TRIVIALITY OF VECTOR BUNDLES ON TWISTED IND-GRASSMANNIANS

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Abstract. Twisted ind-Grassmannians are ind-varieties $\mathbf{G}$ obtained as direct limits of Grassmannians $G(i_m, V^{n_m})$ for $m \in \mathbb{Z}_{>0}$, under embeddings $\varphi_m : G(i_m, V^{n_m}) \to G(i_{m+1}, V^{n_{m+1}})$ of degree greater than one. It has been conjectured in [PT] and [DP] that any vector bundle of finite rank on a twisted ind-Grassmannian is trivial. We prove this conjecture.

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1. Introduction and statement of the main result

An *ind-Grassmannian* $\mathbf{G} = \varinjlim G(i_m, V^{n_m})$ is an ind-variety obtained as the direct limit of a chain of embeddings

$$G(i_1, V^{n_1}) \stackrel{\varphi_1}{\hookrightarrow} G(i_2, V^{n_2}) \stackrel{\varphi_2}{\hookrightarrow} \dots \stackrel{\varphi_{m-1}}{\hookrightarrow} G(i_m, V^{n_m}) \stackrel{\varphi_m}{\hookrightarrow} \dots, \tag{1}$$

where $G(i, V)$ denotes the Grassmanian of i-dimensional subspaces in a finite dimensional vector space V. Each embedding φ_m has a well defined degree $\deg \varphi_m$, and the ind-Grassmannian $\mathbf{G}$ is *twisted* if $\deg \varphi_m > 1$ for infinitely many m. A *vector bundle* $\mathbf{E}$ *of rank* $\mathbf{r} \in \mathbb{Z}_{>0}$ on $\mathbf{G}$ is the inverse limit $\varprojlim E_m$ of an inverse system of vector bundles E_m or rank $\mathbf{r}$ on $G(i_m, V^{n_m})$ (i.e. a system of vector bundles E_m with fixed isomorphisms $\psi_m : E_m \cong \varphi_m^* E_{m+1}$).

In the special case when $i_m = 1$ and $\deg \varphi_m = 1$ for all m, the study of finite rank vector bundles on ind-Grassmannians goes back to W. Barth, A. Van de Ven and A. N. Tyurin, [BV], [T]. In this case $\mathbf{G}$ is just the infinite projective space $\mathbf{P}^\infty$, and the remarkable Barth-Van de Ven-Tyurin Theorem claims that any vector bundle of finite rank on $\mathbf{P}^\infty$ is isomorphic to a direct sum of line bundles. Historically this is the first manifestation of a general phenomenon that seems to take place for ind-varieties defined via sequences of embeddings similar to (1) with $G(i_m, V_m)$ replaced by arbitrary flag varieties: in all cases known, the restriction of any finite rank vector bundle on the ind-variety to a large enough finite-dimensional flag subvariety is equivariant. Around the same time this phenomenon occured also in the work of E. Sato who gave an independent proof of the Barth-Van de Ven-Tyurin Theorem, [S1]. Shortly after that Sato established a more general result which applies in particular to the ind-Grassmannian $\mathbf{G}(i, V)$ of i-dimensional subspaces in a countable-dimensional vector space V [S2].

More recently the subject has been revisited in the papers [DP], [CT] and [PT]. In particular, in [PT] a general conjecture about finite rank vector bundles on ind-Grassmannians $\mathbf{G}$ has been stated. In fact, as we show in [PT], if $\mathbf{G}$ is not twisted (which is equivalent to assuming that $\deg \varphi_m = 1$ for all m), this conjecture is a relatively straightforward corollary of Sato's result. This leaves open the case of a twisted ind-Grassmannian $\mathbf{G}$, where the conjecture claims simply that a finite-rank vector bundle on $\mathbf{G}$ is trivial. So far this latter conjecture was established in the following three cases: for a rank-two bundle on any twisted ind-Grassmannian [PT], for any finite-rank bundle on any twisted projective ind-space (a twisted projective ind-space can be defined via the sequence (1) for $i_m = 1$ and $\deg \varphi_m > 1$ for all m) [DP], and for an arbitrary finite-rank bundle on some special twisted ind-Grassmannians (for which the embeddings φ_m are twisted extensions as defined in [DP]).

In the present paper we prove the conjecture, i.e. the following theorem.

Theorem 1.1. *A finite-rank vector bundle* $\mathbf{E} = \varprojlim E_m$ *on any twisted ind-Grassmannian* $\mathbf{G} = \varinjlim G(i_m, V^{n_m})$ *is trivial.*

Here is a brief description of the main ingredients in the proof of Theorem 1.1. First of all, without loss of generality we can assume that $\mathbf{E}$ is self-dual. This is achieved by possibly replacing of $\mathbf{E}$ with $\mathcal{E}nd\mathbf{E}$. The ultimate goal of the proof is to construct, for large m, subsheafs $\mathcal{F}_m$ of the vector bundles E_m with $c_1(\mathcal{F}_m) > 0$ under the assumption that E_m is nontrivial. This then easily leads to a contradiction since the facts that $\mathbf{G}$ is twisted and E_m is infinitely extendable force $c_1(\mathcal{F}_m)$ to be infinite. The general idea of such a construction goes back to Barth-Van de Ven and Tyurin in the case of $\mathbf{P}^\infty$.

The construction of $\mathcal{F}_m$ combines several ideas and is based on a study of the variety of maximal jumping lines of the vector bundle $\mathbf{E}$. In our case we investigate the variety of maximal jumping lines of E_m on $G(i_m, V^{n_m})$. We reduce the problem to the study of a similar variety for projective space by using a birational isomorphism of $G(i_m, V^{n_m})$ with a fibred space X_m with fibre a projective space. A key result in this connection is the existence of universal bounds for the degree and codimension of the variety of maximal jumping lines through a point of a vector bundle on a projective space.

The paper is organized as follows. Section 3 is a study of varieties of bounded degree and codimension in projective spaces of growing dimension. The main result here is that any two points of such a variety can be connected by chain of projective subspaces of growing dimension. This result is close in spirit to a classical result of A. Predonzan, and is part of the present paper due to the lack of a suitable reference.

In section 4 we give a sufficient condition on an integer m for a given vector bundle E on $\mathbb{P}^n$ to be m-regular in the sense of Mumford-Castelnuovo, i.e. that $H^i(E(m-i)) = 0$ for $i \geq 1$. This condition on m is needed for the estimate of the degree of the variety of maximal jumping lines through a point of a vector bundle on a projective space, given in section 5. This estimate (see Theorem 5.3) is given in terms of rank, second Chern class, maximal jump and dimension of the projective space, under the assumption that the first Chern class vanishes.

The final section 6 is devoted to the construction of the subsheaf $\mathcal{F}_m$ of E_m, where $\mathbf{E} = \varprojlim E_m$ is a self-dual vector bundle on $\mathbf{G}$. Here we replace $G(i_m, V^{n_m})$ by a fibred space X_m, to the fibres of which we apply all above results on vector bundles on projective spaces. The construction of $\mathcal{F}_m$ then quickly leads to a contradiction with the nontriviality of E_m as explained above.

We conclude this introduction with an example of a twisted ind-Grassmannian for which our theorem provides a nontrivial statement. In this example a twisted ind-Grassmannian arises naturally as a homogeneous space of a locally linear ind-group. Various further examples of twisted ind-Grassmannians can be found in the earlier papers [DP] and [PT].

An interesing ind-group is the ind-group $\mathbf{SL}(n, \mathrm{Adj})$. Fix n and consider the embedding

$$SL(n) \to SL(n^2 - 1)$$

defined by the requirement that the natural representation of $SL(n^2-1)$ becomes the adjoint representation when restricted to $SL(n)$. Setting $\mathcal{G}_1 := SL(n)$, $\mathcal{G}_2 := SL(n^2-1)$, and iterating this construction we obtain the ind-group $\mathbf{SL}(n, \mathrm{Adj})$ as the direct limit $\varinjlim \mathcal{G}_m$. Fix a subspace $V_1 \subset \mathbb{C}^n$. Then $V_2 := V_1 \otimes (\mathbb{C}^n/V_1)^\vee$ ($^\vee$ indicates dual space) is a well-defined subspace of the adjoint representation of $\mathcal{G}_2$. Iteration of this construction yields a subspace V_m of the natural representation of $\mathcal{G}_m$ for each m. The stabilizers $\mathcal{P}_m \subset \mathcal{G}_m$ of the spaces V_m form a direct system of parabolic subgroups $\mathcal{P}_1 \subset \mathcal{P}_2 \subset ...$ with the property $\mathcal{P}_m \cap \mathcal{G}_{m-1} = \mathcal{P}_{m-1}$. This defines closed embeddings $\zeta_{m-1} : \mathcal{G}_{m-1}/\mathcal{P}_{m-1} \hookrightarrow \mathcal{G}_m/\mathcal{P}_m$, and hence an ind-variety $\varinjlim \mathcal{G}_m/\mathcal{P}_m$. Since each $\mathcal{G}_m/\mathcal{P}_m$ is a Grassmannian, $\varinjlim \mathcal{G}_m/\mathcal{P}_m$ is an ind-Grassmannian. Moreover, the restriction of the tautological bundle on $\mathcal{G}_m/\mathcal{P}_m$ to $\mathcal{G}_{m-1}/\mathcal{P}_{m-1}$ is isomorphic the cotangent bundle of $\mathcal{G}_{m-1}/\mathcal{P}_{m-1}$. This shows that the degree of ζ_m equals the dimension of the natural representation of $\mathcal{G}_m$. Hence $\varinjlim \mathcal{G}_m/\mathcal{P}_m$ is a twisted ind-Grassmannian. It is an exercise to check that the ind-group $\varinjlim \mathcal{P}_m$ has no non-trivial finite-dimensional representations. Therefore $\varinjlim \mathcal{G}_m/\mathcal{P}_m = \mathbf{SL}(n, \mathrm{Adj})/(\varinjlim \mathcal{P}_m)$ admits no non-trivial $\mathbf{SL}(n, \mathrm{Adj})$-equivariant vector bundles of finite rank. Theorem 1.1, however, yields the much stronger result that any finite rank vector bundle on $\varinjlim \mathcal{G}_m/\mathcal{P}_m$ is trivial.

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2. Notation and Conventions

Our notation is mostly standard. The ground field is $\mathbb{C}$. All vector bundles considered are assumed to have finite rank. We do not make a distinction between locally free sheaves of finite rank and vector bundles. We use the term *algebraic variety* or simply *variety* as shorthand for a reduced noetherian scheme. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an algebraic variety or a scheme X, $\mathcal{F}^j$ denotes the direct sum of j copies of $\mathcal{F}$, $H^j(\mathcal{F})$ denotes the j^{th} cohomology group of $\mathcal{F}$, $h^j(\mathcal{F}) := \dim H^j(\mathcal{F})$, and $\mathcal{F}^\vee$ stands for the dual sheaf, i. e. $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. Sym^j and $\wedge^j$ denote respectively j-th symmetric and exterior power. If $Z \subset X$ is a subvariety, $\mathcal{I}_{Z,X}$ denotes the sheaf of ideals corresponding to Z. By $\mathbb{P}(E)$ we denote the projectivization of a vector bundle E (in particular, of a vector space).

By a *projective subspace* $\mathbb{P}^k$ in $G(i, V)$ we mean linearly embedded projective subspace, i.e. the set of i-dimensional subspaces W of V with $V_0 \subset W \subset V_1$, where $V_0 \subset V_1$ is a fixed flag of subspaces of V of dimensions $i-1$ and $i+k$, or $i-k$ and $i+1$ respectively. In particular, a *line* in $G(i, V)$ is determined by a flag $V_1 \subset V_2$ of subspaces in V with $\dim V_1 = i-1$, $\dim V_2 = i+1$.

If $C \subset X$ is a smooth irreducible rational curve in an algebraic variety X and E is a vector bundle on X, then by a classical theorem of Grothendieck, $E|_C$ is isomorphic to $\bigoplus_i \mathcal{O}_C(\delta_i)$ for some $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{\mathrm{rk}E}$. We call the ordered $\mathrm{rk}E$-tuple $(\delta_1, \ldots, \delta_{\mathrm{rk}E})$ *the splitting type* of $E|_C$.

Let E be a vector bundle on $G(i, V)$. For an arbitrary rational curve C in $G(i, V)$ consider the splitting type $(\delta_1, \ldots, \delta_{\mathrm{rk}E})$ of the bundle $E|_C$ and set

$$\delta_A(E|_C) := \delta_1, \quad \delta_B(E|_C) := \delta_{\mathrm{rk}E}, \quad \delta(E|_C) := \delta_A(E|_C) - \delta_B(E|_C),$$

$$\kappa_A(E|_C) := \max\{k | 1 \leq k \leq \mathrm{rk}E,\ \delta_k = \delta_A(E|_C)\}.$$

Furthermore, set

$$\delta_A(E) := \max_l \delta_A(E|_l), \qquad \delta_B(E) := \min_l \delta_B(E|_l),$$

where l runs over all lines in $G(i, V)$,

$$\delta(E) := \delta_A(E) - \delta_B(E),$$

$$\kappa_A(E) := \max\{\kappa_A(E|_l) \mid l \text{ is a line in } G(i, V) \text{ such that } \delta_A(E|_l) = \delta_A(E)\}.$$

It is essential to note that $\delta_A(E|_C)$ and $\kappa_A(E|_C)$ are semicontinuous functions of C, where C belongs to any fixed flat family of rational curves in $G(i, V)$ [H, Ch. III, Thm. 12.8].

We need also a notation concerning polynomials. For an arbitrary nonzero polynomial $f(y_1, ..., y_q) = \sum \frac{a_{i_1 ... i_q}}{b_{i_1 ... i_q}} y_1^{i_1} ... y_q^{i_q} \in \mathbb{Q}[y_1, ..., y_q]$ with coprime $a_{i_1 ... i_q} \in \mathbb{Z}$ and $b_{i_1 ... i_q} \in \mathbb{Z}$ for all $i_1, ..., i_q$, we denote by $f(y_1, ..., y_q)^+ \in \mathbb{Z}[y_1, ..., y_q]$ the polynomial $\sum a^2_{i_1 ... i_q} y_1^{2i_1} ... y_q^{2i_q}$. Note that $-f(y_1, ..., y_q)^+ \leq f(y_1, ..., y_q) \leq f(y_1, ..., y_q)^+$ for all $y_1, ..., y_q \in \mathbb{Z}$.

3. Projective subspaces in varieties of bounded codimension and degree

In this section we prove that any two points of a subvariety of bounded codimension and degree in a projective space of growing dimension can be connected by a chain of projective subspaces of growing dimension lying on this subvariety. This is a chapter of the theory of Fano schemes in the spirit of Altman and Kleiman [AK], and is also close to Predonzan's Theorem (1948), a modern presentation of which can be found in [BM]. Thoughout the section $d \in \mathbb{Z}_{\geq 2}$ is fixed and $n \in \mathbb{Z}_{\geq 6}$ is variable. The integer $k \in \mathbb{Z}_{\geq 1}$ is variable and satisfies

$$n \geq d\binom{k+d}{d} + k, \tag{2}$$

for instance, one may set $k = k(n) := \left[\ \sqrt[d+1]{n/d}\ \right]$.

3.1. Projective subspaces in hypersurfaces of bounded degree and growing dimension. Consider the projective space $\mathbb{P}^n = \mathbb{P}(V)$ where V is a vector space of dimension $n+1$. Let

$$\mathbb{P}^s := |\mathcal{O}_{\mathbb{P}^n}(d)|, \quad s = \binom{n+d}{d} - 1,$$

be the complete linear series of hypersurfaces of given degree d in $\mathbb{P}^n$. Consider the natural diagram

$$G(k+1, V) \overset{\tilde{p}}{\leftarrow} \Gamma \overset{\tilde{q}}{\rightarrow} \mathbb{P}^s, \tag{3}$$

where $\Gamma = \{(\mathbb{P}^k, H) \in G(k+1, V) \times \mathbb{P}^s \mid \mathbb{P}^k \subset H\}$ and we interpret $G(k+1, V)$ as the Grassmannian of k-dimensional projective subspaces in $\mathbb{P}^n$. For each pair $(\mathbb{P}^k, H) \in \Gamma$ choose homogeneous coordinates $(x_0 : x_1 : ... : x_n)$ in $\mathbb{P}^n$ such that $\mathbb{P}^k = \{x_{k+1} = ... = x_n = 0\}$. Let $H = \{f(x_0, ..., x_n) = 0\}$, $\quad f \in H^0(\mathcal{O}_{\mathbb{P}^n}(d))$, and

$$\Phi_i(x_0, x_1, ..., x_k) := \frac{\partial f}{\partial x_{k+i}}(x_0, x_1, ..., x_k, 0, ..., 0), \quad 1 \leq i \leq n-k.$$

Assume that H is smooth. Then $\overset{n-k}{\underset{i=1}{\cap}} \{\Phi_i(x_0, x_1, ..., x_k) = 0\} = \emptyset$ and we have an exact sequence of normal bundles on $\mathbb{P}^k$

$$0 \to N_{\mathbb{P}^k/H} \to \mathcal{O}_{\mathbb{P}^k}(1)^{n-k} \overset{\epsilon_k}{\to} \mathcal{O}_{\mathbb{P}^k}(d) \to 0, \qquad \epsilon_k = (\cdot\Phi_1, ..., \cdot\Phi_{n-k}). \tag{4}$$

Assume H is generic in the sense that

$$\mathrm{Span}(\Phi_1, ..., \Phi_{n-k}) = H^0(\mathcal{O}_{\mathbb{P}^k}(d-1)). \tag{5}$$

Then the exact sequence obtained from (4) via twisting by $\mathcal{O}_{\mathbb{P}^k}(-1)$ induces a surjective homomorphism $H^0(\mathcal{O}_{\mathbb{P}^k}^{n-k}) \to H^0(\mathcal{O}_{\mathbb{P}^k}(d-1))$, and it is easy to see that, after twisting back by $\mathcal{O}_{\mathbb{P}^k}(1)$, we get a surjective homomorphism $h^0(\epsilon_k) : H^0(\mathcal{O}_{\mathbb{P}^k}(1)^{n-k}) \to H^0(\mathcal{O}_{\mathbb{P}^k}(d))$. Therefore

$$h^0(N_{\mathbb{P}^k/H}) = (k+1)(n-k) - \binom{k+d}{d} > 0, \quad h^1(N_{\mathbb{P}^k/H}) = h^1(N_{\mathbb{P}^k/H}(-1)) = 0 \tag{6}$$

(the inequality follows from (2)).

Note that $\tilde{p} : \Gamma \to G(k+1, V)$ is a projective bundle with fibre $\mathbb{P}(H^0(\mathcal{I}_{\mathbb{P}^k,\mathbb{P}^n}(d)))$, hence Γ smooth and irreducible. Therefore $\dim \tilde{q}^{-1}(H) \geq \dim \Gamma - s = \dim G(k+1, V) + \dim \mathbb{P}(H^0(\mathcal{I}_{\mathbb{P}^k,\mathbb{P}^n}(d))) - s = (k+1)(n-k) + (s - \binom{k+d}{d}) - s = (k+1)(n-k) - \binom{k+d}{d}$. From this and (6) we obtain by deformation theory that

$$B_H := \tilde{p}(\tilde{q}^{-1}(H))$$

has dimension

$$\dim B_H = h^0(N_{\mathbb{P}^k/H}) = (k+1)(n-k) - \binom{k+d}{d} \tag{7}$$

and is smooth at the point $\mathbb{P}^k$ for a generic smooth $H \in \mathbb{P}^s$. Moreover, the projective morphism $\tilde{q}$ is dominant. Since the image of a projective morphism is closed [H, Ch. II, §4, Thm. 4.9], this implies that $\tilde{q}$ is surjective.

Lemma 3.1. *For a smooth generic (in the sense of (5)) hypersurface $H \in \mathbb{P}^s$, B_H is a smooth irreducible variety of dimension $(k+1)(n-k) - \binom{k+d}{d}$.*

Proof. The smoothness of B_H follows from the fact that B_H is a generic fibre of the surjective morphism $\tilde{q} : \Gamma \to \mathbb{P}^s$ of smooth varieties [H, Ch. III, §10, Cor.10.7].

Let S_{k+1} be the rank-$(k+1)$ tautological bundle on $G(k+1, V)$. By [AK, Thm. 1.3] B_H is the zero-scheme of a regular section $\sigma \in H^0(T^\vee)$, where $T := Sym^d S_{k+1}$. Moreover, we have the standard Koszul resolution of the sheaf $\mathcal{O}_{B_H}$

$$0 \to \wedge^{\mathrm{rk}T} T \to ... \to \wedge^2 T \to T \xrightarrow{\sigma^\vee} \mathcal{O}_{G(k+1,V)} \to \mathcal{O}_{B_H} \to 0. \tag{8}$$

We will show that

$$H^0(T) = H^j(\wedge^j T) = 0, \quad 1 \le j \le \mathrm{rk}T. \tag{9}$$

For this, consider the incidence diagram

$$G(i+1, V) \xleftarrow{p_i} Fl(i, i+1, V) \xrightarrow{q_i} G(i, V), \quad 1 \le i \le k. \tag{10}$$

On $Fl(i, i+1, V)$ one has an exact sequence of vector bundles

$$0 \to q_i^* S_i \xrightarrow{\theta_i} p_i^* S_{i+1} \to q_i^* \mathcal{O}_{G(i,V)}(1) \otimes p_i^* \mathcal{O}_{G(i+1,V)}(-1) \to 0, \tag{11}$$

where S_i be the rank-i tautological bundle on $G(i, V)$. Restricting (11) to a fibre $\mathbb{P}^{n-i}_y := q_i^{-1}(y)$ for $y \in G(i, V)$, we obtain an exact triple

$$0 \to q_i^* S_i|_{\mathbb{P}^{n-i}_y} \xrightarrow{\theta_i|_{\mathbb{P}^{n-i}_y}} p_i^* S_{i+1}|_{\mathbb{P}^{n-i}_y} \to \mathcal{O}_{\mathbb{P}^{n-i}_y}(-1) \to 0, \qquad q_i^* S_i|_{\mathbb{P}^{n-i}_y} \simeq (\mathcal{O}_{\mathbb{P}^{n-i}_y})^i.$$

Passing to symmetric powers and setting $s_i := \binom{d+i}{d-1}$, $t_i := \binom{d+i-1}{d}$, we have

$$0 \to q_i^* Sym^d S_i|_{\mathbb{P}^{n-i}_y} \to p_i^* Sym^d S_{i+1}|_{\mathbb{P}^{n-i}_y} \to \bigoplus_{p=1}^{s_i} \mathcal{O}_{\mathbb{P}^{n-i}_y}(a_p) \to 0, \quad -d \le a_p \le -1, \ \ 1 \le p \le s_i, \tag{12}$$

$$q_i^* Sym^d S_i|_{\mathbb{P}^{n-i}_y} \simeq (\mathcal{O}_{\mathbb{P}^{n-i}_y})^{t_i} \ . \tag{13}$$

Consider the exact triples

$$0 \to q_i^* \wedge^j (Sym^d S_i) \xrightarrow{\Theta_{ij}} p_i^* \wedge^j (Sym^d S_{i+1}) \to \Lambda_{ij} \to 0, \quad \Lambda_{ij} := \mathrm{coker}\, \Theta_{ij}, \quad 1 \le j \le \mathrm{rk}T, \tag{14}$$

where Θ_{ij} are the monomorphisms induced by θ_i in (11). After restriction to $\mathbb{P}^{n-i}_y$, using (12) and (13) we obtain

$$\Lambda_{ij}|_{\mathbb{P}^{n-i}_y} \simeq \bigoplus_{q=1}^{u_{ij}} \mathcal{O}_{\mathbb{P}^{n-i}_y}(b_q), \quad -jd \le b_q \le -1, \quad 1 \le q \le u_{ij}, \tag{15}$$

where $u_{ij} := \binom{s_i+t_i}{j} - \binom{t_i}{j}$. The key observation is that (2) and (15) imply that $H^a(\Lambda_{ij}|_{\mathbb{P}^{n-i}_y}) = 0,\ a \ge 0,\ 1 \le j \le \mathrm{rk}T,\ 1 \le i \le k$. This shows that the Leray spectral sequence $E_2^{aa'} = H^a(R^{a'} q_{i*} \Lambda_{ij}) \Rightarrow H^{\cdot}(\Lambda_{ij})$ degenerates and thus gives

$$H^a(\Lambda_{ij}) = 0,\ a \ge 0,\ 1 \le j \le \mathrm{rk}T,\ 1 \le i \le k. \tag{16}$$

Since (it is well known that) $H^a(\wedge^j(Sym^d S_i)) = H^a(q_i^* \wedge^j (Sym^d S_i))$, $H^a(\wedge^j(Sym^d S_{i+1})) = H^a(p_i^* \wedge^j (Sym^d S_{i+1}))$, $a \geq 0$, we derive from (16) and (14) that

$$H^a(\wedge^j(Sym^d S_{i+1})) = H^a(\wedge^j(Sym^d S_i)), \quad 1 \leq i \leq k. \tag{17}$$

Moreover, setting $j_i := \mathrm{rk} Sym^d S_i$, we obtain $\wedge^{j_i}(Sym^d S_i) \simeq \mathcal{O}_{G(i,V)}(-\binom{d+i-1}{i})$, so that, similarly to (16), $H^a(\wedge^{j_i}(Sym^d S_i)) = 0, \quad a \geq 0,\ 1 \leq i \leq k.$ This together with (17) yields (9).

Now (8) and (9) show that $h^0(\mathcal{O}_{B_H}) = h^0(\mathcal{O}_{G(k+1,V)}) = 1$. Hence, B_H is connected. This together with the smoothness of B_H yields its irreducibility. □

Consider the graph of incidence $\Sigma_H = \{(x, \mathbb{P}^k) \in H \times B_H \mid x \in \mathbb{P}^k\}$ with projections

$$H \overset{\pi_1}{\leftarrow} \Sigma_H \overset{\pi_2}{\rightarrow} B_H. \tag{18}$$

Since the fibers of π_2 are isomorphic to $\mathbb{P}^k$, the irreducibility of B_H implies the irreducibility of Σ_H.

Lemma 3.2. *Let $H \subset \mathbb{P}^n$ be a smooth hypersurface which is generic in the sense of (5). Then*

(i) H is filled by the subspaces $\mathbb{P}^k$ of the family B_H, and for an arbitrary $x \in H$ the set $B_H(x) := \pi_2(\pi_1^{-1}(x))$ is equidimensional of dimension

$$\dim B_H(x) = k(n-k) - \binom{k+d}{d} + 1; \tag{19}$$

moreover, for a generic $x \in H$ $B_H(x)$ is an irreducible subvariety of B_H;

(ii) the subset $K_{H,k}(x) := \pi_1(\pi_2^{-1}(B_H(x))) = \underset{\mathbb{P}^k \in B_H(x)}{\cup} \mathbb{P}^k$ of H has dimension

$$\dim K_{H,k}(x) \geq n - d; \tag{20}$$

moreover, for a generic $x \in H$ $K_{H,k}(x)$ is an irreducible subvariety of H.

Proof. (i) Let $(x, \mathbb{P}^k) \in \Sigma_H$. Consider the standard Koszul resolution of the ideal sheaf $\mathcal{I}_{x,\mathbb{P}^k}$

$$0 \to \mathcal{O}_{\mathbb{P}^k}(-k) \to \dots \to \mathcal{O}_{\mathbb{P}^k}(-i)^{\binom{k}{i}} \to \dots \to \mathcal{O}_{\mathbb{P}^k}(-1)^k \to \mathcal{I}_{x,\mathbb{P}^k} \to 0. \tag{21}$$

Twisting (21) by $N_{\mathbb{P}^k/H}$, we obtain the exact sequence

$$0 \to N_{\mathbb{P}^k/H}(-k) \to \dots \to N_{\mathbb{P}^k/H}(-i)^{\binom{k}{i}} \to \dots \to N_{\mathbb{P}^k/H}(-1)^k \to \mathcal{I}_{x,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H} \to 0. \tag{22}$$

Since $h^i(N_{\mathbb{P}^k/H}(-i)) = 0, \quad 1 \leq i \leq k$ (for $i > 1$ this follows immediately from (4); for $i = 1$ see (6)), (22) gives

$$h^1(\mathcal{I}_{x,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H}) = 0. \tag{23}$$

Next, consider the exact triple

$$0 \to \mathcal{I}_{x,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H} \to N_{\mathbb{P}^k/H} \to \mathbb{C}_x \otimes N_{\mathbb{P}^k/H} \to 0. \tag{24}$$

Since $\mathbb{C}_x \otimes N_{\mathbb{P}^k/H} \simeq \mathbb{C}^{n-1-k}$, it follows from (6), (24) and (23) that

$$h^0(\mathcal{I}_{x,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H}) = k(n-k) - \binom{k+d}{d} + 1. \tag{25}$$

Note that $H^0(\mathcal{I}_{x,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H})$ is the Zariski tangent space to $B_H(x)$ at the point $\mathbb{P}^k$. Moreover, (23) and (25) imply via deformation theory the smoothness of $B_H(x)$ at $\mathbb{P}^k$ and the equidimensionality of $B_H(x)$ together with the equality (19). This latter equality shows that $\dim \pi_1(\Sigma_H) = \dim H$. Since H is irreducible, $\pi_1 : \Sigma_H \to H$ is surjective as it is a projective morphism of projective varieties. This means that H is filled by the spaces $\mathbb{P}^k \in B_H$.

(ii) Now let y be an arbitrary point of $\mathbb{P}^k$ distinct from x and let $\mathbb{P}^1$ be a projective line in $\mathbb{P}^k$ joining the points x and y. Twisting (4) by the sheaves $\mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k}$ and $\mathcal{O}_{\mathbb{P}^1}(-2)$ yields the exact triples

$$0 \to \mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H} \to \mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes \mathcal{O}_{\mathbb{P}^k}(1)^{n-k} \xrightarrow{\epsilon_k} \mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes \mathcal{O}_{\mathbb{P}^k}(d) \to 0, \tag{26}$$

$$0 \to N_{\mathbb{P}^k/H} \otimes \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-1)^{n-k} \to \mathcal{O}_{\mathbb{P}^1}(d-2) \to 0. \tag{27}$$

Consider the morphism ϵ_k in (26). Passing to sections, we obtain the homomorphism $H^0(\epsilon_k) : H^0(\mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes \mathcal{O}_{\mathbb{P}^k}(1)^{n-k}) \to H^0(\mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes \mathcal{O}_{\mathbb{P}^k}(d))$. To show that $H^0(\epsilon_k)$ is an epimorphism, consider the standard Koszul resolution of the sheaf $\mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes \mathcal{O}_{\mathbb{P}^k}(1)$

$$0 \to \mathcal{O}_{\mathbb{P}^k}(2-k) \to \dots \to \mathcal{O}_{\mathbb{P}^k}(-1)^{\binom{k-1}{2}} \to \mathcal{O}_{\mathbb{P}^k}^{k-1} \xrightarrow{e_1} \mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes \mathcal{O}_{\mathbb{P}^k}(1) \to 0.$$

Passing to cohomology, we obtain the epimorphism in sections $H^0(e_1) : H^0(\mathcal{O}_{\mathbb{P}^k}^{k-1}) \to H^0(\mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes \mathcal{O}_{\mathbb{P}^k}(1))$. Twisting the above resolution by $\mathcal{O}_{\mathbb{P}^k}(d-1)$ and again passing to cohomology, we obtain an epimorphism $H^0(e_d) : H^0(\mathcal{O}_{\mathbb{P}^k}(d-1)^{k-1}) \to H^0(\mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes \mathcal{O}_{\mathbb{P}^k}(d))$. Now the homomorphisms $H^0(\epsilon_k)$, $H^0(e_1)$ and $H^0(e_d)$ fit in a commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes \mathcal{O}_{\mathbb{P}^k}(1)^{n-k}) & \xrightarrow{H^0(\epsilon_k)} & H^0(\mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes \mathcal{O}_{\mathbb{P}^k}(d)) \\ {\scriptstyle H^0(e_1)^{n-k}} \uparrow & & \uparrow {\scriptstyle H^0(e_d)} \\ H^0(\mathcal{O}_{\mathbb{P}^k}^{(k-1)(n-k)}) & \xrightarrow{H^0(\epsilon_k)^{k-1}} & H^0(\mathcal{O}_{\mathbb{P}^k}(d-1)^{k-1}), \end{array}$$

in which the surjectivity of the lower horizontal map $H^0(\epsilon_k)^{k-1}$ follows from (5). Hence $H^0(\epsilon_k)$ is an epimorphism. Thus the cohomology sequence of (26) yields

$$h^0(\mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H}) = (k-1)(n-k) - \binom{k+d}{d} + d + 1, \quad h^1(\mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H}) = 0. \tag{28}$$

Next, (27) implies $h^0(N_{\mathbb{P}^k/H} \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$, $h^1(N_{\mathbb{P}^k/H} \otimes \mathcal{O}_{\mathbb{P}^1}(-2)) = d-1$. This together with (28) and the exact triple

$$0 \to \mathcal{I}_{\mathbb{P}^1,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H} \to \mathcal{I}_{x\cup y,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H} \to \mathcal{O}_{\mathbb{P}^1}(-1) \otimes N_{\mathbb{P}^k/H} \to 0. \tag{29}$$

yields

$$h^0(\mathcal{I}_{x\cup y,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H}) = (k-1)(n-k) - \binom{k+d}{d} + d + 1, \quad h^1(\mathcal{I}_{x\cup y,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H}) = d-1. \tag{30}$$

Put $\Sigma_H(x) := \pi_2^{-1}(B_H(x))$, $\pi_1(x) := \pi_1|_{\Sigma_H(x)}$, and let

$$K_{H,k}(x) \xleftarrow{\pi_1(x)} \Sigma_H(x) \xrightarrow{\pi_2(x)} B_H(x).$$

be the diagram of projections. For any $y \in K_{H,k}(x)$, $y \neq x$, consider the fibre $B_{H,x}(y) := \pi_1(x)^{-1}(y)$ as lying in $B_H(x)$. The Zariski tangent space to $B_{H,x}(y)$ at the point $\mathbb{P}^k$ coincides with $H^0(\mathcal{I}_{x\cup y,\mathbb{P}^k} \otimes N_{\mathbb{P}^k/H})$, hence by (30) and deformation theory we have

$$(k-1)(n-k) - \binom{k+d}{d} + d + 1 \geq \dim B_{H,x}(y) \geq (k-1)(n-k) - \binom{k+d}{d} + 2. \tag{31}$$

Clearly $\dim B_{H,x}(y) > 0$, hence $\pi_1(x)$ is surjective. Since the fibre of $\pi_2(x)$ is $\mathbb{P}^k$, this, together with (19), (31) and the irreducibility of $B_H(x)$, implies (20) and the irreducibility of $K_{H,k}(x)$. □

As a corollary of this lemma we obtain the following theorem.

Theorem 3.3. *Any hypersurface H of degree d in $\mathbb{P}^n$ is filled by subspaces $\mathbb{P}^k \subset \mathbb{P}^n$.*

Proof. Consider the graphs of incidence $\Pi := \{(\mathbb{P}^k, x) \in G(k+1, V) \times \mathbb{P}^n \ | x \in \mathbb{P}^k\}$ and $\tilde{H} := \{(H, x) \in \mathbb{P}^s \times \mathbb{P}^n \mid x \in H\}$ fitting in the commutative diagram

$$\begin{array}{ccccc} \Pi & \xleftarrow{pr_1} & \Pi_\Gamma & \xrightarrow{pr_2} & \tilde{H} \\ \downarrow & & \downarrow & & \downarrow \\ G(k+1,V)\times\mathbb{P}^n & \xleftarrow{\tilde{p}\times id} & \Gamma\times\mathbb{P}^n & \xrightarrow{\tilde{q}\times id} & \mathbb{P}^s\times\mathbb{P}^n, \end{array} \tag{32}$$

where Γ, $\tilde{p}$ and $\tilde{q}$ were defined in (3), $\Pi_\Gamma = (\tilde{p} \times id)^{-1}(\Pi)$ and pr_1 and pr_2 are the induced projections. Since a generic smooth $H \in \mathbb{P}^s$ is filled by projective subspaces $\mathbb{P}^k \subset \mathbb{P}^n$ (Lemma 3.2(i)), pr_2 is dominant. Hence pr_2 is surjective since all varieties and morphisms in (32) are projective. This implies the statement. □

3.2. Projective subspaces in varieties of bounded codimension and degree and of growing dimension. Let $X \subset \mathbb{P}^n$ be an irreducible projective variety satisfying the conditions

$$1 \le c := \operatorname{codim}_{\mathbb{P}^n} X, \quad \deg X \le d, \tag{33}$$

where c is a constant. Assume that $\mathbb{P}^n = \operatorname{Span} X$. Then it is well known that $\deg X \ge c+1$. If $c \ge 2$, starting with $X_0 := X$ one can construct inductively a sequence of projective varieties $X_i \subset \mathbb{P}^{n-i}$, $0 \le i \le c-1$, of respective codimensions $c - i$, together with linear projections

$$p_{x_i} : \mathbb{P}^{n-i} \dashrightarrow \mathbb{P}^{n-i-1}, \quad 0 \le i \le c-2,$$

with centers at points $x_i \in X_i \smallsetminus \operatorname{Sing} X_i$ such that each restriction

$$p_i := p_{x_i}|_{X_i} : X_i \dashrightarrow X_{i+1}, \quad 0 \le i \le c-2. \tag{34}$$

is a birational isomorphism. For this, it suffices to fix $x_i \in X_i \smallsetminus \operatorname{Sing} X_i$ and let X_{i+1} be the closure of $p_{x_i}(X_i)$ in $\mathbb{P}^{n-i}$. Then $\deg X_{i+1} = \deg X_i - 1$. The fact that p_i is birational is standard.

Next, using the notation (2), we set

$$k_{c-1}(n) := k(n-c+1), \quad k_{c-1-i}(n) := \underbrace{\left[\frac{1}{2} ... \left[\frac{1}{2}\left[\frac{1}{2} k_{c-1}(n)\right]\right]...\right]}_{i}, \quad 1 \le i \le c-1.$$

We now argue by reverse induction that $X = X_0$ is filled by projective subspaces of dimension $k_0(n)$. By definition, X_{c-1} is a hypersurface in $\mathbb{P}^{n-(c-1)}$ of degree

$$\deg X_{c-1} = \deg X - (c-1) \le d. \tag{35}$$

Hence, by Theorem 3.3, X_{c-1} is filled by subspaces $\mathbb{P}^{k_{c-1}(n)}$ of $\mathbb{P}^{n-(c-1)}$. This settles the base of induction.

For the induction step, consider the birational map (34). Assume that X_{i+1} is filled by subspaces $\mathbb{P}^{k_{i+1}(n)} \subset \mathbb{P}^{n-i-1}$. Let B be an irreducible component of the base of all such subspaces, with the property that the subspaces in B fill X_{i+1}. Take a generic space $\mathbb{P}^{k_{i+1}(n)} \in B$ and consider the closure $Y_{i+1} := \overline{p_i^{-1}(\mathbb{P}^{k_{i+1}(n)})}$. Since $\mathbb{P}^{k_{i+1}}$ is a generic point of B, the rational map $\tilde{p} := p_i|_{Y_{i+1}} : Y_{i+1} \dashrightarrow \mathbb{P}^{k_{i+1}(n)}$ is a linear projection from the point $x_i \in Y_{i+1}$, and one of the following alternatives holds.

(i) Y_{i+1} is an irreducible quadric and

$$\tilde{p} : Y_{i+1} \dashrightarrow \mathbb{P}^{k_{i+1}(n)}$$

is a birational (stereographic) projection from the point $x_i \in Y_{i+1}$.

(ii) Y_{i+1} is a reducible quadric containing as a component a certain $k_{i+1}(n)$-dimensional space $\tilde{\mathbb{P}}^{k_{i+1}(n)}$ mapping isomorphically onto $\mathbb{P}^{k_{i+1}(n)}$,

$$\tilde{p} : \tilde{\mathbb{P}}^{k_{i+1}(n)} \xrightarrow{\simeq} \mathbb{P}^{k_{i+1}(n)}.$$

Consider these two cases.

In case (i) Y_{i+1} is an irreducible quadric of dimension $k_{i+1}(n)$ filled by projective spaces of dimension $\left[\frac{1}{2}k_{i+1}(n)\right] = k_i(n)$. Since $\mathbb{P}^{k_{i+1}(n)}$ is a generic point of B, the quadrics $Y_{i+1}(\mathbb{P}^{k_{i+1}(n)})$, $\mathbb{P}^{k_{i+1}(n)} \in B$, fill the variety X_i. Hence, the subspaces $\mathbb{P}^{k_i(n)}$ fill X_i.

In case (ii) the irreducibility of X_i and the birationality of p_i imply that the subspaces $\tilde{\mathbb{P}}^{k_{i+1}(n)}$ fill X_i. Moreover, each $\tilde{\mathbb{P}}^{k_{i+1}(n)}$ is filled by subspaces $\mathbb{P}^{k_i(n)}$. Hence X_i is filled by these $\mathbb{P}^{k_i(n)}$'s as well.

Finally note that $\lim_{n\to\infty} k_0(n) = \infty$. We have thus proved the following theorem.

Theorem 3.4. *Let $X \subset \mathbb{P}^n$ be an irreducible projective variety satisfying the conditions (33) and* $\mathrm{Span}X = \mathbb{P}^n$. *Then X is filled by projective subspaces $\mathbb{P}^{k_0(n)} \subset \mathbb{P}^n$ with $\lim_{n\to\infty} k_0(n) = \infty$.*

3.3. Chains of projective subspaces connecting the points of varieties of bounded codimension and degree. Let again H be a smooth hypersurface of degree $d \geq 2$ in $\mathbb{P}^n$ and $x \in H$. Denote by $\mathbb{P}^{n-1}(x)$ the hyperplane in $\mathbb{P}^n$ tangent to H at the point x. Take an affine subspace $\mathbb{A}^{n-1}(x)$ of $\mathbb{P}^{n-1}(x)$ containing x, together with affine coordinates $(y_1, ..., y_{n-1})$ around x in $\mathbb{A}^{n-1}(x)$. The intersection $Y_H(x) := H \cap \mathbb{A}^{n-1}(x)$ is a hypersurface in $\mathbb{A}^{n-1}(x)$ and is given by an equation $\Psi_x = 0$ for some polynomial $\Psi_x = \Psi_x(y_1, ..., y_{n-1})$ of degree d. Decompose Ψ_x into a sum of its homogeneous components

$$\Psi_x = \sum_{p=2}^{d} \Psi_p(y_1, ..., y_{n-1}), \quad \deg \Psi_p = p. \tag{36}$$

Consider $(y_1 : y_2 : ... : y_{n-1})$ as homogeneous coordinates in $\mathbb{P}^{n-2}$; respectively, consider Ψ_p as forms $\Psi_p \in H^0(\mathcal{O}_{\mathbb{P}^{n-2}}(p))$. Define the closed subset

$$X_x = \bigcap_{p=2}^{d} \{\Psi_p(y_1, ..., y_{n-1}) = 0\}, \quad \deg \Psi_p = p.$$

in $\mathbb{P}^{n-2}$. Then Bezout's Theorem implies

$$\mathrm{codim}_{\mathbb{P}^{n-2}} X \leq d - 1, \quad \deg X \leq d! \tag{37}$$

for any irreducible component X of X_x. Therefore $n - 2 \geq \dim \mathrm{Span}X \geq n - d - 1$. In particular, the codimension and degree of X are bounded by constants not depending on n, hence Theorem 3.4 applies to X. This proves the following lemma.

Lemma 3.5. *There exists $n(d) \in \mathbb{Z}_{>0}$ such that, for $n \geq n(d)$, the variety X_x is connected and any irreducible component X of X_x is filled by subspaces $\mathbb{P}^{\tilde{k}(n)} \subset \mathbb{P}^{n-2}$ with $\lim_{n\to\infty} \tilde{k}(n) = \infty$.*

Let $K_H(x)$ be the cone in $\mathbb{A}^{n-1}(x)$ over X_x. By Lemma 3.5 the closure $\overline{K_H(x)}$ of $K_H(x)$ in $\mathbb{P}^{n-1}(x)$ is filled by subspaces $\mathbb{P}^{\tilde{k}(n)}$. Now consider the subvariety $K_{H,k}(x)$ of H defined in Lemma 3.2(ii). By definition, $K_{H,k}(x)$ is filled by those subspaces $\mathbb{P}^k$ filling H which pass through x. Clearly,

$$K_H(x) \supset K_{H,k}(x) \quad \text{for} \quad k = \tilde{k}(n). \tag{38}$$

Assume that $H \subset \mathbb{P}^n$ is a generic smooth hypersurface of degree d and x is a generic point of H. In particular, the forms Ψ_p are generic points of the spaces $H^0(\mathcal{O}_{\mathbb{P}^{n-2}}(p))$. Hence, in this case $X = X_x$ is a smooth and irreducible complete intersection of $d-1$ hypersurfaces $\{\Psi_p = 0\}$, $p = 2, ..., d$ in $\mathbb{P}^{n-2}$, and the inequalities (37) become equalities. This together with (38) and (20) implies $K_H(x) = K_{H,k}(x)$. In addition, the sheaf $\mathcal{O}_X$ has a standard Koszul resolution $0 \to \mathcal{O}_{\mathbb{P}^{n-1}}(2 - d(d+1)/2) \to ... \to \bigoplus_{p=2}^{d} \mathcal{O}_{\mathbb{P}^{n-2}}(1-p) \to \mathcal{O}_{\mathbb{P}^{n-2}}(1) \overset{res}{\to} \mathcal{O}_X(1) \to 0$. This resolution together with (2) shows that the restriction map $H^0(res) : H^0(\mathcal{O}_{\mathbb{P}^{n-2}}(1)) \to H^0(\mathcal{O}_X(1))$ is an isomorphism. Therefore $\mathrm{Span}X = \mathbb{P}^{n-2}$ and, consequently,

$$\mathrm{Span}K_{H,k}(x) = \mathbb{P}^{n-1}(x). \tag{39}$$

We now define a sequence of irreducible subvarieties $x \in X_1 \subset X_2 \subset ... \subset X_i \subset ... \subset H$ by induction:

1) $X_1 := K_{H,k}(x)$;

2) $X_{i+1} := \pi_1(\pi_2^{-1}(Y_i))$ for $i \geq 1$, Y_i being any irreducible component of $\pi_2(\pi_1^{-1}(X_i))$, where π_1 and π_2 are introduced in diagram (18).

Since X is irreducible, this sequence stabilizes, i.e.

$$X_1 \subset X_2 \subset ... \subset X_{i_0} = X_{i_0+1} ... \subset H. \tag{40}$$

for some i_0. Consider the dense open subset $U := \{x' \in H \mid K_{H,k}(x')$ is irreducible and $\mathrm{Span}(K_{H,k}(x')) = \mathbb{P}^{n-1}(x')\} \subset H$. By construction, $x \in U$, hence $X_{i_0} \cap U$ is a dense open subset of X_{i_0}. Moreover, by the definition of X_{i_0} we have

$$K_{H,k}(x') \subset X_{i_0} \tag{41}$$

for $x' \in X_{i_0} \cap U$. Denote by $H(x')$ the projective subspace of $\mathbb{P}^n$ tangent to X_{i_0} at the point $x' \in (X_{i_0} \smallsetminus \mathrm{Sing}\, X_{i_0}) \cap U$. Since $K_{H,k}(x')$ is by definition filled by projective subspaces on H through x', it follows from (41) that $K_{H,k}(x') \subset H(x') \subset \mathbb{P}^{n-1}(x')$. On the other hand, since $x' \in U$, it follows that $\mathrm{Span}K_{H,k}(x) = H(x')$. As $H(x')$ is a subspace of $\mathbb{P}^{n-1}(x')$, by (39) we have $H(x') = \mathbb{P}^{n-1}(x')$. Hence, since x' is a nonsingular point of X_{i_0}, we obtain $\dim X_{i_0} = \dim H$, so that

$$X_{i_0} = H. \tag{42}$$

This equality and the construction of the chain (40) shows that the point $x \in H$ can be joined with any point $x' \in H$ by a chain of subspaces $\mathbb{P}^k_1, \mathbb{P}^k_2, ..., \mathbb{P}^k_{i_0}$. We thus have

$$x \in \mathbb{P}^k_1 \subset \mathbb{P}^k_1 \cup \mathbb{P}^k_2 \cup ... \cup \mathbb{P}^k_{i_0} \supset \mathbb{P}^k_{i_0} \ni x'. \tag{43}$$

Finally, we will show that (43) holds also without the genericness assumption on H and x. This is done by essentially the same argument as in the proof of Theorem 3.3. Indeed, consider the Grassmannian $G := G(k+1, V)$, the incidence variety

$$Inc^{i_0}(G) := \{(\mathbb{P}^k_1, ..., \mathbb{P}^k_{i_0}) \in G^{\times i_0} \mid \mathbb{P}^k_1, ..., \mathbb{P}^k_{i_0} \text{ is a chain of subspaces of } \mathbb{P}^n\},$$

and the graphs of incidence

$$\Pi_{i_0} := \{(\mathbb{P}^k_1, ..., \mathbb{P}^k_{i_0}, x, x') \in Inc^{i_0}(G) \times \mathbb{P}^n \times \mathbb{P}^n \mid x \in \mathbb{P}^k_1,\ x' \in \mathbb{P}^k_{i_0}\},$$

$$\widetilde{\widetilde{H}} := \{(H, x, x') \in \mathbb{P}^s \times \mathbb{P}^n \times \mathbb{P}^n \mid x, x' \in H\},$$

$$\Gamma_{i_0} := \{(\mathbb{P}^k_1, ..., \mathbb{P}^k_{i_0}, G) \in Inc^{i_0}(G) \times \mathbb{P}^s \mid \mathbb{P}^k_1, ..., \mathbb{P}^k_{i_0} \subset H\}$$

with natural projections

$$Inc^{i_0}(G) \overset{\tilde{p}_{i_0}}{\longleftarrow} \Gamma_{i_0} \overset{\tilde{q}_{i_0}}{\longrightarrow} \mathbb{P}^s.$$

We have the commutative diagram

$$\begin{array}{ccccc}
\Pi_{i_0} & \xleftarrow{\quad pr_1 \quad} & \Pi_{\Gamma_{i_0}} & \xrightarrow{\quad pr_2 \quad} & \widetilde{\widetilde{H}} \\
\downarrow & & \downarrow & & \downarrow \\
Inc^{i_0}(G) \times \mathbb{P}^n \times \mathbb{P}^n & \xleftarrow{\tilde{p}_{i_0} \times id} & \Gamma_{i_0} \times \mathbb{P}^n \times \mathbb{P}^n & \xrightarrow{\tilde{q}_{i_0} \times id} & \mathbb{P}^s \times \mathbb{P}^n \times \mathbb{P}^n,
\end{array}$$

where pr_1 and pr_2 are the induced projections. As a generic smooth $H \in \mathbb{P}^s$ is filled by projective subspaces $\mathbb{P}^k \subset \mathbb{P}^n$ (Lemma 3.2(i)), the morphism pr_2 is dominant. Hence pr_2 is surjective since all varieties and morphisms in the above diagram are projective. This is equivalent to (43) for any $H \in \mathbb{P}^s$ and any $x, x' \in H$.

We thus have proved the following lemma.

Lemma 3.6. *Let H be a hypersurface of degree d in $\mathbb{P}^n$. Any two distinct points $x, x' \in H$ can be joined by a chain (43) of subspaces $\mathbb{P}^k$ of H.*

Finally, Lemma 3.6 together with Theorem 3.4 leads to our main result in section 3.

Theorem 3.7. *Under the assumptions of Theorem 3.4, any two distinct points $x, x' \in X$ can be joined by a chain (43) of subspaces $\mathbb{P}^{k_0(n)}$ of X with $\lim\limits_{n\to\infty} k_0(n) = \infty$.*

4. A sufficient condition on m for a vector bundle on $\mathbb{P}^N$ to be m-regular

Recall that a vector bundle E on a scheme Y is called *ample* if the invertible Grothendieck sheaf $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)$ on $\mathbb{P}(E^\vee)$ is ample. The following result is well known - see, e.g., [L, Prop. 6.3.56].

Lemma 4.1. *Let E be a vector bundle on $\mathbb{P}^N$. Then $E(a)$ is ample for any $a \in \mathbb{Z}_{\geq a_0}$, a_0 being some fixed integer.*

Lemma 4.2. *Let E be a vector bundle on $\mathbb{P}^1$. Then $E(a)$ is ample for $a \geq 1 - \delta_B(E)$.*

Proof. By Grothendieck's theorem, $E \simeq \overset{r}{\underset{j=1}{\oplus}} \mathcal{O}_{\mathbb{P}^1}(a_j)$, where $\delta_B(E) = a_1 \leq a_2 \leq ... \leq a_r$, $r = \mathrm{rk}E$. Hence, for $a \geq 1 - \delta_B(E)$, the bundle $E(a)$ is a direct sum of ample line bundles. By [L, Prop. 6.1.12(i)] $E(a)$ is itself ample. ☐

We now recall the notion of *degree* of a vector bundle $\mathcal{E}$ on a 1-dimensional scheme Y. If Y is a smooth irreducible curve, $\deg \mathcal{E} := \chi(\mathcal{E}) - \chi(\mathcal{O}_Y)\mathrm{rk}\mathcal{E}$. If Y is irreducible, but not necessarily smooth, the degree $\deg \mathcal{E}$ is defined as the degree of the pullback of $\mathcal{E}$ to the normalization of Y. If Y is a general 1-dimensional scheme with irreducible components $Y_1, ..., Y_q$, then the multiplicities $k_i \in \mathbb{Z}_{>0}$ of Y_i in Y are well defined (see [F, 1.5]), and we set

$$\deg \mathcal{E} = \sum_i k_i \deg(\mathcal{E}|_{Y_i}). \tag{44}$$

Lemma 4.3. *Let E be a vector bundle on $\mathbb{P}^N$ and let $pr : \mathbb{P}(E^\vee) \to \mathbb{P}^N$ be the projection. Let Y be a 1-dimensional subscheme of $\mathbb{P}(E^\vee)$ such that $Y_{red} \subset pr^{-1}(\mathbb{P}^1)$ for some line $\mathbb{P}^1 \subset \mathbb{P}^N$. Consider the line bundle $L_0 = \mathcal{O}_{\mathbb{P}(E^\vee)}(1) \otimes pr^*\mathcal{O}_{\mathbb{P}^1}(a)$ on $\mathbb{P}(E^\vee)$ for $a \geq 1 - \delta_B(E)$. Then*

$$\deg(L_0|_Y) > 0. \tag{45}$$

Proof. By (44),

$$\deg(L_0|_Y) = \sum_i k_i \deg(L_0|_{Y_i}), \quad k_i > 0, \tag{46}$$

where Y_i are the irreducible components of Y. Since $\delta_B(E|_{\mathbb{P}^1}) \geq \delta_B(E)$, it follows from Lemma 4.2 that the sheaf $L_0|_{pr^{-1}(\mathbb{P}^1)}$ is ample. Hence $\deg(L_0|_{Y_i}) > 0$ for each Y_i above, and (46) implies (45). ☐

Let Z_1 be an arbitrary reduced irreducible curve in $\mathbb{P}^N$ with $N \geq 3$. Pick a projective line $l_0 \subset \mathbb{P}^N$ and a subspace $\mathbb{P}^{N-2} \subset \mathbb{P}^N$ such that

$$l_0 \cap Z_1 = \mathbb{P}^{N-2} \cap Z_1 = \emptyset. \tag{47}$$

Fix homogeneous coordinates $(x_0 : ... : x_N)$ in $\mathbb{P}^N$ so that $l_0 = \{x_2 = ... = x_N = 0\}$, $\mathbb{P}^{N-2} = \{x_0 = x_1 = 0\}$, and fix the isomorphism

$$\Lambda : \mathbb{C}^* \times \mathbb{P}^N \xrightarrow{\sim} \mathbb{C}^* \times \mathbb{P}^N, (t, (x_0 : ... : x_N)) \mapsto (t, (x_0 : x_1 : tx_2 : ... : tx_N)),$$

where $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$. Set $\Gamma^* := \Lambda(\mathbb{C}^* \times Z_1)$ and consider the Hilbert scheme $\mathcal{H} := \mathrm{Hilb}^{P_{Z_1}}(\mathbb{P}^N)$, where P_{Z_1} is the Hilbert polynomial $P_{Z_1}(n) = \chi(\mathcal{O}_{\mathbb{P}^N}(n)|_{Z_1})$. By construction $\Gamma^* \to \mathbb{C}^*$ is a flat family of curves over $\mathbb{C}^*$, hence it defines a morphism $g : \mathbb{C}^* \to \mathcal{H}$ such that $\Gamma^* = \Gamma_{\mathcal{H}} \times_{\mathcal{H}} \mathbb{C}^*$, where $\Gamma_{\mathcal{H}} \subset \mathbb{P}^N \times \mathcal{H}$ is the universal family of curves. The coordinate t on $\mathbb{C}^*$ identifies $\mathbb{C}^*$ with $\mathbb{P}^1 \smallsetminus \{z_0, z_\infty\}$, where $z_0 = \{t = 0\}$, $z_\infty = \{t = \infty\}$, and since the Hilbert scheme $\mathcal{H}$ is projective, the morphism g extends to a morphism $\tilde{g} : \mathbb{P}^1 \to H$. We thus obtain a flat family $\varphi : \Gamma = \Gamma_{\mathcal{H}} \times_{\mathcal{H}} \mathbb{P}^1 \to \mathbb{P}^1$ of curves over $\mathbb{P}^1$ such that $Z_1 = \varphi^{-1}(z_1)$ for $z_1 := \{t = 1\}$, and $(\varphi^{-1}(z_0))_{red} = l_0$.

Let again E be a vector bundle of rank $\mathrm{rk}E \geq 2$ on $\mathbb{P}^N$ and let $pr : \mathbb{P}(E^\vee) \to \mathbb{P}^N$ be the projection. Consider the projection $q : \Gamma \to \mathbb{P}^N$ and the scheme $\Gamma^E := \mathbb{P}(q^*E^\vee) = \mathbb{P}(E^\vee) \times_{\mathbb{P}^N} \Gamma$ with projections $\mathbb{P}(E^\vee) \overset{q'}{\leftarrow} \Gamma^E \overset{pr'}{\rightarrow} \Gamma$ and $\rho = \varphi \circ pr' : \Gamma^E \to \mathbb{P}^1$. Note that, by Lemma 4.1 there exists $a_0 \in \mathbb{Z}$ such that the line bundle $A = \mathcal{O}_{\mathbb{P}(E^\vee)}(1) \otimes pr^* O_{\mathbb{P}^N}(a_0)$ is ample on $\mathbb{P}(E^\vee)$; hence the line bundle q'^*A is ρ-ample on Γ^E.

Fix an irreducible curve Y_1 in Γ^E such that $pr'(Y_1) = Z_1$, and denote by P_{Y_1} the Hilbert polynomial $P_{Y_1}(n) := \chi(q'^*A^{\otimes n}|_{Y_1})$. Consider the relative Hilbert scheme $\mathcal{H}_{\mathbb{P}^1} = \mathrm{Hilb}^{P_{Y_1}}(\Gamma^E/\mathbb{P}^1)$, together with the natural surjective projective morphism $f : \mathcal{H}_{\mathbb{P}^1} \to \mathbb{P}^1$ and the universal family $\Sigma \hookrightarrow \Gamma^E \times_{\mathbb{P}^1} \mathcal{H}_{\mathbb{P}^1}$ with projections $\Gamma^E \overset{p''}{\leftarrow} \Sigma \overset{q''}{\rightarrow} \mathcal{H}_{\mathbb{P}^1}$. By definition, there is a point $y_1 \in \mathcal{H}_{\mathbb{P}^1}$ such that

$$q''^{-1}(y_1) \underset{\sim}{\xrightarrow{p''}} Y_1 \tag{48}$$

and $f(y_1) = a_1$. Next, consider the normalization $\nu : Z \to Z_1$ of Z_1 and the surfaces $\mathcal{S} = \mathbb{P}(\nu^*(E^\vee|_{Z_1}))$ and $\mathcal{S}_1 = \mathbb{P}(E^\vee|_{Z_1}) \subset X_\Gamma$ with their projections $pr_{\mathcal{S}} : \mathcal{S} \to Z$ and $pr_{\mathcal{S}_1} : \mathcal{S}_1 \to Z_1$. By construction, the morphism ν lifts to the normalization $\tilde{\nu} : \mathcal{S} \to \mathcal{S}_1$ such that $pr_{\mathcal{S}_1} \circ \tilde{\nu} = \nu \circ pr_{\mathcal{S}}$, and the curve $Y = \tilde{\nu}^{-1}(Y_1)$ is a multisection of the projection $pr_{\mathcal{S}}$.

Consider the Hilbert polynomial $P_Y(n) := \chi(\tilde{\nu}^* q'^* A^{\otimes n}|_{Y_1})$. Since $\mathcal{S}$ is a smooth surface, the Hilbert scheme $\mathrm{Hilb}^{P_Y}(\mathcal{S})$ coincides with the linear series $|\mathcal{O}_{\mathcal{S}}(Y)| \simeq \mathbb{P}^h$, $h = h^0(\mathcal{O}_{\mathcal{S}}(Y)) - 1$, and there is a bijective morphism $\mathbb{P}^h = \mathrm{Hilb}^{P_Y}(\mathcal{S}) \to \mathrm{Hilb}^{P_{Y_1}}(\mathcal{S}_1) = f^{-1}(a_1) : C \mapsto \tilde{\nu}(C)$. Thus the fibre $f^{-1}(a_1)$ is irreducible.

Since the morphism $f : \mathcal{H}_{\mathbb{P}^1} \to \mathbb{P}^1$ is projective, the scheme $\mathcal{H}_{\mathbb{P}^1}$ is projective as well. Therefore, in view of the surjectivity and flatness of f and the irreducibility of the fibre $f^{-1}(a_1)$, there exists a smooth irreducible curve T and a morphism $\theta : T \to \mathcal{H}_{\mathbb{P}^1}$ such that $\theta_T = f \circ \theta : T \to \mathbb{P}^1$ is surjective. Hence

$$\theta_T(t_0) = z_0 \tag{49}$$

for some $t_0 \in T$, and, since $f(y_1) = z_1$,

$$\theta(t_1) = y_1, \quad \theta_T(t_1) = z_1 \tag{50}$$

for some $t_1 \in T$.

Consider the fibre product $\Sigma_T = \Sigma \times_{\mathbb{P}^1} T$ with projections $p_T : \Sigma_T \to T$, $q_T : \Sigma_T \to \Sigma \overset{p''}{\rightarrow} X_\Gamma \overset{q''}{\rightarrow} \mathbb{P}(E^\vee)$, and the embedding $i = (q_T, p_T) : \Sigma \hookrightarrow \mathbb{P}(E^\vee) \times T$. The family $p_T : \Sigma_T \to T$ is a flat family of curves in $\mathbb{P}(E^\vee)$ with base T such that the fibre $p_T^{-1}(t_1)$ coincides with Y_1, and the reduced fibre $(Y_0)_{red} := (p_T^{-1}(t_0))_{red}$ lies in $pr^{-1}(l_0)$. Next, consider the line bundle $L_T = i^*(L_0 \boxtimes \mathcal{O}_T)$ on Σ_T, where L_0 is the line bundle on X defined in Lemma 4.3. The degree $\deg(L_T|_{p_T^{-1}(t)})$ does not depend on $t \in T$ by the principle of continuity [F, Thm. 10.2]. In particular, since $\deg(L_0|_{Y_0}) > 0$ by Lemma 4.3, we obtain

$$\deg(L_0|_{Y_1}) > 0. \tag{51}$$

Lemma 4.4. *Let E and pr be as in Lemma 4.3.*

(i) The line bundle $L := \mathcal{O}_{\mathbb{P}(E^\vee)}(1) \otimes pr^*\mathcal{O}_{\mathbb{P}^N}(2 - \delta_B(E))$ on $\mathbb{P}(E^\vee)$ is ample.

(ii) The line bundle

$$A_i := L^{r+1} \otimes pr^*\mathcal{O}_{\mathbb{P}^N}(i) \simeq \mathcal{O}_{\mathbb{P}(E^\vee)}(r+1) \otimes pr^*\mathcal{O}_{\mathbb{P}^N}((r+1)((2 - \delta_B(E)) + i)), \tag{52}$$

where $r = \mathrm{rk}E$, is also ample for any $i \geq 0$.

Proof. (i) We note first that the line bundle $L_0 := L \otimes pr^*\mathcal{O}_{\mathbb{P}^N}(-1)$ is numerically effective, i.e. the degree of its restriction to any curve in $\mathbb{P}(E^\vee)$ is positive. Indeed, let Y be an irreducible curve in $\mathbb{P}(E^\vee)$. If $pr(Y)$ is a curve, then our claim follows from (51). If $pr(Y)$ is a point z, then $Y \subset pr^{-1}(z) \simeq \mathbb{P}^{r-1}$ and $\deg(L_0|_Y) = \deg(\mathcal{O}_{\mathbb{P}^{r-1}}(1)|_Y)$ is again positive.

The numerically effective divisor class $c_1(L_0)$ equals $W + (1 - \delta_B(E))H$, where $W := c_1(\mathcal{O}_{\mathbb{P}(E^\vee)}(1))$, $H := pr^*c_1(\mathcal{O}_{\mathbb{P}^N}(1))$. By Lemma 4.1 the divisor class $W + a_0H$ on $\mathbb{P}(E^\vee)$ is ample for $a_0 - 2 + \delta_B(E)$ large enough. Moreover, a corollary of Kleiman's Theorem [L, Cor. 1.4.9] implies that the divisor class $(a_0 - 2 + \delta_B(E))c_1(L_0) + W + a_0H = (a_0 - 1 + \delta_B(E))(W + (2 - \delta_B(E))H)$ is ample. Hence $W + (2 - \delta_B(E))H$ is also ample.

(ii) is a direct corollary of (i). □

Recall that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^N$ is $m - regular$ in the sense of Mumford-Castelnuovo if $H^i(\mathcal{F}(m - i)) = 0$ for $i \geq 1$.

Theorem 4.5. *Let E be a vector bundle of rank r on $\mathbb{P}^N$.*

(i) E is m-regular for $m \geq m_0 := c_1(E) + (1 + r)(2 - \delta_B(E)) - 1$. Furthermore, $E(m)$ is generated by global sections for $m \geq m_0$.

(ii) For any hyperplane $\mathbb{P}^{N-1}$ in $\mathbb{P}^N$ the vector bundle $E(m)|_{\mathbb{P}^{N-1}}$, $m \geq m_0$, is generated by global sections and

$$h^0(E(m)|_{\mathbb{P}^{N-1}}) \leq \frac{r}{(N-1)!}\big(\delta_A(E) + m + N - 1\big)^{N-1}. \tag{53}$$

Proof. We keep the notations of Lemmas 4.3 and 4.4. The dualizing sheaf $\omega_{\mathbb{P}(E^\vee)}$ of $\mathbb{P}(E^\vee)$ is given by the standard formula

$$\omega_{\mathbb{P}(E^\vee)} \simeq \mathcal{O}_{\mathbb{P}(E^\vee)}(-r) \otimes pr^*\mathcal{O}_{\mathbb{P}^N}(c_1(E) - N - 1). \tag{54}$$

Therefore (52) and (54) imply $\omega_{\mathbb{P}(E^\vee)} \otimes A_i \simeq \mathcal{O}_{\mathbb{P}(E^\vee)}(1) \otimes pr^*\mathcal{O}_{\mathbb{P}^N}(m_0 - N + i)$. Since A_i is ample for $i \geq 0$ by Lemma 4.4, the Kodaira vanishing theorem yields

$$0 = H^j(\omega_{\mathbb{P}(E^\vee)} \otimes A_i) = H^j(\mathcal{O}_{\mathbb{P}(E^\vee)}(1) \otimes pr^*\mathcal{O}_{\mathbb{P}^N}(m_0 - N + i)), \quad i \geq 0, \quad j \geq 1. \tag{55}$$

In addition, clearly

$$pr_*(\omega_{\mathbb{P}(E^\vee)} \otimes A_i) \simeq E(m_0 - N + i), \qquad R^j pr_*(\omega_{\mathbb{P}(E^\vee)} \otimes A_i) = 0, \quad j \geq 1, i \geq 0.$$

Thus the Leray spectral sequence $E_2^{aa'} = H^a(R^{a'}pr_*(\omega_{\mathbb{P}(E^\vee)} \otimes A_i)) \Rightarrow H^{a+a'}(\omega_{\mathbb{P}(E^\vee)} \otimes A_i)$ degenerates and yields (via (55)) $H^j(E(m_0 - N + i)) = 0, \quad i \geq 0, \quad j \geq 1$. This shows that E is m-regular for $m \geq m_0$. The fact that, if E is m-regular then $E(m)$ is generated by global sections, is well known [HL, Lem. 1.7.2]. Assertion (i) is proved.

Assertion (ii) follows from Le Potier-Simpson's Theorem - see [HL, Lem. 3.3.2] and substitute $X = \mathbb{P}^N$, $\deg(X) = 1$, $F = E(m)$, $\nu = N - 1$, $X_\nu = \mathbb{P}^{N-1}$, $X_1 = \mathbb{P}^1$, $\mu_{max}(E(m)|_{\mathbb{P}^1}) = \delta_A(E(m)) = \delta_A(E) + m$. □

5. An upper bound for the degree of the variety of maximal jumping lines through a point of a vector bundle E on $\mathbb{P}^N$

5.1. The transformation L_0 of a vector bundle E under a linear projection. Let

$$p_x : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$$

be the rational linear projection with center at a point $x \in \mathbb{P}^N$ and let $\tilde{\mathbb{P}}^N$ be the closure in $\mathbb{P}^N \times \mathbb{P}^{N-1}$ of the graph of p_x. We have the following obvious diagram of projections

$$\mathbb{P}^N \xleftarrow{\sigma} \tilde{\mathbb{P}}^N \xrightarrow{\pi} \mathbb{P}^{N-1}. \tag{56}$$

In this section E will denote a vector bundle of rank r on $\mathbb{P}^N$ with the additional condition

$$\delta_B(E) = 0. \tag{57}$$

Set $L_0 := \pi_*\sigma^*E$.

Theorem 5.1. *(i) L_0 is a vector bundle of rank*

$$\rho_0 := rkL_0 = c_1(E) + r \tag{58}$$

on $\mathbb{P}^{N-1}$, and its construction is compatible with base change, i.e. for any cartesian square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\tilde{g}} & \tilde{\mathbb{P}}^N \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathcal{Y} & \xrightarrow{g} & \mathbb{P}^{N-1} \end{array} \tag{59}$$

there is a base change isomorphism

$$\beta_0 : \tilde{\pi}_*\tilde{g}^*\sigma^*E \xrightarrow{\sim} g^*\pi_*\sigma^*E = g^*L_0;$$

moreover, the natural evaluation map $ev : \pi^*L_0 \to \sigma^*E$ is an epimorphism.

(ii) $c_1(L_0) = P(c_1(E), c_2(E))$, where $P(x,y) := \frac{1}{2}x(x+1) - y \in \mathbb{Q}[x,y]$.

(iii) $\delta_A(E) \geq \delta_A(L_0)$.

(iv) The following inequalities hold:

$$\delta_A(L_0) \geq -(P(c_1(E), c_2(E)))^2, \tag{60}$$

$$\delta_B(L_0) \geq Q(r, \delta_A(E), c_1(E), c_2(E)), \tag{61}$$

where $Q(x,y,z,t) := -(x+z)y + P(z,t) - (P(z,t))^2 \in \mathbb{Q}[x,y]$ and the polynomial P is defined in (ii).

Proof. (i) Consider an arbitrary point $y \in \mathbb{P}^{N-1}$ and set $\mathbb{P}^1_y := \pi^{-1}(y)$. It follows immediately from (57) that $h^1(E|_{\mathbb{P}^1_y}) = 0$, hence $h^0(E|_{\mathbb{P}^1_y}) = \chi(E|_{\mathbb{P}^1_y}) = c_1(E) + r$. These equalities and the Base Change Theorem [H, Ch. 3, Thm. 12.11] imply (58), the equality $R^1\pi_*\sigma^*E = 0$ and the existence of the isomorphism β_0. Moreover, by (57) the sheaf $\sigma^*E|_{\mathbb{P}^1_y}$ is generated by global sections. This means that there is an epimorphism $ev_y : H^0(\sigma^*E|_{\mathbb{P}^1_y}) \otimes \mathcal{O}_{\mathbb{P}^1_y} \twoheadrightarrow \sigma^*E|_{\mathbb{P}^1_y}$. Moreover, the evaluation map $ev : \pi^*L_0 \to \sigma^*E$ is compatible with base change, i.e. we have a commutative diagram

$$\begin{array}{ccc} \pi^*L_0 \otimes \mathbb{C}_y & \xrightarrow{ev\otimes\mathbb{C}_y} & \sigma^*E \otimes \mathbb{C}_y \\ {\scriptstyle\pi^*\beta_0}\downarrow & & \| \\ H^0(\sigma^*E|_{\mathbb{P}^1_y}) \otimes \mathcal{O}_{\mathbb{P}^1_y} & \xrightarrow{ev_y} & \sigma^*E|_{\mathbb{P}^1_y}, \end{array} \tag{62}$$

where $\pi^*\beta_0$ is an isomorphism; whence the evaluation map $ev : \pi^*L_0 \to \sigma^*E$ is epimorphic.

(ii) For the duration of the proof, fix an arbitrary line $\mathbb{P}^1 \subset \mathbb{P}^{N-1}$ and consider the surface $S := \pi^{-1}(\mathbb{P}^1)$ and the projective plane $\mathbb{P}^2 := \sigma(S) \subset \mathbb{P}^N$. This plane passes through the center x of the projection $p_x : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$, and diagram (56) extends to the commutative diagram

$$\begin{array}{ccccc} \mathbb{P}^2 & \xleftarrow{\tilde{\sigma}} & S & \xrightarrow{\tilde{\pi}} & \mathbb{P}^1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^N & \xleftarrow{\sigma} & \tilde{\mathbb{P}}^N & \xrightarrow{\pi} & \mathbb{P}^{N-1}, \end{array} \tag{63}$$

where $\tilde{\pi} = \pi|_S$, $\tilde{\sigma} = \sigma|_S : S \to \mathbb{P}^2$ is the blowing up of $\mathbb{P}^2$ at the point x, and the vertical arrows are the inclusions.

Set $\mathcal{O}_S(\tau) := \tilde{\sigma}^*\mathcal{O}_{\mathbb{P}^2}(1)$, $\quad \mathcal{O}_S(h) := \tilde{\pi}^*\mathcal{O}_{\mathbb{P}^1}(1)$. Note that the relations

$$\tau^2 = \tau h = [pt] \tag{64}$$

hold in the Chow ring $A(S)$. Furthermore, $R^1\pi_*\sigma^*E = 0$ implies via base change $R^1\tilde{\pi}_*\tilde{\sigma}^*E|_{\mathbb{P}^2} = 0$. Hence Riemann-Roch (see, e.g., [F, Ex. 15.2.8]) yields

$$\mathrm{ch}(L_0|_{\mathbb{P}^1}) = \tilde{\pi}_*(\mathrm{td}(T_{S/\mathbb{P}^1}) \cdot \mathrm{ch}(\tilde{\sigma}^*E|_{\mathbb{P}^2})). \tag{65}$$

Here $T_{S/\mathbb{P}^1} \simeq \mathcal{O}_S(2\tau - h)$. Therefore, setting $c_i := c_i(E),\ i = 1, 2$, and using the relations (64), we obtain in $A(S)$

$$\mathrm{td}(T_{S/\mathbb{P}^1}) \cdot \mathrm{ch}(\tilde{\sigma}^*E|_{\mathbb{P}^2}) = 1 + (r + c_1)\tau - \frac{1}{2}rh + P(c_1, c_2)[pt]. \tag{66}$$

In $A(\mathbb{P}^1)$ we have, respectively,

$$\tilde{\pi}_*(\mathrm{td}(T_{S/\mathbb{P}^1}) \cdot \mathrm{ch}(\tilde{\sigma}^*E|_{\mathbb{P}^2})) = (r + c_1) \cdot 1 + P(c_1, c_2)[pt]. \tag{67}$$

Whence (ii) follows.

(iii) Set $\tilde{E} := E(-\delta_A(E) - 1)$. Then $\delta_A(\tilde{E}) = -1$ and

$$\pi_*\sigma^*\tilde{E} = 0, \quad h^0(\tilde{E}) = h^0(\sigma^*\tilde{E}) = 0. \tag{68}$$

In addition, (57) and the condition $-1 = \delta_A(\tilde{E}) \geq \delta_B(\tilde{E})$ give

$$D := \delta_A(E) + 1 = -\delta_B(\tilde{E}) > 0. \tag{69}$$

Let $l_0 = \tilde{\sigma}^{-1}(x)$ be the exceptional line on S. Since $\mathcal{O}_S(l_0) = \mathcal{O}_S(\tau - h)$, there is an exact triple

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(D(\tau - h)) \to \mathcal{O}_{Dl_0}(Dl_0) \to 0. \tag{70}$$

Here Dl_0 is the standard multiplicity D scheme structure on l_0 as a divisor in S, and we have the following exact triples of sheaves of $\mathcal{O}_{Dl_0}$-modules for $D \geq 2$:

$$\begin{gathered} 0 \to \mathcal{O}_{l_0}(-1) \to \mathcal{O}_{Dl_0}(Dl_0) \to \mathcal{O}_{(D-1)l_0}(Dl_0) \to 0, \\ 0 \to \mathcal{O}_{l_0}(-2) \to \mathcal{O}_{(D-1)l_0}(Dl_0) \to \mathcal{O}_{(D-2)l_0}(Dl_0) \to 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 \to \mathcal{O}_{l_0}(1 - D) \to \mathcal{O}_{2l_0}(Dl_0) \to \mathcal{O}_{l_0}(-D) \to 0. \end{gathered} \tag{71}$$

Note that, similarly to (68),

$$h^0(\tilde{\sigma}^*\tilde{E}|_{\mathbb{P}^2}) = 0. \tag{72}$$

Next, $l_0 = \tilde{\sigma}^{-1}(x)$ implies $(\tilde{\sigma}^*\tilde{E}|_{\mathbb{P}^2})|_{ml_0} \simeq \mathcal{O}^r_{ml_0}$ for $m \geq 1$. Thus, twisting the triples (71) by $\tilde{\sigma}^*\tilde{E}|_{\mathbb{P}^2}$, we obtain for $D \geq 2$

$$h^0(\tilde{\sigma}^*\tilde{E}|_{\mathbb{P}^2} \otimes \mathcal{O}_{Dl_0}(Dl_0)) = 0. \tag{73}$$

Moreover, (73) is evident for $D = 1$, hence it holds for $D \geq 1$.

Twisting (70) by $\tilde{\sigma}^*\tilde{E}|_{\mathbb{P}^2}$ we obtain the exact triple $0 \to \tilde{\sigma}^*\tilde{E}|_{\mathbb{P}^2} \to \tilde{\sigma}^*\tilde{E}|_{\mathbb{P}^2}(D(\tau-h)) \to \tilde{\sigma}^*\tilde{E}|_{\mathbb{P}^2} \otimes \mathcal{O}_{Dl_0}(Dl_0) \to 0$, and (72) and (73) imply

$$0 = h^0(\tilde{\sigma}^*\tilde{E}|_{\mathbb{P}^2})(D(\tau-h)) = h^0(\tilde{\pi}_*\tilde{\sigma}^*\tilde{E}|_{\mathbb{P}^2}(D(\tau-h))). \tag{74}$$

Applying the base change isomorphism β_0 to the right square of the diagram (63) and using the projection formula, we get $\tilde{\pi}_*(\tilde{\sigma}^*\tilde{E}|_{\mathbb{P}^2}(D(\tau-h))) \simeq (\pi_*\sigma^*\tilde{E}(D)|_{\mathbb{P}^1})(-D) \simeq (\pi_*\sigma^*E|_{\mathbb{P}^1})(-D) \simeq L_0(-D)|_{\mathbb{P}^1}$. Therefore (74) implies $h^0(L_0(-D)|_{\mathbb{P}^1}) = 0$, or equivalently $\delta_A(L_0) < D$ as $\mathbb{P}^1$ is an arbitrary line in $\mathbb{P}^{N-1}$. This together with (69) yields (iii).

(iv) Let $L_0|_{\mathbb{P}^1} \simeq \bigoplus_{i=1}^{\rho_0} \mathcal{O}_{\mathbb{P}^1}(a_i)$. Clearly, $\delta_A(L_0) \geq c_1(L_0)/\rho_0$ as $a_i \leq \delta_A(L_0)$ for $1 \leq i \leq \rho_0$. It is clear also that $c_1(L_0)/\rho_0 \geq -(c_1(L_0))^2$. Therefore $\delta_A(L_0) \geq -(c_1(L_0))^2$. On the other hand, by (ii), $-(c_1(L_0))^2 = -(P(c_1,c_2))^2$. Hence (60) holds.

Finally, set $\tilde{L}_0 := L_0(-\delta_A(L_0)-1)$. We have

$$\delta_A(\tilde{L}_0) = -1, \quad \delta_B(L_0) = \delta_B(\tilde{L}_0) + \delta_A(L_0) + 1. \tag{75}$$

Assume in addition that the line $\mathbb{P}^1$ in $\mathbb{P}^{N-1}$ is chosen in such a way that $\delta_B(\tilde{L}_0|_{\mathbb{P}^1}) = \delta_B(\tilde{L}_0)$. Then $h^0(\tilde{L}_0|_{\mathbb{P}^1}) = 0$, hence Riemann-Roch yields

$$h^1(\tilde{L}_0|_{\mathbb{P}^1}) = -c_1(\tilde{L}_0) - \mathrm{rk}\tilde{L}_0 = -c_1(\tilde{L}_0) - \rho_0. \tag{76}$$

On the other hand, since $\delta_B(\tilde{L}_0) \leq \delta_A(\tilde{L}_0) = -1$, we have $-\delta_B(\tilde{L}_0) - 1 = h^1(\mathcal{O}_{\mathbb{P}^1}(\delta_B(\tilde{L}_0))) \leq h^1(\tilde{L}_0|_{\mathbb{P}^1})$. The last two inequalities, together with (76), imply

$$-1 \geq \delta_B(\tilde{L}_0) \geq c_1(\tilde{L}_0) + \rho_0 - 1. \tag{77}$$

In addition, the definition of $\tilde{L}_0$ and statements (ii) and (iii) imply $c_1(\tilde{L}_0)+\rho_0-1 = -\rho_0(\delta_A(L_0)+1) + c_1(L_0) + \rho_0 - 1 \geq -\rho_0(\delta_A(E)+1) + P(c_1,c_2) + \rho_0 - 1$. Substituting this together with (77) into (75), and using (60) and (58), we obtain

$$\delta_B(L_0) \geq -\rho_0\delta_A(E) + P(c_1,c_2) + \delta_A(L_0) \geq -(r+c_1)\delta_A(E) + P(c_1,c_2) - (P(c_1,c_2))^2,$$

i.e. (61). □

5.2. An estimate for the transformed kernel of the evaluation map $\pi^*L_0 \to \sigma^*E$. Assume in addition

$$\delta_A(E) = 2\delta, \quad c_1(E) = r\delta \tag{78}$$

for some $\delta \in \mathbb{Z}_{>0}$. Set

$$\gamma := c_2(E) - \frac{1}{2}r(r-1)\delta^2. \tag{79}$$

Then Theorem 5.1 yields

$$\rho_0 = \mathrm{rk}L_0 = r(1+\delta), \tag{80}$$

$$c_1(L_0) = P_1(r,\gamma,\delta), \tag{81}$$

$$Q_1(r,\gamma,\delta) \leq \delta_B(L_0) \leq \delta_A(L_0) \leq 2\delta, \tag{82}$$

where $P_1(r,\gamma,\delta) := P(r\delta, \gamma + r(r-1)\delta^2/2)$, $Q_1(r,\gamma,\delta) := Q(r,\ 2\delta,\ r\delta,\ \gamma + r(r-1)\delta^2/2)$.

By Theorem 5.1(i) we have an exact triple of vector bundles

$$0 \to F \xrightarrow{\iota} \pi^*L_0 \xrightarrow{ev} \sigma^*E \to 0, \tag{83}$$

where $F := \mathrm{Ker}\, ev$. Restriction to S yields an exact triple

$$0 \to F|_S \to \tilde{\pi}^*(L_0|_{\mathbb{P}^1}) \to \tilde{\sigma}^*E|_{\mathbb{P}^2} \to 0 \tag{84}$$

and its twisted version

$$0 \to (F|_S)(jh) \to \tilde{\pi}^*(L_0|_{\mathbb{P}^1}(j)) \to \tilde{\sigma}^* E|_{\mathbb{P}^2}(jh) \to 0. \tag{85}$$

Base change implies $L_0|_{\mathbb{P}^1} \simeq \tilde{\pi}_* \tilde{\sigma}^* E|_{\mathbb{P}^2}$. Therefore $L_0|_{\mathbb{P}^1}(j) \simeq (\tilde{\pi}_* \tilde{\sigma}^* E|_{\mathbb{P}^2})(jh),\ j \in \mathbb{Z}$. Since $H^1(\tilde{\pi}^*(L_0|_{\mathbb{P}^1}(j))) = H^1(L_0|_{\mathbb{P}^1}(j)) = 0, \quad j \geq -\delta_B(L_0)$, and the morphism $H^0(\tilde{\pi}^*(L_0|_{\mathbb{P}^1}(j))) \to H^0(\tilde{\sigma}^* E|_{\mathbb{P}^2})(jh)),\ j \in \mathbb{Z}$, induced by (85) is an isomorphism, we obtain

$$h^1(F|_S(jh)) = 0, \quad j \geq -Q_1(r, \gamma, \delta) \tag{86}$$

(see (82)).

Next, the triple (84) implies via (80)-(81)

$$\mathrm{rk} F = \mathrm{rk}(F|_S) = r\delta,$$

$$c_1(F|_S) = (r\delta)\tau - P_1(r, \gamma, \delta)h, \quad c_2(F|_S) = P_2(r, \gamma, \delta)[pt], \tag{87}$$

where $P_2(r, \gamma, \delta) := -\gamma - r(r-1)\delta^2/2 - r^2\delta^2 + r\delta P_1(r, \gamma, \delta)$ and we use the relations (64).

Set

$$b := -\min\{\delta_B(F|_{\mathbb{P}^1_y}) \mid y \in \mathbb{P}^{N-1}\} \tag{88}$$

and observe that $b \geq 0$ in view of the monomorphism ι in (83). To obtain an upper bound for b, take a point $y \in \mathbb{P}^{N-1}$ such that $\delta_B(F|_{\mathbb{P}^1_y}) = -b$. Then

$$F|_{\mathbb{P}^1_y} \simeq \bigoplus_{i=1}^{r\delta} \mathcal{O}_{\mathbb{P}^1}(b_i) = \mathcal{O}_{\mathbb{P}^1}(-b) \oplus \bigoplus_{i=2}^{r\delta} \mathcal{O}_{\mathbb{P}^1}(b_i), \tag{89}$$

where $-b = b_1 \leq b_2 \leq \ldots\ldots \leq b_{r\delta} \leq 0$. Restricting (83) onto $\mathbb{P}^1_y$ and using (80), we obtain the triple

$$0 \to F|_{\mathbb{P}^1_y} \to \mathcal{O}_{\mathbb{P}^1_y}^{r(1+\delta)} \to \sigma^* E|_{\mathbb{P}^1_y} \to 0.$$

Moreover, (78) yields $\chi(\sigma^* E|_{\mathbb{P}^1_y}) = \mathrm{rk} E + c_1(E) = r(1+\delta)$. Therefore $0 = \chi(F|_{\mathbb{P}^1_y}) = -b + \sum_{i=2}^{r\delta} b_i + r(1+\delta)$. Since $\sum_{i=2}^{r\delta} b_i \leq 0$, this gives the following upper bound for b:

$$b = \sum_{i=2}^{r\delta} b_i + r(1+\delta) \leq r(1+\delta). \tag{90}$$

Consider the vector bundles

$$\mathcal{O}_\pi(1) := \sigma^* \mathcal{O}_{\mathbb{P}^N}(1), \quad F_b := F \otimes \mathcal{O}_\pi(b), \quad L_1 := \pi_* F_b.$$

Note that

$$R^1 \pi_* F_b = 0. \tag{91}$$

Furthermore, (87) implies

$$c_1(F_b|_S) = r\delta(1+b)\tau - hP_1(r, \gamma, \delta), \tag{92}$$

$$c_2(F_b|_S) = c_2(F|_S) + (r\delta - 1)(r\delta\tau - hP_1(r, \gamma, \delta))b\tau + \frac{1}{2} r b^2 \delta(r\delta - 1)[pt]. \tag{93}$$

Base change, together with (91), yields

$$\tilde{\pi}_*(F_b|_S) = L_1|_{\mathbb{P}^1}, \quad R^1\tilde{\pi}_*(F_b|_S) = 0. \tag{94}$$

Hence, by Riemann-Roch (cf. (65))

$$\mathrm{ch}(L_1|_{\mathbb{P}^1}) = \tilde{\pi}_*(\mathrm{td}(T_{S/\mathbb{P}^1}) \cdot \mathrm{ch}(F_b|_S)). \tag{95}$$

Substituting (92) and (93) into (95) and proceeding as in (66) and (67), we obtain

$$\mathrm{rk} L_1 = r\delta(2+b), \tag{96}$$

(97) $$c_1(L_1) = F_1(r,b,\gamma,\delta) := \binom{r\delta(1+b)+1}{2} - (r\delta(1+b)+1)P_1(r,\gamma,\delta) - F(r,b,\gamma,\delta).$$

Moreover, (94) implies

$$\tilde{\pi}_*(F_b|_S(jh)) = L_1(j)|_{\mathbb{P}^1}, \quad R^1\tilde{\pi}_*(F_b|_S(jh)) = 0, \quad j \in \mathbb{Z}.$$

Therefore the Leray spectral sequence $E_2^{aa'} = H^a(R^{a'}\tilde{\pi}_*(F_b|_S(jh)) \Rightarrow H^{a+a'}(F_b|_S(jh))$ degenerates and

(98) $$H^1(F_b|_S(jh)) = H^1(L_1(j)|_{\mathbb{P}^1}), \quad j \in \mathbb{Z}.$$

We are now ready to prove the following lemma.

Lemma 5.2. *There exist polynomials $R(x,y,z)$ and $S(x,y,z)$ in $\mathbb{Z}[x,y,z]$ such that*

(99) $$-R(r,\gamma,\delta) \le \delta_B(L_1) \le \delta_A(L_1) \le S(r,\gamma,\delta).$$

Proof. Fix a line $l \subset \mathbb{P}^2$ not passing through x (recall that x is the center of the blow-up $\tilde{\sigma} : S \to \mathbb{P}^2$), and let $\mathbb{P}^1_\tau := \tilde{\sigma}^{-1}(l)$. Then $\mathcal{O}_S(\mathbb{P}^1_\tau) \simeq \mathcal{O}_S(\tau)$. Restricting the triple (85) onto $\mathbb{P}^1_\tau$, we obtain an exact triple on $\mathbb{P}^1_\tau \simeq \mathbb{P}^1$

$$0 \to F|_{\mathbb{P}^1_\tau}(j) \to \tilde{\pi}^*(L_0|_{\mathbb{P}^1}(j))|_{\mathbb{P}^1_\tau} \to \tilde{\sigma}^*(E(j)|_l) \to 0.$$

Since $\tilde{\pi}^*(L_0|_{\mathbb{P}^1}(j))|_{\mathbb{P}^1_\tau} \simeq L_0|_{\mathbb{P}^1}(j)$, we have $\chi(E(j)|_l) = r(1+\delta+j)$. Moreover, (80), (81) and Riemann-Roch yield $\chi(L_0|_{\mathbb{P}^1}(j)) = P_1(r,\gamma,\delta) + r(1+\delta)(j+1)$. Hence

(100) $$\chi(F|_{\mathbb{P}^1_\tau}(j)) = P_1(r,\gamma,\delta) + r\delta j.$$

Next, (82) implies

(101) $$Q_1(r,\gamma,\delta) + j \le \delta_B(L_0|_{\mathbb{P}^1}(j)) \le \delta_A(L_0|_{\mathbb{P}^1}(j)) \le 2\delta + j.$$

On the other hand, $F|_{\mathbb{P}^1_\tau}(j) \simeq \overset{r\delta}{\underset{i=1}{\oplus}} \mathcal{O}_{\mathbb{P}^1}(e_i)$, where $\delta_B(F|_{\mathbb{P}^1_\tau}(j)) = e_1 \le e_2 \le ...... \le e_{r\delta}$. Therefore (100) yields

(102) $$P_1(r,\gamma,\delta) + r\delta j = \chi(F|_{\mathbb{P}^1_\tau}(j)) = \delta_B(F|_{\mathbb{P}^1_\tau}(j)) + r\delta + \sum_{i=2}^{r\delta} e_i.$$

Note that, since $F|_{\mathbb{P}^1_\tau}(j)$ is a subbundle of $\tilde{\pi}^*(L_0|_{\mathbb{P}^1}(j))|_{\mathbb{P}^1_\tau} \simeq L_0|_{\mathbb{P}^1}(j)$, (101) implies $e_2 \le ...... \le e_{r\delta} \le \delta_A(L_0(j)|_{\mathbb{P}^1}) \le 2\delta + j$, so that $\sum_{i=2}^{r\delta} e_i \le (2\delta+j)(r\delta-1)$. This together with (102) shows that

$$\delta_B(F|_{\mathbb{P}^1_\tau}(j)) = P_1(r,\gamma,\delta) + r\delta(j-1) - \sum_{i=2}^{r\delta} e_i \ge \delta(r - 2r\delta + 2\delta) - P_1(r,\gamma,\delta) + j.$$

Hence

(103) $$\delta_B(F|_{\mathbb{P}^1_\tau}(j)) \ge 0, \quad j \ge P_2(r,\gamma,\delta) := P_1(r,\gamma,\delta) + \delta(-r + 2r\delta - 2\delta),$$

and this establishes the implication

(104) $$j \ge P_2(r,\gamma,\delta) \quad \Rightarrow \quad h^1((F|_{\mathbb{P}^1_\tau})(j)) = 0.$$

Consider now the sequence of exact triples

$$0 \to F|_S(jh) \to F|_S(jh+\tau) \to F|_{\mathbb{P}^1_\tau}(jh+\tau) \to 0,$$
$$0 \to F|_S(jh+\tau) \to F|_S(jh+2\tau) \to F|_{\mathbb{P}^1_\tau}(jh+2\tau) \to 0,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$0 \to F|_S(jh+(b-1)\tau) \to F|_S(jh+b\tau) \to F|_{\mathbb{P}^1_\tau}(jh+b\tau) \to 0,$$

where

$$j \geq \max\{-Q_1(r,\gamma,\delta),\ P_2(r,\gamma,\delta)\}. \tag{105}$$

Since $\mathcal{O}_S(\tau)|_{\mathbb{P}^1_\tau} \simeq \mathcal{O}_S(\tau)|_{\mathbb{P}^1_\tau} \simeq \mathcal{O}_{\mathbb{P}^1_\tau}(1)$, it follows from (86) and (104) that

$$h^1(F|_S(jh)) = h^1(F|_{\mathbb{P}^1_\tau}(jh+i\tau)) = 0, \quad 0 \leq i \leq b,$$

for j as in (105). Substituting these equalities subsequently into the triples in the above sequence and keeping in mind that $(F|_S)(jh+b\tau) = (F_b|_S)(jh)$, we eventually obtain

$$h^1(F_b|_S(jh)) = 0, \quad j \geq \max\{-Q_1(r,\gamma,\delta),\ P_2(r,\gamma,\delta)\}. \tag{106}$$

Set $R(x,y,z) := (-Q_1(x,y,z)^+) + P_2(x,y,z)^+$ (the notation $(\cdot)^+$ is introduced in section 2). Then (98) and (106) imply $h^1(L_1(j)|_{\mathbb{P}^1}) = 0, \quad j \geq R(r,\gamma,\delta)$. Hence, since $\mathbb{P}^1$ is an arbitrary line in $\mathbb{P}^{N-1}$, it follows that

$$-R(r,\gamma,\delta) \leq \delta_B(L_1). \tag{107}$$

This establishes the left-hand side of the inequality (99).

To obtain the right-hand side, consider a line $\mathbb{P}^1 \subset \mathbb{P}^{N-1}$ in diagram (63) with $\delta_A(L_1|_{\mathbb{P}^1}) = \delta_A(L_1)$ and

$$L_1|_{\mathbb{P}^1} \simeq \bigoplus_{i=1}^{r\delta(2+b)} \mathcal{O}_{\mathbb{P}^1}(a_i),$$

where $\delta_A(L_1) = a_1 \geq a_2 \geq ... \geq a_{r\delta(2+b)} \geq \delta_B(L_1)$ and $\mathrm{rk} L_1 = r\delta(2+b)$ by (96). Note that (96), (97) and Riemann-Roch yield

$$\chi(L_1|_{\mathbb{P}^1}) = \mathrm{rk} L_1 + c_1(L_1) = r\delta(2+b) + F_1(r,b,\gamma,\delta) =: F_2(r,b,\gamma,\delta). \tag{108}$$

On the other hand, $\chi(L_1|_{\mathbb{P}^1}) = \delta_A(L_1) + \sum_{i=2}^{r\delta(2+b)} a_i$. Whence, in view of (107), we obtain $\delta_A(L_1) = \chi(L_1|_{\mathbb{P}^1}) - \sum_{i=2}^{r\delta(2+b)} a_i \leq \chi(L_1|_{\mathbb{P}^1}) - (r\delta(2+b)-1)\delta_B(L_1) \leq \chi(L_1|_{\mathbb{P}^1}) + (r\delta(2+b)-1)R(r,\gamma,\delta)$. Combined with (108), this yields

$$\delta_A(L_1) \leq R_1(r,b,\gamma,\delta) := F_2(r,b,\gamma,\delta) + (r\delta(2+b)-1)R(r,\gamma,\delta). \tag{109}$$

Recall that, according to (90),

$$0 \leq b \leq r(1+\delta). \tag{110}$$

Setting $S(x,y,z) := R_1(r, r(1+\delta), \gamma, \delta)^+$, we obtain from (109) the desired right-hand side of (99).

□

5.3. An estimate for the degree of the variety of maximal jumping lines $B^\kappa_\delta(E^0,x,\mathbb{P}^{N-1})$. Note that (89) and (88) imply that $F_b|_{\mathbb{P}^1_y}$ is generated by global sections for any $y \in \mathbb{P}^{N-1}$. Hence base change yields an epimorphism $\pi^*L_1 \twoheadrightarrow F_b$ and its twist

$$ev_1 : \pi^*L_1 \otimes \mathcal{O}_\pi(-b) \twoheadrightarrow F. \tag{111}$$

Combining (84) with (111) we get the exact sequence

$$\pi^*L_1 \otimes \mathcal{O}_\pi(-b) \xrightarrow{\Psi} \pi^*L_0 \to \sigma^*E \to 0, \tag{112}$$

where $\Psi := i \circ ev_1$. Twisting (112) by the π-relative dualizing sheaf $\omega_\pi \simeq \mathcal{O}_\pi(-2) \otimes \pi^*\mathcal{O}_{\mathbb{P}^{N-1}}(1)$, and applying $R^1\pi_*$ we obtain the exact sequence

$$L_1 \otimes A_b \xrightarrow{\Phi} L_0 \to R^1\pi_*(\sigma^*E \otimes \omega_\pi) \to 0, \tag{113}$$

where

(114) $A_b := (\pi_*\mathcal{O}_\pi(b))^\vee \simeq S^b(\mathcal{O}_{\mathbb{P}^{N-1}}(-1)\oplus\mathcal{O}_{\mathbb{P}^{N-1}}) = \mathcal{O}_{\mathbb{P}^{N-1}}(-b)\oplus\mathcal{O}_{\mathbb{P}^{N-1}}(-b+1)\oplus...\oplus\mathcal{O}_{\mathbb{P}^{N-1}}.$

Set $E^0 := E(-\delta)$. Then (78) and (79) yield

(115) $$c_1(E^0) = 0, \quad c_2(E^0) = \gamma, \quad \delta_A(E^0) = \delta = -\delta_B(E^0).$$

We set also

(116) $$B^\kappa_\delta(E^0, x, \mathbb{P}^{N-1}) := \{y \in \mathbb{P}^{N-1} \mid \dim_{\mathbb{C}_y}(\mathbb{C}_y \otimes R^1\pi_*(\sigma^*E\otimes\omega_\pi)) = \kappa\}$$

for $x \in \mathbb{P}^N$.

Note that $\kappa \le \mathrm{rk}E = r$. Hence in view of (80),

(117) $$r(1+\delta) = \rho_0 \ge \rho_0 - \kappa \ge r\delta \ge 0.$$

Next, denote

(118) $$\rho_1 := \mathrm{rk}(L_1\otimes A_b) = \mathrm{rk}L_1\cdot\mathrm{rk}A_b = r\delta(2+b)(1+b)$$

(we use (96) and (114) here). Observe that (114) implies $\delta_A(L_1\otimes A_b) = \delta_A(L_1) - b$, $\delta_B(L_1\otimes A_b) = \delta_B(L_1) - b$, so that

$$j\delta_B(L_1)-jb = j\delta_B(L_1\otimes A_b) \le \delta_B(\wedge^j(L_1\otimes A_b)) \le \delta_A(\wedge^j(L_1\otimes A_b)) \le j\delta_A(L_1\otimes A_b) = j\delta_A(L_1)-jb$$

for any $j \in \mathbb{Z}_{>0}$. This, together with Lemma 5.2 and (110), gives the inequalities

(119) $$-jR(r,\gamma,\delta) - jr(1+\delta) \le \delta_B(\wedge^j(L_1\otimes A_b)) \le \delta_A(\wedge^j(L_1\otimes A_b)) \le jS(r,\gamma,\delta).$$

In a similar way (82) gives

(120) $$jQ_1(r,\gamma,\delta) \le j\delta_B(L_0) \le \delta_B(\wedge^jL_0) \le \delta_A(\wedge^jL_0) \le j\delta_A(L_0) \le 2j\delta.$$

Notice now that the locally free resolution (113) of the sheaf $R^1\pi_*(E\otimes\omega_\pi)$ shows that the κ-th Fitting ideal sheaf [1] $\mathcal{I} := \mathcal{F}itt_\kappa(R^1\pi_*(\sigma^*E\otimes\omega_\pi))$ of the sheaf $R^1\pi_*(E\otimes\omega_\pi)$ coincides with the image of the morhism

$$\Lambda :\ \mathcal{E} := \wedge^{(\rho_0-\kappa)}(L_1\otimes A_b)\otimes\wedge^{(\rho_0-\kappa)}L_0^\vee \to \mathcal{O}_{\mathbb{P}^{N-1}}$$

induced by the morphism Φ in (113). We thus have an epimorphism

(121) $$\mathcal{E} \twoheadrightarrow \mathcal{I}.$$

Denote by $V^\kappa_\delta(x)$ the subscheme of $\mathbb{P}^{N-1}$ defined by the ideal sheaf $\mathcal{I}$, i.e.

(122) $$\mathcal{O}_{V^\kappa_\delta(x)} := \mathcal{O}_{\mathbb{P}^{N-1}}/\mathcal{I} = \mathrm{coker}\,\Lambda.$$

Now (116) implies

(123) $$B^\kappa_\delta(E^0, x, \mathbb{P}^{N-1}) = \mathrm{Supp}(\mathrm{coker}\,\Lambda) = V^\kappa_\delta(x)_{red}.$$

Clearly,

$$\delta_B(\wedge^{(\rho_0-\kappa)}(L_1\otimes A_b)) + \delta_B(\wedge^{(\rho_0-\kappa)}L_0^\vee) \le \delta_B(\mathcal{E}) \le \delta_A(\mathcal{E}) \le \delta_A(\wedge^{(\rho_0-\kappa)}(L_1\otimes A_b)) + \delta_A(\wedge^{(\rho_0-\kappa)}L_0^\vee).$$

Substituting here (119) and (120) with $j = \rho_0 - \kappa$ and using (117), we obtain

(124) $$T_1(r,\gamma,\delta) \le \delta_B(\mathcal{E}) \le \delta_A(\mathcal{E}) \le T_2(r,\gamma,\delta),$$

where

$$T_1(r,\gamma,\delta) = -r(1+\delta)(Q_1(r,\gamma,\delta) - R(r,\gamma,\delta) - r(1+\delta))^+,$$
$$T_2(r,\gamma,\delta) = r(1+\delta)(S(r,\gamma,\delta) + 2p\delta)^+.$$

Furthermore, taking into account (80) and (118), we obtain

(125) $$\mathrm{rk}\mathcal{E} = I_0(r,b,\rho_0-\kappa,\delta) := \binom{r\delta(2+b)(1+b)}{\rho_0-\kappa}\binom{r(1+\delta)}{\rho_0-\kappa}.$$

[1] For the definition of Fitting ideals see for instance [E, p. 492].

Therefore, using (110) and (117) we obtain

$$\mathrm{rk}\mathcal{E} \le I(r,\delta) := I_0(r, r(1+\delta), r(1+\delta), \delta)^+. \tag{126}$$

Similarly, (81) and (97) yield

$$c_1(\wedge^{(\rho_0-\kappa)} L_0^\vee) = U_0(r,\gamma,\delta) := P_1(r,\gamma,\delta)\binom{r(1+\delta)-1}{\rho_0-\kappa-1},$$

$$c_1(\wedge^{(\rho_0-\kappa)} L_1 \otimes A_b) = U_1(r,b,\gamma,\delta) := (b+1)\left(F_1(r,b,\gamma,\delta) + \binom{r(1+\delta)-1}{\rho_0-\kappa-1} r\delta b(b+1)(b+2)/2\right);$$

hence

$$c_1(\mathcal{E}) = J_0(r,b,\gamma,\delta) := U_0(r,\gamma,\delta) r\delta(2+b)(1+b) + U_1(r,\gamma,\delta) r(1+\delta).$$

Then

$$c_1(\mathcal{E}) \le J(r,\gamma,\delta) := J_0(r, r(1+\delta), \gamma, \delta)^+. \tag{127}$$

Apply now Theorem 4.5 to the bundle $\mathcal{E}$. From (124), (126) and (127) we obtain that $\mathcal{E}(m_0)$ is globally generated for

$$m_0 = m_0(r,\gamma,\delta) := J(r,\gamma,\delta) + (1 + I(r,\delta))(2 - T_1(r,\gamma,\delta)) - 1. \tag{128}$$

We thus have an epimorphism $\mathcal{O}_{\mathbb{P}^{N-1}}^{t_0} \twoheadrightarrow \mathcal{E}(m_0)$, where

$$t_0 = t_0(r,\gamma,\delta,N) := I(r,\delta)\big(T_2(r,\gamma,\delta) + m + N - 1\big)^{N-1}. \tag{129}$$

Hence, by (121), we have an epimorphism $\mathcal{O}_{\mathbb{P}^{N-1}}^{t_0} \twoheadrightarrow \mathcal{I}(m_0)$. This epimorphism and the Bezout Theorem show that the degree[2] of the reduced closed subscheme

$$B_\delta^\kappa(E^0, x, \mathbb{P}^{N-1}) = V_\delta^\kappa(x)_{red}. \tag{130}$$

of $\mathbb{P}^{N-1}$ satisfies the inequality

$$\deg B_\delta^\kappa(E^0, x, \mathbb{P}^{N-1}) \le \deg V_\delta^\kappa(x) \le m_0^{t_0}.$$

Substituting here (128) and (129) and using the relations (115), we obtain the following main result of this section.

Theorem 5.3. *Let E^0 be a rank-r vector bundle on $\mathbb{P}^N$ with $c_1(E^0) = 0$, $\delta_A(E^0) = \delta = -\delta_B(E^0)$ and $\kappa_A(E^0) = \kappa$. Let l be a line in $\mathbb{P}^N$ with $\delta_A(E^0|_l) = \delta$ and $\kappa_A(E^0|_l) = \kappa$ and let x be any point on l. Let $\mathbb{P}^{N-1}$ be the base of the family of lines through x in $\mathbb{P}^N$. Then the degree of the reduced closed subscheme $B_\delta^\kappa(E^0, x, \mathbb{P}^{N-1})$ of $\mathbb{P}^{N-1}$ satisfies the inequality*

$$\deg B_\delta^\kappa(E^0, x, \mathbb{P}^{N-1}) \le m_0(r, c_2(E^0), \delta)^{t_0(r, c_2(E^0), \delta, N)}, \tag{131}$$

where $m_0(x_1,x_2,x_3)$ and $t_0(x_1,x_2,x_3,x_4)$ are given by (128) and (129), respectively.

[2]By the degree of a closed reduced subscheme of $\mathbb{P}^{N-1}$ we mean the sum of degrees of its irreducible components.

6. Proof of Theorem 1.1

In the rest of the paper we fix a twisted ind-Grassmannian $\mathbf{G} = \varinjlim G(i_m, V^{n_m})$ given by a sequence of embeddings (1), and assume that $1 < i_m \leq n_m - i_m$ for all m. We set $G_m := G(i_m, V^{n_m})$ and $\tilde{\varphi}_m := \varphi_{m-1} \circ ... \circ \varphi_1$. We fix also a self-dual vector bundle $\mathbf{E} = \varprojlim E_m$ on $\mathbf{G}$ (this means that $E_m \simeq E_m^\vee$ for each m) of rank $\mathbf{r} \in \mathbb{Z}_{>0}$. Then

$$c_1(E_m) = 0, \quad \delta(E_m) = 2\delta_A(E_m), \quad m \geq 1. \tag{132}$$

Note that it suffices to prove Theorem 1.1 for self-dual bundles $\mathbf{E}$. Indeed, consider an arbitrary finite-rank vector bundle $\mathbf{E}' = \varprojlim E'_m$ on $\mathbf{G}$. Set $\mathbf{E} = \mathcal{E}nd\ \mathbf{E}'$. Since $\mathbf{E}$ is self-dual, we can assume that Theorem 1.1 holds for $\mathbf{E}$. Therefore, for any m and any line l in $G(i_m, V^{n_m})$, $(\mathcal{E}nd E'_m)|_l$ is a trivial bundle. Grothendieck's theorem for vector bundles on $\mathbb{P}^1$ implies immediately that $E'_m|_l \otimes L_{m,l}$ is a trivial bundle for a suitable line bundle $L_{m,l}$ on l. Since $c_1(L_{m,l}) = -c_1(E'_m)/\mathrm{rk} E'_m$ does not depend on l, the line bundles $L_{m,l}$ define a line bundle L_m on G_m. Now a standard result in [PT] (Prop. 1.4.1) shows that $E'_m \otimes L_m$ is trivial for any m. Thus $\varprojlim (E'_m \otimes L_m)$ is trivial. To see that $\mathbf{E}'$ itself is trivial, note that the line bundles L_m define a line bundle $\mathbf{L} = \varprojlim L_m$ on $\mathbf{G}$. As $\mathbf{G}$ is twisted, for every m the Chern class $c_1(\mathbf{L}|_{G_m}) = c_1(L_m)$ must be divisible by $\deg(\varphi_n \circ ... \circ \varphi_m)$ for all $n > m$. Since $\lim_{n \to \infty} \deg(\varphi_n \circ ... \circ \varphi_m) = \infty$, it follows that $c_1(L_m) = 0$, and hence $\mathbf{L}$ is trivial. Therefore $\mathbf{E}' \simeq \varprojlim (E'_m \otimes L_m)$ is trivial.

6.1. A first observation on $c_2(\mathbf{E})$. Note that the embeddings $\varphi_m : G_m \to G_{m+1}$ define homomorphisms $\varphi_m^* : A^2(G_{m+1}) \to A^2(G_m)$, and the second Chern class of $\mathbf{E}$ is, by definition, the projective system $\{c_2(E_m) = \varphi_m^* c_2(E_{m+1})\}_{m \geq 1}$. Here $A(G_m) = \underset{i \geq 0}{\oplus} A^i(G_m)$ stands for the Chow ring of G_m, and we recall some standard facts about $A(G_m)$ - cf [F, 14.7]:

(i) $A^1(G_m) = \mathrm{Pic}(G_m) = \mathbb{Z}[\mathbb{V}_m]$, $A^2(G_m) = \mathbb{Z}[\mathbb{W}_{1,m}] \oplus \mathbb{Z}[\mathbb{W}_{2,m}]$, where $\mathbb{V}_m, \mathbb{W}_{1,m}, \mathbb{W}_{2,m}$ are the Schubert varieties of the form $\mathbb{V}_m = \{V^{i_m} \in G_m |\ \dim(V^{i_m} \cap V_0^{n_m - i_m - 1}) \geq 1$ for a fixed subspace $V_0^{n_m - i_m - 1}$ of $V^{n_m}\}$, $\mathbb{W}_{1,m} = \{V^{i_m} \in G_m |\ \dim(V^{i_m} \cap V_0^{n_m - i_m - 1}) \geq 1$ for a fixed subspace $V_0^{n_m - i_m - 1}$ in $V^{n_m}\}$, $\mathbb{W}_{2,m} = \{V^{i_m} \in G_m |\ \dim(V^{i_m} \cap V_0^{n_m - i_m + 1}) \geq 2$ for a fixed subspace $V_0^{n_m - i_m + 1}$ of $V^{n_m}\}$;

(ii) $[\mathbb{V}_m]^2 = [\mathbb{W}_{1,m}] + [\mathbb{W}_{2,m}]$ in $A^2(G_m)$;

(iii) there exist integers $a_{ij}(m) \geq 0$, $\ i, j = 1, 2$, such that

$$\varphi_m^*[\mathbb{W}_{1,m+1}] = a_{11}(m)[\mathbb{W}_{1,m}] + a_{21}(m)[\mathbb{W}_{2,m}], \quad \varphi_m^*[\mathbb{W}_{2,m+1}] = a_{12}(m)[\mathbb{W}_{1,m}] + a_{22}(m)[\mathbb{W}_{2,m}], \tag{133}$$

$$a_{11}(m) + a_{12}(m) = a_{21}(m) + a_{22}(m) = (\deg \varphi_m)^2, \quad m \geq 1. \tag{134}$$

Lemma 6.1. *Given $\mathbf{E} = \varprojlim E_m$, there exists an infinite subsequence of the sequence of Grassmannians G_m such that the coordinates of $c_2(E_m)$ in the basis $\{[\mathbb{W}_{1,m}], [\mathbb{W}_{2,m}]\}$ are constants $\lambda_1 \in \mathbb{Z}$ and $\lambda_2 \in \mathbb{Z}$. Moreover, if $\lambda_1\lambda_2 \neq 0$, then $\lambda_1\lambda_2 < 0$.*

Proof. Let

$$c_2(E_m) = \lambda_{1m}[\mathbb{W}_{1,m}] + \lambda_{2m}[\mathbb{W}_{2,m}]. \tag{135}$$

Consider the 2×2-matrix $A(m) = (a_{ij}(m))$ and the column vector $\Lambda_m = (\lambda_{1m}, \lambda_{2m})^t$. Relations (135) and (133) give

$$\Lambda_m = A(m)\Lambda_{m+1}. \tag{136}$$

Set $\gamma_m := \lambda_{1m} - \lambda_{2m}$. Then, substituting (134) in (136) we compute

$$\gamma_m = (a_{11}(m) - a_{21}(m))\gamma_{m+1} = \gamma_{m+m'+1}\prod_{i=1}^{m'}(a_{11}(m+i) - a_{21}(m+i)), \quad m, m' \geq 1. \tag{137}$$

Assume that $\gamma_{m_0} \neq 0$ for some $m_0 \geq 1$. Then (137) implies $a_{11}(m) - a_{21}(m) \neq 0$, $\gamma_m \neq 0$, $m \geq m_0$. Furthermore, if $|a_{11}(m) - a_{21}(m)| > 1$ for an infinite number of values of $m \geq m_0$, then the the right-hand side of (137) grows to infinity when $m' \to \infty$, a contradiction. Hence $|a_{11}(m) - a_{21}(m)| > 1$ for at most a finite number of values of $m \geq m_0$. Removing the Grassmannians G_m with these values of m from our ind-Grassmannian $\mathbf{G}$ (and taking as new embeddings the corresponding compositions of old embeddings) we may assume that $|a_{11}(m) - a_{21}(m)| = 1$ for all $m \geq m_0$. Since for an infinite number of values of m the numbers $a_{11}(m) - a_{21}(m)$ have the same sign, the sequence $\{\gamma_m\}$ has an infinite constant subsequence. Hence, again by removing appropriate m's in the construction of $\mathbf{G}$, we may assume

$$\gamma := \gamma_m = \lambda_{1m} - \lambda_{2m} \neq 0, \quad m \geq m_0. \tag{138}$$

Let $\gamma > 0$. (The case $\gamma < 0$ is treated similarly.) As it was shown in [PT, section 5], for m large enough, say, for $m \geq m_0$, λ_{1m} and λ_{2m} cannot be both nonzero of the same sign. (The argument is carried out in [PT] for rank-2 bundles but applies to bundles E_m of any rank.) This property and (138) imply that

$$\gamma \geq \lambda_{1m} \geq 0, \quad 0 \geq \lambda_{2m} \geq -\gamma, \quad m \geq m_0.$$

Thus, there exist infinite constant subsequences $\{\lambda_{1,m'} =: \lambda_1 \geq 0\}_{m' \geq m_0}$ and $\{\lambda_{2,m'} =: \lambda_2 \leq 0\}_{m' \geq m_0}$, of the sequences $\{\lambda_{1m}\}_{m \geq m_0}$ and $\{\lambda_{2m}\}_{m \geq m_0}$, respectively. Thus, again without loss of generality we may assume that the sequences $\{\lambda_{1m}\}_{m \geq m_0}$ and $\{\lambda_{2m}\}_{m \geq m_0}$ are constant:

$$0 \leq \lambda_1 = \lambda_{1m}, \quad 0 \geq \lambda_2 = \lambda_{2m}, \quad m \geq m_0. \tag{139}$$

□

In what follows we assume that the coordinates of $c_2(E_m)$ in the basis $\{[\mathbb{W}_{1,m}], [\mathbb{W}_{2,m}]\}$ are constant for our fixed sequence of Grassmannians G_m.

Recall that there are two families of projective subspaces of maximal dimension in G_m: family I consisting of subspaces $\mathbb{P}^{i_m} = \{V^{i_m} \in G_m \mid V^{i_m} \subset V_0^{i_m+1}\}$, $V_0^{i_m+1} \in G(i_m+1, V^{n_m})$, and family II consisting of subspaces $\mathbb{P}^{n_m-i_m} = \{V^{i_m} \in G_m \mid V^{i_m} \supset V_0^{i_m-1}\}$, $V_0^{i_m-1} \in G(i_m-1, V^{n_m})$. Lemma 6.1 implies therefore the following.

Corollary 6.2. *In the notations of Lemma 6.1, we have*

$$c_2(E_m|_{\mathbb{P}^{i_m}}) = \lambda_2 \qquad \text{for any } \mathbb{P}^{i_m} \text{ in family I,}$$
$$c_2(E_m|_{\mathbb{P}^{n_m-i_m}}) = \lambda_1 \qquad \text{for any } \mathbb{P}^{n_m-i_m} \text{ in family II.} \tag{140}$$

6.2. The variety of maximal jumping lines of E_m passing through a point. For a fixed m, consider the natural diagram

$$G_m = G(i_m, V^{n_m}) \overset{\pi_1}{\leftarrow} \Gamma_m \overset{\pi_2}{\to} Fl_m, \tag{141}$$

where $\Gamma_m := Fl(i_m-1, i_m, i_m+1, V^{n_m})$, and $Fl_m := Fl(i_m-1, i_m+1, V^{n_m})$ is the base of the family of (projective) lines on G_m. Set

$$Z_a(E_m) := \{l \in Fl_m \mid \delta_A(E_m|_l) \geq a\}, \quad B_a(E_m) := \pi_2^{-1}(Z_a(E_m)), \quad a \in \mathbb{Z}_{>0}.$$

The semicontinuity of $\delta_A(E_m|_l)$ as a function of l implies that $Z_a(E_m)$ is closed in Fl_m; respectively, $B_a(E_m)$ is closed in Γ_m. Next, set

$$\Delta := \min\{a \mid \operatorname{Im}(\pi_1(B_a(E_m)) \neq G_m\} - 1. \tag{142}$$

We then have $Y := \pi_1(B_{\Delta+1}(E_m)) \neq G_m,\ \pi_1(B_\Delta(E_m)) = G_m$, and

$$G'_m := G_m \smallsetminus Y = \left\{ x \in G_m \;\middle|\; \Delta = \max\{\delta_A(E_m|_l) |\ l \text{ is a line on } G_m \text{ through } x\} \right\}$$

is a dense open subset of G_m.

Denote $p_{\Delta,E_m} := \pi_1|_{B_\Delta(E_m)}$. Then $B_\Delta(E_m)' := p^{-1}_{\Delta,E_m}(G'_m)$ is closed in $\pi_1^{-1}(G'_m)$ and the morphism $p_{\Delta,E_m} : B_\Delta(E_m)' \to G'_m$ is projective and surjective. Similarly, for each $a,\ 1 \leq a \leq \mathbf{r}$,

$$B^a_\Delta(E_m) := \{(x,l) \in B_\Delta(E_m) \mid l \in Z_\Delta(E_m),\quad \kappa_A(E_m|_l) \geq a\},$$

is a closed subset in $B_\Delta(E_m)$; respectively, $p_{\Delta,E_m}(B^a_\Delta(E_m))$ is closed in G_m. Since $\kappa_A(E_m|_l) \geq 1$ for any $l \in Z_\Delta(E_m)$, it follows that $\pi_1(B^1_\Delta(E_m)) = G_m$.

If $\pi_1(B^{\mathbf{r}}_\Delta(E_m)) \neq G_m$, we put

$$K := \min\{2 \leq a \leq \mathbf{r} \mid \pi_1(B^a_\Delta(E_m)) \neq G_m\} - 1, \quad T := \pi_1(B^{K+1}_\Delta(E_m)), \tag{143}$$

$$G^0_m := G'_m \smallsetminus T, \quad B^K_\Delta(E_m)^0 := \pi_1^{-1}(G^0_m) \cap B^K_\Delta(E_m); \tag{144}$$

if $\pi_1(B^{\mathbf{r}}_\Delta(E_m)) = G_m$, we put

$$K := \mathbf{r}, \quad G^0_m := G'_m, \quad B^K_\Delta(E_m)^0 := B_\Delta(E_m). \tag{145}$$

By definition, G^0_m is a dense open subset of G'_m, hence of G_m, and the morphism $p^K_{\Delta,E_m} := \pi_1|_{B^K_\Delta(E_m)^0} : B^K_\Delta(E_m)^0 \to G^0_m$ is projective and surjective.

6.3. A bound for the codimension of $B^K_\Delta(E_m)$. The semicontinuity of $\delta_A(E_m|_l)$ (respectively, of $\delta(E_m|_l)$) forces the minimal value of of $\delta_A(E_m|_l)$ (respectively, of $\delta(E_m|_l)$) to be attained on a dense open set of lines $l \in Fl_m$. In what follows we denote this minimal value by $\delta^{gen}_A(E_m)$ (respectively, by $\delta^{gen}(E_m)$).

Lemma 6.3. *Assume $\delta^{gen}_A(E_m) > \frac{1}{2}\mathbf{r}$. Then there exists a subsheaf $\mathcal{F}_m$ of E_m with $c_1(\mathcal{F}_m) > 0$.*

Proof. The inequality $\delta^{gen}_A(E_m) > \frac{1}{2}\mathbf{r}$ and the vanishing of $c_1(E_m)$ imply, for any line $l \subset G_m$ with splitting type $(\delta_1, ..., \delta_{\mathbf{r}})$ of $E|_l$, that $\delta_s - \delta_{s+1} \geq 2$ for some $s,\ 1 \leq s \leq \mathbf{r} - 1$. Thus, the assumption of Theorem 1.4.2 in [PT] (which is version of the Descent Lemma of [OSS, Ch. II, Lemma 2.1.2] for a Grassmannian) is satisfied, and this theorem yields a subsheaf $\mathcal{F}_m$ of E_m. Since E_m is self-dual, the vanishing of $c_1(E_m)$ forces the integer δ_s to be positive, hence by the construction of $\mathcal{F}_m$ we have $c_1(\mathcal{F}_m) = \delta_1 + ... + \delta_s > 0$. □

Lemma 6.4. *For sufficiently large m there are no subsheaves $\mathcal{F}_m \subset E_m$ with $c_1(\mathcal{F}_m) > 0$.*

Proof. Set $\tilde{d}_m := \deg \tilde{\varphi}_m$. Consider the polynomial $P_m(t) := \tilde{d}_m t + 1$ and let

$$\mathcal{H}_m := \{C \in \mathrm{Hilb}^{P_m(t)}\, G_m \mid C \text{ is a smooth irreducible rational curve of degree } \tilde{d}_m \text{ on } G_m\}.$$

It is well known after Strømme [St] that $\mathcal{H}_m$ is a smooth irreducible variety of dimension $n_m\tilde{d}_m + i_m(n_m - i_m) - 3$.

Assume that $\mathcal{F}_m$ is a subsheaf of E_m with $c_1(\mathcal{F}_m) > 0$. Then $\mathrm{codim}_{G_m} \mathrm{Sing}\, \mathcal{F}_m \geq 2$ as E_m is locally free [OSS, Ch. II, Cor. 1.1.9]. Furthermore, since G_m is a homogeneous space, $\mathcal{H}^*_m := \{C \in \mathcal{H}_m \mid C \cap \mathrm{Sing}\mathcal{F}_m = \emptyset\}$ is a dense open subset of $\mathcal{H}_m$.

Set $a_m := \min\limits_{C \in \mathcal{H}_m} \{\delta_A(E_m|_C\}$. Since $\delta_A(E_m|_C)$ is semicontinuous as a function of C, $\mathcal{H}^0_m := \{C \in \mathcal{H}_m \mid \delta_A(E_m|_C) = a_m\}$ is a dense open subset of $\mathcal{H}_m$, and, for any projective line $l \subset G_1$,

$$\delta_A(E_1|_l) = \delta_A(E_m|_{C_1}) \geq a_m, \tag{146}$$

where $C_1 := \tilde{\varphi}_m(l) \in \mathcal{H}_m$. Now assume that $c_1(\mathcal{F}_m) \geq 1$ and consider any curve $C \in \mathcal{H}^*_m \cap \mathcal{H}^0_m$ such that $\delta_A(E_m|_C) = a_m$. Since $\mathcal{F}_m|_C$ is a locally free subsheaf of $E_m|_C$ with $c_1(\mathcal{F}_m) \geq 1$, it follows that $a_m = \delta_A(E_m|_C) \geq \delta_A(\mathcal{F}_m|_C) \geq c_1(\mathcal{F}_m|C)/\mathbf{r} = \tilde{d}_m c_1(\mathcal{F}_m)/\mathbf{r} \geq \tilde{d}_m/\mathbf{r}$. Combining this with (146) we obtain $\delta_A(E_1|_l) \geq \tilde{d}_m/\mathbf{r}$, in particular,

$$b_1 := \max_{l' \in \mathcal{H}_1} \{\delta_A(E_1|_{l'})\} \geq \tilde{d}_m/\mathbf{r}.$$

If $\mathcal{F}_m$ exists for infinitely many values of m, the right-hand side of the last inequality tends to infinity for $m \to \infty$ as $\lim\limits_{m\to\infty} \tilde{d}_m = \infty$, a contradiction. □

Corollary 6.5. *For sufficiently large m, $\delta_A^{gen}(E_m) \leq \frac{1}{2}\mathbf{r}$.*

Fix $x \in G_m$ and for any $d \in \mathbb{Z}_{>0}$ consider the locally closed subset

$$B_a^0(x) := \{l \in \pi_1^{-1}(x) \mid \delta(E_m|_l) = a\}$$

of $\pi_1^{-1}(x)$. Let $B_a(x)$ be its closure in $\pi_1^{-1}(x)$. The semicontinuity of $\delta(E_m|_l)$ implies $B_a^0(x) = B_a(x) \smallsetminus (\bigcup\limits_{a'>a} B_{a'}(x))$, $\ a > 0$. Denote

$$\delta := \delta_A(E_m), \quad \kappa := \kappa_A(E_m).$$

Then

$$B_\delta(x) = B_\delta^0(x). \tag{147}$$

Furthermore, put

$$B_\delta^\kappa(x) := \{l \in B_\delta(x) \mid \kappa_A(E_m|_l) = \kappa\} \tag{148}$$

and note that $B_\delta^\kappa(x)$ is a closed subset of $B_\delta(x)$. The following result is proved by A.Tyurin [T, Ch. 2, §2, Lemmas 3 and 4].

Lemma 6.6. *If $B_\delta^\kappa(x) \neq \emptyset$, then* $\operatorname{codim}_{\pi_1^{-1}(x)} B \leq \mathbf{r}(\mathbf{r}-1)\delta(E_m)$ *for any irreducible component B of $B_\delta^\kappa(x)$.*

Since E_m is self-dual, it follows that $\delta(E_m) = 2\delta$, hence Lemma 6.6 implies

$$\operatorname{codim}_{\pi_1^{-1}(x)} B \leq 2\mathbf{r}(\mathbf{r}-1)\delta \tag{149}$$

whenever $B_\delta^\kappa(x) \neq \emptyset$.

Consider the closed subset

$$W := \{x \in G_m \mid B^0_{\delta_A^{gen}(E_m)}(x) = \emptyset\}$$

and set

$$G_m^* := (G_m \smallsetminus W) \cap G_m^0,$$

where G_m^0 was defined in (144) and (145).

Clearly, $W \cap l = \emptyset$ for a generic line $l \subset G_m$, hence G_m^* is a dense open subset of G_m and for any $x \in G_m^*$ there exists a line $l \subset G_m$ through x with $\delta_A(E_m|_l) = \delta_A^{gen}(E_m)$.

We need one more result of Tyurin. Lemma 5 in [T, Ch. 2, §2] implies directly the following.

Corollary 6.7. *There exists a polynomial $F \in \mathbb{Q}[x_1, x_2]$ such that, if E is a self-dual vector bundle on $\mathbb{P}^3$ and P is an arbitrary plane on $\mathbb{P}^3$, then*

$$\delta_A(E|_l) \leq F(\delta^{\rm gen}(E), \chi(E|_P))$$

for any line $l \subset \mathbb{P}^3$.

Now fix a point $x \in G_m$ and let $K_m(x)$ be the subvariety of G_m filled by projective subspaces of maximal dimension in G_m passing through x. It is well known that $K_m(x)$ is a cone over the cartesian product $\mathbb{P}^{i_m-1} \times \mathbb{P}^{n_m-i_m-1}$. Corollary 6.7 implies that, for any line $l \in p_1^{-1}(x)$,

$$\delta_A(E_m|_l) \leq F(\delta^{\rm gen}(E_m), \chi(E_m|_P)) \tag{150}$$

for some polynomial $F \in \mathbb{Q}[x_1, x_2]$ and some projective plane plane $P \subset K_m(x)$. The class of P in the Chow ring $A(G_m)$ coincides with the class of a plane contained in a projective subspace of family I or II. Hence, since $c_1(E_m) = 0$, the Riemann-Roch theorem and Corollary 6.2 imply that $\chi(E|_P)$ coincides with $\mathbf{r} - \lambda_1$ or $\mathbf{r} - \lambda_2$. Substituting this, together with Corollary 6.5, into (150) we see that there exists a constant Δ not depending on m which bounds $\delta_A(E_m|_l)$ from above for any line $l \in p_1^{-1}(x)$ and any $x \in G_m^*$.

Passing from the sequence $(G_m, E_m)_{m\geq 1}$ to its appropriate subsequence $(G_{m'}, E_{m'})_{m'\geq 1}$, and replacing the original sequence by this subsequence, we obtain in view of (142), (143)-(145), Lemma 6.6 and (149), the following result.

Proposition 6.8. *There exist constants Δ, K and $m_0 \geq 1$ such that for any $m \geq m_0$ there is a dense open subset G_m^* of G_m such that the following statements hold for any $x \in G_m^*$.*

(1) $\delta_A(E_m|_l) \leq \Delta$, $\kappa_A(E_m|_l) \leq K$ for any $l \in B_m(x)$, and

$$\delta_A(E_m|_l) = \Delta, \qquad \kappa_A(E_m|_l) = K$$

for some $l \in B_m(x)$. Therefore, $B_\Delta^K(x) \neq \emptyset$ and

$$\mathrm{codim}_{\pi_1^{-1}(x)} B \leq 2\mathbf{r}(\mathbf{r}-1)\Delta$$

for any irreducible component B of $B_\Delta^K(x)$ according to Lemma 6.6.

(2) Set $B_\Delta^K(E_m)^* := (p_{\Delta,E_m}^K)^{-1}(G_m^*)$. Then $p_{\Delta,E_m}^K : B_\Delta^K(E_m)^* \to G_m^*$ is a projective surjective morphism such that

$$(p_{\Delta,E_m}^K)^{-1}(x) = B_\Delta^K(x). \tag{151}$$

6.4. Final arguments.

It remains to prove the following.

Theorem 6.9. *In the framework of Proposition 6.8 assume $\Delta > 0$. Then there exists a subsheaf $\mathcal{F}_m$ of E_m with $c_1(\mathcal{F}_m) > 0$.*

Proof. Consider the relative Grassmannian $g : G(K, E_m) \to G_m$ with fibre $g^{-1}(x) = G(K, E_m|_x)$ for $x \in G_m$. Set $G(K, E_m)^* := g^{-1}(G_m^*)$. According to Proposition 6.8 for any point $(x, l) \in B_\Delta^K(E_m)^*$ there is a subbundle

$$F(x,l) \simeq \mathcal{O}_{\mathbb{P}^1}(\Delta)^K. \tag{152}$$

of $E|_l$. This yields a morphism

$$\Phi : B_\Delta^K(E_m)^* \to G(K, E_m)^*, (x,l) \mapsto F(x,l)|_x \tag{153}$$

which clearly fits in the commutative diagram

$$\begin{array}{ccc} B_\Delta^K(E_m)^* & \xrightarrow{\Phi} & G(K,E_m)^* \\ & \searrow^{p_{\Delta,E_m}^K} & \downarrow g \\ & & G_m^* \end{array} . \tag{154}$$

In the rest of the proof we assume that $i_m \geq 2$. The remaining case is the case of a twisted ind-projective space, and we leave it as an exercise to the reader. Note that (as $i_m \geq 2$) $G_m = G(i_m, V^{n_m})$ fits into the diagram (10) for $V = V^{n_m}$, $i = i_m - 1$, and set $p := p_{i_m-1}$, $q := q_{i_m-1}$:

$$G_m \xleftarrow{p} Fl(i_m - 1, i_m, V^{n_m}) \xrightarrow{q} G(i_m - 1, V^{n_m}). \tag{155}$$

Furthermore, fix a subspace $V_0^{n_m-1}$ in V^{n_m} and put $Y := q^{-1}(G(i_m - 1, V_0^{n_m-1}))$. The projection $\sigma := p|_Y : Y \to G_m$ is nothing but a blow-up of G_m with center at the subvariety

$$Z_0 := G(i_m, V_0^{n_m-1}), \qquad \mathrm{codim}_{G_m} Z_0 = i_m \geq 2. \tag{156}$$

Fix an arbitrary point $x \in G_m^* \smallsetminus Z_0$ and consider the projective subspace

$$\mathbb{P}_x^{n_m-i_m} := \sigma(q^{-1}(q(\sigma^{-1}(x)))) \subset G_m \tag{157}$$

passing through x. Note that the fibre $B^K_\Delta(x) = (p^K_{\Delta,E_m})^{-1}(x)$ of the projection $p^K_{\Delta,E_m} : B^K_\Delta(E_m)^* \to G^*_m$ lies in $p_1^{-1}(x)$. Next, setting $\mathbb{P}^{n_m-i_m-1}(x) := \{\mathbb{P}^{i_m}$ belongs to family I $\mid x \in \mathbb{P}^{i_m}\}$ and $\mathbb{P}^{i_m-1}(x) := \{\mathbb{P}^{n_m-i_m}$ belongs to family II $\mid x \in \mathbb{P}^{n_m-i_m}\}$ we obtain the isomorphism

$$\mathbb{P}^{i_m-1}(x) \times \mathbb{P}^{n_m-i_m-1}(x) \xrightarrow{\simeq} \pi_1^{-1}(x),\ (\mathbb{P}^{n_m-i_m}, \mathbb{P}^{i_m}) \mapsto l = \mathbb{P}^{n_m-i_m} \cap \mathbb{P}^{i_m}. \tag{158}$$

Consider the projections

$$\mathbb{P}^{n_m-i_m-1}(x) \xleftarrow{pr_1} \pi_1^{-1}(x) \xrightarrow{pr_2} \mathbb{P}^{i_m-1}(x).$$

By the construction of σ, the base $\mathbb{P}^{n_m-i_m-1}_x$ of the family of lines through x lying in the subspace $\mathbb{P}^{n_m-i_m}_x$ is a fibre of the projection $\pi_1^{-1}(x) \xrightarrow{pr_2} \mathbb{P}^{i_m-1}(x)$ over a certain point determined by x.

Consider the closed subset

$$B^K_{\Delta,x} := B^K_\Delta(x) \cap \mathbb{P}^{n_m-i_m-1}_x \tag{159}$$

in $\mathbb{P}^{n_m-i_m-1}_x$. Proposition 6.8 implies

$$\operatorname{codim}_{\mathbb{P}^{n_m-i_m-1}_x} X \leq 2\mathbf{r}(\mathbf{r}-1)\Delta \tag{160}$$

for any fixed irreducible component X of $B^K_{\Delta,x}$. Set

$$N := 2\mathbf{r}(\mathbf{r}-1)\Delta + 1. \tag{161}$$

Take a projective subspace $\mathbb{P}^{N-1} \subset \mathbb{P}^{n_m-i_m-1}_x$ and let $\mathbb{P}^N \subset \mathbb{P}^{n_m-i_m}_x$ be the subspace filled by the lines from $\mathbb{P}^{N-1}$. Put $E^0 := E_m|_{\mathbb{P}^N}$. Then $\delta_A(E^0) = \Delta$, $\kappa_A(E^0) = K$ by Proposition 6.8 and $c_2(E^0) = \lambda_1$ by (140). In addition, comparing (116) with (159) and (151), we obtain

$$B^K_\Delta(E^0, x, \mathbb{P}^{N-1}) = B^K_{\Delta,x} \cap \mathbb{P}^{N-1},$$

and (160) and (161) imply that

$$\deg B^K_\Delta(E^0, x, \mathbb{P}^{N-1}) = \deg B^K_{\Delta,x}$$

for a generic choice of the subspace $\mathbb{P}^{N-1}$ in $\mathbb{P}^{n_m-i_m-1}_x$. Applying Theorem 5.3 to the vector bundle E^0 with $\delta_A(E^0) = \Delta$ and $\kappa_A(E^0) = K$, we obtain $\deg B^K_\Delta(E^0, x, \mathbb{P}^{N-1}) \leq d$, where d is a constant not depending on m. We thus obtain

$$\deg B^K_{\Delta,x} \leq d. \tag{162}$$

Suppose next that m is large enough so that the estimate (160) for any irreducible component of $B^K_{\Delta,x}$ together with the condition $\lim\limits_{m\to\infty}(n_m - i_m) = \infty$ ensure that $B^K_{\Delta,x}$ is connected. Then $\deg X \leq d$ by (162). We can assume without loss of generality that $\mathbb{P}^{n_m-i_m-1}_x = \mathrm{Span}X$. Therefore, Theorem 3.7 applied to X implies that the following statement holds. For large enough m any two points of X can be joined by a chain of subspaces $\mathbb{P}^{\mathbf{k}_0} \subset X$, where $\mathbf{k}_0 > \dim G(K, E_m|_x)$. Thus all such subspaces $\mathbb{P}^{\mathbf{k}_0}$ are mapped by Φ into the same point. Consequently $\Phi(X)$ is a point, and since $B^K_{\Delta,x}$ is connected, $\Phi(B^K_{\Delta,x}) = \Phi(X)$. This defines a morphism

$$G^*_m \smallsetminus Z_0 \to G(K, E_m|_x), x \mapsto \Phi(B^K_{\Delta,x}),$$

hence a subbundle $\mathcal{F}'_m$ of $E_m|_{G^*_m \smallsetminus Z_0}$.

The following well-known construction shows that $\mathcal{F}'_m$ extends to a subsheaf $\mathcal{F}_m$ of E_m. The epimorphism of locally free sheaves $E^\vee_m|_{G^*_m \smallsetminus Z_0} \twoheadrightarrow (\mathcal{F}'_m)^\vee$ defines the following composition of embeddings $\zeta : \mathbb{P}(\mathcal{F}'_m) \hookrightarrow \mathbb{P}(E_m|_{G^*_m \smallsetminus Z_0}) \hookrightarrow \mathbb{P}(E_m)$. Let U be the closure of $\zeta(\mathbb{P}(\mathcal{F}'_m))$ in $\mathbb{P}(E_m)$. Set $A := \mathcal{O}_{\mathbb{P}(E_m)/G_m}(1)$ and let $\theta : \mathbb{P}(E_m) \to G_m$ be the structure morphism. Applying the functor $R^{\cdot}\theta_*$ to an exact triple $0 \to \mathcal{I}_{U,\mathbb{P}(E_m)} \otimes A \to A \to A|_U \to 0$ we obtain the exact sequence $E^\vee_m \xrightarrow{\epsilon} \theta_*(A|_U) \to R^1\theta_*(\mathcal{I}_{U,\mathbb{P}(E_m)} \otimes A)$. The morphism $\epsilon|_{G^*_m \smallsetminus Z_0}$ is an epimorphism, hence $\epsilon^\vee : \mathcal{F}_m := (\theta_*(A|_U))^\vee \to E_m$ is a monomorphim and $\mathcal{F}_m|_{G^*_m \smallsetminus Z_0} \simeq \mathcal{F}'_m$.

It remains to show that $c_1(\mathcal{F}_m) > 0$. By (152), if $x \in G^*_m \smallsetminus Z_0$, for any point $(x, l_0) \in B^K_{\Delta,x}$ we have a subbundle $\mathcal{F}(x, l_0) \simeq \mathcal{O}_{\mathbb{P}^1}(\Delta)^K$ of $E_m|_l$. Hence $c_1(\mathcal{F}(x, l_0)) > 0$ as $\Delta > 0$. The line

l_0 lies in the projective subspace $\mathbb{P}_x^{n_m-i_m}$ of G_m defined in (157), so the definition (159) shows that $(y, l_0) \in B^K_{\Delta,y}$ for any y in the dense open subset $U := l_0 \cap (G_m^* \smallsetminus Z_0)$ of l_0. Therefore $\mathcal{F}(x,l_0)|_U = \mathcal{F}_m|_U$, and consequently $\mathcal{F}(x,l_0)$ is isomorphic to a locally free quotient of $\mathcal{F}_m|_{l_0}$, i.e. $\mathcal{F}(x,l_0) \simeq (\mathcal{F}_m|_{l_0})/\mathrm{Torsion}(\mathcal{F}_m|_{l_0})$.

Since $\mathcal{F}_m$ is torsion free being a subsheaf of E_m, it follows that $\mathrm{codim}_{G_m} \mathrm{Sing}\,\mathcal{F}_m \geq 2$, so that $\mathrm{codim}_{Fl_m}\ \pi_2(\pi_1^{-1}(\mathrm{Sing}\,\mathcal{F}_m)) \geq 1$. We thus can find a smooth affine curve $C \subset Fl_m$ with a marked point $c \in C$ such that $\pi_1(\pi_2^{-1}(c)) = l_0$ and $(C \smallsetminus \{c\}) \subset (Fl_m \smallsetminus \pi_2(\pi_1^{-1}(\mathrm{Sing}\,\mathcal{F}_m)))$. Consider the ruled surface $S := \pi_2^{-1}(C) \xrightarrow{\pi_2} C$ and set $\mathcal{F}_S := (\pi_1^*\mathcal{F}_m|_S)/\mathrm{Torsion}(\pi_1^*\mathcal{F}_m|_S)$, $\tilde{l}_0 := \pi_2^{-1}(c)$, $\mathcal{F}_{\tilde{l}_0} := (\mathcal{F}_S|_{\tilde{l}_0})/\mathrm{Torsion}(\mathcal{F}_S|_{\tilde{l}_0})$, $\tilde{l}_t := \pi_2^{-1}(t)$, $t \in C \smallsetminus \{c\}$. The condition $\pi_1(\pi_2^{-1}(c)) = l_0$ implies that $\pi_1|_{\tilde{l}_0} : \tilde{l}_0 \to l_0$ is an isomorphism, hence $(\pi_1|_{\tilde{l}_0})^*\mathcal{F}(x,l_0) \simeq \mathcal{F}_{\tilde{l}_0}$. Consequently,

$$c_1(\mathcal{F}(x,l_0)) = c_1(\mathcal{F}_{\tilde{l}_0}). \tag{163}$$

Furthermore, as $\mathcal{F}_m|_{\pi_1(\tilde{l}_t)}$ is locally free for $t \in C \smallsetminus \{c\}$ by the inclusion $(C \smallsetminus \{c\}) \subset (Fl_m \smallsetminus \pi_2(\pi_1^{-1}(\mathrm{Sing}\,\mathcal{F}_m)))$, it follows that

$$c_1(\mathcal{F}_S|_{\tilde{l}_t}) = c_1(\mathcal{F}_m), \qquad t \in C \smallsetminus \{c\}. \tag{164}$$

We claim that

$$c_1(\mathcal{F}_{\tilde{l}_0}) \leq c_1(\mathcal{F}_S|_{\tilde{l}_t}), \qquad t \in C \smallsetminus \{c\}. \tag{165}$$

Indeed, as $\mathcal{F}_S$ is torsion free, using a filtration of $\mathcal{F}_S$ with rank-1 torsion free consequtive quotients, and removing, if necessary, a finite number of points from $C \smallsetminus \{c\}$, we reduce the proof of (165) to the case when $\mathrm{rk}\mathcal{F}_S = 1$. Here $\mathcal{F}_S \simeq \mathcal{I}_{Y,S} \otimes L$ for some line bundle L on S and for some subscheme Y of S of dimension ≤ 0. Consider the scheme $Y_0 := Y \cap \tilde{l}_0$ of length $\chi(\mathcal{O}_{Y_0}) \geq 0$ with support on $\tilde{l}_0$. Then $\mathcal{F}_{\tilde{l}_0} \simeq L|_{\tilde{l}_0}(-\chi(\mathcal{O}_{Y_0}))$, hence $c_1(\mathcal{F}_{\tilde{l}_0}) = c_1(L|_{\tilde{l}_0}) - \chi(\mathcal{O}_{Y_0}) = c_1(L|_{\tilde{l}_t}) - \chi(\mathcal{O}_{Y_0}) = c_1(\mathcal{F}_S|_{\tilde{l}_t}) - \chi(\mathcal{O}_{Y_0}) \leq c_1(\mathcal{F}_S|_{\tilde{l}_t})$, $t \in C \smallsetminus \{c\}$.

Finally, (163)-(165) imply $c_1(\mathcal{F}_m) \geq c_1(\mathcal{F}(x,l_0)) > 0$. ☐

Corollary 6.10. *For all $m > 0$ E_m is a trivial vector bundle on G_m, and Theorem 1.1 follows.*

Proof. If for sufficiently large m, $\Delta > 0$, then Theorem 6.9 contradicts to Lemma 6.4. Hence $\Delta = 0$. We are going to show now that this implies the triviality of E_m.

Consider diagram (155). Note that for any $x \in G_m$ the projective subspace $\mathbb{P}^{i_m-1}(x)$ of $G(i_m-1, V^{n_m})$ introduced in the proof of Theorem 6.9 equals $q(p^{-1}(x))$. Similarly, $\mathbb{P}^{n_m-i_m}(y) := p(q^{-1}(y))$ is a projective subspace of G_m for $y \in G(i_m-1, V^{n_m})$. Moreover, it is easy to see that the cone $K(x) := \underset{y \in \mathbb{P}^{i_m-1}(x)}{\cup} \mathbb{P}^{n_m-i_m}(y)$ with vertex at x, considered as a reduced subscheme of G_m, has the same Zariski tangent space at x as G_m:

$$T_xK(x) = T_xG_m. \tag{166}$$

Since $\Delta = 0$, in the notations of the proof of Theorem 6.9 we have $B^K_\Delta(x) = \pi^{-1}(x)$ and $K = \mathbf{r}$ for any $x \in G_m^*$, i.e. $E_m|_l$ is trivial for any projective line $l \subset G_m$ passing through x. This implies that $E_m|_{\mathbb{P}^{n_m-i_m}(y)}$ is trivial for any point $y \in \mathbb{P}^{i_m-1}(x)$ (see, e.g., [OSS, Ch. I, Thm. 3.2.1]), and hence $\mathbb{P}^{n_m-i_m}(y) \subset G_m^*$.

We claim that $G_m^* = G_m$. Indeed, if $G_m^* \neq G_m$, then for any $x \in (G_m \smallsetminus G_m^*)$ and any $y \in \mathbb{P}^{i_m-1}(x)$ we have $\mathbb{P}^{n_m-i_m}(y) \subset (G_m \smallsetminus G_m^*)$. Hence $K(x) \subset (G_m \smallsetminus G_m^*)$, and by (166), $T_xG_m = T_xK(x) \subset T_x(G_m \smallsetminus G_m^*)$, where we consider $(G_m \smallsetminus G_m^*)$ as a reduced subscheme of G_m. Whence $(G_m \smallsetminus G_m^*) = G_m$, contrary to the fact that G_m^* is a dense open subset of G_m.

We have shown that $G_m^* = G_m$ for sufficiently large m. As $\Delta = 0$, this means that $E_m|_l$ is trivial for any line l in G_m. By [PT, Prop. 1.4.1] this is sufficient to conclude that E_m is trivial for large enough m, and hence for all $m > 0$. ☐

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