gives a contradiction since $\dim \mathrm{LHS} = \operatorname{codim} C_j > \dim \mathrm{RHS} = \dim C_j$.

We are left with the case where $\overline{\psi_i} \in \mathrm{PGL}(V)$, $\overline{\psi_j} \in fl \circ \mathrm{PGL}(V)$ and $2\dim C_j = \dim V$ for some $i, j \in \{1, \dots, m\}$. Here we have 2 possibilities:

- If $C_i \subset C_j$ then

$$\overline{\psi_i}(C_j) = \sum_{g \in \mathrm{GL}(V), g(C_j)=C_j} \overline{\psi_i}(g(C_i)) \subset \overline{\psi_j}(C_j)$$

  By comparing dimensions we have $\overline{\psi_i}(C_j) = \overline{\psi_j}(C_j)$, so that $\overline{\psi_j} = \overline{\psi_i}$ as elements of $\mathrm{Aut}(\mathrm{Gr}(\dim C_j, V))$ as above. Contradiction.

- If $C_j \subset C_i$ then

$$\overline{\psi_i}(C_j) = \bigcap_{g \in \mathrm{GL}(V), g(C_j)=C_j} \overline{\psi_i}(g(C_i)) \supset \overline{\psi_j}(C_j)$$

  By comparing dimensions we have $\overline{\psi_i}(C_j) = \overline{\psi_j}(C_j)$, so that $\overline{\psi_j} = \overline{\psi_i}$ as elements of $\mathrm{Aut}(\mathrm{Gr}(\dim C_j, V))$ as above. Contradiction.

The above analysis allows us to conclude that the automorphism group $\mathrm{Aut}\,\mathcal{F}\ell(F, V)$ is as claimed. □

5 Infinite Version

In this section, V will be a countable-dimensional vector space. Recall that E is a fixed basis of V. We will only consider generalized flags that are E-compatible and consist of spaces of finite dimension or finite codimension. We begin by first proving a stronger version of lemma 4.2.

Lemma 5.1. *Let $F = (0 \subsetneq C \subset C' \subsetneq V)$ be a generalized flag of length 2. Let $f : \mathcal{F}\ell(F, E) \to \mathcal{F}\ell(F, E)$ be the automorphism*

$$(0 \subset U \subset U' \subset V) \mapsto (0 \subset \phi(U) \subset \psi(U') \subset V)$$

for some $\phi \in \mathrm{GL}(V)$ and $\psi \in \mathrm{Aut}\,\mathrm{Gr}(C', E)$. Then $\psi(U') = \phi(U')$ and thus f is determined by ϕ. Furthermore, the same holds if $\psi \in \mathrm{GL}(V)$ and $\phi \in \mathrm{Aut}\,\mathrm{Gr}(C, E)$.

Proof. Fix a point $(0 \subset W \subset W' \subset V) \in \mathcal{F}\ell(F, E)$, and note that

$$\sum_{(U,U')\in\mathcal{F}\ell(F,E),U'=W'} \phi(U) = \phi(\sum_{(U,U')\in\mathcal{F}\ell(F,E),U'=W'} U) = \phi(W') \subset \psi(W')$$

If $\dim W' < \infty$, then since the LHS and RHS have the same dimension, we get $\phi(W') = \psi(W')$. Analogously when $\operatorname{codim} W' < \infty$ we get $\phi(W') = \psi(W')$. For the second case of $f = (\psi, \phi)$ we have

$$\psi(W) \subset \bigcap_{(U,U')\in\mathcal{F}\ell(F,E),U=W} \phi(U') = \phi(\bigcap_{(U,U')\in\mathcal{F}\ell(F,E),U=W} U') = \phi(W)$$

and the conclusion follows. □

Definition 5.2. We define the *Mackey* group $M(V, V_*)$ to be the group consisting of $\{f \in \mathrm{GL}(V) \mid f^*(V_*) = V_*\} \subset \mathrm{GL}(V)$

Note that this definition implies that $M(V, V_*)$ is isomorphic to $M(V_*, V)$ canonically.

Lemma 5.3. *Let $F = (0 \subset C \subset V)$ with $\operatorname{codim} C < \infty$. Let $\phi \in \mathrm{GL}(V)$ be such that it induces a well-defined automorphism*

$$(0 \subset U \subset V) \in \mathcal{F}\ell(F, E) \mapsto (0 \subset \phi(U) \subset V) \in \mathcal{F}\ell(F, E). \qquad (\star)$$

Then $\phi \in M(V, V_*)$. Consequently, $M(V, V_*)$ is the maximal subgroup of $\mathrm{GL}(V)$ that acts on $\mathcal{F}\ell(F, E)$ via $(\star)$.

Proof. We have $(0 \subset U \subset V) \in \mathcal{F}\ell(F, E)$ iff $U = U' \oplus \mathrm{Span}\{e_{n+1}, \cdots\}$ for some $U' \subset V_n$ with $\operatorname{codim}_{V_n} U' = \operatorname{codim}_V C$. Equivalently $U^\perp = \{v \in V^* \mid v(U) = 0\}$ is contained in V_* and $\dim_{V_*} U^\perp = \dim_{V_*} C^\perp$. Denote $\psi := \phi^{-1}$. Then ψ also satisfies the assumptions in the statement, furthermore we have

$$\psi(U)^\perp = \{v \in V^* \mid v(\psi(U)) = (\psi^*(v))(U) = 0\} = (\psi^*)^{-1}(U^\perp) = \phi^*(U^\perp) \subset V_*.$$

By varying U we obtain

$$V_* \supset \sum_{(0\subset U\subset V)\in\mathcal{F}\ell(F,E)} \phi^*(U^\perp) = \phi^*(\sum_{(0\subset U\subset V)\in\mathcal{F}\ell(F,E)} U^\perp) = \phi^*(V_*).$$

Hence $\phi \in M(V,V_*)$. For the converse, let $\phi \in M(V,V_*)$, then $\phi(U)^\perp = (\phi^*)^{-1}(U^\perp) \subset (\phi^*)^{-1}(V_*) = V_*$. Using that $(\phi^*)^{-1}$ is invertible we have $\dim_{V_*}\phi(U)^\perp = \dim_{V_*}(\phi^*)^{-1}(U^\perp) = \dim_{V_*} U^\perp = \dim_{V_*} C^\perp$. Thus ϕ induces a well-defined automorphism of $\mathcal{F}\ell(F,E)$. $\square$

Corollary 5.4. *Let $F = (0 \subset C \subset C' \subset V)$ be a generalized flag of length 2 such that $\dim C, \operatorname{codim} C' < \infty$. Assume that the pair $(\phi,\psi) \in \mathrm{GL}(V)\times \mathrm{GL}(V_*)$ induces a well-defined automorphism of $\mathcal{F}\ell(F,E)$*

$$(0\subset U\subset U'\subset V) \mapsto (0\subset \phi(U)\subset fl\circ\psi\circ fl(U')\subset V) \in \mathcal{F}\ell(F,E).$$

Then ϕ belongs to $M(V,V_*)$.

Proof. Follows from lemma 5.1 and lemma 5.3. $\square$

In the rest of the thesis F stands for a generalized flag consisting of subspaces $\{C_\alpha\}_{\alpha\in I}$ such that $\dim C_\alpha < \infty$ or $\operatorname{codim} C_\alpha < \infty$ for all $\alpha \in I$. We shall choose the linearly ordered index set I so that $I \subset \mathbb{Z}_{>0} \sqcup \mathbb{Z}_{<0}$, $\alpha < \beta$ whenever $\alpha \in \mathbb{Z}_{>0}$, $\beta \in \mathbb{Z}_{<0}$, $\dim C_\alpha < \infty$ for $\alpha \in I\cap\mathbb{Z}_{>0}$, and $\operatorname{codim} C_\alpha < \infty$ for $\alpha \in I \cap \mathbb{Z}_{<0}$.

Consider the Picard groups of the finite-dimensional flag varieties $\operatorname{Pic}\mathcal{F}\ell(F_n,V_n) \cong \bigoplus_i \mathbb{Z}[L_i^n]$ where $L_i^n = (p_i^n)^*(\mathcal{O}_{Gr(d_i,V_n)}(1))$ (as in section 4) for each $i \in I$. Recall the closed embeddings $\iota_n : \mathcal{F}\ell(F_n,V_n) \hookrightarrow \mathcal{F}\ell(F_{n+1},V_{n+1})$ (introduced in section 2). Note that $\iota_n^* L_i^{n+1} = L_i^n$ and $\iota_n^* : \operatorname{Pic}\mathcal{F}\ell(F_{n+1},V_{n+1}) \to \operatorname{Pic}\mathcal{F}\ell(F_n,V_n)$ is surjective. This allows us to claim that the Picard group of the ind-variety $\operatorname{Pic}\mathcal{F}\ell(F,E)$ is naturally isomorphic to the inverse limit $\varprojlim \operatorname{Pic}\mathcal{F}\ell(F_n,V_n) \cong \prod \mathbb{Z}[L_i]$, where $L_i := \varprojlim L_i^n$.

This point of view also corresponds to considering the morphism of ind-varieties $p_i : \mathcal{F}\ell(F,E) \to \mathrm{Gr}(C_i,E)$ and $L_i := p_i^*(\mathcal{O}_{\mathrm{Gr}(C_i,E)}(1))$.

Definition 5.5. We call a generalized flag F *symmetric* if it is of the form $(0 \subset C_1 \subset C_2 \subset \cdots \subset C_n \subset \cdots \subset C_{-n} \subset \cdots \subset C_{-1} \subset V)$ (possibly finite length) with $\dim C_i = \operatorname{codim} C_{-i} < \infty$.

Theorem 5.6. *Let F be a generalized flag. Then the following statements hold.*

- *If F is symmetric then $\operatorname{Aut}\mathcal{F}\ell(F,E)$ is the semi-direct product of a group $\mathrm{P}A_F \subset \mathrm{P}M(V,V_*)$ and the group $\mathbb{Z}_2$ corresponding to the 'flip' morphism*

$$fl : U_i \mapsto U_i^\perp = \{v \in V_* \mid v(U_i) = 0\} \subset V_* \cong V.$$

- *If* $\dim C < \infty$ *for every* $C \in F$ *then* $\operatorname{Aut} \mathcal{F}\ell(F, E) \cong \mathrm{P}A_F \subset \mathrm{PGL}(V)$.

- *If* $\operatorname{codim} C < \infty$ *for every* $C \in F$ *then* $\operatorname{Aut} \mathcal{F}\ell(F, E) \cong \mathrm{P}A_F \subset \mathrm{PGL}(V_*)$.

- *In the remaining cases,* $\operatorname{Aut} \mathcal{F}\ell(F, E) \cong \mathrm{P}A_F \subset \mathrm{P}M(V, V_*)$.

Proof. Let $\phi \in \operatorname{Aut} \mathcal{F}\ell(F, E)$. Then ϕ induces an automorphism ϕ^* on $\operatorname{Pic} \mathcal{F}\ell(F, E)$. Fix $\mathcal{L} := L_i$ and let $\mathcal{L}' := \phi^*\mathcal{L} = \prod \alpha_j L_j$. Since $\mathcal{L}$ is generated by its global sections we must have the same for $\mathcal{L}'$, thus $\alpha_j \geq 0$ for all $j \in I$.

Consider the set $S := \bigoplus_i \mathbb{Z}_{\geq 0}[L_i] \subset \operatorname{Pic} \mathcal{F}\ell(F, E)$ consisting of elements that are generated by their global sections. Note that S is closed under addition, hence $\phi^*(S) = S$. Consider $S' := \{x \in S | x = y + z \text{ for some } y, z \in S \implies y = 0 \text{ or } z = 0\}$, then we check that $\phi^*(S') = S'$ and $S' = \{0\} \cup \{L_i\}$. Therefore ϕ^* permutes the basis $\{L_i\}$. Thus $\mathcal{L}' = L_j$ for some $j \in I$.

Note that the automorphism $\phi : \mathcal{F}\ell(F, E) \to \mathcal{F}\ell(F, E)$ of ind-varieties is induced by closed immersions $\phi_n : \mathcal{F}\ell(F_n, V_n) \to \mathcal{F}\ell(F_{N(n)}, V_{N(n)})$ of varieties. By corollary 4.4 we get morphisms $\theta_j^n : \mathrm{Gr}(\dim(C_j \cap V_n), V_n) \to \mathrm{Gr}(\dim(C_i \cap V_{N(n)}), V_{N(n)})$.

Analogously, considering ϕ^{-1} we obtain morphisms $\psi_i^n : \mathrm{Gr}(\dim(C_i \cap V_n), V_n) \to \mathrm{Gr}(\dim(C_j \cap V_{M(n)}), V_{M(n)})$. The morphism $\theta_j^{N(n)} \circ \psi_i^n : \mathrm{Gr}(\dim(C_i \cap V_n), V_n) \to \mathrm{Gr}(\dim(C_i \cap V_{N(M(n))}), V_{N(M(n))})$ induces the ind-variety morphism $\theta_j \circ \psi_i : \mathrm{Gr}(C_i, E) \to \mathrm{Gr}(C_i, E)$. Since ϕ is an isomorphism, we conclude that $\theta_j \circ \psi_i = \varinjlim \theta_j^{N(n)} \circ \psi_i^n = \mathrm{id}_{\mathrm{Gr}(C_i, E)}$, and $\psi_i := \varinjlim \psi_i^n : \mathrm{Gr}(C_i, E) \to \mathrm{Gr}(C_j, E)$ is an isomorphism of ind-varieties. By example 2.5 we have either $i = j$ or $\dim C_i = \operatorname{codim} C_j$ and $\operatorname{codim} C_i = \dim C_j$.

If $\phi^*(L_i) = L_j$ with $\dim C_i = \operatorname{codim} C_j < \infty$ then $\widetilde{\psi_i} := fl \circ \psi_i \in \operatorname{Aut} \mathrm{Gr}(C_i, E) \cong \mathrm{PGL}(V)$, and thus ψ_i is induced by $fl \circ \widetilde{\psi_i}$.

If $\phi^*(L_i) = L_i$ then $i = j$, $\psi_i \in \operatorname{Aut} \mathrm{Gr}(C_i, E)$, and we get two cases:

- $\dim C_i < \infty$, and ψ_i is induced by $\widetilde{\psi_i} \in \mathrm{PGL}(V)$;

- $\operatorname{codim} C_i < \infty$, and ψ_i is induced by $fl \circ \widetilde{\psi_i} \circ fl$ with $\widetilde{\psi_i} \in \mathrm{PGL}(V_*)$.

If $\phi^*(L_i) = L_i$ for every $i \in I$ then by lemma 5.1 the automorphism ϕ is induced globally by the action of an element in $\mathrm{GL}(V)$ (or $\mathrm{GL}(V_*)$). In particular, considering all such elements, we obtain a subgroup $A_F \subset \mathrm{GL}(V)$ (respectively, $A_F \subset \mathrm{GL}(V_*)$).

If however F is of the form $(C_1 \subset C_2 \subset \cdots \subset C_n \subset \cdots \subset C_{-n} \subset \cdots \subset C_{-1})$ with $\dim C_i = \operatorname{codim} C_{-i}$ and $\phi^*(L_i) = L_{-i}$ for all i then $(\phi \circ fl)^*(L_i) = L_i$ for all $i \in I$, thus $\phi \circ fl \in A_F$ by the above.

We are left with the 'mixed' case when $\phi^*(L_j) = L_j$ and $\phi^*(L_i) = L_{-i}$ for some $j, i, -i \in I$ with $\dim C_i = \operatorname{codim} C_{-i} < \infty$. We have four possibilities:

- If $C_j \subset C_i \subset C_{-i}$, then by the definition of ϕ we have

$$\begin{aligned} C_j \subset C_i \subset C_{-i} \mapsto \psi_j(C_j) \subset (\psi_i)^{-1}(C_{-i}) \subset \psi_i(C_i) = \\ \widetilde{\psi_j}(C_j) \subset (\widetilde{\psi_i}^{-1} \circ fl)(C_{-i}) \subset (fl \circ \widetilde{\psi_i})(C_i). \end{aligned}$$

  By considering the action of $\mathrm{GL}(E,V)$ on the flag $C_j \subset C_{-i}$, we obtain

$$\widetilde{\psi_j}(C_{-i}) = \sum_{g \in \mathrm{GL}(E,V), g(C_{-i}) = C_{-i}} \widetilde{\psi_j}(g(C_j)) \subset (\widetilde{\psi_i}^{-1} \circ fl)(C_{-i}),$$

  which gives a contradiction since $\dim \mathrm{LHS} = \infty > \dim \mathrm{RHS} = \operatorname{codim} C_{-i}$.

- If $C_i \subset C_j \subset C_{-i}$ with $\dim C_j < \infty$, then by the definition of ϕ we have

$$\begin{aligned} C_i \subset C_j \subset C_{-i} \mapsto (\psi_i)^{-1}(C_{-i}) \subset \psi_j(C_j) \subset \psi_i(C_i) = \\ (\widetilde{\psi_i}^{-1} \circ fl)(C_{-i}) \subset \widetilde{\psi_j}(C_j) \subset (fl \circ \widetilde{\psi_i})(C_i). \end{aligned}$$

  By considering the action of $\mathrm{GL}(E,V)$ on the flag $C_j \subset C_{-i}$, we obtain

$$(\widetilde{\psi_i}^{-1} \circ fl)(C_j) = \sum_{g \in \mathrm{GL}(E,V), g(C_j) = C_j} (\widetilde{\psi_i}^{-1} \circ fl)(g(C_{-i})) \subset \widetilde{\psi_j}(C_j),$$

  which gives a contradiction since $\dim \mathrm{LHS} = \infty > \dim \mathrm{RHS} = \dim C_j$.

- If $C_i \subset C_j \subset C_{-i}$ with $\operatorname{codim} C_j < \infty$, then by the definition of ϕ we have

$$\begin{aligned} C_i \subset C_j \subset C_{-i} \mapsto (\psi_i)^{-1}(C_{-i}) \subset \psi_j(C_j) \subset \psi_i(C_i) = \\ (\widetilde{\psi_i}^{-1} \circ fl)(C_{-i}) \subset \widetilde{\psi_j}(C_j) \subset (fl \circ \widetilde{\psi_i})(C_i). \end{aligned}$$

  By considering the action of $\mathrm{GL}(E,V)$ on the flag $C_i \subset C_j$, we obtain

$$\widetilde{\psi_j}(C_j) \subset \bigcap_{g \in \mathrm{GL}(E,V), g(C_j) = C_j} (fl \circ \widetilde{\psi_i})(g(C_i)) = (fl \circ \widetilde{\psi_i})(C_j),$$

  which gives a contradiction since $\dim \mathrm{LHS} = \infty > \dim \mathrm{RHS} = \operatorname{codim} C_j$.

- If $C_i \subset C_{-i} \subset C_j$, then by the definition of ϕ we have

$$\begin{aligned} C_i \subset C_{-i} \subset C_j \mapsto (\psi_i)^{-1}(C_{-i}) \subset \psi_i(C_i) \subset \psi_j(C_j) = \\ (\widetilde{\psi_i}^{-1} \circ fl)(C_{-i}) \subset (fl \circ \widetilde{\psi_i})(C_i) \subset \widetilde{\psi_j}(C_j). \end{aligned}$$

  By considering the action of $\mathrm{GL}(E, V)$ on the flag $C_i \subset C_j$, we obtain

$$(fl \circ \widetilde{\psi_i})(C_i) \subset \bigcap_{g \in \mathrm{GL}(E,V), g(C_i) = C_i} \widetilde{\psi_j}(g(C_j)) = \widetilde{\psi_j}(C_i),$$

  which gives a contradiction since $\dim \mathrm{LHS} = \infty > \dim \mathrm{RHS} = \dim C_i$.

Thus we get a contradiction to the 'mixed' case and we are done. $\square$

Since the action of $A_F \subset \mathrm{GL}(V)$ (or $A_F \subset \mathrm{GL}(V_*)$) is effective, it consists of all maps $\phi \in \mathrm{GL}(V)$ (or $\mathrm{GL}(V_*)$) whose usual action on $\mathcal{F}\ell(F, E)$ is well-defined (i.e. $\phi(G) \in \mathcal{F}\ell(F, E)$ for any $G \in \mathcal{F}\ell(F, E)$).

Similarly to example 2.5, we denote by $P_{(F,E)} \subset \mathrm{GL}(V)$ the subgroup of linear maps that fix F.

For a subgroup $H \subset \mathrm{GL}(V_*)$ we set

$$H^* := \{h^* : (V_*)^* \to (V_*)^* \mid h \in H\} \subset \mathrm{End}((V_*)^*).$$

We shall use the following result:

Proposition 5.7. *Let $\phi \in \mathrm{GL}(V)$ be such that $\phi(G) \in \mathcal{F}\ell(F, E)$ for any $G \in \mathcal{F}\ell(F, E)$. Then $\phi \in \mathrm{GL}(E, V) \cdot P_{(F,E)}$.*

Proof. Since $\phi(F) \in \mathcal{F}\ell(F, E)$, there is $\psi \in \mathrm{GL}(E, V)$ such that $\psi \circ \phi(F) = F$, i.e. $\psi \circ \phi \in P_{(F,E)}$. Hence $\phi \in \mathrm{GL}(E, V) \cdot P_{(F,E)}$. $\square$

The following is a summary of our results.

Theorem 5.8. *Let F be a generalized flag, denote $F_+ := \{C \in F \mid \dim C < \infty\} \cup \{V\}$ and $F_- := \{C \in F \mid \operatorname{codim} C < \infty\} \cup \{0\}$ to be the generalized sub-flags of the finite dimension and cofinite dimension components respectively. Then the following table describes the group A_F:*

	$F_+ = (0 \subset V)$	$\lvert F_+\rvert < \infty$	$\lvert F_+\rvert = \infty$
$F_- = (0 \subset V)$	0	$\mathrm{GL}(V)$	$\mathrm{GL}(E, V) \cdot P_{(F_+,E)}$
$\lvert F_-\rvert < \infty$	$\mathrm{GL}(V_*)$	$M(V, V_*)$	$\mathrm{GL}(E, V) \cdot P_{(F_+,E)} \cap M(V, V_*)$
$\lvert F_-\rvert = \infty$	$\mathrm{GL}(E^*, V_*) \cdot P_{(F_-,E^*)}$	$\mathrm{GL}(E^*, V_*) \cdot P_{(F_-,E^*)} \cap M(V_*, V)$	$(\mathrm{GL}(E^*, V_*) \cdot P_{(F_-,E^*)})^* \cap \mathrm{GL}(E, V) \cdot P_{(F_+,E)} \cap M(V, V_*)$

The automorphism group $\mathrm{Aut}\, \mathcal{F}\ell(F, E)$ *is then isomorphic to* $\mathrm{P}\left(\langle A_F, fl\rangle\right)$ *when F is symmetric and* $\mathrm{P}A_F$ *otherwise.*

6 Conclusion and further questions

We have described in this thesis the automorphism groups of ind-varieties of generalized flags under the condition that the generalized flags consist of subspaces of finite dimension and finite codimension. Describing the automorphism group of a general ind-variety of generalized flags is an open problem.

A starting question would be to consider a general ind-grassmannian and calculate its automorphism group.

Another question of interest is to describe the isomorphism classes of the automorphism groups described here. As we have seen, the connected components of unity of the automorphism groups of the Grassmannian and of the flag ind-variety $\mathcal{F}\ell(F, E)$ for $F = (0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset \dots)$ are not isomorphic. This is drastically different from the classical case where all automorphism groups of the flag varieties have isomorphic connected components of unity.

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