Jacobs University September 18, 2017
Fall 2017

Advanced Calculus (Element of Analysis)

Homework 2

Due on September 25, 2017

Problem 1 [6 points]
What is the limit as $n \to \infty$ of the following sequences? Explain your reasoning.

(a) $a_n = x^{1/n}$ for some $x > 0$; (*Hint: Bernoulli's inequality from the last homework sheet might be helpful.*)

(b) $b_n = \left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n + \frac{1}{3}}$;

(c) $c_n = \frac{10^n}{n!}$.

Problem 2 [2 points]
Consider the sequence $(a_n)_{n\geq 0}$ with

$$a_n = \begin{cases} \frac{n+6}{n+4} & \text{, for } n \text{ even} \\ -\left(\frac{2}{n+3}\right) & \text{, for } n \text{ odd.} \end{cases}$$

What are $\liminf_{n\to\infty} a_n$, $\limsup_{n\to\infty} a_n$, $\inf\{a_n\}$ and $\sup\{a_n\}$?

Problem 3 [2 points]
Prove that the function

$$f(x) = \begin{cases} 0 & \text{, for } x = 0 \\ \sin\left(\frac{1}{x}\right) & \text{, for } x \neq 0 \end{cases}$$

is not continuous at $x = 0$ by constructing a sequence $(x_n)_{n\geq 0}$ that converges to 0, but for which $f(x_n)$ does not converge to $f(0)$.

Problem 4 [2 points]
Using the definition of limits of functions via converging sequences, prove that for

$$f(x) = \begin{cases} 0 & \text{, for } x = 0 \\ \frac{x}{|x|} & \text{, for } x \neq 0 \end{cases}$$

we have $\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0^-} f(x) = -1$, and that $\lim_{x\to 0} f(x)$ does not exist.

Problem 5 [2 points]
Let
$$f(x) = \frac{x^2 + 3x - 10}{x - 2}.$$
What is the domain of f? What is $\lim_{x \to 2} f(x)$?

Problem 6 [2 points]
Compute
$$\sum_{k=1}^{n} \frac{1}{k(k+1)},$$
e.g., by using the difference method. *Hint: Decompose the summands into partial fractions, i.e., find a, b such that*
$$\frac{1}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}.$$

Problem 7 [4 points]
Prove that for $|x| < 1$ and any natural number $m \geq 1$, we have
$$(1-x)^{-m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k.$$
Hint: You could proceed by induction, using the Cauchy product formula and an identity from the last homework sheet.