TENSOR REPRESENTATIONS OF CLASSICAL LOCALLY FINITE LIE ALGEBRAS

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Abstract. The structure of tensor representations of the classical finite-dimensional Lie algebras was described by H. Weyl. In this paper we extend Weyl's results to the classical infinite-dimensional locally finite Lie algebras $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$, and study important new features specific to the infinite-dimensional setting.

Let $\mathfrak{g}$ be one of the above locally finite Lie algebras and let $\mathbf{V}$ be the natural representation of $\mathfrak{g}$. The tensor representations of $\mathfrak{g}$ have the form $\mathbf{V}^{\otimes d}$ for the cases $\mathfrak{g} = \mathfrak{sp}_\infty, \mathfrak{so}_\infty$, and the form $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$ for the cases $\mathfrak{g} = \mathfrak{gl}_\infty, \mathfrak{sl}_\infty$, where $\mathbf{V}_*$ is the restricted dual of $\mathbf{V}$. In contrast with the finite-dimensional case, these tensor representations are not semisimple. We explicitly describe their Jordan-Hölder constituents, socle filtrations, and indecomposable direct summands.

0. Introduction

One of H. Weyl's beautiful results is the description of the structure of the space of tensors of given rank as a module over the various classical groups of transformations of the underlying space. In this paper we extend Weyl's constructions to the locally-finite Lie algebras $\mathfrak{gl}_\infty, \mathfrak{sl}_\infty, \mathfrak{so}_\infty$ and $\mathfrak{sp}_\infty$, and highlight some important differences and new features arising in the infinite-dimensional case.

Each of the Lie algebras $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty, \mathfrak{so}_\infty$ and $\mathfrak{sp}_\infty$ can be defined as the union of the respective series of finite-dimensional Lie algebras under the obvious inclusions. We refer to $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty, \mathfrak{so}_\infty$ and $\mathfrak{sp}_\infty$ as the classical infinite-dimensional locally finite Lie algebras. The structure of these Lie algebras and their representations have been studied actively in recent years; an incomplete list of references for this field is [B, BB, BS, DP1, DP2, DPS, DPW, N, NS, Na, NP, O1, PS, PZ]. Nevertheless, the structure of the tensor representations of the classical infinite-dimensional locally finite Lie algebras has not yet been sufficiently explored.

Recall that for the simple classical Lie groups SL_n, SO_n and Sp_{2n} every irreducible finite-dimensional module can be realized inside the covariant tensor algebra of the natural representation V. In particular, for the dual representation V^* we have $V^* \cong V$ in the cases of SO_n and Sp_{2n}, and $V^* \cong \bigwedge^{n-1} V$ in the case of SL_n. However, for the reductive Lie group GL_n not every finite-dimensional irreducible representation occurs in the tensor algebra $\mathrm{T}(V)$. For example, the dual representation V^* does not appear in $\mathrm{T}(V)$. Thus, in order to obtain all irreducible finite-dimensional GL_n-modules, we must consider mixed tensors, i.e. the tensor algebra $\mathrm{T}(V \oplus V^*)$.

A similar phenomenon occurs for the simple Lie algebra $\mathfrak{sl}_\infty$. Let $\mathbf{V}$ denote the natural representation of $\mathfrak{sl}_\infty$. One can show that the (restricted) dual representation $\mathbf{V}_*$ does not occur in the tensor algebra $\mathrm{T}(\mathbf{V})$, and we study the space of mixed tensors $\mathrm{T}(\mathbf{V} \oplus \mathbf{V}_*)$. The main new feature of these tensor representations is their failure to be completely reducible. For $\mathfrak{so}_\infty$ and $\mathfrak{sp}_\infty$ the natural representation $\mathbf{V}$ is self-dual, and we consider just the tensor algebra $\mathrm{T}(\mathbf{V})$, which also fails to be a completely reducible $\mathfrak{so}_\infty$- or $\mathfrak{sp}_\infty$-module.

The main purpose of this paper is to describe the structure of the tensor algebras $\mathrm{T}(\mathbf{V}\oplus\mathbf{V}_*)$ and $\mathrm{T}(\mathbf{V})$ as explicitly as possible. First, we decompose the tensor algebra into indecomposable submodules of finite length. The answer is very simple: for $\mathfrak{g} = \mathfrak{gl}_\infty, \mathfrak{sl}_\infty$ the indecomposables are of the form $\mathbf{\Gamma}_{\lambda;0} \otimes \mathbf{\Gamma}_{0;\mu}$, where $\mathbf{\Gamma}_{\lambda;0}$ and $\mathbf{\Gamma}_{0;\mu}$ are the simple irreducible $\mathfrak{sl}_\infty$-submodules of the tensor algebras $\mathrm{T}(\mathbf{V})$ and $\mathrm{T}(\mathbf{V}_*)$ respectively, and for $\mathfrak{g} = \mathfrak{so}_\infty, \mathfrak{sp}_\infty$ the indecomposable modules are the $\mathfrak{gl}_\infty$-modules $\mathbf{\Gamma}_{\lambda;0}$, considered as $\mathfrak{g}$-modules.

We then explicitly describe the socle filtration of $\mathbf{\Gamma}_{\lambda;0} \otimes \mathbf{\Gamma}_{0;\mu}$ as an $\mathfrak{sl}_\infty$-module, and the socle filtrations of $\mathbf{\Gamma}_{\lambda;0}$ both as an $\mathfrak{so}_\infty$- and an $\mathfrak{sp}_\infty$-module. In particular, we prove that each indecomposable module has a simple socle.

In the last section of the paper we extend our description of tensor representations to a wider class of Lie algebras $\mathfrak{g} = \mathfrak{k} \ni\!\!\!\!+ \mathfrak{m}$, where the ideal $\mathfrak{k}$ is a classical simple locally-finite Lie algebra, and $\mathfrak{m}$ satisfies a certain technical condition. This class of Lie algebras includes, in particular, the root-reductive Lie algebras. The main result in this case states that the natural representation $\mathbf{V}$ of $\mathfrak{k}$ can be equipped with a $\mathfrak{g}$-module structure, and the socle filtration of any tensor representation of $\mathfrak{g}$ coincides with its socle filtration as a $\mathfrak{k}$-module.

1. Preliminaries

By $\mathbb{N}$ we denote the set of positive integers. In this paper we work with a field $\Bbbk$ of characteristic zero. All vector spaces, Lie algebras, and their modules are assumed to be defined over $\Bbbk$. Tensor products are also over $\Bbbk$. All algebras and modules are assumed to be at most countable dimensional over $\Bbbk$. The sign $\ni\!\!\!\!+$ stands for semidirect sum of Lie algebras: in $\mathfrak{k} \ni\!\!\!\!+ \mathfrak{m}$ the Lie algebra $\mathfrak{k}$ is an ideal.

Let V be a finite length module over some algebra. The *socle* of V, denoted $\operatorname{soc} V$, is the maximal semisimple submodule of V. Equivalently, $\operatorname{soc} V$ is the sum of all simple submodules of V. The *socle filtration* of V is defined inductively by

$$\operatorname{soc}^{(0)} V = 0, \qquad \operatorname{soc}^{(i+1)} V / \operatorname{soc}^{(i)} V = \operatorname{soc}(V/\operatorname{soc}^{(i)} V), \quad i \geq 0.$$

The smallest positive integer l such that $\operatorname{soc}^{(l)} V = V$ is called the *Loewy length* of V. The semisimple modules $\overline{\operatorname{soc}}^{(i)} V \overset{\text{def}}{=} \operatorname{soc}^{(i)} V / \operatorname{soc}^{(i-1)} V$ are called the *layers* of the socle filtration.

A *partition* λ is by definition a finite nonstrictly decreasing sequence of positive integers:

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k).$$

For any partition λ we set $|\lambda| \overset{\text{def}}{=} \lambda_1 + \cdots + \lambda_k$, and use the convention that $\lambda_i = 0$ if $i > k$. The empty partition is denoted by 0.

For any partition λ we denote by 2λ the partition determined by $(2\lambda)_i = 2\lambda_i$ for all i, and by $\lambda^\top$ the partition determined by $(\lambda^\top)_i = \#\{j \mid \lambda_j \geq i\}$ for all i. The Young diagram for $\lambda^\top$ is obtained by transposing the Young diagram for λ.

Irreducible representations of the symmetric group $\mathfrak{S}_d$ are parameterized by partitions λ satisfying $|\lambda| = d$, and can be realized inside the regular representation $\Bbbk[\mathfrak{S}_d]$. We denote

$$H_\lambda \stackrel{\text{def}}{=} \Bbbk[\mathfrak{S}_d]\, c_\lambda,$$

where $c_\lambda \in \Bbbk[\mathfrak{S}_d]$ is the Young projector corresponding to the standard Young tableau of shape λ (see e.g. [FH] for details). The left regular action makes H_λ an irreducible $\mathfrak{S}_d$-module, and any irreducible representation of $\mathfrak{S}_d$ can be obtained in this way.

For any vector space V, the symmetric group $\mathfrak{S}_d$ acts in $V^{\otimes d}$ by permuting the tensor factors. For any partition λ with $|\lambda| = d$ we denote

$$\mathbb{S}_\lambda V = \operatorname{im}\Big(c_\lambda : V^{\otimes d} \to V^{\otimes d}\Big)$$

The correspondence $V \rightsquigarrow \mathbb{S}_\lambda V$ is called the *Schur functor* corresponding to λ.

The *Littlewood-Richardson coefficients* $\mathrm{N}^\nu_{\lambda,\mu}$ are nonnegative integers, determined for any partitions λ, μ, ν by the relation $S_\lambda\, S_\mu = \sum_\nu \mathrm{N}^\nu_{\lambda,\mu}\, S_\nu$, where S_λ denotes the Schur symmetric polynomial corresponding to the partition λ. In particular, $\mathrm{N}^\nu_{\lambda,\mu} = 0$ unless $|\nu| = |\lambda| + |\mu|$.

2. Tensor representations of $\mathfrak{gl}_\infty$ and $\mathfrak{sl}_\infty$

Let $\mathbf{V}, \mathbf{V}_*$ be countable dimensional vector spaces, and let $\langle\cdot,\cdot\rangle : \mathbf{V} \otimes \mathbf{V}_* \to \Bbbk$ be a non-degenerate pairing. The Lie algebra $\mathfrak{gl}_\infty$ is defined as the space $\mathbf{V} \otimes \mathbf{V}_*$, equipped with the Lie bracket

$$[u \otimes u^*, v \otimes v^*] = \langle u^*, v\rangle\, u \otimes v^* - \langle v^*, u\rangle\, v \otimes u^*, \qquad u, v \in \mathbf{V},\ u^*, v^* \in \mathbf{V}_*. \tag{2.1}$$

The kernel of the map $\langle\cdot,\cdot\rangle$ is a Lie subalgebra of $\mathfrak{gl}_\infty$, which we denote $\mathfrak{sl}_\infty$.

As observed by G. Mackey [M], there always exist dual bases $\{\xi_i\}_{i\in\mathfrak{I}}$ of $\mathbf{V}$ and $\{\xi_i^*\}_{i\in\mathfrak{I}}$ of $\mathbf{V}_*$, indexed by a countable set $\mathfrak{I}$, so that we have $\langle\xi_j^*, \xi_i\rangle = \delta_{i,j}$ for $i, j \in \mathfrak{I}$. This gives another, more straightforward coordinate definition of $\mathfrak{gl}_\infty$ as the Lie algebra with a linear basis $\{E_{i,j} = \xi_i \otimes \xi_j^*\}_{i,j\in\mathfrak{I}}$ satisfying the usual commutation relations $[E_{i,j}, E_{k,l}] = \delta_{j,k}E_{i,l} - \delta_{i,l}E_{k,j}$.

We call $\mathbf{V}$ the *natural representation* of $\mathfrak{gl}_\infty$ and $\mathfrak{sl}_\infty$, and $\mathbf{V}_*$ its *restricted dual.* For any nonnegative integers p, q we define the *tensor representation* $\mathbf{V}^{\otimes(p,q)}$ as the vector space

$\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$, equipped with the following $\mathfrak{gl}_\infty$-module structure:

$$(u \otimes u^*) \cdot (v_1 \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots \otimes v_q^*)$$
$$= \sum_{i=1}^{p} \langle u^*, v_i \rangle \, v_1 \otimes \ldots \otimes v_{i-1} \otimes u \otimes v_{i+1} \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots \otimes v_q^*$$
$$- \sum_{j=1}^{q} \langle v_j^*, u \rangle \, v_1 \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots \otimes v_{j-1}^* \otimes u^* \otimes v_{j+1} \otimes \ldots \otimes v_q^*$$

for $u, v_1, \ldots, v_p \in V$ and $u^*, v_1^*, \ldots, v_q^* \in \mathbf{V}_*$. The product of symmetric groups $\mathfrak{S}_p \times \mathfrak{S}_q$ acts in $\mathbf{V}^{\otimes(p,q)}$ by permuting the factors, and this action commutes with the action of $\mathfrak{gl}_\infty$. We express this by saying that $\mathbf{V}^{\otimes(p,q)}$ is a $(\mathfrak{gl}_\infty, \mathfrak{S}_p \times \mathfrak{S}_q)$-module, and use similar notation throughout the paper.

Our main goal in this section is to reveal the structure of the tensor representations as a $\mathfrak{gl}_\infty$-module, and in particular to identify the Jordan-Hölder constituents of $\mathbf{V}^{\otimes(p,q)}$. We describe these modules explicitly as appropriately defined highest weight $\mathfrak{gl}_\infty$-modules. Consider the direct sum decomposition $\mathfrak{gl}_\infty = \mathfrak{h}_{\mathfrak{gl}} \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathfrak{gl}}} \Bbbk X_\alpha^{\mathfrak{gl}} \right)$, where

$$\mathfrak{h}_{\mathfrak{gl}} = \bigoplus_{i \in \mathfrak{I}} \Bbbk E_{i,i}, \qquad \Delta_{\mathfrak{gl}} = \left\{ \varepsilon_i - \varepsilon_j \,\middle|\, i, j \in \mathfrak{I},\ i \neq j \right\}, \qquad X^{\mathfrak{gl}}_{\varepsilon_i - \varepsilon_j} = E_{i,j}\ ,$$

and ε_i denotes the functional on $\mathfrak{h}_{\mathfrak{gl}}$ determined by $\varepsilon_i(E_{j,j}) = \delta_{i,j}$. We have $[H, X_\alpha^{\mathfrak{gl}}] = \alpha(H) X_\alpha^{\mathfrak{gl}}$ for any $\alpha \in \Delta_{\mathfrak{gl}}$ and $H \in \mathfrak{h}_{\mathfrak{gl}}$. As usual, we refer to elements of $\Delta_{\mathfrak{gl}}$ as *roots*, and to functionals on $\mathfrak{h}_{\mathfrak{gl}}$ as *weights*.

Remark 1. For an algebraically closed field $\Bbbk$ the general notion of a Cartan subalgebra of $\mathfrak{gl}_\infty$ has been defined and studied in [NP, DPS]. The subalgebra $\mathfrak{h}_{\mathfrak{gl}}$ is an example of a *splitting Cartan* subalgebra, and all splitting Cartan subalgebras of $\mathfrak{gl}_\infty$ are conjugated.

From now on we identify the index set $\mathfrak{I}$ with $\mathbb{Z} \setminus \{0\}$, and consider the polarization of the root system $\Delta_{\mathfrak{gl}} = \Delta_{\mathfrak{gl}}^+ \coprod -\Delta_{\mathfrak{gl}}^+$, where the set $\Delta_{\mathfrak{gl}}^+$ of *positive roots* is given by

$$\Delta_{\mathfrak{gl}}^+ = \{\varepsilon_i - \varepsilon_j \,|\, 0 < i < j\} \bigcup \{\varepsilon_i - \varepsilon_j \,|\, i < j < 0\} \bigcup \{\varepsilon_i - \varepsilon_j \,|\, j < 0 < i\}.$$

We denote $\mathfrak{n}_{\mathfrak{gl}} = \bigoplus_{\alpha \in \Delta_{\mathfrak{gl}}^+} \mathfrak{g}_\alpha$ and define $\mathfrak{b}_{\mathfrak{gl}}$ as the Lie subalgebra of $\mathfrak{gl}_\infty$, generated by $\mathfrak{n}_{\mathfrak{gl}}$ and $\mathfrak{h}_{\mathfrak{gl}}$. It is clear that $\mathfrak{n}_{\mathfrak{gl}}$ is a Lie subalgebra of $\mathfrak{gl}_\infty$, that $\mathfrak{b}_{\mathfrak{gl}} = \mathfrak{n}_{\mathfrak{gl}} \in\!\!\!\!+ \mathfrak{h}_{\mathfrak{gl}}$, and that $[\mathfrak{b}_{\mathfrak{gl}}, \mathfrak{b}_{\mathfrak{gl}}] = \mathfrak{n}_{\mathfrak{gl}}$.

Let V be a $\mathfrak{gl}_\infty$-module, and let $v \in V$. We say that v is a *highest weight vector* if it generates a one-dimensional $\mathfrak{b}_{\mathfrak{gl}}$-module. Any such v must satisfy

$$\mathfrak{n}_{\mathfrak{gl}} \, v = 0, \qquad H\, v = \chi(H)\, v \qquad \forall H \in \mathfrak{h}_{\mathfrak{gl}},$$

for some $\chi \in \mathfrak{h}_{\mathfrak{gl}}^*$. We say that V is a *highest weight module* if V is generated by a highest weight vector v as above; the functional χ is then called the *highest weight of* V. As in the finite-dimensional case, it is easy to prove that for each $\chi \in \mathfrak{h}_{\mathfrak{gl}}^*$ there exists a unique irreducible highest weight $\mathfrak{gl}_\infty$-module with highest weight χ.

Remark 2. The subalgebra $\mathfrak{b}_{\mathfrak{gl}}$ is an example of a *splitting Borel* subalgebra of $\mathfrak{gl}_\infty$, see [DP2] for a classification of the splitting Borel subalgebras in $\mathfrak{gl}_\infty$ over algebraically closed fields. In this paper we only consider the notion of a highest weight module, associated with $\mathfrak{b}_{\mathfrak{gl}}$. This choice has the important property that all Jordan-Hölder constituents of all tensor representations are highest weight modules.

In particular, the natural representation $\mathbf{V}$ is a highest weight module with highest weight ε_1, generated by the highest weight vector ξ_1; similarly, $\mathbf{V}_*$ is a highest weight module with highest weight $-\varepsilon_{-1}$, generated by the highest weight vector ξ^*_{-1}.

We now describe the Weyl construction for $\mathfrak{gl}_\infty$. For any pair of indices $I = (i, j)$ with $i \in \{1, 2, \dots, p\}$ and $j \in \{1, 2, \dots, q\}$, define the contraction

$$\Phi_I : \mathbf{V}^{\otimes(p,q)} \to \mathbf{V}^{\otimes(p-1,q-1)},$$
$$v_1 \otimes \dots \otimes v_p \otimes v_1^* \otimes \dots \otimes v_q^* \mapsto \langle v_j^*, v_i\rangle\, v_1 \otimes \dots \otimes \hat{v}_i \otimes \dots \otimes v_p \otimes v_1^* \otimes \dots \otimes \hat{v}_j^* \otimes \dots \otimes v_q^*,$$

and consider the $(\mathfrak{gl}_\infty, \mathfrak{S}_p \times \mathfrak{S}_q)$-submodule $\mathbf{V}^{\{p,q\}}$ of $\mathbf{V}^{\otimes(p,q)}$,

$$\mathbf{V}^{\{p,q\}} \stackrel{\text{def}}{=} \bigcap_I \ker\Big(\Phi_I : \mathbf{V}^{\otimes(p,q)} \to \mathbf{V}^{\otimes(p-1,q-1)}\Big).$$

Set also $\mathbf{V}^{\{p,0\}} \stackrel{\text{def}}{=} \mathbf{V}^{\otimes p}$ and $\mathbf{V}^{\{0,q\}} \stackrel{\text{def}}{=} \mathbf{V}_*^{\otimes q}$. For any partitions λ, μ such that $|\lambda| = p$ and $|\mu| = q$, define the $\mathfrak{gl}_\infty$-submodule of $\mathbf{V}^{\otimes(p,q)}$

$$\mathbf{\Gamma}_{\lambda;\mu} \stackrel{\text{def}}{=} \mathbf{V}^{\{p,q\}} \cap (\mathbb{S}_\lambda \mathbf{V} \otimes \mathbb{S}_\mu \mathbf{V}_*).$$

Theorem 2.1. *For any p, q there is an isomorphism of $(\mathfrak{gl}_\infty, \mathfrak{S}_p \times \mathfrak{S}_q)$-modules*

$$\mathbf{V}^{\{p,q\}} \cong \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} \mathbf{\Gamma}_{\lambda;\mu} \otimes (H_\lambda \otimes H_\mu). \tag{2.2}$$

For any partitions λ, μ, the $\mathfrak{gl}_\infty$-module $\mathbf{\Gamma}_{\lambda;\mu}$ is an irreducible highest weight module with highest weight $\chi \stackrel{\text{def}}{=} \sum_{i\in\mathbb{N}} \lambda_i\, \varepsilon_i - \sum_{i\in\mathbb{N}} \mu_i\, \varepsilon_{-i}$. Furthermore, $\mathbf{\Gamma}_{\lambda;\mu}$ is irreducible when regarded by restriction as an $\mathfrak{sl}_\infty$-module.

Proof. Pick an enumeration $\mathfrak{I} = \{k_1, k_2, k_3, \dots\}$ of the index set $\mathfrak{I} = \mathbb{Z}\setminus\{0\}$, such that $k_i = i$ for $1 \le i \le p$, and $k_{p+i} = -i$ for $1 \le i \le q$. For each n denote by V_n the subspace of $\mathbf{V}$ spanned by $\xi_{k_1}, \dots, \xi_{k_n}$, and by V_n^* the subspace of $\mathbf{V}_*$ spanned by $\xi^*_{k_1}, \dots, \xi^*_{k_n}$; the pairing between $\mathbf{V}$ and $\mathbf{V}_*$ restricts to a nondegenerate pairing between V_n and V_n^*.

Denote by $\mathfrak{g}_n$ the Lie subalgebra of $\mathfrak{gl}_\infty$ generated by $\{E_{k_i,k_j}\}_{1\le i,j\le n}$, and set $\mathfrak{b}_n \stackrel{\text{def}}{=} \mathfrak{b}_{\mathfrak{gl}} \cap \mathfrak{g}_n$, $\mathfrak{h}_n \stackrel{\text{def}}{=} \mathfrak{h}_{\mathfrak{gl}} \cap \mathfrak{g}_n$. It is clear that $\mathfrak{g}_n \cong \mathfrak{gl}_n$, and that $\mathfrak{h}_n$ (respectively, $\mathfrak{b}_n$) is a Cartan (respectively, Borel) subalgebra of $\mathfrak{g}_n$. Moreover, the inclusion $\mathfrak{g}_n \hookrightarrow \mathfrak{g}_{n+1}$ restricts to inclusions $\mathfrak{h}_n \hookrightarrow \mathfrak{h}_{n+1}$ and $\mathfrak{b}_n \hookrightarrow \mathfrak{b}_{n+1}$.

The tensor representations $(V_n)^{\otimes(p,q)}$ and the contractions $\Phi_I^{(n)} : (V_n)^{\otimes(p,q)} \to (V_n)^{\otimes(p-1,q-1)}$ are defined by analogy with their infinite-dimensional counterparts. We set

$$(V_n)^{\{p,q\}} \stackrel{\text{def}}{=} \bigcap_I \ker \Phi_I^{(n)}, \qquad \mathit{\Gamma}_{\lambda;\mu}^{(n)} \stackrel{\text{def}}{=} (V_n)^{\{p,q\}} \cap (\mathbb{S}_\lambda V_n \otimes \mathbb{S}_\mu V_n^*).$$

A version of the finite-dimensional Weyl construction (see Appendix for more details) implies that for $n \geq p+q$ the $\mathfrak{gl}_n$-module $\mathit{\Gamma}_{\lambda;\mu}^{(n)}$ is an irreducible highest weight module with highest weight χ, regarded by restriction as a functional on $\mathfrak{h}_n$. Futhermore, we have $(\mathfrak{g}_n, \mathfrak{S}_p \times \mathfrak{S}_q)$-module isomorphisms

$$(V_n)^{\{p,q\}} \cong \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} \mathit{\Gamma}_{\lambda;\mu}^{(n)} \otimes (H_\lambda \otimes H_\mu). \tag{2.3}$$

It is clear from the definitions that we have the following diagram of inclusions:

$$\begin{array}{ccccccccccc} (V_{p+q})^{\{p,q\}} & \subset & (V_{p+q+1})^{\{p,q\}} & \subset & \cdots & \subset & (V_n)^{\{p,q\}} & \subset & \cdots & \subset & \bigcup_{n\in\mathbb{N}} (V_n)^{\{p,q\}} & = & \mathbf{V}^{\{p,q\}} \\ \cup & & \cup & & & & \cup & & & & \cup & & \cup \\ \mathit{\Gamma}_{\lambda;\mu}^{(p+q)} & \subset & \mathit{\Gamma}_{\lambda;\mu}^{(p+q+1)} & \subset & \cdots & \subset & \mathit{\Gamma}_{\lambda;\mu}^{(n)} & \subset & \cdots & \subset & \bigcup_{n\in\mathbb{N}} \mathit{\Gamma}_{\lambda;\mu}^{(n)} & = & \mathbf{\Gamma}_{\lambda;\mu} \end{array}. \tag{2.4}$$

The irreducibility of the $\mathfrak{gl}_n$-module $\mathit{\Gamma}_{\lambda;\mu}^{(n)}$ for each n implies the irreducibility of the $\mathfrak{gl}_\infty$-module $\mathbf{\Gamma}_{\lambda;\mu}$. Similarly, the isomorphism (2.2) follows from the isomorphisms (2.3), and it remains to show that $\mathbf{\Gamma}_{\lambda;\mu}$ is a highest weight $\mathfrak{gl}_\infty$-module with highest weight χ.

For each $n \geq p+q$ the highest weight subspace $\mathit{\Gamma}_{\lambda;\mu}^{(n)}[\chi]$ is one-dimensional, and therefore

$$\mathit{\Gamma}_{\lambda;\mu}^{(p+q)}[\chi] = \mathit{\Gamma}_{\lambda;\mu}^{(p+q+1)}[\chi] = \cdots = \mathit{\Gamma}_{\lambda;\mu}^{(n)}[\chi] = \cdots = \bigcup_{n\in\mathbb{N}} \mathit{\Gamma}_{\lambda;\mu}^{(n)}[\chi] = \mathbf{\Gamma}_{\lambda;\mu}[\chi].$$

We also have $\mathfrak{b}_{\mathfrak{gl}} \cdot \mathbf{\Gamma}_{\lambda;\mu}[\chi] = \bigcup_{n\in\mathbb{N}} \left(\mathfrak{b}_n \cdot \mathit{\Gamma}_{\lambda;\mu}^{(n)}[\chi]\right) = \bigcup_{n\in\mathbb{N}} \mathit{\Gamma}_{\lambda;\mu}^{(n)}[\chi] = \mathbf{\Gamma}_{\lambda;\mu}[\chi]$, which means that any nonzero $v \in \mathbf{\Gamma}_{\lambda;\mu}[\chi]$ is a highest weight vector generating the $\mathfrak{gl}_\infty$-module $\mathbf{\Gamma}_{\lambda;\mu}[\chi]$. This completes the proof of the theorem. □

Applying Theorem 2.1 to the special cases $p = 0$ and $q = 0$, we obtain a version of Schur-Weyl duality for $\mathfrak{gl}_\infty$:

$$\mathbf{V}^{\otimes p} \cong \bigoplus_{|\lambda|=p} \mathbf{\Gamma}_{\lambda;0} \otimes H_\lambda, \qquad \mathbf{V}_*^{\otimes q} \cong \bigoplus_{|\mu|=q} \mathbf{\Gamma}_{0;\mu} \otimes H_\mu. \tag{2.5}$$

Next, we study the structure of the $\mathfrak{gl}_\infty$-module $\mathbf{V}^{\otimes(p,q)}$. The main feature and the crucial difference from the finite-dimensional case is that $\mathbf{V}^{\otimes(p,q)}$ fails to be completely reducible.

Theorem 2.2. *Let p, q be nonnegative integers, and let $\ell = \min(p,q)$. Then the Loewy length of the $\mathfrak{gl}_\infty$-module $\mathbf{V}^{\otimes(p,q)}$ equals $\ell+1$, and*

$$\operatorname{soc}^{(r)} \mathbf{V}^{\otimes(p,q)} = \bigcap_{I_1,\ldots,I_r} \ker\left(\Phi_{I_1,\ldots,I_r} : \mathbf{V}^{\otimes(p,q)} \to \mathbf{V}^{\otimes(p-r,q-r)}\right), \qquad r = 1,\ldots,\ell. \tag{2.6}$$

Moreover, the socle filtration of $\mathbf{V}^{\otimes(p,q)}$*, regarded as an* $\mathfrak{sl}_\infty$*-module, coincides with* (2.6).

Proof. Denote by $\mathbf{F}^{(r)}$ the subspaces of $\mathbf{V}^{\otimes(p,q)}$ on the right side of (2.6), and for each n set

$$F_n^{(r)} = \bigcap_{I_1,\dots,I_r} \ker\left(\Phi_{\{I_1,\dots,I_r\}} : (V_n)^{\otimes(p,q)} \to (V_n)^{\otimes(p-r,q-r)}\right), \qquad r = 1,\dots,\ell,$$

where $\{V_n\}_{n\in\mathbb{N}}$ is the exhaustion of $\mathbf{V}$ used in the proof of Theorem 2.1. Then we have the commutative diagram of $(\mathfrak{gl}_\infty, \mathfrak{S}_p \times \mathfrak{S}_q)$-modules

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
F_n^{(1)} & \subset & F_n^{(2)} & \subset & \cdots & \subset & F_n^{(\ell)} & \subset & (V_n)^{\otimes(p,q)} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
F_{n+1}^{(1)} & \subset & F_{n+1}^{(2)} & \subset & \cdots & \subset & F_{n+1}^{(\ell)} & \subset & (V_{n+1})^{\otimes(p,q)} \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\mathbf{F}^{(1)} & \subset & \mathbf{F}^{(2)} & \subset & \cdots & \subset & \mathbf{F}^{(\ell)} & \subset & \mathbf{V}^{\otimes(p,q)}
\end{array},$$

in which the vertical arrows are obtained as restrictions of the inclusions in the rightmost column, and yield exhaustions of $\mathbf{F}^{(r)}$. Denote for convenience $\mathbf{F}^{(0)} = 0$, $\mathbf{F}^{(\ell+1)} = \mathbf{V}^{\otimes(p,q)}$, and similarly $F_n^{(0)} = 0$, $F_n^{(\ell+1)} = (V_n)^{\otimes(p,q)}$ for all n. It is easy to check that the induced maps $F_n^{(r+1)}/F_n^{(r)} \to F_{n+1}^{(r+1)}/F_{n+1}^{(r)}$ are injective for all n, r, and therefore for each $r = 0, 1, \dots, \ell$ the layer $\mathbf{F}^{(r+1)}/\mathbf{F}^{(r)}$ is the union of quotients $F_n^{(r+1)}/F_n^{(r)}$.

It is a standard exercise (see Appendix for more details) to show that for each r there exists a $(\mathfrak{gl}_n, \mathfrak{S}_p \times \mathfrak{S}_q)$-module isomorphism

$$F_n^{(r+1)}/F_n^{(r)} \cong \bigoplus_{|\lambda|=p-r} \bigoplus_{|\mu|=q-r} \Gamma_{\lambda;\mu}^{(n)} \otimes H(\lambda,\mu;r), \tag{2.7}$$

where $H(\lambda,\mu;r)$ are some $\mathfrak{S}_p \times \mathfrak{S}_q$-modules. As in the proof of Theorem 2.1, it follows that

$$\mathbf{F}^{(r+1)}/\mathbf{F}^{(r)} \cong \bigoplus_{|\lambda|=p-r} \bigoplus_{|\mu|=q-r} \mathbf{\Gamma}_{\lambda;\mu} \otimes H(\lambda,\mu;r).$$

In particular, this shows that $\mathbf{V}^{\otimes(p,q)}$ has a finite Jordan-Hölder series with irreducible constituents of the form $\mathbf{\Gamma}_{\lambda;\mu}$ for appropriate λ, μ. Moreover, $\mathbf{F}^{(r)}$ is characterized as the unique submodule of $\mathbf{V}^{\otimes(p,q)}$ such that for any λ, μ

$$[\mathbf{F}^{(r)} : \mathbf{\Gamma}_{\lambda;\mu}] = \begin{cases} [\mathbf{V}^{\otimes(p,q)} : \mathbf{\Gamma}_{\lambda;\mu}] & \text{if } |\lambda| > p-r \text{ and } |\mu| > q-r, \\ 0 & \text{otherwise .} \end{cases}$$

We use induction on r to prove the main statement of the theorem: $\mathbf{F}^{(r)} = \operatorname{soc}^{(r)} \mathbf{V}^{\otimes(p,q)}$. The base of induction $r = 0$ is trivial since $\mathbf{F}^{(0)} = \operatorname{soc}^{(0)} \mathbf{V}^{\otimes(p,q)} = 0$.

Suppose that $\mathbf{F}^{(r)} = \mathrm{soc}^{(r)}\,\mathbf{V}^{\otimes(p,q)}$ for some r. The quotient $\mathbf{F}^{(r+1)}/\mathbf{F}^{(r)}$ is a semisimple $\mathfrak{gl}_\infty$-module, hence $\mathbf{F}^{(r+1)} \subset \mathrm{soc}^{(r+1)}\,\mathbf{V}^{\otimes(p,q)}$. Now let $\mathbf{U}$ be a simple submodule of $\mathbf{V}^{\otimes(p,q)}/\mathbf{F}^{(r)}$; then $\mathbf{U} \cong \mathbf{\Gamma}_{\lambda;\mu}$ with λ, μ satisfying $|\lambda| = p - s$ and $|\mu| = q - s$ for some $s \geq r$. In particular $\mathbf{U} \subset \mathbf{F}^{(s+1)}/\mathbf{F}^{(r)}$. Our goal is to show that in fact $s = r$; indeed, we would then conclude that $\mathrm{soc}^{(r+1)}\,\mathbf{V}^{\otimes(p,q)} \subset \mathbf{F}^{(r+1)}$, proving the induction step.

Fix a vector $u \in \mathbf{V}^{\otimes(p,q)}$ of weight $\chi = \sum_i \lambda_i \varepsilon_i - \sum_i \mu_i \varepsilon_{-i}$, such that the image of u under the projection $\mathbf{V}^{\otimes(p,q)} \to \mathbf{V}^{\otimes(p,q)}/\mathbf{F}^{(r)}$ is generates $\mathbf{U}$. Fix a large enough $m \in \mathbb{N}$ such that $u \in (V_m)^{\otimes(p,q)}$. We may further assume without loss of generality that u generates a $\mathfrak{gl}_m$-submodule of $(V_m)^{\otimes(p,q)}$ isomorphic to $\mathit{\Gamma}^{(m)}_{\lambda;\mu}$.

For each $n > m$ denote by $\pi_n : F_n^{(s+1)} \to \mathit{\Gamma}^{(n)}_{\lambda;\mu} \otimes H(\lambda, \mu; s)$ the projection corresponding to the decomposition (2.7). The map π_n can be described explicitly, and we show in Proposition 6.3 in the Appendix that $u - \pi_n(u) \notin F_n^{(s-1)}$ for infinitely many n. On the other hand, u generates a submodule of $F_n^{(s+1)}/F_n^{(r)}$ isomorphic to $\mathit{\Gamma}^{(n)}_{\lambda;\mu}$, which implies that $u - \pi_n(u) \in F_n^{(r)}$ for all n. We conclude that $s \leq r$, which implies that in fact $s = r$. The induction is now complete, and thus $\mathbf{F}^{(r)} = \mathrm{soc}^{(r)}\,\mathbf{V}^{\otimes(p,q)}$ for all r.

Finally, we need to verify that the filtration $\{\mathbf{F}^{(r)}\}$ is also the socle filtration of $\mathbf{V}^{\otimes(p,q)}$ regarded as an $\mathfrak{sl}_\infty$-module. The irreducibility of $\mathbf{\Gamma}_{\lambda;\mu}$ as an $\mathfrak{sl}_\infty$-module implies that the layers of the filtration $\mathbf{F}^{(r)}$ remain semisimple. Therefore $\mathbf{F}^{(r)} \subset \mathrm{soc}^{(r)}\,\mathbf{V}^{\otimes(p,q)}$ for all r. The proof of the opposite inclusion, given above for $\mathfrak{gl}_\infty$, works without any alterations for $\mathfrak{sl}_\infty$ as well, and this completes the proof of the theorem. □

Next, we describe the indecomposable constituents of the tensor representations of $\mathfrak{gl}_\infty$.

Theorem 2.3. *For any partitions λ, μ the $\mathfrak{gl}_\infty$-module $\mathbf{\Gamma}_{\lambda;0} \otimes \mathbf{\Gamma}_{0;\mu}$ is indecomposable, and*

$$\overline{\mathrm{soc}}^{(r+1)}(\mathbf{\Gamma}_{\lambda;0} \otimes \mathbf{\Gamma}_{0;\mu}) \cong \bigoplus_{\lambda',\mu'} \left(\sum_{|\gamma|=r} \mathrm{N}^{\lambda}_{\lambda',\gamma} \mathrm{N}^{\mu}_{\mu',\gamma} \right) \mathbf{\Gamma}_{\lambda';\mu'}. \tag{2.8}$$

The same applies to $\mathbf{\Gamma}_{\lambda;0} \otimes \mathbf{\Gamma}_{0;\mu}$ regarded as an $\mathfrak{sl}_\infty$-module.

Proof. Schur-Weyl duality (2.5) for $\mathfrak{gl}_\infty$ implies that

$$\mathbf{V}^{\otimes(p,q)} \cong \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} (\mathbf{\Gamma}_{\lambda;0} \otimes \mathbf{\Gamma}_{0;\mu}) \otimes (H_\lambda \otimes H_\mu).$$

Hence the $\mathfrak{gl}_\infty$-module $\mathbf{\Gamma}_{\lambda;0} \otimes \mathbf{\Gamma}_{0;\mu}$ is realized as the direct summand $(\mathbb{S}_\lambda \mathbf{V} \otimes \mathbb{S}_\mu \mathbf{V}_*)$ of the $\mathfrak{gl}_\infty$-module $\mathbf{V}^{\otimes(p,q)}$. Therefore

$$\mathrm{soc}^{(r)}\,\mathbf{\Gamma}_{\lambda;\mu} = \mathbf{\Gamma}_{\lambda;\mu} \cap \mathrm{soc}^{(r)}\,\mathbf{V}^{\otimes d}.$$

It is known that for any partitions $\lambda, \mu, \lambda', \mu'$ we have $[\mathit{\Gamma}^{(n)}_{\lambda;0} \otimes \mathit{\Gamma}^{(n)}_{0;\mu} : \mathit{\Gamma}^{(n)}_{\lambda';\mu'}] = \sum_\gamma \mathrm{N}^{\lambda}_{\lambda',\gamma} \mathrm{N}^{\mu}_{\mu',\gamma}$ provided n is large enough, see e.g. [HTW]. This yields

$$[\mathbf{\Gamma}_{\lambda;0} \otimes \mathbf{\Gamma}_{0;\mu} : \mathbf{\Gamma}_{\lambda';\mu'}] = \sum_\gamma \mathrm{N}^{\lambda}_{\lambda',\gamma} \mathrm{N}^{\mu}_{\mu',\gamma}, \tag{2.9}$$

and combining (2.9) with the description of the socle filtration of $\mathbf{V}^{\otimes(p,q)}$, we obtain (2.8). In particular, $\mathrm{soc}(\mathbf{\Gamma}_{\lambda;0} \otimes \mathbf{\Gamma}_{0;\mu}) \cong \mathbf{\Gamma}_{\lambda;\mu}$, and the simplicity of the socle implies the indecomposability of the $\mathfrak{gl}_\infty$-module $\mathbf{\Gamma}_{\lambda;0} \otimes \mathbf{\Gamma}_{0;\mu}$.

□

Corollary 2.4. *The decomposition of $\mathbf{V}^{\otimes(p,q)}$ into indecomposable $\mathfrak{gl}_\infty$-modules is given by the isomorphism*

$$\mathbf{V}^{\otimes(p,q)} \cong \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} (\dim H_\lambda \ \dim H_\mu) \ \ \mathbf{\Gamma}_{\lambda;0} \otimes \mathbf{\Gamma}_{0;\mu}.$$

Examples. We begin with the description of tensors of rank 2. Purely covariant and purely contravariant tensor representations of $\mathfrak{gl}_\infty$ and $\mathfrak{sl}_\infty$ are completely reducible, and are decomposed by the Schur-Weyl duality (2.2). For tensors of rank 2 these decompositions correspond to the isomorphisms

$$\mathbf{V} \otimes \mathbf{V} \cong S^2\mathbf{V} \oplus \Lambda^2\mathbf{V}, \qquad \mathbf{V}_* \otimes \mathbf{V}_* \cong S^2\mathbf{V}_* \oplus \Lambda^2\mathbf{V}_*.$$

The tensor representation of mixed type $\mathbf{V} \otimes \mathbf{V}_*$ is the adjoint representation of $\mathfrak{gl}_\infty$, for which we have the non-splitting short exact sequence of $\mathfrak{gl}_\infty$-modules

$$0 \to \mathfrak{sl}_\infty \to \mathfrak{gl}_\infty \to \Bbbk \to 0.$$

In other words, the socle filtration of the adjoint $\mathfrak{gl}_\infty$-module has length 2, with a simple socle isomorphic to $\mathfrak{sl}_\infty$, and a simple top isomorphic to $\Bbbk$. We summarize the structure of these modules graphically by

$$\begin{aligned}
\mathbf{V}^{\otimes(2,0)} &\sim \boxed{\mathbf{\Gamma}_{(2);(0)}} \oplus \boxed{\mathbf{\Gamma}_{(1,1);(0)}}\,, \\
\mathbf{V}^{(1,1)} &\sim \begin{array}{|c|}\hline \mathbf{\Gamma}_{(0);(0)} \\ \hline \mathbf{\Gamma}_{(1);(1)} \\ \hline \end{array}\,, \\
\mathbf{V}^{\otimes(0,2)} &\sim \boxed{\mathbf{\Gamma}_{(0);(2)}} \oplus \boxed{\mathbf{\Gamma}_{(0);(1,1)}}\,.
\end{aligned}$$

In these diagrams each tower represents an indecomposable direct summand of a module, and the vertical arrangement of boxes in each tower represents the structure of the layers of the socle filtration, with the bottom box corresponding to the socle.

Similarly, Theorem 2.3 yields the following diagrams for tensor representations of rank 3:

$$\begin{aligned}
\mathbf{V}^{\otimes(3,0)} &\sim \boxed{\mathbf{\Gamma}_{(3);(0)}} \oplus 2\,\boxed{\mathbf{\Gamma}_{(2,1);(0)}} \oplus \boxed{\mathbf{\Gamma}_{(1,1,1);(0)}}\,, \\
\mathbf{V}^{\otimes(2,1)} &\sim \begin{array}{|c|}\hline \mathbf{\Gamma}_{(1);(0)} \\ \hline \mathbf{\Gamma}_{(2);(1)} \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \mathbf{\Gamma}_{(1);(0)} \\ \hline \mathbf{\Gamma}_{(1,1);(1)} \\ \hline \end{array}\,, \\
\mathbf{V}^{\otimes(0,3)} &\sim \boxed{\mathbf{\Gamma}_{(0);(3)}} \oplus 2\,\boxed{\mathbf{\Gamma}_{(0);(2,1)}} \oplus \boxed{\mathbf{\Gamma}_{(0);(1,1,1)}}\,,
\end{aligned}$$

and for tensor representations of rank 4:

$$\begin{aligned}
\mathbf{V}^{\otimes(4,0)} &\sim \boxed{\mathbf{\Gamma}_{(4);(0)}} \oplus 3\,\boxed{\mathbf{\Gamma}_{(3,1);(0)}} \oplus 2\,\boxed{\mathbf{\Gamma}_{(2,2);(0)}} \oplus 3\,\boxed{\mathbf{\Gamma}_{(2,1,1);(0)}} \oplus \boxed{\mathbf{\Gamma}_{(1,1,1,1);(0)}}\,,\\
\mathbf{V}^{\otimes(3,1)} &\sim \begin{array}{|c|}\hline \mathbf{\Gamma}_{(2);(0)}\\\hline \mathbf{\Gamma}_{(3);(1)}\\\hline\end{array} \oplus 2\begin{array}{|c|}\hline \mathbf{\Gamma}_{(2);(0)}\oplus\mathbf{\Gamma}_{(1,1);(0)}\\\hline \mathbf{\Gamma}_{(2,1);(1)}\\\hline\end{array} \oplus \begin{array}{|c|}\hline \mathbf{\Gamma}_{(1,1);(0)}\\\hline \mathbf{\Gamma}_{(1,1,1);(1)}\\\hline\end{array}\,,\\
\mathbf{V}^{\otimes(2,2)} &\sim \begin{array}{|c|}\hline \mathbf{\Gamma}_{(0);(0)}\\\hline \mathbf{\Gamma}_{(1);(1)}\\\hline \mathbf{\Gamma}_{(2);(2)}\\\hline\end{array} \oplus \begin{array}{|c|}\hline \mathbf{\Gamma}_{(1);(1)}\\\hline \mathbf{\Gamma}_{(2);(1,1)}\\\hline\end{array} \oplus \begin{array}{|c|}\hline \mathbf{\Gamma}_{(1);(1)}\\\hline \mathbf{\Gamma}_{(1,1);(2)}\\\hline\end{array} \oplus \begin{array}{|c|}\hline \mathbf{\Gamma}_{(0);(0)}\\\hline \mathbf{\Gamma}_{(1);(1)}\\\hline \mathbf{\Gamma}_{(1,1);(1,1)}\\\hline\end{array}\,,\\
\mathbf{V}^{\otimes(1,3)} &\sim \begin{array}{|c|}\hline \mathbf{\Gamma}_{(0);(2)}\\\hline \mathbf{\Gamma}_{(1);(3)}\\\hline\end{array} \oplus 2\begin{array}{|c|}\hline \mathbf{\Gamma}_{(0);(2)}\oplus\mathbf{\Gamma}_{(0);(1,1)}\\\hline \mathbf{\Gamma}_{(1);(2,1)}\\\hline\end{array} \oplus \begin{array}{|c|}\hline \mathbf{\Gamma}_{(0);(1,1)}\\\hline \mathbf{\Gamma}_{(1);(1,1,1)}\\\hline\end{array}\,,\\
\mathbf{V}^{\otimes(0,4)} &\sim \boxed{\mathbf{\Gamma}_{(0);(4)}} \oplus 3\,\boxed{\mathbf{\Gamma}_{(0);(3,1)}} \oplus 2\,\boxed{\mathbf{\Gamma}_{(0);(2,2)}} \oplus 3\,\boxed{\mathbf{\Gamma}_{(0);(2,1,1)}} \oplus \boxed{\mathbf{\Gamma}_{(0);(1,1,1,1)}}\,.
\end{aligned}$$

3. Tensor representations of $\mathfrak{sp}_\infty$

Let $\mathbf{V}$ be a countable dimensional vector space, and let $\Omega : \mathbf{V}\otimes\mathbf{V} \to \Bbbk$ be a non-degenerate anti-symmetric bilinear form on $\mathbf{V}$. We realize the Lie algebra $\mathfrak{gl}_\infty$ as in Section 2 by taking $\mathbf{V}_* = \mathbf{V}$ and Ω as the pairing $\langle\cdot,\cdot\rangle$. The Lie algebra $\mathfrak{sp}_\infty$ is defined as the Lie subalgebra of $\mathfrak{gl}_\infty$, which preserves the form Ω, i.e.

$$\mathfrak{sp}_\infty = \left\{ X \in \mathfrak{gl}_\infty \;\middle|\; \Omega(Xu, v) + \Omega(u, Xv) = 0 \text{ for all } u, v \in \mathbf{V} \right\}.$$

It is always possible to pick a basis $\{\xi_i\}$ of $\mathbf{V}$, indexed as before by the set $\mathfrak{I} = \mathbb{Z}\setminus\{0\}$, such that $\Omega(\xi_i, \xi_j) = \operatorname{sign}(i)\,\delta_{i+j,0}$. In the coordinate realization of $\mathfrak{gl}_\infty$, the Lie algebra $\mathfrak{sp}_\infty$ has a linear basis $\{\operatorname{sign}(j)E_{i,j} - \operatorname{sign}(i)E_{-j,-i}\}_{i,j\in\mathfrak{I}}$. Since the dual basis $\{\xi_i^*\}_{i\in\mathfrak{I}}$ is given by $\xi_i^* = \operatorname{sign}(i)\,\xi_{-i}$, it follows that $\mathfrak{sp}_\infty = S^2\mathbf{V}$, and the Lie bracket is induced by (2.1).

We call $\mathbf{V}$, regarded as an $\mathfrak{sp}_\infty$-module by restriction, the *natural representation* of $\mathfrak{sp}_\infty$. It is easy to see that the $\mathfrak{sp}_\infty$-action on the $\mathfrak{gl}_\infty$-module $\mathbf{V}^{\otimes(p,q)}$ coincides with the $\mathfrak{sp}_\infty$-action on the $\mathfrak{gl}_\infty$-module $\mathbf{V}^{\otimes(p+q)}$. Therefore it suffices to study the tensor representations $\mathbf{V}^{\otimes d}$.

To define the notion of a highest weight $\mathfrak{sp}_\infty$-module, we consider the direct sum decomposition $\mathfrak{sp}_\infty = \mathfrak{h}_{\mathfrak{sp}} \oplus \left(\bigoplus_{\alpha\in\Delta_{\mathfrak{sp}}} \Bbbk\, X_\alpha^{\mathfrak{sp}}\right)$, where

$$\mathfrak{h}_{\mathfrak{sp}} = \bigoplus_{i\in\mathfrak{I}} \Bbbk\,(E_{i,i} - E_{-i,-i}), \qquad \Delta_{\mathfrak{sp}} = \{\pm(\varepsilon_i + \varepsilon_j) \mid i,j\in\mathbb{N}\} \bigcup \{\varepsilon_i - \varepsilon_j \mid i,j\in\mathbb{N} \text{ and } i\neq j\},$$

$$X^{\mathfrak{sp}}_{\varepsilon_i+\varepsilon_j} = E_{i,-j} + E_{j,-i}, \qquad X^{\mathfrak{sp}}_{-\varepsilon_i-\varepsilon_j} = E_{-i,j} + E_{-j,i}, \qquad X^{\mathfrak{sp}}_{\varepsilon_i-\varepsilon_j} = E_{i,j} - E_{-j,-i}.$$

The subalgebra $\mathfrak{b}_{\mathfrak{sp}}$ and its ideal $\mathfrak{n}_{\mathfrak{sp}}$ are defined as before using the polarization of the root system $\Delta_{\mathfrak{sp}} = \Delta^+_{\mathfrak{sp}} \coprod -\Delta^+_{\mathfrak{sp}}$, where

$$\Delta^+_{\mathfrak{sp}} = \{\varepsilon_i + \varepsilon_j \mid i,j\in\mathbb{N} \text{ and } i\leq j\} \bigcup \{\varepsilon_i - \varepsilon_j \mid i,j\in\mathbb{N} \text{ and } i<j\}.$$

An $\mathfrak{sp}_\infty$-module V is called a *highest weight module* with highest weight $\chi\in\mathfrak{h}^*_{\mathfrak{sp}}$, if it is generated by a vector $v\in V$ satisfying $\mathfrak{n}_{\mathfrak{sp}}\,v = 0$ and $H\,v = \chi(H)\,v$ for all $H\in\mathfrak{h}_{\mathfrak{sp}}$. In particular, the natural representation $\mathbf{V}$ is a highest weight $\mathfrak{sp}_\infty$-module with highest weight ε_1, generated by the highest weight vector ξ_1.

For any pair $I = (i, j)$ of integers such that $1 \leq i < j \leq d$, the form Ω determines a contraction

$$\Phi_{\langle I\rangle} : \mathbf{V}^{\otimes d} \to \mathbf{V}^{\otimes(d-2)},$$
$$v_1 \otimes \ldots \otimes v_d \mapsto \Omega(v_i, v_j)\, v_1 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes \hat{v}_j \otimes \ldots \otimes v_d.$$

We set $\mathbf{V}^{\langle 0\rangle} \stackrel{\text{def}}{=} \Bbbk$, $\mathbf{V}^{\langle 1\rangle} \stackrel{\text{def}}{=} \mathbf{V}$ and

$$\mathbf{V}^{\langle d\rangle} \stackrel{\text{def}}{=} \bigcap_I \ker\Big(\Phi_{\langle I\rangle} : \mathbf{V}^{\otimes d} \to \mathbf{V}^{\otimes(d-2)}\Big)$$

for $d \geq 2$. For any partition λ with $|\lambda| = d$ we define the $\mathfrak{sp}_\infty$-submodule $\mathbf{\Gamma}_{\langle\lambda\rangle}$ of $\mathbf{V}^{\otimes d}$,

$$\mathbf{\Gamma}_{\langle\lambda\rangle} \stackrel{\text{def}}{=} \mathbf{V}^{\langle d\rangle} \cap \mathbb{S}_\lambda \mathbf{V}.$$

Theorem 3.1. *For any nonnegative integer d there is an isomorphism of $(\mathfrak{sp}_\infty, \mathfrak{S}_d)$-modules*

$$\mathbf{V}^{\langle d\rangle} \cong \bigoplus_{|\lambda|=d} \mathbf{\Gamma}_{\langle\lambda\rangle} \otimes H_\lambda. \tag{3.1}$$

For every partition λ, the $\mathfrak{sp}_\infty$-module $\mathbf{\Gamma}_{\langle\lambda\rangle}$ is an irreducible highest weight module with highest weight $\omega = \sum_{i\in\mathbb{N}} \lambda_i\, \varepsilon_i$.

Proof. For each n, denote by V_n the $2n$-dimensional subspace of $\mathbf{V}$ spanned by $\xi_{\pm1}, \ldots, \xi_{\pm n}$, and identify the Lie subalgebra of $\mathrm{End}(V_n)$ which preserves the restriction of the form Ω to V_n with the Lie algebra $\mathfrak{sp}_{2n}$. The tensor representations $(V_n)^{\otimes d}$ of $\mathfrak{sp}_{2n}$ and the contractions $\Phi^{(n)}_{\langle I\rangle} : (V_n)^{\otimes d} \to (V_n)^{\otimes(d-2)}$ are defined as before. We set

$$(V_n)^{\langle d\rangle} \stackrel{\text{def}}{=} \bigcap_I \ker \Phi^{(n)}_{\langle I\rangle}, \qquad \mathit{\Gamma}^{(n)}_{\langle\lambda\rangle} \stackrel{\text{def}}{=} (V_n)^{\langle d\rangle} \cap \mathbb{S}_\lambda V_n.$$

The Weyl construction for $\mathfrak{sp}_{2n}$ implies that there are isomorphisms of $(\mathfrak{sp}_{2n}, \mathfrak{S}_d)$-modules

$$(V_n)^{\langle d\rangle} \cong \bigoplus_{|\lambda|=d} \mathit{\Gamma}^{(n)}_{\langle\lambda\rangle} \otimes H_\lambda, \tag{3.2}$$

and that for $n \geq d$ the module $\mathit{\Gamma}^{(n)}_{\langle\lambda\rangle}$ is an irreducible highest weight module with highest weight $\sum_{i\in\mathbb{N}} \lambda_i\, \varepsilon_i$. As in the proof of Theorem 2.1, the diagram of inclusions

$$\begin{array}{ccccccccccc} (V_d)^{\langle d\rangle} & \subset & (V_{d+1})^{\langle d\rangle} & \subset & \cdots & \subset & (V_n)^{\langle d\rangle} & \subset \cdots \subset & \bigcup_{n\in\mathbb{N}} (V_n)^{\langle d\rangle} & = & \mathbf{V}^{\langle d\rangle} \\ \cup & & \cup & & & & \cup & & \cup & & \cup \\ \mathit{\Gamma}^{(d)}_{\langle\lambda\rangle} & \subset & \mathit{\Gamma}^{(d+1)}_{\langle\lambda\rangle} & \subset & \cdots & \subset & \mathit{\Gamma}^{(n)}_{\langle\lambda\rangle} & \subset \cdots \subset & \bigcup_{n\in\mathbb{N}} \mathit{\Gamma}^{(n)}_{\langle\lambda\rangle} & = & \mathbf{\Gamma}_{\langle\lambda\rangle} \end{array} \tag{3.3}$$

establishes both the irreducibility of the $\mathfrak{sp}_\infty$-module $\mathbf{\Gamma}_{\langle\lambda\rangle}$ and the isomorphism (3.1), and the equalities

$$\mathit{\Gamma}^{(d)}_{\langle\lambda\rangle}[\chi] = \mathit{\Gamma}^{(d+1)}_{\langle\lambda\rangle}[\chi] = \cdots = \mathit{\Gamma}^{(n)}_{\langle\lambda\rangle}[\chi] = \cdots = \bigcup_{n\in\mathbb{N}} \mathit{\Gamma}^{(n)}_{\langle\lambda\rangle}[\chi] = \mathbf{\Gamma}_{\langle\lambda\rangle}[\chi]$$

yields the characterization of $\mathbf{\Gamma}_{\langle\lambda\rangle}[\chi]$ as a highest weight $\mathfrak{sp}_\infty$-module. □

Next, we describe the socle filtration of the tensor representations of $\mathfrak{sp}_\infty$.

Theorem 3.2. *For any nonnegative integer d the $\mathfrak{gl}_\infty$-module $\mathbf{V}^{\otimes d}$, regarded as an $\mathfrak{sp}_\infty$-module, has Loewy length $\left[\frac{d}{2}\right]+1$, and*

$$\operatorname{soc}^{(r)} \mathbf{V}^{\otimes d} = \bigcap_{I_1,\dots,I_r} \ker\Big(\Phi_{\langle I_1,\dots,I_r\rangle} : \mathbf{V}^{\otimes d} \to \mathbf{V}^{\otimes (d-2r)}\Big), \qquad r = 1,\dots,\left[\frac{d}{2}\right]. \tag{3.4}$$

Proof. The proof is obtained as a minor variation of the proof of Theorem 2.2. Denoting by $\mathbf{F}^{(r)}$ the module on the right side of (3.4), we argue as before that

$$\mathbf{F}^{(r+1)}/\mathbf{F}^{(r)} \cong \bigoplus_{|\lambda|=p-2r} \mathbf{\Gamma}_{\langle\lambda\rangle} \otimes H(\lambda; r)$$

for some $\mathfrak{S}_d$-modules $H(\lambda;r)$. The semisimplicity of these layers shows that $\mathbf{F}^{(r)} \subset \operatorname{soc}^{(r)} \mathbf{V}^{\otimes d}$ for all r, and the opposite inclusion is proved by an obvious modification of Corollary 6.3. □

Theorem 3.3. *For any partition λ the $\mathfrak{sp}_\infty$-module $\mathbf{\Gamma}_{\lambda;0}$ is indecomposable, and*

$$\overline{\operatorname{soc}}^{(r+1)}\mathbf{\Gamma}_{\lambda;0} = \bigoplus_{\mu} \left(\sum_{|\gamma|=r} \mathrm{N}^{\lambda}_{\mu,\,(2\gamma)^\top}\right) \mathbf{\Gamma}_{\langle\mu\rangle}, \qquad r = 1,\dots,\left[\frac{d}{2}\right]. \tag{3.5}$$

Proof. By construction, the $\mathfrak{gl}_\infty$-module $\mathbf{\Gamma}_{\lambda;0}$ is realized as the direct summand $\mathbb{S}_\lambda\mathbf{V}$ of the $\mathfrak{gl}_\infty$-module $\mathbf{V}^{\otimes d}$. It remains a direct summand when $\mathbf{V}^{\otimes d}$ is regarded as an $\mathfrak{sp}_\infty$-module, therefore $\operatorname{soc}^{(r)} \mathbf{\Gamma}_{\lambda;0} = \mathbf{\Gamma}_{\lambda;0} \cap \operatorname{soc}^{(r)} \mathbf{V}^{\otimes d}$.

It is known that for any partitions λ, μ we have $[\mathit{\Gamma}^{(2n)}_\lambda : \mathit{\Gamma}^{(n)}_{\langle\mu\rangle}] = \sum_\gamma \mathrm{N}^\lambda_{\mu,\,(2\gamma)^\top}$ provided n is large enough, see e.g. [HTW]. This implies that $[\mathbf{\Gamma}_{\lambda;0} : \mathbf{\Gamma}_{\langle\mu\rangle}] = \sum_\gamma \mathrm{N}^\lambda_{\mu,\,(2\gamma)^\top}$, and combining it with the description of the socle filtration of $\mathbf{V}^{\otimes d}$, we get (3.5). In particular, $\operatorname{soc}\mathbf{\Gamma}_{\lambda;0} \cong \mathbf{\Gamma}_{\langle\lambda\rangle}$, and the simplicity of the socle implies the indecomposability of $\mathbf{\Gamma}_{\lambda;0}$ as an $\mathfrak{sp}_\infty$-module. □

Examples. We begin by describing the structure of $\mathbf{V}\otimes\mathbf{V} = \Lambda^2\mathbf{V}\oplus S^2\mathbf{V}$ as an $\mathfrak{sp}_\infty$-module. The symmetric square $S^2\mathbf{V} = \mathbf{\Gamma}_{(2);(0)}$ is the irreducible adjoint $\mathfrak{sp}_\infty$-module, isomorphic to $\mathbf{\Gamma}_{\langle(2)\rangle}$. For the exterior square $\Lambda^2\mathbf{V} = \mathbf{\Gamma}_{(1,1);(0)}$ one has the short exact sequence of $\mathfrak{sp}_\infty$-modules

$$0 \to \mathbf{\Gamma}_{\langle(1,1)\rangle} \to \Lambda^2\mathbf{V} \xrightarrow{\Omega} \Bbbk \to 0$$

which does not split. Therefore the structure of $\mathbf{V}\otimes\mathbf{V}$ is graphically represented as

$$\mathbf{V}\otimes\mathbf{V} \quad \sim \quad \boxed{\mathbf{\Gamma}_{\langle(2)\rangle}} \oplus \begin{array}{|c|}\hline \mathbf{\Gamma}_{\langle(0)\rangle} \\ \hline \mathbf{\Gamma}_{\langle(1,1)\rangle} \\ \hline \end{array}.$$

Similarly, the structure of the tensor representation of rank 3 is represented as

$$\mathbf{V}^{\otimes 3} \sim \begin{array}{|c|}\hline \boldsymbol{\Gamma}_{\langle(3)\rangle} \\ \hline\end{array} \oplus 2 \begin{array}{|c|}\hline \boldsymbol{\Gamma}_{\langle(1)\rangle} \\ \hline \boldsymbol{\Gamma}_{\langle(2,1)\rangle} \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \boldsymbol{\Gamma}_{\langle(1)\rangle} \\ \hline \boldsymbol{\Gamma}_{\langle(1,1,1)\rangle} \\ \hline\end{array},$$

and the structure of the tensor representation of rank 4 as

$$\mathbf{V}^{\otimes 4} \sim \begin{array}{|c|}\hline \boldsymbol{\Gamma}_{\langle(4)\rangle} \\ \hline\end{array} \oplus 3 \begin{array}{|c|}\hline \boldsymbol{\Gamma}_{\langle(2)\rangle} \\ \hline \boldsymbol{\Gamma}_{\langle(3,1)\rangle} \\ \hline\end{array} \oplus 2 \begin{array}{|c|}\hline \boldsymbol{\Gamma}_{\langle(0)\rangle} \\ \hline \boldsymbol{\Gamma}_{\langle(1,1)\rangle} \\ \hline \boldsymbol{\Gamma}_{\langle(2,2)\rangle} \\ \hline\end{array} \oplus 3 \begin{array}{|c|}\hline \boldsymbol{\Gamma}_{\langle(2)\rangle} \oplus \boldsymbol{\Gamma}_{[(1,1)]} \\ \hline \boldsymbol{\Gamma}_{\langle(2,1,1)\rangle} \\ \hline\end{array} \oplus \begin{array}{|c|}\hline \boldsymbol{\Gamma}_{\langle(0)\rangle} \\ \hline \boldsymbol{\Gamma}_{\langle(1,1)\rangle} \\ \hline \boldsymbol{\Gamma}_{\langle(1,1,1,1)\rangle} \\ \hline\end{array}.$$

4. Tensor representations of $\mathfrak{so}_\infty$

Let $\mathbf{V}$ be a countable dimensional vector space, and let $Q : \mathbf{V} \otimes \mathbf{V} \to \Bbbk$ be a non-degenerate symmetric bilinear form on $\mathbf{V}$. We realize the Lie algebra $\mathfrak{gl}_\infty$ as in Section 2 by taking $\mathbf{V}_* = \mathbf{V}$ and Q as the pairing $\langle\cdot,\cdot\rangle$. The Lie algebra $\mathfrak{so}_\infty^{(Q)}$ is defined as the Lie subalgebra of this $\mathfrak{gl}_\infty$, which preserves the form Q, i.e.

$$\mathfrak{so}_\infty^{(Q)} = \left\{ X \in \mathfrak{gl}_\infty \;\middle|\; Q(Xu,v) + Q(u,Xv) = 0 \text{ for all } u,v \in \mathbf{V} \right\}.$$

It is customary when dealing with the orthogonal groups to work over algebraically closed fields. Throughout this section we assume that $\Bbbk$ is algebraically closed. Then all non-degenerate forms Q on $\mathbf{V}$ are equivalent (e.g. they can all be transformed to the standard sum of squares), and as a consequence the corresponding Lie algebras $\mathfrak{so}_\infty^{(Q)}$ are isomorphic. Hence we drop Q from the notation, and denote our Lie algebra simply by $\mathfrak{so}_\infty$.

It is convenient to pick a basis $\{\xi_i\}$ of $\mathbf{V}$, indexed as before by the set $\mathfrak{I} = \mathbb{Z} \setminus \{0\}$, such that $Q(\xi_i, \xi_j) = \delta_{i+j,0}$. In the coordinate realization of $\mathfrak{gl}_\infty$, the Lie algebra $\mathfrak{so}_\infty$ has a linear basis $\{E_{i,j} - E_{-j,-i}\}_{i,j\in\mathfrak{I}}$. Since the dual basis $\{\xi_i^*\}_{i\in\mathfrak{I}}$ is given by $\xi_i^* = \xi_{-i}$, it follows that $\mathfrak{so}_\infty = \bigwedge^2 \mathbf{V}$, and the Lie bracket is induced by (2.1).

We call $\mathbf{V}$, regarded as an $\mathfrak{so}_\infty$-module by restriction, the *natural representation* of $\mathfrak{so}_\infty$. It is easy to see that the $\mathfrak{so}_\infty$-action on the $\mathfrak{gl}_\infty$-module $\mathbf{V}^{\otimes(p,q)}$ coincides with the $\mathfrak{so}_\infty$-action on the $\mathfrak{gl}_\infty$-module $\mathbf{V}^{\otimes(p+q)}$. Therefore it suffices to study the tensor representations $\mathbf{V}^{\otimes d}$.

To define the notion of a highest weight $\mathfrak{so}_\infty$-module, we consider the direct sum decomposition $\mathfrak{so}_\infty = \mathfrak{h}_{\mathfrak{so}} \oplus \left(\bigoplus_{\alpha\in\Delta_{\mathfrak{sp}}} \Bbbk X_\alpha^{\mathfrak{so}}\right)$, where

$$\mathfrak{h}_{\mathfrak{so}} = \bigoplus_{i\in\mathfrak{I}} \Bbbk\,(E_{i,i} - E_{-i,-i}), \qquad \Delta_{\mathfrak{so}} = \{\pm\varepsilon_i \pm \varepsilon_j) \mid i,j \in \mathbb{N} \text{ and } i \neq j\},$$

$$X^{\mathfrak{so}}_{\varepsilon_i+\varepsilon_j} = E_{i,-j} - E_{j,-i}, \qquad X^{\mathfrak{so}}_{\varepsilon_i-\varepsilon_j} = E_{i,j} - E_{-j,-i}, \qquad X^{\mathfrak{so}}_{-\varepsilon_i-\varepsilon_j} = E_{-j,i} - E_{-i,j}.$$

The subalgebras $\mathfrak{b}_{\mathfrak{so}}$ and its ideal $\mathfrak{n}_{\mathfrak{so}}$ are defined as before using the polarization of the root system $\Delta_{\mathfrak{so}} = \Delta^+_{\mathfrak{so}} \coprod -\Delta^+_{\mathfrak{so}}$, where

$$\Delta^+_{\mathfrak{so}} = \{\varepsilon_i \pm \varepsilon_j \mid i,j \in \mathbb{N} \text{ and } i < j\}.$$

An $\mathfrak{so}_\infty$-module V is called a *highest weight module* with highest weight $\chi \in \mathfrak{h}^*_{\mathfrak{so}}$, if it is generated by a vector $v \in V$ satisfying $\mathfrak{n}_{\mathfrak{so}}\, v = 0$ and $H\,v = \chi(H)\,v$ for all $H \in \mathfrak{h}_{\mathfrak{so}}$. In particular, the natural representation $\mathbf{V}$ is a highest weight module with highest weight ε_1, generated by a highest weight vector ξ_1.

Remark 3. Our choice of a splitting Cartan subalgebra $\mathfrak{h}_{\mathfrak{so}}$ leads to a root decomposition of type D_∞, corresponding to the "even" infinite orthogonal series of Lie algebras $\mathfrak{so}_{2n}$. It is also possible to pick another splitting Cartan subalgebra $\tilde{\mathfrak{h}}_{\mathfrak{so}}$ which leads to a root decomposition of type B_∞, corresponding to the "odd" infinite orthogonal series $\mathfrak{so}_{2n+1}$. These two types of splitting Cartan subalgebras are clearly not conjugated, and in [DPS] it is proved that any splitting Cartan subalgebra of $\mathfrak{so}_\infty$ is either "even" or "odd".

For any pair $I = (i, j)$ of integers, satisfying $1 \leq i < j \leq d$, define the contraction

$$\begin{gathered} \Phi_{[I]} : \mathbf{V}^{\otimes d} \to \mathbf{V}^{\otimes (d-2)}, \\ v_1 \otimes \ldots \otimes v_d \mapsto Q(v_i, v_j)\, v_1 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes \hat{v}_j \otimes \ldots \otimes v_d, \end{gathered} \tag{4.1}$$

We set $\mathbf{V}^{[0]} \overset{\text{def}}{=} \Bbbk$, $\mathbf{V}^{[1]} \overset{\text{def}}{=} \mathbf{V}$ and

$$\mathbf{V}^{[d]} \overset{\text{def}}{=} \bigcap_I \ker\left(\Phi_{[I]} : \mathbf{V}^{\otimes d} \to \mathbf{V}^{\otimes (d-2)}\right) \tag{4.2}$$

for $d \geq 2$. For any partition λ, define the $\mathfrak{so}_\infty$-submodule $\mathbf{\Gamma}_{[\lambda]}$ of $\mathbf{V}^{\otimes d}$,

$$\mathbf{\Gamma}_{[\lambda]} \overset{\text{def}}{=} \mathbf{V}^{[d]} \cap \mathbb{S}_\lambda \mathbf{V}.$$

Theorem 4.1. *For any $d \in \mathbb{N}$ there is an isomorphism of $(\mathfrak{so}_\infty, \mathfrak{S}_d)$-modules*

$$\mathbf{V}^{[d]} \cong \bigoplus_{|\lambda| = d} \mathbf{\Gamma}_{[\lambda]} \otimes H_\lambda. \tag{4.3}$$

For every partition λ, the $\mathfrak{so}_\infty$-module $\mathbf{\Gamma}_{[\lambda]}$ is an irreducible highest weight module with highest weight $\omega = \sum_{i \in \mathbb{N}} \lambda_i\, \varepsilon_i$.

Proof. For each n denote by V_n the $2n$-dimensional subspace of $\mathbf{V}$ spanned by $\xi_{\pm 1}, \ldots, \xi_{\pm n}$. Let $\mathfrak{g}_n$ be the Lie subalgebra of $\mathrm{End}(V_n)$ which preserves the restriction of Q to V_n. It is clear that $\mathfrak{g}_n \cong \mathfrak{so}_{2n}$. The tensor representations $(V_n)^{\otimes d}$ of $\mathfrak{g}_n$ and the contractions $\Phi^{(n)}_{[I]} : (V_n)^{\otimes d)} \to (V_n)^{\otimes (d-2)}$ are defined as before. We set

$$(V_n)^{[d]} \overset{\text{def}}{=} \bigcap_I \ker \Phi^{(n)}_{[I]}, \qquad \Gamma^{(n)}_{[\lambda]} \overset{\text{def}}{=} (V_n)^{[d]} \cap \mathbb{S}_\lambda V_n.$$

The Weyl construction for $\mathfrak{so}_{2n}$ implies that there are isomorphisms of $(\mathfrak{so}_{2n}, \mathfrak{S}_d)$-modules

$$(V_n)^{[d]} \cong \bigoplus_{|\lambda| = d} \Gamma^{(n)}_{[\lambda]} \otimes H_\lambda, \tag{4.4}$$

and that for $n \geq d$ the module $\Gamma^{(n)}_{[\lambda]}$ is an irreducible highest weight module with highest weight $\sum_{i\in\mathbb{N}} \lambda_i \varepsilon_i$. As in the proof of Theorem 2.1, the diagram of inclusions

$$\begin{array}{ccccccccccc} (V_1)^{[d]} & \subset & (V_2)^{[d]} & \subset \cdots \subset & (V_n)^{[d]} & \subset \cdots \subset & \bigcup_{n\in\mathbb{N}}(V_n)^{[d]} & = & \mathbf{V}^{[d]} \\ \cup & & \cup & & \cup & & \cup & & \cup \\ \Gamma^{(1)}_{[\lambda]} & \subset & \Gamma^{(2)}_{[\lambda]} & \subset \cdots \subset & \Gamma^{(n)}_{[\lambda]} & \subset \cdots \subset & \bigcup_{n\in\mathbb{N}}\Gamma^{(n)}_{[\lambda]} & = & \mathbf{\Gamma}_{[\lambda]} \end{array} \tag{4.5}$$

establishes both the irreducibility of the $\mathfrak{so}_\infty$-module $\mathbf{\Gamma}_{\langle\lambda\rangle}$ and the isomorphism (4.3), and the equalities

$$\Gamma^{(d)}_{\langle\lambda\rangle}[\chi] = \Gamma^{(d+1)}_{\langle\lambda\rangle}[\chi] = \cdots = \Gamma^{(n)}_{\langle\lambda\rangle}[\chi] = \cdots = \bigcup_{n\in\mathbb{N}} \Gamma^{(n)}_{\langle\lambda\rangle}[\chi] = \mathbf{\Gamma}_{\langle\lambda\rangle}[\chi]$$

yield the characterization of $\mathbf{\Gamma}_{[\lambda]}[\chi]$ as a highest weight $\mathfrak{so}_\infty$-module. □

Theorem 4.2. *For any nonnegative integer d the Loewy length of $\mathbf{V}^{\otimes d}$, regarded as an $\mathfrak{so}_\infty$-module, equals $\left[\frac{d}{2}\right]+1$, and*

$$\operatorname{soc}^{(r)} \mathbf{V}^{\otimes d} = \bigcap_{I_1,\ldots,I_r} \ker\left(\Phi_{[I_1,\ldots,I_r]} : \mathbf{V}^{\otimes d} \to \mathbf{V}^{\otimes(d-2r)}\right), \qquad r = 1, \ldots, \left[\frac{d}{2}\right]. \tag{4.6}$$

Proof. The proof is a minor variation of the proof of Theorem 2.2. Denoting by $\mathbf{F}^{(r)}$ the module on the right side of (4.6), we argue as before that

$$\mathbf{F}^{(r+1)}/\mathbf{F}^{(r)} \cong \bigoplus_{|\lambda|=p-2r} \mathbf{\Gamma}_{[\lambda]} \otimes H(\lambda; r)$$

for some $\mathfrak{S}_d$-modules $H(\lambda; r)$. The semisimplicity of these layers shows that $\mathbf{F}^{(r)} \subset \operatorname{soc}^{(r)} \mathbf{V}^{\otimes d}$ for all r, and the opposite inclusion is proved by an obvious modification of Corollary 6.3. □

Theorem 4.3. *Let λ be a partition of a positive integer d. Then the $\mathfrak{so}_\infty$-module $\mathbf{\Gamma}_{\lambda;0}$ is indecomposable, and the layers of its socle filtration satisfy*

$$\overline{\operatorname{soc}}^{(r+1)}\mathbf{\Gamma}_{\lambda;0} \cong \bigoplus_{\mu} \left(\sum_{|\gamma|=r} \mathrm{N}^{\lambda}_{\mu,\,2\gamma} \right) \mathbf{\Gamma}_{[\mu]}. \tag{4.7}$$

Proof. By construction, the $\mathfrak{gl}_\infty$-module $\mathbf{\Gamma}_{\lambda;0}$ is realized as the direct summand $\mathbb{S}_\lambda\mathbf{V}$ of the $\mathfrak{gl}_\infty$-module $\mathbf{V}^{\otimes d}$. It remains a direct summand when $\mathbf{V}^{\otimes d}$ is regarded as an $\mathfrak{so}_\infty$-module, therefore $\operatorname{soc}^{(r)} \mathbf{\Gamma}_{\lambda;0} = \mathbf{\Gamma}_{\lambda;0} \cap \operatorname{soc}^{(r)} \mathbf{V}^{\otimes d}$.

It is known that $[\Gamma^{(2n)}_{\lambda} : \Gamma^{(n)}_{[\mu]}] = \sum_\gamma \mathrm{N}^{\lambda}_{\mu,\,2\gamma}$ for any partitions λ, μ, provided n is large enough, see e.g. [HTW]. Hence $[\mathbf{\Gamma}_{\lambda;0} : \mathbf{\Gamma}_{[\mu]}] = \sum_\gamma \mathrm{N}^{\lambda}_{\mu,\,2\gamma}$, and combining this with the description of the socle filtration of $\mathbf{V}^{\otimes d}$, we get (4.7). In particular, $\operatorname{soc}\mathbf{\Gamma}_{\lambda;0} \cong \mathbf{\Gamma}_{[\lambda]}$, and simplicity of the socle implies the indecomposability of $\mathbf{\Gamma}_{\lambda;0}$ as an $\mathfrak{so}_\infty$-module. □

Corollary 4.4. *The decomposition of* $\mathbf{V}^{\otimes d}$ *into indecomposable* $\mathfrak{so}_\infty$*-modules is given by the isomorphism*

$$\mathbf{V}^{\otimes d} \cong \bigoplus_{|\lambda|=d} (\dim H_\lambda)\, \mathbf{\Gamma}_{\lambda;0}.$$

Examples. We begin by describing the structure of $\mathbf{V}\otimes\mathbf{V} = \Lambda^2\mathbf{V}\oplus S^2\mathbf{V}$ as an $\mathfrak{so}_\infty$-module. The exterior square $\Lambda^2\mathbf{V} = \mathbf{\Gamma}_{(1,1);(0)}$ is the irreducible adjoint $\mathfrak{so}_\infty$-module, isomorphic to $\mathbf{\Gamma}_{[(1,1)]}$. For the symmetric square $S^2\mathbf{V} = \mathbf{\Gamma}_{(2);(0)}$ one has the short exact sequence of $\mathfrak{so}_\infty$-modules

$$0 \to \mathbf{\Gamma}_{[(2)]} \to S^2\mathbf{V} \xrightarrow{Q} \Bbbk \to 0$$

which does not split. Therefore, the structure of $\mathbf{V}\otimes\mathbf{V}$ is graphically represented as

$$\mathbf{V}\otimes\mathbf{V} \quad \sim \quad \begin{array}{|c|}\hline \mathbf{\Gamma}_{[(0)]} \\ \hline \mathbf{\Gamma}_{[(2)]} \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \mathbf{\Gamma}_{[(1,1)]} \\ \hline \end{array}.$$

Similarly, the structure of the tensor representation of rank 3 is represented as

$$\mathbf{V}^{\otimes 3} \quad \sim \quad \begin{array}{|c|}\hline \mathbf{\Gamma}_{[(1)]} \\ \hline \mathbf{\Gamma}_{[(3)]} \\ \hline \end{array} \oplus 2 \begin{array}{|c|}\hline \mathbf{\Gamma}_{[(1)]} \\ \hline \mathbf{\Gamma}_{[(2,1)]} \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \mathbf{\Gamma}_{[(1,1,1)]} \\ \hline \end{array},$$

and the structure of the tensor representation of rank 4 as

$$\mathbf{V}^{\otimes 4} \quad \sim \quad \begin{array}{|c|}\hline \mathbf{\Gamma}_{[(0)]} \\ \hline \mathbf{\Gamma}_{[(2)]} \\ \hline \mathbf{\Gamma}_{[(4)]} \\ \hline \end{array} \oplus 3 \begin{array}{|c|}\hline \mathbf{\Gamma}_{[(2)]} \oplus \mathbf{\Gamma}_{[(1,1)]} \\ \hline \mathbf{\Gamma}_{[(3,1)]} \\ \hline \end{array} \oplus 2 \begin{array}{|c|}\hline \mathbf{\Gamma}_{[(0)]} \\ \hline \mathbf{\Gamma}_{[(2)]} \\ \hline \mathbf{\Gamma}_{[(2,2)]} \\ \hline \end{array} \oplus 3 \begin{array}{|c|}\hline \mathbf{\Gamma}_{[(1,1)]} \\ \hline \mathbf{\Gamma}_{[(2,1,1)]} \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \mathbf{\Gamma}_{[(1,1,1,1)]} \\ \hline \end{array}.$$

5. Tensor representations of root-reductive Lie algebras

It is known that over an algebraically closed field any infinite-dimensional simple locally finite Lie algebra which admits a root decomposition is classical, i.e. isomorphic to $\mathfrak{sl}_\infty, \mathfrak{sp}_\infty$ or $\mathfrak{so}_\infty$, see [PS, NS]. Here we discuss a generalization of our results for $\mathfrak{sl}_\infty, \mathfrak{so}_\infty$ and $\mathfrak{sp}_\infty$ to a more general class of infinite-dimensional Lie algebras.

Let $\mathfrak{k}$ be one of the Lie algebras $\mathfrak{sl}_\infty, \mathfrak{sp}_\infty$ or $\mathfrak{so}_\infty$, and let $\mathfrak{h}_\mathfrak{k}$ denote its splitting Cartan subalgebra introduced in previous sections. Let $\mathbf{V}$ denote the natural representation of $\mathfrak{k}$. Suppose that a Lie algebra $\mathfrak{g}$ is the semidirect sum of $\mathfrak{k}$ and some Lie algebra $\mathfrak{m}$,

$$\mathfrak{g} = \mathfrak{k} \subset\!\!\!\!+\, \mathfrak{m},$$

and suppose furthermore that $\mathfrak{g}$ has a subalgebra $\mathfrak{h}$ such that

$$\mathfrak{h} = \mathfrak{h}_\mathfrak{k} \oplus \mathfrak{m}.$$

Theorem 5.1. *The socle filtration of the tensor representation of* $\mathbf{V}^{\otimes(p,q)}$ *as a* $\mathfrak{g}$*-module coincides with the socle filtration of* $\mathbf{V}^{\otimes(p,q)}$ *as a* $\mathfrak{k}$*-module.*

Proof. Since $\mathfrak{m}$ commutes with $\mathfrak{h}_{\mathfrak{k}}$, the action of $\mathfrak{m}$ preserves the $\mathfrak{h}_{\mathfrak{k}}$-weight subspaces of any $\mathfrak{k}$-module. The $\mathfrak{h}_{\mathfrak{k}}$-weight subspaces of $\mathbf{V}$ are one-dimensional, hence any $H \in \mathfrak{m}$ acts in any $\mathbf{V}[\chi]$ as a scalar. Thus $\mathbf{V}$ admits a $\mathfrak{h}$-weight subspace decomposition, and the same is true for the weight subspaces of tensor representations $\mathbf{V}^{\otimes(p,q)}$.

This shows that each $\mathfrak{k}$-submodule of $\mathbf{V}^{\otimes(p,q)}$ is automatically a $\mathfrak{g}$-module, and thus the socle filtration of $\mathbf{V}^{\otimes(p,q)}$ as a $\mathfrak{k}$-module is a filtration by $\mathfrak{g}$-submodules. Moreover, the layers of the socle filtration for $\mathfrak{k}$ remain semisimple as $\mathfrak{g}$-modules, and the statement follows.

□

Our main application of Theorem 5.1 is to the class of infinite-dimensional root-reductive Lie algebras, studied in [DP1, PS]. Recall the following structural theorem from [DP1].

Theorem 5.2. *Let $\mathfrak{g}$ be a root reductive Lie algebra. Set $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Then*

$$\mathfrak{s} = \bigoplus_{i \in \mathcal{I}} \mathfrak{s}^{(i)},$$

where each $\mathfrak{s}^{(i)}$ is isomorphic either to $\mathfrak{sl}_\infty, \mathfrak{so}_\infty, \mathfrak{sp}_\infty$, or to a simple finite-dimensional Lie algebra, and $\mathcal{I}$ is an at most countable index set. Moreover, the short exact sequence of Lie algebras

$$0 \to \mathfrak{s} \to \mathfrak{g} \to \mathfrak{a} \to 0 \tag{5.1}$$

splits. In other words, $\mathfrak{g} \cong \mathfrak{s} \in\!\!\!+ \mathfrak{a}$.

To apply Theorem 5.1 to a root-reductive Lie algebra $\mathfrak{g}$, we identify $\mathfrak{g}$ with a semidirect sum $\mathfrak{g} \cong \mathfrak{s} \in\!\!\!+ \mathfrak{a}$, pick an infinite-dimensional direct summand $\mathfrak{k} = \mathfrak{s}^{(j)}$ of $\mathfrak{s}$, and set $\mathfrak{m} = \left(\bigoplus_{i \neq j} \mathfrak{s}^{(i)}\right) \in\!\!\!+ \mathfrak{a}$. The Lie algebra $\mathfrak{g}$ acts in the natural representation $\mathbf{V}$ of $\mathfrak{s}^{(j)}$: $\mathfrak{s}^{(i)}$ annihilate $\mathbf{V}$ for $i \neq j$, and $\mathfrak{a}$ acts by scalars in each $\mathfrak{h}_{\mathfrak{k}}$-weight subspace.

In contrast with the finite-dimensional reductive Lie algebras, there exist root-reductive $\mathfrak{g}$, such that $\mathfrak{g} \not\cong \mathfrak{s} \oplus \mathfrak{a}$. For example, for $\mathfrak{g} = \mathfrak{gl}_\infty$ one has $\mathfrak{s} = \mathfrak{sl}_\infty$ and $\mathfrak{a} = \Bbbk$, but $\mathfrak{gl}_\infty \not\cong \mathfrak{sl}_\infty \oplus \Bbbk$. Another interesting example is the Lie algebra $\tilde{\mathfrak{g}}$, constructed via the following root injections

$$\mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_{n+2}, \qquad A \mapsto \begin{pmatrix} \frac{\mathrm{Tr}(A)}{n} & & \\ & A & \\ & & 0 \end{pmatrix}.$$

Then $\tilde{\mathfrak{g}}$ is not isomorphic to $\mathfrak{gl}_\infty$, although it can still be included in a short exact sequence of Lie algebras $0 \to \mathfrak{sl}_\infty \to \tilde{\mathfrak{g}} \to \Bbbk \to 0$, see [DPS]. However, Theorem 5.1 still applies and describes the socle filtration of the tensor representations of $\tilde{\mathfrak{g}}$.

6. Appendix

For completeness, we discuss the details of Weyl's duality approach, see [W, FH].

Let p, q, n be nonnegative integers such that $n > p + q$. Let $V_n = \Bbbk^n$ be the natural representation of the Lie algebra $\mathfrak{gl}_n$. For partitions λ, μ such that $|\lambda| = p$ and $|\mu| = q$ we denote by $\Gamma^{(n)}_{\lambda;\mu}$ the standard irreducible highest weight $\mathfrak{gl}_n$-module with highest weight $\omega = (\lambda_1, \ldots, \lambda_p, 0, \ldots, 0, -\mu_q, \ldots, -\mu_1)$.

Proposition 6.1. *For any $n > p+q$ there is an isomorphism* (2.3)

$$(V_n)^{\{p,q\}} \cong \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} \Gamma^{(n)}_{\lambda;\mu} \otimes (H_\lambda \otimes H_\mu).$$

Proof. For any $I=(i,j)$ with $i \in \{1,2,\dots,p\}$ and $j \in \{1,2,\dots,q\}$, we define the inclusion

$$\Psi^{(n)}_I : (V_n)^{\otimes(p-1,q-1)} \to (V_n)^{\otimes(p,q)},$$

$$v_1 \otimes \ldots \otimes v_{p-1} \otimes v_1^* \otimes \ldots \otimes v_{q-1}^* \mapsto \frac{1}{n}\sum_{k=1}^{n} \ldots \otimes v_{i-1} \otimes \zeta_k \otimes v_{i+1} \otimes \ldots \otimes v_{j-1}^* \otimes \zeta_k^* \otimes v_{j+1}^* \otimes \ldots,$$

where $\{\zeta_k\}$ and $\{\zeta_k^*\}$ are any dual bases of V_n and V_n^* respectively. Set $\theta^{(n)}_I = \Psi^{(n)}_I \Phi^{(n)}_I$. The operators $\theta^{(n)}_I$ are idempotent, and $(V_n)^{\{p,q\}} = \bigcap_I \ker \theta^{(n)}_I$.

Let $\widetilde{\mathcal{A}}$ denote the subalgebra of endomorphisms of $(V_n)^{\otimes(p,q)}$, generated by the images of elements of $\mathfrak{g}_n$, and let $\widetilde{\mathcal{B}}$ denote its commutator subalgebra, i.e. the set of all endomorphisms of $(V_n)^{\otimes(p,q)}$, commuting with the action of $\mathfrak{g}_n$. From invariant theory it is known that $\widetilde{\mathcal{B}}$ is generated by $\theta^{(n)}_I$ for various I, and by permutation maps corresponding to elements from $\mathfrak{S}_p \times \mathfrak{S}_q$. A general result from the theory of semisimple finite-dimensional algebras implies that, conversely, $\widetilde{\mathcal{A}}$ is the commutator subalgebra of $\widetilde{\mathcal{B}}$. In other words, any endomorphism of $(V_n)^{\otimes(p,q)}$, commuting with all $\theta^{(n)}_I$ and all permutations from $\mathfrak{S}_p \times \mathfrak{S}_q$, must lie in $\widetilde{\mathcal{A}}$.

Let $\mathcal{A}$ denote the subalgebra of endomorphisms of $(V_n)^{\{p,q\}}$, generated by the images of elements of $\mathfrak{g}_n$, and let $\mathcal{B}$ denote the subalgebra of endomorphisms of $(V_n)^{\{p,q\}}$, generated by the permutations from $\mathfrak{S}_p \times \mathfrak{S}_q$. We claim that $\mathcal{A}$ and $\mathcal{B}$ are each other's commutator subalgebras in $\operatorname{End}\big((V_n)^{\{p,q\}}\big)$. Indeed, suppose that L lies in the commutator subalgebra of $\mathcal{B}$. Using the $(\mathfrak{g}_n, \mathfrak{S}_p \times \mathfrak{S}_q)$-module isomorphism (see e.g. [FH] for details)

$$(V_n)^{\otimes(p,q)} = (V_n)^{\{p,q\}} \oplus \sum_I \operatorname{im} \theta^{(n)}_I,$$

we construct an endomorphism $\widetilde{L}$ of $(V_n)^{\otimes(p,q)}$ by extending L trivially on the second direct summand. It is clear that $\widetilde{L}$ commutes with all permutations from $\mathfrak{S}_p \times \mathfrak{S}_q$, and also with the operators $\theta^{(n)}_I$, all of which act on $(V_n)^{\{p,q\}}$ by zero. Hence $\widetilde{L}$ belongs to the commutator subalgebra of $\widetilde{\mathcal{B}}$, i.e. $\widetilde{L} \in \widetilde{\mathcal{A}}$, and by restriction $L \in \mathcal{A}$. Thus $\mathcal{A}$ is the commutator subalgebra of $\mathcal{B}$, and it follows that $\mathcal{B}$ is also the commutator subalgebra of $\mathcal{A}$.

The general theory of dual pairs and the fact that $\{H_\lambda \otimes H_\mu\}_{|\lambda|=p, |\mu|=q}$ is a complete list of irreducible $\mathfrak{S}_p \times \mathfrak{S}_q$-modules imply the existence of an isomorphism

$$(V_n)^{\{p,q\}} \cong \bigoplus_{|\lambda|=p} \bigoplus_{|\mu|=q} \Gamma(\lambda,\mu) \otimes (H_\lambda \otimes H_\mu)$$

for some irreducible $\mathfrak{gl}_n$-modules $\Gamma(\lambda,\mu)$. To identify these modules explicitly, we note that Schur-Weyl duality yields

$$(V_n)^{\otimes p} \cong \sum_{|\lambda|=p} \mathit{\Gamma}^{(n)}_{\lambda;0} \otimes H_\lambda, \qquad (V_n^*)^{\otimes q} \cong \sum_{|\mu|=q} \mathit{\Gamma}^{(n)}_{0;\mu} \otimes H_\mu,$$

and therefore $\Gamma(\lambda,\mu)$ must be a submodule of $\mathit{\Gamma}^{(n)}_{\lambda;0} \otimes \mathit{\Gamma}^{(n)}_{0;\mu}$. On the other hand, the submodule $\mathit{\Gamma}^{(n)}_{\lambda;\mu}$ of this tensor product does not occur as a submodule of $(V_n)^{\otimes(p-1,q-1)}$, and thus lies in the kernel of all operators Φ_I. We conclude that $\mathit{\Gamma}^{(n)}_{\lambda;\mu} \subset \Gamma(\lambda,n)$, and the irreducibility of $\Gamma(\lambda,\mu)$ yields the desired statement. □

Finally, to prove the technical statement used in the proof of Theorem 2.2, we need a preparatory lemma.

Define the contractions $\Phi_{I_1,\ldots,I_r} : \mathbf{V}^{\otimes(p,q)} \to \mathbf{V}^{\otimes(p-r,q-r)}$ as the r-fold convolutions between copies of $\mathbf{V}$ and $\mathbf{V}_*$ indicated by the pairwise disjoint collection of index pairs $I_1,\ldots,I_r$. For $r=1,\ldots,\ell$, define the inclusions $\Psi^{(n)}_{I_1,\ldots,I_r} : (V_n)^{\otimes(p-r,q-r)} \to (V_n)^{\otimes(p,q)}$, by analogy with $\Phi_{I_1,\ldots,I_r}$, as the r-fold insertions of the canonical element of $V_n \otimes V_n^*$ into positions specified by disjoint pair of indices $I_1,\ldots,I_r$. Set also

$$(V_n)^{\{p,q\}}_r = \sum_{I_1,\ldots,I_r} \mathrm{im}\left(\Psi^{(n)}_{I_1,\ldots,I_r} : (V_n)^{\otimes(p-r,q-r)} \to (V_n)^{\otimes(p,q)}\right).$$

It is a standard exercise to show that for each n one has the direct sum decomposition

$$(V_n)^{\otimes(p,q)} = (V_n)^{\{p,q\}}_0 \oplus (V_n)^{\{p,q\}}_1 \oplus (V_n)^{\{p,q\}}_2 \oplus \cdots \oplus (V_n)^{\{p,q\}}_\ell.$$

For any $I=(i,j)$ we consider the linear map

$$\Xi^{(n)}_I : (V_n)^{\otimes(p,q)} \to (V_n)^{\otimes(p-1,q-1)},$$

$$v_1 \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots \otimes v_q^* \mapsto\ n\,\langle \xi_n^*, v_i\rangle\langle v_j^*, \xi_n\rangle\ v_1 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes v_p \otimes v_1^* \otimes \ldots \otimes \hat{v}_j^* \otimes \ldots \otimes v_q^*.$$

Lemma 6.2. *For any $v \in \mathbf{V}^{\{p,q\}}$ we have*

$$\lim_{n\to\infty} \Xi^{(n)}_{J_1}\,\Phi^{(n)}_{J_2,\ldots,J_r}\,\Psi^{(n)}_{I_1,\ldots,I_r} v = \begin{cases} v, & \text{if } \{I_1,\ldots,I_r\} = \{J_1,\ldots,J_r\} \text{ as sets} \\ 0, & \text{otherwise} \end{cases}.$$

Proof. We use induction on s. The base of induction $s=1$ states that

$$\lim_{n\to\infty} \Xi^{(n)}_J\,\Psi^{(n)}_I v = \begin{cases} v, & \text{if } I=J \\ 0, & \text{otherwise} \end{cases},$$

which is clear from the definition. Assume now that $r \geq 2$, and let $J_r = (i,j)$.

Case 1. There exists k such that $I_k = (i,j)$; we may assume that $k=r$. Then

$$\Xi^{(n)}_{J_1}\,\Phi^{(n)}_{J_2,\ldots,J_r}\,\Psi^{(n)}_{I_1,\ldots,I_r} v = \Xi^{(n)}_{J_1}\,\Phi^{(n)}_{J_2,\ldots,J_{r-1}}\,\Psi^{(n)}_{I_1,\ldots,I_{r-1}}\,\Phi_{I_r}\,\Psi_{J_r} v = \Xi^{(n)}_{J_1}\,\Phi^{(n)}_{J_2,\ldots,J_{r-1}}\,\Psi^{(n)}_{I_1,\ldots,I_{r-1}} v,$$

and the desired statement follows from the induction hypothesis.

Case 2. There exist k, l such that $I_k = (i,a)$ and $I_l = (b,j)$; we may assume that $k = r$ and $l = r-1$. Setting $I' = (b,a)$ and using the identity $\Phi^{(n)}_{(i,j)} \Psi^{(n)}_{(i,a),(b,j)} = \frac{1}{n}\Psi^{(n)}_{b,a}$, we get

$$\Xi^{(n)}_{J_1} \Phi^{(n)}_{J_2,\dots,J_r} \Psi^{(n)}_{I_1,\dots,I_r} v = \Xi^{(n)}_{J_1} \Phi^{(n)}_{J_2,\dots,J_{r-1}} \Psi^{(n)}_{I_1,\dots,I_{r-2}} \Phi^{(n)}_{J_r} \Psi^{(n)}_{I_{r-1},I_r} v = \frac{1}{n} \Xi^{(n)}_{J_1} \Phi^{(n)}_{J_2,\dots,J_{r-1}} \Psi^{(n)}_{I_1,\dots,I_{r-2},I'} v.$$

Applying the induction hypothesis, in both cases we obtain $\lim_{n\to\infty} \Xi^{(n)}_{J_1} \Phi^{(n)}_{J_2,\dots,J_r} \Psi^{(n)}_{I_1,\dots,I_r} v = 0$.

Case 3. There exists k such that $I_k = (i,a)$, but j never occurs in the second position of any I_l; we may assume that $k = r$. Using the identity $\Phi^{(n)}_{(i,j)} \Psi^{(n)}_{(i,a)} v = \frac{m}{n} \Phi^{(m)}_{(i,j)} \Psi^{(m)}_{(i,a)} v$, we get

$$\begin{aligned} \lim_{n\to\infty} \Xi^{(n)}_{J_1} \Phi^{(n)}_{J_2,\dots,J_r} \Psi^{(n)}_{I_1,\dots,I_r} v &= \lim_{n\to\infty} \Xi^{(n)}_{J_1} \Phi^{(n)}_{J_2,\dots,J_{r-1}} \Psi^{(n)}_{I_1,\dots,I_{r-1}} \Phi^{(n)}_{J_r} \Psi^{(n)}_{I_r} v \\ &= \lim_{n\to\infty} \frac{m}{n} \Xi^{(n)}_{J_1} \Phi^{(n)}_{J_2,\dots,J_{r-1}} \Psi^{(n)}_{I_1,\dots,I_{r-1}} \left(\Phi^{(m)}_{J_r} \Psi^{(m)}_{I_r} v \right). \end{aligned}$$

Applying the induction hypothesis to $\Phi^{(m)}_{J_r} \Psi^{(m)}_{I_r} v \in \mathbf{V}^{\{p,q\}}$, we see that the desired statement holds.

Case 4. The index i never occurs in the first position of any I_k, and j never occurs in the second position of any I_l. Then for all n we have

$$\Xi^{(n)}_{J_1} \Phi^{(n)}_{J_2,\dots,J_r} \Psi^{(n)}_{I_1,\dots,I_r} v = \Xi^{(n)}_{J_1} \Phi^{(n)}_{J_2,\dots,J_{r-1}} \Psi^{(n)}_{I_1,\dots,I_r} \Phi_{I_r} v = 0.$$

□

We are now ready to prove the following assertion, used in the proof of Theorem 2.2.

Proposition 6.3. *There exist infinitely many n such that $u - \pi_n(u) \notin F^{(s-1)}_n$.*

Proof. Assume that, on the contrary, $u - \pi_n(u) \in F^{(s-1)}_n$ for all $n \gg m$. Let $J_1, \dots, J_s$ be any collection of pairwise disjoint indices. Since $\Phi_{J_2,\dots,J_r}(u - \pi_n(u)) = 0$, we obtain for all n

$$\Xi^{(n)}_{J_1} \Phi_{J_2,\dots,J_r} \pi_n(u) = \Xi^{(n)}_{J_1} \Phi_{J_2,\dots,J_r} u = 0.$$

On the other hand, the vector $\pi_n(u)$ can be represented as

$$\pi_n(u) = \sum_{I_1,\dots,I_s} \Psi_{I_1,\dots,I_s} \zeta_{I_1,\dots,I_s}$$

for some collection $\{\zeta_{I_1,\dots,I_s}\}$ of vectors from $(V_n)^{\{p,q\}}$, and according to Lemma 6.2

$$\lim_{n\to\infty} \Xi^{(n)}_{J_1} \Phi_{J_2,\dots,J_r} \pi_n(u) = \zeta_{J_1,\dots,J_n}.$$

It follows that $\zeta_{J_1,\dots,J_n} = 0$, and thus $\pi_n(u) = 0$. This contradicts the assumption that $\pi_n(u)$ generates a submodule of $F^{(s+1)}_n / F^{(r)}_n$ isomorphic to $\Gamma^{(n)}_{\lambda;\mu}$. □

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