Non-rigidity of injective indecomposable tensor modules of $\mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$

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Abstract

In this paper we focus on the tensor representations of the classical locally finite Lie algebra $\mathfrak{sp}_\infty$. The first part of the paper is a review of known results on the tensor representations of $\mathfrak{sp}_\infty$. In particular, we discuss the decomposition of a tensor power of the natural representation of $\mathfrak{sp}_\infty$ into indecomposable direct summands. In the second part we prove that not all of these indecomposable direct summands are rigid. It has been proven that the category of tensor representations of $\mathfrak{sp}_\infty$ and the category of tensor representations of another classical locally finite Lie algebra $\mathfrak{so}_\infty$ are equivalent. Therefore, our result also applies to injective indecomposable tensor modules of $\mathfrak{so}_\infty$.

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Contents

1 Introduction

The purpose of this bachelor thesis research project is to provide information about the structure of modules over the locally finite Lie algebra $\mathfrak{sp}_\infty$. Any finite-length module M over a Lie algebra $\mathfrak{g}$ has well-defined socle and radical filtrations. If these two filtrations coincide we say that the module M is *rigid*. The following questions about an arbitrary module are interesting: What are the socle and radical filtrations of M? What are the lengths of these filtrations? Is M rigid? The answers to these questions help to estabilish links between these filtrations and other filtrations of M, and therefore can provide essential information on the structure of M.
We are interested in filtrations of the tensor modules of $\mathfrak{sp}_\infty$. The results obtained in [**styrk**], [**weiss**] and [**cohen**] are going to be particularly influential in proving our main result. [**styrk**] extends some results of Weyl to the classical locally finite Lie algebras $\mathfrak{gl}_\infty$, $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$ and presents beautiful descriptions of the structure of their tensor representations. One of the most interesting results in [**styrk**] is the decomposition of tensor representations of the classical locally finite Lie algebras $\mathfrak{gl}_\infty$, $\mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$ into indecomposable direct summands. Based on V.Serganova's observations, [**weiss**] proves that some indecomposable direct summands of $\mathfrak{gl}_\infty$-tensor modules are non-rigid. In the paper [**styrk**], it is proven that the tensor modules of the classical locally finite Lie algebras $\mathfrak{gl}_\infty$ and $\mathfrak{sp}_\infty$ are injective. This beautiful argument is going to help us to show that some indecomposable direct summands of tensor representations of $\mathfrak{sp}_\infty$ are not rigid.
We started this project by studying the theory we need to understand the classical Lie algebra $\mathfrak{sp}_\infty$ and its tensor modules. Later we performed computations of their socle filtrations using the results from [**styrk**]. The main results along with the computed diagrams are presented at the end of this paper.
Throughout this paper we are going to work over the field of complex numbers $\mathbb{C}$.

2 Preliminaries

We start our work by recalling the notions we will need about semisimple modules. Let M be a module over a ring or a Lie algebra R.

Definition 2.1. *M is called **simple** if it has no proper submodules, and M is called **semisimple** if it is a direct sum of simple submodules.*

Definition 2.2. *Let M be a nonzero module. M is said to be **indecomposable** if it cannot be expressed as a direct sum of two nonzero submodules. Otherwise it is called **decomposable**.*

Theorem 2.1. *For any module M the following statements are equivalent:*

(1) M is a direct sum of simple submodules.

(2) M is the sum of all its simple submodules.

(3) Any submodule $N \subset M$ admits a direct complement in M i.e a submodule $K \subset M$ such that $M = N \oplus K$.

Proof: **??** $\implies$ **??** is obvious.
?? $\implies$ **??** Consider the set A of all submodules of M which intersect N trivially

$$A := \{L \subset M \mid L \cap N = \{0\}\}.$$

A is non-empty since $\{0\} \in A$, A is partially ordered by inclusion, and every chain of submodules in A has an upper bound (the union of all submodules in the chain is an element of A). By Zorn's

lemma, A contains a maximal element K. By assumption, $M = \Sigma S_i$ for all simple submodules $S_i \subset M$. We claim that $N \oplus K = M$. Indeed, if there is a simple submodule $S \subset M$ such that $S \not\subset N \oplus K$, then $(N \oplus K) \cap S = \{0\}$ by the simplicity of S. Hence, $(S + K) \cap N = \{0\}$, therefore $S + K$ is contained in A. On the other hand, $S + K$ properly contains K, which contradicts the maximality of K.
?? $\implies$ **??** First we show that a stronger version of (3) applies: if $N \subset M$ is a submodule, then for every submodule $L \subset N$ there exists a submodule $L' \subset N$ such that $L \oplus L' = N$. Indeed, as $L \subset M$, L has a complement L'' in M, i.e, $L \oplus L'' = M$. Consider $L' = L'' \cap N \subset L''$. Then $L' \cap L = \{0\}$ and $L' + L = N$. Therefore, $N = L' \oplus L$.
Now consider

$$B := \{L \subset M \mid L = \oplus S_i \text{ where } S_i \subset L \text{ are simple}\}.$$

The set B is non-empty and is partially ordered: $L = \oplus S_i < L' = \oplus S'_i$ if for any i, there exists i' such that $S_i = S'_{i}$. Moreover, Zorn's condition is clearly satisfied. Applying Zorn's lemma to B yields a maximal element K of B. We claim that $K = M$. Indeed, if $K \subsetneq M$, by (3) there is $N \subset M$ such that $K \oplus N = M$. If N contains a simple submodule $N' \subset N$, then $K \oplus N' \in B$, contradicting the maximality of K.
Therefore, in order to finish the proof we need to show that N contains a simple submodule. If N is simple there is nothing to prove. If N is not simple, fix a nonzero $x \in N$ such that

$$C := \{P \subset N | P \text{ is nonzero and } P \not\ni x\}$$

is non-empty. Then C is ordered by inclusion and satisfies Zorn's condition. Let Y be a maximal element of C. Then $N = X \oplus Y$ for some submodule $X \subset N$. Clearly, X is generated by x. But then X is simple as otherwise $Y \oplus X' \in C$ for any submodule $X' \subset X$, $x \notin X'$. This shows that N contains a simple submodule, and the proof is complete. ■

2.1 Loewy filtrations

Let M be an $R-$module.

Definition 2.3. *A **filtration** of M is a chain of nested submodules of M*

$$\cdots \subset M_i \subset M_{i+1} \subset \dots \tag{1}$$

*A filtration is called **finite** if there are only finitely many distinct modules in the set $\{M_i\}$. The number of distinct non-zero proper submodules of M in the chain (1) is called the **length** of the filtration.*

Definition 2.4. *A filtration is **semisimple** if the quotients M_{i+1}/M_i are semisimple whenever nonzero.*

In this research we are interested only in semisimple filtrations.

Definition 2.5. *We call a finite filtration of M a **Loewy filtration** if there is no semisimple finite filtration of M with smaller length.*

Below we are going to give important examples of Loewy filtrations.

Let R be a $\mathbb{C}$-algebra and M be an R-module.

Definition 2.6. *The **socle** of M is defined to be the largest semisimple submodule of M, in other words, $socM$ is the sum of all simple submodules of M. If there are no simple submodules of M, we set $socM = 0$.*

We define $soc^0 M := 0$, $soc^1 M := socM$, and $soc^i M := \pi_i^{-1}(soc(M/soc^{i-1}M))$, where $\pi_i : M \to M/soc^i M$, for $i \in \mathbb{Z}_{>0}$. One can see that the following holds

$$soc^i M/soc^{i-1}M = soc(M/soc^{i-1}M),$$

and it implies that the quotients $soc^i M/soc^{i-1}M$ are semisimple. Then the following semisimple filtration of M is called the **socle filtration** of M

$$0 = soc^0 M \subset soc^1 M \subset \dots .$$

The quotients are called the **socle layers** and are denoted as

$$\overline{soc^i M} = soc^i M/soc^{i-1}M.$$

Definition 2.7. *The **radical** of M is the smallest submodule N of M such that the quotient M/N is semisimple.*

One could also verify that the following is true for the radical of a module M :

$$radM = \bigcap_{\Phi:M\to W} ker\Phi,$$

where Φ denotes a homomorphism from M to an arbitrary semisimple R-module W. We define $rad^0 M := M$, $rad^{i+1}M := rad(rad^i M)$, for $i \in \mathbb{Z}_{>0}$. It follows that the quotients $rad^i M/rad^{i+1}M$ are semisimple and we obtain the following filtration of M:

$$\dots \subset rad^2 M \subset rad^1 M \subset rad^0 M = M.$$

This filtration is called the **radical filtration** of M, if for some $n \in \mathbb{Z}_{>0}$, $rad^n M = 0$. The quotients of consecutive submodules are called the **radical layers** and are denoted as

$$\overline{rad^{i+1}M} := rad^i M/rad^{i+1}M.$$

The following theorem explains the relation between the socle and radical filtrations of a given module:

Theorem 2.2. *Let M be a module over a ring or a Lie algebra R and suppose that one of the following holds for some $n \in \mathbb{Z}^+$*

- $soc^n M = M$,
- $\mathrm{rad}^n M = 0$

but neither holds for any smaller n. Then the other equation is also true and the filtrations satisfy

$$\mathrm{rad}^{n-i} M \subseteq soc^i M$$

*for all $0 \leq i \leq n$. In other words, if one of the filtrations exhausts M after a finite number of steps, then both filtrations are finite and have the same length. This length is called the **Loewy length** of M.*

Proof: Assume $soc^n M = M$. We apply induction on the length of socle filtration of M. For $i = 0$, $rad^0 M = M = soc^n M$. Suppose that for some $0 < k < n$, $rad^k M \subseteq soc^{n-k} M$. Then,

$$rad^{k+1} M = \bigcap_{\Phi:rad^k M\to W} ker\Phi \subseteq \bigcap_{\Phi:soc^{n-k} M\to W} ker\Phi$$

The map $\pi : soc^{n-k}M \to (soc^{n-k}M)/(soc^{n-k-1}M)$ is a homomorphism with a semisimple module as its codomain (the socle layers are semisimple). Therefore,

$$rad^{k+1}M \subseteq ker\pi = soc^{n-k-1}M.$$

Hence, we have proven that $rad^{n-i}M \subseteq soc^{i}M$ for all positive integers $0 \le i \le n$. In particular, $rad^n M \subseteq soc^0 M = 0$.

Now assume $rad^n M = 0$. We apply induction on the length of the radical filtration of M. It is obvious that $rad^n M = 0 = soc^0 M$. Suppose that $rad^k M \subseteq soc^{n-k}M$ for some positive integer $0 < k < n$. Consider the homomorphisms $\pi_1 : M/rad^k M \to M/soc^{n-k}M$, $\pi_2 : M \to M/rad^k M$ and $\pi : M \to M/soc^{n-k}M$. Clearly, $\pi = \pi_1 \circ \pi_2$. By definition,

$$soc^{n-k+1}M = \pi^{-1}(soc(M/soc^{n-k}M)).$$

On the other hand,

$$rad^{k-1}M \subseteq \pi_2^{-1}(rad^{k-1}M)/(rad^k M) = \pi_2^{-1}(\pi_1^{-1}(soc(M/soc^{n-k}M))) = \pi^{-1}(soc(M/soc^{n-k}M)).$$

It follows that $rad^{k-1}M \subseteq soc^{n-k+1}M$, which implies that

$$rad^i M \subseteq soc^{n-i}M$$

for all positive integers $0 \le i \le n$. Moreover, $M = rad^0 M \subseteq soc^n M$, therefore $M = soc^n M$. ∎
Under the assumption of Theorem 2.2 both the radical and socle filtrations of a module M are Loewy filtrations. Their length coincide and is called the *Loewy length* of M. If the radical and socle filtrations of M coincide, we say that M is a **rigid** module.

2.2 Tensor representations of $\mathfrak{gl}_\infty$ and $\mathfrak{sp}_\infty$

In this section we define the infinite dimensional locally finite Lie algebras $\mathfrak{gl}_\infty$ and $\mathfrak{sp}_\infty$, and introduce their tensor representations.

In order to describe the tensor representations of $\mathfrak{sp}_\infty$ first we need to understand the tensor representations of $\mathfrak{gl}_\infty$. For this reason I am going to review some results from [**styrk**].

Let V and V_* be countable dimensional vector spaces and $\langle \cdot, \cdot \rangle : V \otimes V_* \to \mathbb{C}$ be a non-degenerate pairing. The Lie algebra $\mathfrak{gl}_\infty$ is defined as the vector space $V \otimes V_*$ endowed with the Lie bracket satisfying

$$[u \otimes u^*, v \otimes v^*] = \langle v, u^* \rangle u \otimes v^* - \langle u, v^* \rangle v \otimes u^* \tag{2}$$

for $u, v \in V$ and $u^*, v^* \in V^*$. G. Mackey [**mackey**] has proven that one can find dual bases $\{\xi_i\}_{i \in J}$ of V and $\{\xi_i^*\}_{i \in J}$ of V_*, where $J = \mathbb{Z}/\{0\}$, such that we have $\langle \xi_i, \xi_j^* \rangle = \delta_{i,j}$. Given such a basis of V we can think of $\mathfrak{gl}_\infty$ as the Lie algebra with linear basis $\{E_{i,j} = \xi_i \otimes \xi_j^* | \ i, j \in J\}$. Then (2) becomes

$$[E_{i,j}, E_{k,l}] = \delta_{k,j} E_{i,l} - \delta_{i,l} E_{k,j}.$$

We call V the *natural representation* of $\mathfrak{gl}_\infty$, and V_* the conatural representation of V.

For integers $p, q > 0$ the tensor representation $V^{\otimes(p,q)}$ is defined as $V^{\otimes p} \otimes (V_*)^{\otimes q}$ and is equipped with a $\mathfrak{gl}_\infty$-module structure satisfying

$$(u \otimes u^*)(v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^*) = \Sigma_{i=1}^{p} \langle v_i, u^* \rangle v_1 \otimes \cdots \otimes v_{i-1} \otimes u \otimes v_{i+1} \otimes \cdots \otimes v_p \otimes v_1^* \otimes \cdots \otimes v_q^* -$$

$$\Sigma_{j=1}^{q} \langle u, v_j^* \rangle v_1 \otimes \cdots \otimes v_p \otimes v_1^* \otimes v_{j-1}^* \otimes u^* \otimes v_{j+1}^* \otimes \cdots \otimes v_q^*$$

for $u, v_1, \dots, v_p \in V$ and $u^*, v_1^*, \dots, v_q^* \in V_*$.

The product of symmetric groups $\mathfrak{S}_p \times \mathfrak{S}_q$ acts on $V^{\otimes(p,q)}$ by permuting the factors, and this action commutes with the action of $\mathfrak{gl}_\infty$ on $V^{\otimes(p,q)}$. For this reason we say that $V^{\otimes(p,q)}$ is a $(\mathfrak{gl}_\infty, \mathfrak{S}_p \times \mathfrak{S}_q)$-module.
For any pair of indices $I = (i, j)$ such that $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$ we consider a tensor contraction

$$\Phi_I : V^{\otimes(p,q)} \to V^{\otimes(p-1,q-1)}$$

such that $v_1 \otimes \cdots \otimes v_p \otimes v_1^* \cdots \otimes v_q^* \mapsto \langle v_i, v_j^* \rangle v_1 \otimes \cdots \otimes \hat{v_i} \otimes \cdots \otimes \hat{v_j}^* \otimes \cdots \otimes v_q^*$, where $\hat{v_i}$ and $\hat{v_j^*}$ means that these terms are left out.

We define the $(\mathfrak{gl}_\infty, \mathfrak{S}_p \times \mathfrak{S}_q)$ submodule $V^{\{p,q\}}$ of $V^{\otimes(p,q)}$ as

$$V^{\{p,q\}} := \bigcap_I ker(\Phi_I : V^{\otimes(p,q)} \to V^{\otimes(p-1,q-1)}),$$

and we set $V^{\{p,0\}} := V^{\otimes p}$ and $V^{\{0,q\}} := V^{\otimes q}$ whenever $p = 0$ or $q = 0$.
Now we are ready to define the simple finitary Lie algebra $\mathfrak{sp}_\infty$.
Let V be a countable dimensional vector space and $\Omega : V \otimes V \to \mathbb{C}$ be a non-degenerate anti-symmetric bilinear form on V. Then the Lie algebra $\mathfrak{sp}_\infty$ is defined as the Lie subalgebra of $\mathfrak{gl}_\infty$ under which the bilinear form Ω is invariant. In other words,

$$\mathfrak{sp}_\infty = \{g \in \mathfrak{gl}_\infty | \ \Omega(gu, v) + \Omega(u, gv) = 0\}$$

for all $u, v \in V$. Similarly to the case $\mathfrak{gl}_\infty$, based on an observation of Mackey we can pick a basis $\{\xi_i\}_{i\in\mathbb{Z}/\{0\}}$ of V, such that, $\Omega(\xi_i, \xi_j) = sgn(i)\delta_{i+j,0}$.

We can also view $\mathfrak{sp}_\infty$ as the Lie algebra with linear basis $\{sgn(j)E_{i,j} - sgn(i)E_{-j,-i}\}$, where $E_{i,j} = \{\xi_i \otimes \xi_j | i, j \in J\}$. In this case the dual basis is $\xi_i^* = sgn(i)\xi_i$, therefore $\mathfrak{sp}_\infty = Sym^2 V$.

We call V, considered as an $\mathfrak{sp}_\infty$-module by restriction, the *natural representation* of $\mathfrak{sp}_\infty$. Since $V \cong V_*$ as $\mathfrak{sp}_\infty$-modules, the $\mathfrak{sp}_\infty$- action on the $\mathfrak{gl}_\infty$-module $V^{\otimes(p,q)}$ coincides with the $\mathfrak{sp}_\infty$-action on the $\mathfrak{gl}_\infty$-module $V^{\otimes(p+q)}$. For this reason it suffices to study the tensor representations $V^{\otimes d}$. In other words, we are going to study the decomposition of $V^{\otimes d}$ into a direct sum of indecomposable submodules. As we will see later, the irreducible subquotients of these indecomposable $\mathfrak{sp}_\infty$-modules are highest weight modules.

Now we consider a tensor contraction for any pair of indices $I = (i, j)$ such that $i, j \in \{1, \dots, d\}$

$$\Phi_{\langle I \rangle} : V^{\otimes d} \to V^{\otimes(d-2)},$$

$$v_1 \otimes \cdots \otimes v_d \mapsto \Omega(v_i, v_j)v_1 \otimes \cdots \otimes \hat{v_i} \otimes \cdots \otimes \hat{v_j} \otimes \cdots \otimes v_d.$$

Furthermore, we define $V^{\langle 0 \rangle} := \mathbb{C}$ and $V^{\langle 1 \rangle} := V$ and

$$V^{\langle d \rangle} := \bigcap_I ker(\Phi_{\langle I \rangle} : V^{\otimes d} \to V^{\otimes(d-2)})$$

for all positive integers $d \geq 2$.

2.3 Young tableaux and Schur functors

Definition 2.8. *Given a positive integer d, we define a* ***partition*** *λ of d to be a sequence of positive integers*

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \text{ such that } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \text{ and } d = |\lambda| := \Sigma_1^k \lambda_i.$$

We call k the ***length*** *of the partition λ.*

To each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer d, one associates a **Young diagram** which has k rows and λ_i boxes in the i th row, the rows being lined up on the left.

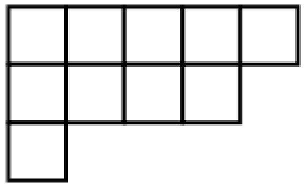


Figure 1 Young diagram of the partition $(5, 4, 1)$

The *conjugate* of a partition λ is the partition λ^T associated to the Young diagram obtained by transposition of the Young diagram of λ. The importance of partitions in representation theory is that they parametrize the irreducible representations of $\mathfrak{S}_d$.
A **Young tableau** is a Young diagram in which boxes are filled with the numbers from 1 to $|\lambda|$. Given a Young tableau of λ, where $|\lambda| = d$, we can define two subgroups of the symmetric group

$$P = P_\lambda := \{g \in \mathfrak{S}_d | \text{ g preserves each row}\},$$

$$Q = Q_\lambda := \{h \in \mathfrak{S}_d | \ h \text{ preserves each column}\}.$$

Consider two elements in the group algebra $\mathbb{C}\mathfrak{S}_d$:

$$a_\lambda = \Sigma_{g \in P} e_g, \ b_\lambda = \Sigma_{h \in Q} sgn(h) \cdot e_h.$$

If V is a vector space over $\mathbb{C}$, and $\mathfrak{S}_d$ acts on $V^{\otimes d}$ by permuting the factors, the image of $a_\lambda : V^{\otimes d} \to V^{\otimes d}$ is the following subspace of $V^{\otimes d}$:

$$Im(a_\lambda) = Sym^{\lambda_1} V \otimes Sym^{\lambda_2} V \otimes \cdots \otimes Sym^{\lambda_k} V$$

Similarly, the image of b_λ is the following subspace of $V^{\otimes d}$:

$$Im(b_\lambda) = \bigwedge^{\mu_1} V \otimes \bigwedge^{\mu_2} V \otimes \cdots \otimes \bigwedge^{\mu_n} V$$

where μ is the partition conjugate to λ. We define the Young symmetrizer c_λ by setting:

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}\mathfrak{S}_d.$$

If we define

$$H_\lambda := \mathbb{C}[\mathfrak{S}_d] c_\lambda,$$

then H_λ is an irreducible representation of $\mathfrak{S}_d$. It can be shown that every irreducible representation of $\mathfrak{S}_d$ is isomorphic to H_λ for some λ with $|\lambda| = d$.
We can also see that c_λ projects $V^{\otimes d}$ onto

$$\mathbb{S}_\lambda V := im(c_\lambda : V^{\otimes d} \to V^{\otimes d})$$

which is an irreducible representation of $\mathfrak{gl}_n$. Here $\mathbb{S}_\lambda$ is called the Schur functor corresponding to a partition λ. More on this can be found in [**fulton**].

2.4 Highest weight modules

In order to give a concrete construction of tensor representations of $\mathfrak{sp}_\infty$, we need to review some background material.

Definition 2.9. *Let $\mathfrak{g}$ be a Lie algebra. Then $x \in \mathfrak{g}$ is said to be semisimple if adx is semisimple (i.e the adjoint map of x is diagonalizable).*

Definition 2.10. *A Lie subalgebra of $\mathfrak{g}$ is said to be* ***toral*** *if it entirely consists of semisimple elements.*

Let M be a $\mathfrak{g}$-module, $\mathfrak{t}$ be a toral subalgebra of $\mathfrak{g}$, and $\mathfrak{t}^*$ be its dual. For each $\alpha \in \mathfrak{t}^*$ let:

$$M^\alpha := \{v \in M | tv = \alpha(t)v \ \ \forall t \in \mathfrak{t}\}.$$

The space M^α is the space of *weight vectors of weight* α. If $M^\alpha \neq 0$, we say that α is a *weight* of M. It can be verified that weight vectors for distinct weights in M are linearly independent, therefore $\Sigma_{\alpha\in\mathfrak{t}^*} M^\alpha$ is a direct sum and $\Sigma_{\alpha\in\mathfrak{t}^*} M^\alpha \subseteq M$. We call M a $\mathfrak{t}$-*weight* $\mathfrak{g}$-*module* if $\Sigma_{\alpha\in\mathfrak{t}^*} M^\alpha = M$. In what follows $\mathfrak{g} = \mathfrak{gl}_\infty, \mathfrak{sp}_\infty$.

Definition 2.11. *A* ***splitting Cartan subalgebra*** *$\mathfrak{h}$ of $\mathfrak{g}$ is a maximal toral subalgebra such that $\mathfrak{g}$ is an $\mathfrak{h}$-weight module.*

If $\mathfrak{h}$ is a splitting Cartan algebra, then $\mathfrak{g}$ is an $\mathfrak{h}$-weight module:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\mathfrak{h}^*/\{0\}} \mathfrak{g}^\alpha.$$

This decomposition of $\mathfrak{g}$ is called the **root decomposition** of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. The nonzero spaces $\mathfrak{g}^\alpha$ for $\alpha \neq 0$ are called the **root spaces** of $\mathfrak{g}$ and $\Delta := \{\alpha \in \mathfrak{h}^*/\{0\} \mid \mathfrak{g}^\alpha \neq 0\}$ is called the **set of roots** of $\mathfrak{g}^\alpha$. The decomposition of the set Δ into two disjoint sets Δ^+ and Δ^- is called a **triangular decomposition** of Δ if and only if $\alpha, \beta \in \Delta^+$, $\alpha + \beta \in \Delta$ implies $\alpha + \beta \in \Delta^+$, and $\alpha \in \Delta^\pm$ implies $-\alpha \in \Delta^\mp$.

Definition 2.12. *A Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is called a* ***Borel subalgebra*** *if there is a triangular decomposition such that $\mathfrak{b} = (\bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^\alpha) \rtimes \mathfrak{h}$.*

Definition 2.13. *Let M be a $\mathfrak{g}$-module and $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra. If $v \in M$ and v generates a one-dimensional $\mathfrak{b}$-submodule, then we call v a* ***$\mathfrak{b}$-singular vector****.*

It follows that any such vector satisfies $(\bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^\alpha)v = 0$ and for all, $x \in \mathfrak{h}$, $xv = \chi(x)v$ for some fixed *weight* $\chi \in \mathfrak{h}^*$.

Definition 2.14. *A module M is called a* ***highest weight module*** *if it is generated by some $\mathfrak{b}$-singular vector v of weight $\chi \in \mathfrak{h}^*$. The weight χ is called the* ***highest weight*** *of M.*

It is a straightforward fact that the highest weight vector v of M is determined uniquely up to scalar multiplication. Moreover, a highest weight module is always a weight module.
When $\mathfrak{g} = \mathfrak{gl}_\infty$, we have the following root decomposition: $\mathfrak{gl}_\infty = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta} \mathbb{C}X_\alpha)$, where

$$\mathfrak{h}_{\mathfrak{gl}} = \bigoplus_{i\in J} \mathbb{C}E_{i,i}, \qquad \Delta_{\mathfrak{gl}} = \{\epsilon_i - \epsilon_j | i,j \in J, i \neq j\}, \qquad X^{\mathfrak{gl}}_{\epsilon_i-\epsilon_j} = E_{i,j}$$

and $\epsilon_i \in \mathfrak{h}^*$ is determined by $\epsilon_i(E_{j,j}) = \delta_{i,j}$. Here J is a countable set. If we put $J = \mathbb{Z}/\{0\}$, we can define the following triangular decomposition

$$\Delta^+ = \{\epsilon_i - \epsilon_j | 0 < i < j\} \cup \{\epsilon_i - \epsilon_j | i < j < 0\} \cup \{\epsilon_i - \epsilon_j | j < i < 0\}, \ \Delta^- = -\Delta^+. \tag{3}$$

Definition 2.15. *For any partitions λ and μ such that $|\lambda| = p$ and $|\mu| = q$, we define the $\mathfrak{gl}_\infty$-submodule $\Gamma_{\lambda,\mu}$ of $V^{\{p,q\}}$ by setting*

$$\Gamma_{\lambda,\mu} := V^{\{p.q\}} \cap (\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V_*).$$

With respect to the Borel subalgebra $\mathfrak{b}$ determined by (3), $\Gamma_{\lambda,\mu}$ is an irreducible highest weight $\mathfrak{gl}_\infty$-module. It was shown in [**styrk**] that as an $\mathfrak{sp}_\infty$-module, $\Gamma_{\lambda,0}$ is not a highest weight module, but only an indecomposable module.
Now let us explore $\mathfrak{sp}_\infty$, the simple locally finite Lie algebra we are interested in. It has the following root decomposition:

$$\mathfrak{sp}_\infty = \mathfrak{h}_{\mathfrak{sp}} \oplus (\bigoplus_{\lambda\in\Delta_{\mathfrak{sp}}} \mathbb{C}X_\lambda^{\mathfrak{sp}}),$$

where

$$\mathfrak{h}_{\mathfrak{sp}} = \bigoplus_{i\in\mathbb{Z}_{>0}} (E_{i,i} - E_{-i,-i}), \qquad \Delta_{\mathfrak{sp}} = \{\pm(\epsilon_i+\epsilon_j)|i,j\in\mathbb{Z}_{>0}\} \cup \{(\epsilon_i-\epsilon_j)|i,j\in\mathbb{Z}_{>0} \text{ and } i\neq j\}$$

$$X^{\mathfrak{sp}}_{\epsilon_i+\epsilon_j} = E_{i,-j}+E_{j,-i}, \qquad X^{\mathfrak{sp}}_{-\epsilon_i-\epsilon_j} = E_{i,j}+E_{-j,i}, \qquad X^{\mathfrak{sp}}_{\epsilon_i-\epsilon_j} = E_{i,j}+E_{-j,-i}.$$

Here $\{E_{i,j}\}$ is the basis of $\mathfrak{gl}_\infty$, and $\epsilon_i \in \mathfrak{h}^*_{\mathfrak{sp}}$ is defined by the equality $\epsilon_i(E_{j,j} - E_{-j,-j}) = \delta_{i,j}$ for $i,j\in\mathbb{Z}_{>0}$.

In this case we fix the following Borel subalgebra of $\mathfrak{sp}_\infty$:

$$\mathfrak{b} = (\bigoplus_{\lambda\in\Delta^+_{\mathfrak{sp}}} \mathfrak{sp}^\lambda_\infty)\oplus\mathfrak{h}_{\mathfrak{sp}} \text{ where } \Delta^+_{\mathfrak{sp}} = \{(\varepsilon_j+\varepsilon_i)|i,j\in\mathbb{Z}_{>0} \text{ and } i\geq j\}\cup\{(\varepsilon_i-\varepsilon_j)|i,j\in\mathbb{Z}_{>0} \text{ and } i<j\}.$$

For any partition λ such that $|\lambda| = d$, we define the $\mathfrak{sp}_\infty$-submodule $\Gamma_{\langle\lambda\rangle}$ of $V^{\otimes d}$ as

$$\Gamma_{\langle\lambda\rangle} := V^{\langle d\rangle} \cap \mathbb{S}_\lambda V.$$

It turns out that $\Gamma_{\langle\lambda\rangle}$ is an irreducible highest weight $\mathfrak{sp}_\infty$-module. We proceed to find out more about this and some other known results from [**styrk**] about the structure of $\mathfrak{sp}_\infty$.

3 Results about tensor representations of $\mathfrak{sp}_\infty$

The following theorem from [**styrk**] states in particular that $\Gamma_{\langle\lambda\rangle}$ is an irreducible highest weight $\mathfrak{sp}_\infty$-module.

Theorem 3.1. *For any non-negative integer d there is an isomorphism of* $(\mathfrak{sp}_\infty, \mathfrak{S}_p)$*-modules*

$$V^{\langle d\rangle} \cong \bigoplus_{|\lambda|=p} \Gamma_{\langle\lambda\rangle} \otimes H_\lambda$$

For every partition λ, the $\mathfrak{sp}_\infty$-module $\Gamma_{\langle\lambda\rangle}$ is an irreducible highest weight module with highest weight $\omega = \Sigma_{i\in\mathbb{N}}\lambda_i\epsilon_i$.

The following fact will be very important for the main result of this project:

Lemma 3.2. *Any automophism ϕ of the $\mathfrak{sp}_\infty$-module $\Gamma_{\langle\lambda\rangle}$ is of the form $\phi(v) = cv$, where $c\in\mathbb{C}\setminus\{0\}$.*

Proof: Since $\Gamma_{\langle\lambda\rangle}$ is a highest weight module, it is generated by some $\mathfrak{b}$-singular vector $v\in\Gamma_\lambda$ which is unique up to a scalar multiplication. Therefore any automorphism ϕ sends v to cv where c is a nonzero complex number. Since v generates $\Gamma_{\langle\lambda\rangle}$, ϕ is already determined uniquely by c. Hence, $\phi(v) = cv$ for every $v\in\Gamma_{\langle\lambda\rangle}$.■
Next we restate the theorem from [**styrk**] that describes the socle filtrations of the tensor representations of $\mathfrak{sp}_\infty$.

Theorem 3.3. *For any non-negative integer d the $\mathfrak{gl}_\infty$-module $V^{\otimes d}$, regarded as an $\mathfrak{sp}_\infty$-module, has Loewy length $[\frac{d}{2}]+1$, and*

$$soc^r V^{\otimes d} = \bigcap_{I_1,I_2,\ldots,I_r} ker(\Phi_{\langle I_1,I_2,\ldots I_r\rangle} : V^{\otimes p} \to V^{\otimes(d-2r)}), \qquad r = 1,2,\ldots[\frac{d}{2}],$$

where $\Phi_{\langle I_1,I_2,\ldots I_r\rangle} := \Phi_{I_1} \circ \Phi_{I_2} \circ \cdots \circ \Phi_{I_r}$.

This result can be used to obtain the following theorem which describes the socle filtration of each indecomposable direct summand Γ_λ.

Theorem 3.4. *For any partition λ, the $\mathfrak{sp}_\infty$-module Γ_λ is indecomposable and*

$$\overline{soc}^{r+1}\Gamma_\lambda \cong \bigoplus_{|\mu|=|\lambda|-|2\gamma|} (\Sigma_{|\gamma|=r} N^\lambda_{\mu,(2\gamma)^T})\Gamma_{\langle\mu\rangle}, \qquad r = 1,\ldots,[\frac{d}{2}].$$

The coefficients $N^\lambda_{\mu,(2\gamma)^T}$ are called *Littlewood-Richardson coefficients* and we are going to learn more about them in the next section.

3.1 Littlewood-Richardson coefficients

The Littlewood-Richardson coefficients are very important when studying the socle filtrations of indecomposable direct summands of $V^{\otimes d}$.

The **Littlewood-Richardson coefficients** $N^\lambda_{\mu,\gamma}$ are non-negative integers which depend on three partitions λ,μ,γ and are determined by relation

$$S_\mu S_\gamma = \Sigma_\lambda N^\lambda_{\mu,\gamma} S_\lambda,$$

where S_μ is the Schur symmetric polynomial corresponding to the partition μ. For more details on this relation see [**fulton**] pages 455-456.

Let λ and μ be partitions such that $\lambda_i \geq \mu_i$ for each i. Consider a partition γ such that $|\gamma| = |\lambda| - |\mu|$.

We call the set-theoretic difference of the Young diagrams of λ and μ a *skew diagram* of shape λ/μ. A *semistandard skew* tableau of shape λ/μ and weight γ is a skew diagram filled with numbers that satisfies the following conditions:

- each positive integer i less than or equal to $|\gamma|$ appears exactly γ_i times,
- the numbers along each column are strictly increasing,
- the numbers along each row are weakly increasing.

If in addition the following condition holds

- after removal of 1 or more leftmost columns we again obtain a semistandard skew tableau,

we call such a semistandard skew tableau a **Littlewood-Richardson tableau**.

Theorem 3.5. *(Littlewood-Richardson Rule) For partitions λ,μ,γ, the Littlewood-Richardson coefficient $N^\lambda_{\mu,\gamma}$ is equal to the number of Littlewood-Richardson tableaux of shape λ/μ and of weight γ.*

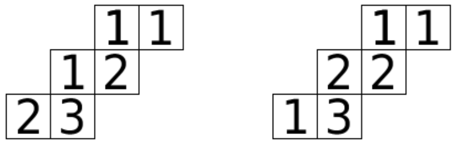


Figure 2 There are only 2 Littlewood-Richardson tableaux of shape $(4,3,2)/(2,1)$ and of weight $(3,2,1)$ therefore $N^{(4,3,2)}_{(2,1),(3,2,1)} = 2$

More information on this theorem can be found in [**fulton**]. The following observation is very easy to verify and is going to be very helpful later.
If λ is a partition of length 1, i.e $\lambda = (d)$ where $d \in \mathbb{Z}^+$, then $N^{\lambda}_{\mu,\gamma} \neq 0$ if and only if γ is of length 1 i.e $\gamma = (d')$ for some $d' \in \mathbb{Z}^+$ such that $d' < d$.

4 Main results

4.1 Non-rigidity of the tensor representations of $\mathfrak{sp}_\infty$ and their indecomposable direct summands

A first observation is that any tensor representation $V^{\otimes d}$ of $\mathfrak{sp}_\infty$ is non-rigid. Indeed, note that the Loewy length of the indecomposable direct summand of $V^{\otimes d}$ corresponding to the partition (d) of d is 1. On the other hand, Theorem 3.4 implies that the remaining indecomposable direct summands are of Loewy length at least 2. Hence, the socle filtration of an arbitrary tensor module $V^{\otimes d}$ does not coincide with its radical filtration. The following example illustrates this argument:

$$V^{\otimes 2} \sim \begin{array}{|c|}\hline \\ \hline \Gamma_{\langle(2)\rangle} \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \Gamma_{\langle(0)\rangle} \\ \hline \Gamma_{\langle(1,1)\rangle} \\ \hline \end{array}.$$

The socle filtration of $V^{\otimes 2}$ is

$$0 \subset soc(V^{\otimes 2}) = \Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle} \subset V^{\otimes 2},$$

while

$$rad(V^{\otimes 2}) = \Gamma_{\langle(1,1)\rangle}.$$

We are interested in answering the following more challenging question: Are all indecomposable direct summands of an arbitrary tensor module $V^{\otimes d}$ rigid? We give the answer to this question at the end of this section. We begin with some preliminaries.

Lemma 4.1. *If all layers except the top layer of a socle filtration of a module M are simple, then M is rigid.*

Proof: We apply induction on the length of M. Suppose the length of the socle filtration of M equals to n, and the length of its top layer equals $k \geq 1$. Then $radM \subseteq soc^{n-1}M$, moreover $radM = soc^{n-1}M$ as otherwise the length of the radical filtration will be at most $n-1$, which contradicts Theorem 2.2. To finish the proof one applies induction to the module $radM = soc^{n-1}M$, whose length equals $n-k < n$ and which satisfies the assumption of the lemma.■
For the proof of our main result we need the following auxiliary fact.

Proposition 4.2. *Let λ be a partition of d and $l := [\frac{d}{2}]$. Assume that the following conditions hold for the socle filtration of the $\mathfrak{sp}_\infty$-module Γ_λ:*

- $\overline{soc}^{l+1}\Gamma_\lambda$ *is nonzero and simple,*

- *the decomposition of $\overline{soc}^l\Gamma_\lambda$ into simple submodules contains at least one submodule with multiplicity at least* 2.

Then Γ_λ is non-rigid.

Proof: We assume, on the contrary, that Γ_λ is rigid. It follows that the quotient $M := \Gamma_\lambda/soc^{l-1}(\Gamma_\lambda)$ is rigid with Loewy length 2. Let us denote the simple top layer of M by $\Gamma_{\langle\lambda''\rangle}$. By assumption the socle of M contains some simple submodule with multiplicity $n \geq 2$; let us denote this submodule by $\Gamma_{\langle\lambda'\rangle}$. If we take the quotient of M by the complement of $n\Gamma_{\langle\lambda'\rangle}$ in $socM$, we obtain the quotient module

$$X \sim \begin{array}{|c|}\hline \Gamma_{\langle\lambda''\rangle} \\ \hline n\Gamma_{\langle\lambda'\rangle} \\ \hline \end{array}.$$

By assumption X is also rigid. In order to prove that this assumption is wrong, it is sufficient to show that X is decomposable.
Set $Y = \Gamma_{\lambda'}$. We claim that there exists an embedding of X into $nY := Y \oplus Y \oplus \cdots \oplus Y$. We can construct this embedding as follows. Observe that $socX = soc(nY) = n\Gamma_{\langle\lambda'\rangle}$. We start by fixing an isomorphism $j' : socX \to nY$. Recall the beautiful argument from **[cohen]** that the tensor representations $V^{\otimes d}$ of $\mathfrak{sp}_\infty$ and their indecomposable direct summands are injective. This fact allows us to extend j' to all of X: there exists an injective homomorphism $j : X \to nY$ which induces the isomorphism j' between $socX$ and $soc(nY)$. Now using this map we construct the following map of quotients:

$$j'' : X/socX \to nY/soc(nY) \cong n(Y/soc(Y)).$$

Furthermore, Theorem 3.4 implies that

$$Y \sim \begin{array}{|c|}\hline \Gamma_{\langle 1\rangle} \\ \hline \Gamma_{\langle\lambda'\rangle} \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|}\hline \Gamma_{\langle 0\rangle} \\ \hline \Gamma_{\langle\lambda'\rangle} \\ \hline \end{array}$$

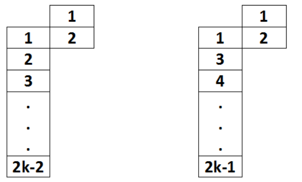
depending on the parity of d (also see the proof of Theorem 4.3). Thus, $Y/soc(Y)$ is simple. Let $Y/soc(Y) \cong A$ (A is either $\Gamma_{\langle 1\rangle}$ or $\Gamma_{\langle 0\rangle}$).
Represent $n(Y/socY)$ as $(A \oplus \cdots \oplus A)$. Then by Lemma 3.2 imj'' has the form $\{c_1 v \oplus c_2 \oplus \cdots \oplus c_n v\}$ where v runs over A and $c_1, c_2, \ldots, c_n$ are fixed complex numbers. If $\widetilde{v}$ is a preimage of v in Y, then the submodule $\widetilde{Y}$ of nY generated by $c_1\widetilde{v} \oplus c_2\widetilde{v} \oplus \cdots \oplus c_n\widetilde{v}$ is isomorphic to Y. Since $imj = \widetilde{Y} + socY$, we see that $X \cong imj$ is decomposable as $\widetilde{Y} \oplus (socY)'$ where $(socY)'$ is a complement to $\widetilde{Y} \cap soc(nY)$ in $socY$. The propostion is proved.■
Now we are ready to answer the main question of our research project.

Theorem 4.3. *For $d \geq 6$, the tensor module $V^{\otimes d}$ contains at least one non-rigid indecomposable direct summand.*

Proof: To prove this theorem, it is sufficient to give an explicit example of a non-rigid indecomposable direct summand that occurs in decomposition of $V^{\otimes d}$ for every $d \geq 6$. Consider the partition $\lambda = (2, 2, 1, \ldots, 1)$ of d for every $d \geq 6$. Our claim is that Γ_λ satisfies the conditions of Proposition 4.2, hence Γ_λ is non-rigid. We will see below by explicit computation that the top layer of the socle filtration of Γ_λ is always simple.
By theorem 3.4 the Loewy length l of $V^{\otimes d}$ is $[\lfloor\frac{d}{2}\rfloor + 1]$. We will consider seperately the cases when d is odd and when d is even.
Case 1 Let $d = 2k + 1 \geq 7$. Then $l = k + 1$ and

$$\overline{soc}^{k+1}\Gamma_\lambda \cong \bigoplus_{|\mu|=|\lambda|-|2\gamma|} (\Sigma_{|\gamma|=k} N^{\lambda}_{\mu,(2\gamma)^T})\Gamma_{\langle\mu\rangle} = (\Sigma_{|\gamma|=k} N^{\lambda}_{(1),(2\gamma)^T})\Gamma_{\langle(1)\rangle}.$$

The second equality holds since $|\mu| = |\lambda| - |2\gamma| = 1$.
Next, we observe that the only Littlewood-Richardson tableaux of shape $(2, 2, 1, \ldots, 1)/(1)$ are



The tableau on the left has weight $(2,2,1,\ldots,1)$ with $|(2,2,1,\ldots,1)| = 2k \geq 6$. The partition $\gamma = (k-1,1)$ satisfies the conditions $(2\gamma)^T = (2,2,1,\ldots,1)$, $|\gamma| = k$. However, for the tableau on the right there is no partition γ satisfying $(2\gamma)^T = (2,1,\ldots,1)$, $|\gamma| = k$ as the number of boxes in the first column of the Young diagram corresponding to the partition $(2,1,\ldots,1)$ is odd (being equal to $2k-1$).
Therefore, $\Sigma_{|\gamma|=k} N^{\lambda}_{(1),(2\gamma)^T} = N^{(2,2,1,\ldots,1)}_{(1),(2,2,1,\ldots,1)}$ and

$$\overline{soc}^{k+1}\Gamma_\lambda \cong (\Sigma_{|\gamma|=k} N^{\lambda}_{(1),(2\gamma)^T})\Gamma_{\langle\mu\rangle}$$

$$= N^{(2,2,1,\ldots,1)}_{(1),(2,2,1\ldots,1)}\Gamma_{\langle(1)\rangle} = \Gamma_{\langle(1)\rangle}.$$

This shows that the top layer of the socle filtration of Γ_λ is a simple module.
We now show that the second top layer of Γ_λ contains a submodule with multiplicity $n \geq 2$. According to Theorem 3.4,

$$\overline{soc}^{k}\Gamma_\lambda \cong \bigoplus_{|\mu|=|\lambda|-|2\gamma|} (\Sigma_{|\gamma|=k-1} N^{\lambda}_{\mu,(2\gamma)^T})\Gamma_{\langle\mu\rangle}.$$

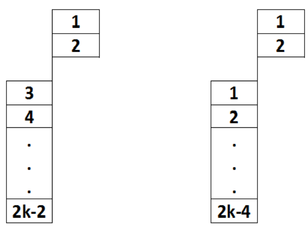
It follows that $|\mu| = |\lambda| - |2\gamma| = 3$. Therefore,

$$\overline{soc}^{k}\Gamma_\lambda \cong (\Sigma_{|\gamma|=k-1} N^{\lambda}_{(2,1),(2\gamma)^T})\Gamma_{\langle(2,1)\rangle} \oplus (\Sigma_{|\gamma|=k-1} N^{\lambda}_{(1,1,1),(2\gamma)^T})\Gamma_{\langle(1,1,1)\rangle}.$$

Consider the partition $\mu = (1,1,1)$. Explicit computations show that

$$\Sigma_{|\gamma|=k-1} N^{\lambda}_{(1,1,1),(2\gamma)^T} = 2.$$

Indeed, there are two Littlewood-Richardson tableaux of shape $(2,2,1,\ldots,1)/(1,1,1)$



(here we use that $d > 5$). The tableau on the left has weight $(1,1,\ldots,1) = (2\gamma)^T$ for $\gamma = (k-1)$. For the right tableau the corresponding partition is $\gamma = (k-2,1)$.

Therefore, in case d is odd, Proposition 4.2 applies to $\Gamma_{(2,2,1,\ldots,1)}$ and shows that $\Gamma_{(2,2,1,\ldots,1)}$ is non-rigid.
Case 2 If $d = 2k \geq 6$, then $l = k+1$. From Theorem 3.4 we obtain

$$\overline{soc}^{k+1}\Gamma_\lambda \cong \bigoplus_{|\mu|=|\lambda|-|2\gamma|} (\Sigma_{|\gamma|=k} N^{\lambda}_{\mu,(2\gamma)^T})\Gamma_{\langle\mu\rangle}.$$

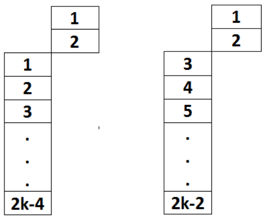
Then we have $|\mu| = |\lambda| - |2\gamma| = 0$, which makes this case easier. Therefore, the top layer of the socle filtration of $\Gamma_{(2,2,1,\ldots,1)}$ is $\Gamma_{\langle(0)\rangle}$. In order to prove that the layer second from the top contains a submodule with multiplicity $n \geq 2$, we observe that

$$\overline{soc}^k \Gamma_\lambda \cong \bigoplus_{|\mu|=|\lambda|-|2\gamma|} (\Sigma_{|\gamma|=k-1} N^{\lambda}_{\mu,(2\gamma)^T}) \Gamma_{\langle\mu\rangle}$$

implies $|\mu| = |\lambda| - |(2\gamma)^T| = 2$. Hence

$$\overline{soc}^k \Gamma_\lambda \cong (\Sigma_{|\gamma|=k-1} N^{\lambda}_{(2),(2\gamma)^T}) \Gamma_{\langle(2)\rangle} \oplus (\Sigma_{|\gamma|=k-1} N^{\lambda}_{(1,1),(2\gamma)^T}) \Gamma_{\langle(1,1)\rangle}.$$

Consider the partition $(1,1)$. There are two Littlewood-Richardson tableaux of shape $(2,2,1,\ldots,1)/(1,1)$ whose weights γ satisfy $|(2\gamma)^T| = 2k-2$



(here we use that $2k-4 \geq 2$). Therefore, $\Sigma_{|\gamma|=k-1} N^{\lambda}_{(1,1),(2\gamma)^T} = 2$, which implies that, for even d, $\Gamma_{\langle(2,2,1,\ldots,1)\rangle}$ is also non-rigid by Proposition 4.2.
Since $\Gamma_{\langle(2,2,1,\ldots,1)\rangle}$ is non-rigid for $d \geq 6$, we have proven that for any $d \geq 6$ the tensor representation $V^{\otimes d}$ of $\mathfrak{sp}_\infty$ contains at least one non-rigid indecomposable direct summand. ■

5 Conclusion and open questions

We have proven that all tensor representations $V^{\otimes d}$ of the simple finitary Lie algebra $\mathfrak{sp}_\infty$ contain at least one non-rigid indecomposable direct summand for $d \geq 6$. Corollary 6.11 from [**cohen**] states that the category of tensor representations of $\mathfrak{sp}_\infty$ is equivalent to the category of tensor representations of $\mathfrak{so}_\infty$. Therefore, our results also apply to the tensor represenations of $\mathfrak{so}_\infty$.

Along with Phillip Weiss' bachelor thesis, "Non-rigidity of tensor representations of $\mathfrak{gl}_\infty$" we have answered the question whether all indecomposable injective tensor modules are rigid for the simple finitary Lie algebras $\mathfrak{sl}_\infty$, $\mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$. However, we have not shown how many non-rigid indecomposable direct summands are contained in the decomposition of $V^{\otimes d}$ for $\mathfrak{sp}_\infty$. This is closely related to the general problem of computing the radical filtrations of the indecomposable direct summands of the $\mathfrak{sp}_\infty$-modules $V^{\otimes d}$ for $d \geq 6$, which could be an interesting avenue for further research.

6 Appendix

In this appendix one can find the results of our computations. The diagrams below represent the socle filtrations of indecomposable direct summands of $V^{\otimes d}$ for $d \leq 10$. More precisely, each tower in these diagrams represents an indecomposable direct summand of the corresponding tensor representation $V^{\otimes d}$. The rows represent the socle layers of the indecomposable submodule, with the bottom row representing the socle itself. The computations were performed using a pre-implemented code in MatLab. The link [**url**] to the code developer's website can be found in the references.

$$V^{\otimes 2} \sim \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(2)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \Gamma_{\langle(0)\rangle} \\ \hline \Gamma_{\langle(1,1)\rangle} \\ \hline\end{array}$$

$$V^{\otimes 3} \sim \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(3)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \Gamma_{\langle(1)\rangle} \\ \hline \Gamma_{\langle(2,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \Gamma_{\langle(1)\rangle} \\ \hline \Gamma_{\langle(1,1,1)\rangle} \\ \hline\end{array}$$

$$V^{\otimes 4} \sim \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \Gamma_{\langle(4)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(2)\rangle} \\ \hline \Gamma_{\langle(3,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \Gamma_{\langle(0)\rangle} \\ \hline \Gamma_{\langle(1,1)\rangle} \\ \hline \Gamma_{\langle(2,2)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle} \\ \hline \Gamma_{\langle(2,1,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \Gamma_{\langle(0)\rangle} \\ \hline \Gamma_{\langle(1,1)\rangle} \\ \hline \Gamma_{\langle(1,1,1,1)\rangle} \\ \hline\end{array}$$

$$V^{\otimes 5} \sim \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \Gamma_{\langle(5)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(3)\rangle} \\ \hline \Gamma_{\langle(4,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \Gamma_{\langle(1)\rangle} \\ \hline \Gamma_{\langle(2,1)\rangle} \\ \hline \Gamma_{\langle(3,2)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(2,1)\rangle} \oplus \Gamma_{\langle(3)\rangle} \\ \hline \Gamma_{\langle(3,1,1)\rangle} \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline \Gamma_{\langle(1)\rangle} \\ \hline \Gamma_{\langle(2,1)\rangle} \oplus \Gamma_{\langle(1,1,1)\rangle} \\ \hline \Gamma_{\langle(2,2,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \Gamma_{\langle(1)\rangle} \\ \hline \Gamma_{\langle(1,1,1)\rangle} \\ \hline \Gamma_{\langle(1,1,1,1,1)\rangle} \\ \hline\end{array}$$

$$V^{\otimes 6} \sim \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \Gamma_{\langle(6)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \Gamma_{\langle(4)\rangle} \\ \hline \Gamma_{\langle(5,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(2)\rangle} \\ \hline \Gamma_{\langle(3,1)\rangle} \\ \hline \Gamma_{\langle(4,2)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \Gamma_{\langle(4)\rangle} \oplus \Gamma_{\langle(3,1)\rangle} \\ \hline \Gamma_{\langle(4,1,1)\rangle} \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(2)\rangle} \\ \hline \Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle} \\ \hline \Gamma_{\langle(3,1,1,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \Gamma_{\langle(0)\rangle} \\ \hline \Gamma_{\langle(1,1)\rangle} \\ \hline \Gamma_{\langle(2,2)\rangle} \\ \hline \Gamma_{\langle(3,3)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(1,1)\rangle} \oplus \Gamma_{\langle(2)\rangle} \\ \hline \Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle} \\ \hline \Gamma_{\langle(3,2,1)\rangle} \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(2)\rangle} \\ \hline \Gamma_{\langle(2,1,1)\rangle} \\ \hline \Gamma_{\langle(2,2,2)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \Gamma_{\langle(0)\rangle} \\ \hline 2\Gamma_{\langle(1,1)\rangle} \\ \hline \Gamma_{\langle(2,2)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle} \\ \hline \Gamma_{\langle(2,2,1,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle} \\ \hline \Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle} \\ \hline \Gamma_{\langle(2,1,1,1,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \Gamma_{\langle(0)\rangle} \\ \hline \Gamma_{\langle(1,1)\rangle} \\ \hline \Gamma_{\langle(1,1,1,1)\rangle} \\ \hline \Gamma_{\langle(1,1,1,1,1,1)\rangle} \\ \hline\end{array}$$

$$V^{\otimes 7} \sim \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \Gamma_{\langle(7)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \Gamma_{\langle(5)\rangle} \\ \hline \Gamma_{\langle(6,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \Gamma_{\langle(5)\rangle} \oplus \Gamma_{\langle(4,1)\rangle} \\ \hline \Gamma_{\langle(5,1,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(3)\rangle} \\ \hline \Gamma_{\langle(4,1)\rangle} \\ \hline \Gamma_{\langle(5,2)\rangle} \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline \Gamma_{\langle(1)\rangle} \\ \hline \Gamma_{\langle(2,1)\rangle} \\ \hline \Gamma_{\langle(3,2)\rangle} \\ \hline \Gamma_{\langle(4,3)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(3)\rangle} \oplus \Gamma_{\langle(2,1)\rangle} \\ \hline \Gamma_{\langle(4,1)\rangle} \oplus \Gamma_{\langle(3,2)\rangle} \oplus \Gamma_{\langle(3,1,1)\rangle} \\ \hline \Gamma_{\langle(4,2,1)\rangle} \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(3)\rangle} \\ \hline \Gamma_{\langle(3,1,1)\rangle} \oplus \Gamma_{\langle(4,1)\rangle} \\ \hline \Gamma_{\langle(4,1,1,1)\rangle} \\ \hline\end{array} \quad \begin{array}{|c|}\hline \ \\ \hline \Gamma_{\langle(2,1)\rangle} \oplus \Gamma_{\langle(3)\rangle} \\ \hline \Gamma_{\langle(2,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1)\rangle} \\ \hline \Gamma_{\langle(3,2,2)\rangle} \\ \hline\end{array}$$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(2,1)\rangle} \oplus \Gamma_{\langle(1,1,1)\rangle}$
$\Gamma_{\langle(3,2)\rangle} \oplus \Gamma_{\langle(2,2,1)\rangle}$
$\Gamma_{\langle(3,3,1)\rangle}$

$\Gamma_{\langle(3)\rangle} \oplus \Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(3,1,1)\rangle} \oplus \Gamma_{\langle(2,,1,1,1)\rangle}$
$\Gamma_{\langle(3,1,1,1,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$2\Gamma_{\langle(2,1)\rangle} \oplus \Gamma_{\langle(1,1,1)\rangle}$
$\Gamma_{\langle(3,2)\rangle} \oplus \Gamma_{\langle(2,2,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1)\rangle} \oplus \Gamma_{\langle(3,1,1)\rangle}$
$\Gamma_{\langle(3,2,1,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(2,1)\rangle} \oplus \Gamma_{\langle(1,,1,1)\rangle}$
$\Gamma_{\langle(2,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1)\rangle}$
$\Gamma_{\langle(2,2,2,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(2,1)\rangle} \oplus 2\Gamma_{\langle(1,1,1)\rangle}$
$\Gamma_{\langle(2,2,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,1,1,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(2,1)\rangle} \oplus \Gamma_{\langle(1,1,1)\rangle}$
$\Gamma_{\langle(2,1,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(1,1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1,1,1)\rangle}$

$V^{\otimes 8} \sim$

$\Gamma_{\langle(8)\rangle}$

$\Gamma_{\langle(6)\rangle}$
$\Gamma_{\langle(7,1)\rangle}$

$\Gamma_{\langle(6)\rangle} \oplus \Gamma_{\langle(5,1)\rangle}$
$\Gamma_{\langle(6,1,1)\rangle}$

$\Gamma_{\langle(4)\rangle}$
$\Gamma_{\langle(5,1)\rangle}$
$\Gamma_{\langle(6,2)\rangle}$

$\Gamma_{\langle(4)\rangle}$
$\Gamma_{\langle(5,1)\rangle} \oplus \Gamma_{\langle(4,1,1)\rangle}$
$\Gamma_{\langle(5,1,1,1)\rangle}$

$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(4)\rangle}$
$\Gamma_{\langle(5,1)\rangle} \oplus \Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(4,1,1)\rangle}$
$\Gamma_{\langle(5,2,1)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(3,1)\rangle}$
$\Gamma_{\langle(4,2)\rangle}$
$\Gamma_{\langle(5,3)\rangle}$

$\Gamma_{\langle(1,1)\rangle} \oplus \Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(3,3)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle}$
$\Gamma_{\langle(4,3,1)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$2\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(4,1,1)\rangle}$
$\Gamma_{\langle(4,2,1,1)\rangle}$

$\Gamma_{\langle(4)\rangle} \oplus \Gamma_{\langle(3,1)\rangle}$
$\Gamma_{\langle(4,1,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle}$
$\Gamma_{\langle(4,1,1,1,1)\rangle}$

$\Gamma_{\langle(4)\rangle} \oplus \Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle}$
$\Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(4,1,1)\rangle}$
$\Gamma_{\langle(4,2,2)\rangle}$

$\Gamma_{\langle(0)\rangle}$
$\Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(2,2)\rangle}$
$\Gamma_{\langle(3,3)\rangle}$
$\Gamma_{\langle(4,4)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(2,2,2)\rangle}$
$\Gamma_{\langle(3,3,2)\rangle}$

$\Gamma_{\langle(0)\rangle}$
$2\Gamma_{\langle(1,1)\rangle}$
$2\Gamma_{\langle(2,2)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,1,1)\rangle} \oplus \Gamma_{\langle(3,3)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle}$
$\Gamma_{\langle(3,3,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle}$
$2\Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle}$
$\Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2)\rangle} \oplus \Gamma_{\langle(2,2,1,1)\rangle}$
$\Gamma_{\langle(3,2,2,1)\rangle}$

$\Gamma_{\langle(1,1)\rangle} \oplus \Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle} \oplus 2\Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,1,1)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,2,1,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(0)\rangle}$
$\Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle}$
$\Gamma_{\langle(2,2,1,1)\rangle}$
$\Gamma_{\langle(2,2,2,2)\rangle}$

$\Gamma_{\langle(1,1)\rangle} \oplus \Gamma_{\langle(2)\rangle}$
$2\Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,2)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1)\rangle}$
$\Gamma_{\langle(2,2,2,1,1)\rangle}$

$\Gamma_{\langle(0)\rangle}$
$2\Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(2,2)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle} \oplus 2\Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,1,1,1,1)\rangle}$

$\Gamma_{\langle(1,1)\rangle} \oplus \Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,1,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(0)\rangle}$
$\Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1,1,1,1)\rangle}$

$V^{\otimes 9} \sim$

$\Gamma_{\langle(9)\rangle}$

$\Gamma_{\langle(7)\rangle}$
$\Gamma_{\langle(8,1)\rangle}$

$\Gamma_{\langle(5)\rangle}$
$\Gamma_{\langle(6,1)\rangle}$
$\Gamma_{\langle(7,2)\rangle}$

$\Gamma_{\langle(7)\rangle} \oplus \Gamma_{\langle(6,1)\rangle}$
$\Gamma_{\langle(7,1,1)\rangle}$

$\Gamma_{\langle(3)\rangle}$
$\Gamma_{\langle(4,1)\rangle}$
$\Gamma_{\langle(5,2)\rangle}$
$\Gamma_{\langle(6,3)\rangle}$

$\Gamma_{\langle(4,1)\rangle} \oplus \Gamma_{\langle(5)\rangle}$
$\Gamma_{\langle(5,2)\rangle} \oplus \Gamma_{\langle(5,1,1)\rangle} \oplus \Gamma_{\langle(6,1)\rangle}$
$\Gamma_{\langle(6,2,1)\rangle}$

$\Gamma_{\langle(5)\rangle}$
$\Gamma_{\langle(6,1)\rangle} \oplus \Gamma_{\langle(5,1,1)\rangle}$
$\Gamma_{\langle(6,1,1,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(3,2)\rangle}$
$\Gamma_{\langle(4,3)\rangle}$
$\Gamma_{\langle(5,4)\rangle}$

$\Gamma_{\langle(2,1)\rangle} \oplus \Gamma_{\langle(3)\rangle}$
$\Gamma_{\langle(4,1)\rangle}$
$\Gamma_{\langle(4,2,1)\rangle} \oplus \Gamma_{\langle(4,3)\rangle}$
$\Gamma_{\langle(5,3,1)\rangle}$

$\Gamma_{\langle(3,2)\rangle} \oplus \Gamma_{\langle(4,1)\rangle} \oplus \Gamma_{\langle(5)\rangle}$
$\Gamma_{\langle(4,2,1)\rangle} \oplus \Gamma_{\langle(5,1,1)\rangle}$
$\Gamma_{\langle(5,2,2)\rangle}$

$\Gamma_{\langle(3)\rangle}$
$\Gamma_{\langle(3,1,1)\rangle} \oplus \Gamma_{\langle(4,1)\rangle}$
$\Gamma_{\langle(4,1,1,1)\rangle} \oplus \Gamma_{\langle(4,2,1)\rangle} \oplus \Gamma_{\langle(5,1,1)\rangle} \oplus \Gamma_{\langle(5,2)\rangle}$
$\Gamma_{\langle(5,2,1,1)\rangle}$

$\Gamma_{\langle(4,1)\rangle} \oplus \Gamma_{\langle(5)\rangle}$
$\Gamma_{\langle(4,1,1,1)\rangle} \oplus \Gamma_{\langle(5,1,1)\rangle}$
$\Gamma_{\langle(5,1,1,1,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(1,1,1)\rangle} \oplus \Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(2,2,1)\rangle} \oplus \Gamma_{\langle(3,2)\rangle}$
$\Gamma_{\langle(3,3,1)\rangle} \oplus \Gamma_{\langle(4,3)\rangle}$
$\Gamma_{\langle(4,4,1)\rangle}$

$\Gamma_{\langle(3)\rangle} \oplus \Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(4,1)\rangle} \oplus \Gamma_{\langle(3,1,1)\rangle} \oplus \Gamma_{\langle(3,2)\rangle} \oplus \Gamma_{\langle(2,2,1)\rangle}$
$\Gamma_{\langle(3,2,2)\rangle} \oplus \Gamma_{\langle(3,3,1)\rangle} \oplus \Gamma_{\langle(4,2,1)\rangle}$
$\Gamma_{\langle(4,3,2)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(1,1,1)\rangle} \oplus 2\Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(2,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1)\rangle} \oplus 2\Gamma_{\langle(3,2)\rangle}$
$\Gamma_{\langle(3,2,1,1)\rangle} \oplus \Gamma_{\langle(3,3,1)\rangle} \oplus \Gamma_{\langle(4,2,1)\rangle} \oplus \Gamma_{\langle(4,3)\rangle}$
$\Gamma_{\langle(4,3,1,1)\rangle}$

$\Gamma_{\langle(2,1)\rangle} \oplus \Gamma_{\langle(3)\rangle}$
$\Gamma_{\langle(2,2,1)\rangle} \oplus 2\Gamma_{\langle(3,1,1)\rangle} \oplus \Gamma_{\langle(3,2)\rangle} \oplus \Gamma_{\langle(4,1)\rangle}$
$\Gamma_{\langle(3,2,1,1)\rangle} \oplus \Gamma_{\langle(3,2,2)\rangle} \oplus \Gamma_{\langle(4,1,1,1)\rangle} \oplus \Gamma_{\langle(4,2,1)\rangle}$
$\Gamma_{\langle(4,2,2,1)\rangle}$

$\Gamma_{\langle(3)\rangle} \oplus \Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(2,1,1,1)\rangle} \oplus 2\Gamma_{\langle(3,1,1)\rangle} \oplus \Gamma_{\langle(3,2)\rangle} \oplus \Gamma_{\langle(4,1)\rangle}$
$\Gamma_{\langle(3,2,1,1)\rangle} \oplus \Gamma_{\langle(4,1,1,1)\rangle} \oplus \Gamma_{\langle(4,2,1)\rangle}$
$\Gamma_{\langle(4,2,1,1,1)\rangle}$

$\Gamma_{\langle(3)\rangle}$
$\Gamma_{\langle(3,1,1)\rangle} \oplus \Gamma_{\langle(4,1)\rangle}$
$\Gamma_{\langle(3,1,1,1,1)\rangle} \oplus \Gamma_{\langle(4,1,1,1)\rangle}$
$\Gamma_{\langle(4,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(3)\rangle}$
$\Gamma_{\langle(3,1,1)\rangle}$
$\Gamma_{\langle(3,2,2)\rangle}$
$\Gamma_{\langle(3,3,3)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(1,1,1)\rangle} \oplus 2\Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(2,1,1,1)\rangle} \oplus 2\Gamma_{\langle(2,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1)\rangle} \oplus \Gamma_{\langle(3,2)\rangle}$
$\Gamma_{\langle(2,2,2,1)\rangle} \oplus \Gamma_{\langle(3,2,1,1)\rangle} \oplus \Gamma_{\langle(3,2,2)\rangle} \oplus \Gamma_{\langle(3,3,1)\rangle}$
$\Gamma_{\langle(3,3,2,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$2\Gamma_{\langle(1,1,1)\rangle} \oplus \Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1)\rangle} \oplus 2\Gamma_{\langle(2,2,1)\rangle} \oplus \Gamma_{\langle(3,2)\rangle}$
$\Gamma_{\langle(2,2,1,1,1)\rangle} \oplus \Gamma_{\langle(3,2,1,1)\rangle}$
$\Gamma_{\langle(3,2,2,2)\rangle}$

$\Gamma_{\langle(1,1,1)\rangle} \oplus 2\Gamma_{\langle(2,1)\rangle} \oplus \Gamma_{\langle(3)\rangle}$
$2\Gamma_{\langle(1,1,1)\rangle} \oplus 2\Gamma_{\langle(2,2,1)\rangle} \oplus 2\Gamma_{\langle(3,2,1)\rangle}$
$\Gamma_{\langle(2,2,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1,1)\rangle} \oplus \Gamma_{\langle(3,2,1,1)\rangle} \oplus \Gamma_{\langle(3,2,2)\rangle}$
$\Gamma_{\langle(3,2,2,1,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(1,1,1)\rangle} \oplus 2\Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1)\rangle} \oplus \Gamma_{\langle(3,2)\rangle}$
$\Gamma_{\langle(2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1,1)\rangle} \oplus \Gamma_{\langle(3,2,1,1)\rangle}$
$\Gamma_{\langle(3,2,1,1,1,1)\rangle}$

$\Gamma_{\langle(2,1)\rangle} \oplus \Gamma_{\langle(3)\rangle}$
$\Gamma_{\langle(2,1,1,1)\rangle} \oplus \Gamma_{\langle(3,1,1)\rangle}$
$\Gamma_{\langle(2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,1,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(1,1,1)\rangle} \oplus \Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1)\rangle}$
$\Gamma_{\langle(2,2,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2,1)\rangle}$
$\Gamma_{\langle(2,2,2,2,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$2\Gamma_{\langle(1,1,1)\rangle} \oplus \Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1)\rangle} \oplus 2\Gamma_{\langle(2,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1)\rangle}$
$\Gamma_{\langle(2,1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2,1)\rangle}$
$\Gamma_{\langle(2,2,2,1,1,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$2\Gamma_{\langle(1,1,1)\rangle} \oplus \Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(1,1,1)\rangle} \oplus \Gamma_{\langle(2,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,1,1,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(1)\rangle}$
$\Gamma_{\langle(1,1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1,1,1,1,1)\rangle}$

$V^{\otimes 10} \sim$

$\Gamma_{\langle(10)\rangle}$

$\Gamma_{\langle(8)\rangle}$
$\Gamma_{\langle(9,1)\rangle}$

$\Gamma_{\langle(6)\rangle}$
$\Gamma_{\langle(7,1)\rangle}$
$\Gamma_{\langle(8,2)\rangle}$

$\Gamma_{\langle(7,1)\rangle} \oplus \Gamma_{\langle(8)\rangle}$
$\Gamma_{\langle(8,1,1)\rangle}$

$\Gamma_{\langle(4)\rangle}$
$\Gamma_{\langle(5,1)\rangle}$
$\Gamma_{\langle(6,2)\rangle}$
$\Gamma_{\langle(7,3)\rangle}$

$\Gamma_{\langle(5,1)\rangle} \oplus \Gamma_{\langle(6)\rangle}$
$\Gamma_{\langle(7,1)\rangle} \oplus \Gamma_{\langle(6,2)\rangle} \oplus \Gamma_{\langle(6,1,1)\rangle}$
$\Gamma_{\langle(7,2,1)\rangle}$

$\Gamma_{\langle(6)\rangle}$
$\Gamma_{\langle(7,1)\rangle} \oplus \Gamma_{\langle(6,1,1)\rangle}$
$\Gamma_{\langle(7,1,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(3,1)\rangle}$
$\Gamma_{\langle(4,2)\rangle}$
$\Gamma_{\langle(5,3)\rangle}$
$\Gamma_{\langle(6,4)\rangle}$

$\Gamma_{\langle(4)\rangle} \otimes \Gamma_{\langle(3,1)\rangle}$
$\Gamma_{\langle(5,1)\rangle} \oplus \Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(4,1,1)\rangle}$
$\Gamma_{\langle(6,2)\rangle} \oplus \Gamma_{\langle(5,3)\rangle} \oplus \Gamma_{\langle(5,2,1)\rangle}$
$\Gamma_{\langle(6,3,1)\rangle}$

$\Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(5,1)\rangle} \oplus \Gamma_{\langle(6)\rangle}$
$\Gamma_{\langle(6,1,1)\rangle} \oplus \Gamma_{\langle(5,2,1)\rangle}$
$\Gamma_{\langle(6,2,2)\rangle}$

$\Gamma_{\langle(4)\rangle}$
$2\Gamma_{\langle(5,1)\rangle} \oplus \Gamma_{\langle(4,1,1)\rangle}$
$\Gamma_{\langle(6,2)\rangle} \oplus \Gamma_{\langle(6,1,1)\rangle} \oplus \Gamma_{\langle(5,1,1,1)\rangle} \oplus \Gamma_{\langle(5,2,1)\rangle}$
$\Gamma_{\langle(6,2,1,1)\rangle}$

$\Gamma_{\langle(5,1)\rangle} \oplus \Gamma_{\langle(6)\rangle}$
$\Gamma_{\langle(6,1,1)\rangle} \oplus \Gamma_{\langle(5,1,1,1)\rangle}$
$\Gamma_{\langle(6,1,1,1,1)\rangle}$

$\Gamma_{\langle(0)\rangle}$
$\Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(2,2)\rangle}$
$\Gamma_{\langle(3,3)\rangle}$
$\Gamma_{\langle(4,4)\rangle}$
$\Gamma_{\langle(5,5)\rangle}$

$\Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(3,3)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle}$
$\Gamma_{\langle(5,3)\rangle} \oplus \Gamma_{\langle(4,4)\rangle} \oplus \Gamma_{\langle(4,3,1)\rangle}$
$\Gamma_{\langle(5,4,1)\rangle}$

$\Gamma_{\langle(4)\rangle} \oplus \Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle}$
$\Gamma_{\langle(5,1)\rangle} \oplus \Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(4,1,1)\rangle} \oplus \Gamma_{\langle(3,3)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle}$
$\Gamma_{\langle(5,2,1)\rangle} \oplus \Gamma_{\langle(4,3,1)\rangle} \oplus \Gamma_{\langle(4,2,2)\rangle}$
$\Gamma_{\langle(5,3,2)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$2\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$2\Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(4,1,1)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle}$
$\Gamma_{\langle(5,3)\rangle} \oplus \Gamma_{\langle(5,2,1)\rangle} \oplus \Gamma_{\langle(4,3,1)\rangle} \oplus \Gamma_{\langle(4,2,1,1)\rangle}$
$\Gamma_{\langle(5,3,1,1)\rangle}$

$\Gamma_{\langle(4)\rangle} \oplus \Gamma_{\langle(3,1)\rangle}$
$\Gamma_{\langle(5,1)\rangle} \oplus \Gamma_{\langle(4,2)\rangle} \oplus 2\Gamma_{\langle(4,1,1)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle}$
$\Gamma_{\langle(5,2,1)\rangle} \oplus \Gamma_{\langle(5,1,1,1)\rangle} \oplus \Gamma_{\langle(4,2,2)\rangle} \oplus \Gamma_{\langle(4,2,1,1)\rangle}$
$\Gamma_{\langle(5,2,2,1)\rangle}$

$\Gamma_{\langle(4)\rangle} \oplus \Gamma_{\langle(3,1)\rangle}$
$\Gamma_{\langle(5,1)\rangle} \oplus \Gamma_{\langle(4,2)\rangle} \oplus 2\Gamma_{\langle(4,1,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle}$
$\Gamma_{\langle(5,2,1)\rangle} \oplus \Gamma_{\langle(5,1,1,1)\rangle} \oplus \Gamma_{\langle(4,2,1,1)\rangle} \oplus \Gamma_{\langle(4,1,1,1,1)\rangle}$
$\Gamma_{\langle(5,2,1,1,1)\rangle}$

$\Gamma_{\langle(4)\rangle}$
$\Gamma_{\langle(5,1)\rangle} \oplus \Gamma_{\langle(4,1,1)\rangle}$
$\Gamma_{\langle(5,1,1,1)\rangle} \oplus \Gamma_{\langle(4,1,1,1,1)\rangle}$
$\Gamma_{\langle(5,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(2,2,2)\rangle}$
$\Gamma_{\langle(4,3,1)\rangle} \oplus \Gamma_{\langle(3,3,2)\rangle}$
$\Gamma_{\langle(4,4,2)\rangle}$

$\Gamma_{\langle(0)\rangle}$
$2\Gamma_{\langle(1,1)\rangle}$
$2\Gamma_{\langle(2,2)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(3,2,1)\rangle} \oplus 2\Gamma_{\langle(3,3)\rangle} \oplus \Gamma_{\langle(2,2,1,1)\rangle}$
$\Gamma_{\langle(4,4)\rangle} \oplus \Gamma_{\langle(4,3,1)\rangle} \oplus \Gamma_{\langle(3,3,1,1)\rangle}$
$\Gamma_{\langle(4,4,1,1)\rangle}$

$\Gamma_{\langle(4)\rangle} \oplus \Gamma_{\langle(3,1)\rangle}$
$\Gamma_{\langle(4,1,1)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle}$
$\Gamma_{\langle(4,2,2)\rangle} \oplus \Gamma_{\langle(3,3,2)\rangle}$
$\Gamma_{\langle(4,3,3)\rangle}$

$\Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle}$
$2\Gamma_{\langle(3,1)\rangle} \oplus 2\Gamma_{\langle(2,2)\rangle} \oplus 2\Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(4,1,1)\rangle} \oplus \Gamma_{\langle(3,3)\rangle} \oplus 3\Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2)\rangle} \oplus \Gamma_{\langle(2,2,1,1)\rangle}$
$\Gamma_{\langle(4,3,1)\rangle} \oplus \Gamma_{\langle(4,2,2)\rangle} \oplus \Gamma_{\langle(4,2,1,1)\rangle} \oplus \Gamma_{\langle(3,3,2)\rangle} \oplus \Gamma_{\langle(3,3,1,1)\rangle} \oplus \Gamma_{\langle(3,2,2,1)\rangle}$
$\Gamma_{\langle(4,3,2,1)\rangle}$

$\Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle} \oplus 2\Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(3,3)\rangle} \oplus 2\Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle}$
$\Gamma_{\langle(4,3,1)\rangle} \oplus \Gamma_{\langle(4,2,1,1)\rangle} \oplus \Gamma_{\langle(3,3,1,1)\rangle} \oplus \Gamma_{\langle(3,2,1,1,1)\rangle}$
$\Gamma_{\langle(4,3,1,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2)\rangle}$
$\Gamma_{\langle(4,2,1,1)\rangle} \oplus \Gamma_{\langle(3,2,2,1)\rangle}$
$\Gamma_{\langle(4,2,2,2)\rangle}$

$\Gamma_{\langle(4)\rangle} \oplus 2\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$2\Gamma_{\langle(4,1,1)\rangle} \oplus 2\Gamma_{\langle(3,2,1)\rangle} \oplus 2\Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1)\rangle}$
$\Gamma_{\langle(4,2,2)\rangle} \oplus \Gamma_{\langle(4,2,1,1)\rangle} \oplus \Gamma_{\langle(4,1,1,1,1)\rangle} \oplus \Gamma_{\langle(3,2,2,1)\rangle} \oplus \Gamma_{\langle(3,2,1,1,1)\rangle}$
$\Gamma_{\langle(4,2,2,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$2\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(4,2)\rangle} \oplus \Gamma_{\langle(4,1,1)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle} \oplus 2\Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle}$
$\Gamma_{\langle(4,2,1,1)\rangle} \oplus \Gamma_{\langle(4,1,1,1,1)\rangle} \oplus \Gamma_{\langle(3,2,1,1,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(4,2,1,1,1,1)\rangle}$

$\Gamma_{\langle(4)\rangle} \oplus \Gamma_{\langle(3,1)\rangle}$
$\Gamma_{\langle(4,1,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle}$
$\Gamma_{\langle(4,1,1,1,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(4,1,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2)\rangle}$
$\Gamma_{\langle(3,3,2)\rangle}\Gamma_{\langle(3,2,2,1)\rangle}$
$\Gamma_{\langle(3,3,3,1)\rangle}$

$2\Gamma_{\langle(1,1)\rangle}$
$2\Gamma_{\langle(2,2)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(3,3)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle} \oplus 2\Gamma_{\langle(2,2,1,1)\rangle}$
$\Gamma_{\langle(3,3,1,1)\rangle} \oplus \Gamma_{\langle(3,2,2,1)\rangle} \oplus \Gamma_{\langle(2,2,2,2)\rangle}$
$\Gamma_{\langle(3,3,2,2)\rangle}$

$\Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle} \oplus 3\Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$2\Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle} \oplus 2\Gamma_{\langle(2,2,2)\rangle} \oplus 2\Gamma_{\langle(2,2,1,1)\rangle} \oplus 2\Gamma_{\langle(2,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,3,2)\rangle} \oplus \Gamma_{\langle(3,3,1,1)\rangle} \oplus \Gamma_{\langle(3,2,2,1)\rangle} \oplus \Gamma_{\langle(3,2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2,1,1)\rangle}$
$\Gamma_{\langle(3,3,2,1,1)\rangle}$

$\Gamma_{\langle(0)\rangle}$
$2\Gamma_{\langle(1,1)\rangle}$
$2\Gamma_{\langle(2,2)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle} \oplus 2\Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(3,3)\rangle} \oplus \Gamma_{\langle(3,2,1)\rangle} \oplus 2\Gamma_{\langle(2,2,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,3,1,1)\rangle} \oplus \Gamma_{\langle(3,2,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,3,1,1,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle} \oplus 2\Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(3,2,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2)\rangle} \oplus 2\Gamma_{\langle(2,2,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,2,2,1)\rangle} \oplus \Gamma_{\langle(3,2,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2,2)\rangle} \oplus \Gamma_{\langle(2,2,2,1,1)\rangle}$
$\Gamma_{\langle(3,2,2,2,1)\rangle}$

$\Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle} \oplus 3\Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(3,2,1)\rangle} \oplus 2\Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2)\rangle} \oplus 2\Gamma_{\langle(2,2,1,1)\rangle} \oplus 2\Gamma_{\langle(2,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,2,2,1)\rangle} \oplus \Gamma_{\langle(3,2,1,1,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,2,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,2,2,1,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,2)\rangle} \oplus 2\Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(3,2,1)\rangle} \oplus 2\Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1)\rangle} \oplus 2\Gamma_{\langle(2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,2,1,1,1)\rangle} \oplus \Gamma_{\langle(3,1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,2,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(3,1)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(3,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(3,1,1,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle}$
$\Gamma_{\langle(2,1,1)\rangle}$
$\Gamma_{\langle(2,2,2)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,2,1,1)\rangle}$
$\Gamma_{\langle(2,2,2,2,2)\rangle}$

$\Gamma_{\langle(0)\rangle}$
$2\Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(2,2)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle} \oplus 2\Gamma_{\langle(1,1,1,1)\rangle}$
$2\Gamma_{\langle(2,2,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,2,2)\rangle} \oplus \Gamma_{\langle(2,2,2,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,2,2,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle}$
$2\Gamma_{\langle(2,1,1)\rangle} \oplus 2\Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,2)\rangle} \oplus \Gamma_{\langle(2,2,1,1)\rangle} \oplus 2\Gamma_{\langle(2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,2,1,1)\rangle} \oplus \Gamma_{\langle(2,2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,2,1,1,1,1)\rangle}$

$\Gamma_{\langle(0)\rangle}$
$2\Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(2,2)\rangle} \oplus \Gamma_{\langle(2,1,1)\rangle} \oplus 2\Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1)\rangle} \oplus 2\Gamma_{\langle(1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(2,1,1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,2,1,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(2)\rangle} \oplus \Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(2,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(2,1,1,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,1,1,1,1,1,1)\rangle} \oplus \Gamma_{\langle(1,1,1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(2,1,1,1,1,1,1,1,1)\rangle}$

$\Gamma_{\langle(0)\rangle}$
$\Gamma_{\langle(1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1,1,1,1)\rangle}$
$\Gamma_{\langle(1,1,1,1,1,1,1,1,1,1)\rangle}$