Jacobs University
Fall 2017

September 25, 2017

Advanced Calculus (Element of Analysis)

Homework 3

Due on October 02, 2017

Problem 1 [6 points]

Determine for each of the following series whether it converges conditionally, converges absolutely or diverges. In case of divergence, does it go to $\pm\infty$ or not? If a series depends on x (power series), determine the radius of convergence ρ and discuss the behavior at $\pm\rho$.

(a)

$$\sum_{k=0}^{\infty} x^k,$$

(b)

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)},$$

(c)

$$\sum_{k=0}^{\infty} \frac{1}{k+5},$$

(d)

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!},$$

(e)

$$\sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} x^{2k+1} \quad \text{(the endpoints might be tricky here)},$$

(f)

$$\sum_{k=0}^{\infty} e^{ky} \quad \text{(distinguish different cases for } y \text{ if necessary)}.$$

Problem 2 [4 points]

In class, the ratio test for a series $\sum_{k=0}^{\infty} a_k$ was introduced. A slightly more general version says that

$$\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1 \quad \Rightarrow \quad \text{series converges absolutely}$$

and

$$\liminf_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1 \quad \Rightarrow \quad \text{series diverges.}$$

Prove these two statements. *Hint 1: Disregard the partial sum up to some very large N. What do the conditions above imply for all $n \geq N$? Hint 2: Geometric series.*

Problem 3 [2 points]

Give an example for functions (be sure to specify domain and range) that are

(a) injective, but not surjective,

(b) surjective, but not injective,

(c) neither surjective nor injective,

(d) bijective.

Problem 4 [3 points]

The hyperbolic sine, hyperbolic cosine and hyperbolic tangent are defined as

$$\begin{aligned} \sinh(x) &= \frac{e^x - e^{-x}}{2}, \\ \cosh(x) &= \frac{e^x + e^{-x}}{2}, \\ \tanh(x) &= \frac{\sinh(x)}{\cosh(x)}. \end{aligned}$$

For each of these functions, determine their domain, range and their inverse.

Problem 5 [5 points]

Using the fact that $\ln(x)$ is the inverse function of e^x, prove the following calculation rules for the logarithm:

(a)

$$\ln(1) = 0,$$

(b)

$$\ln(xy) = \ln(x) + \ln(y), \quad \text{for } x, y > 0,$$

(c)

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y), \quad \text{for } x, y > 0,$$

(d)

$$\ln(x^y) = y\ln(x), \quad \text{for } x > 0,$$

(e)

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)} \quad \text{for } a, b, x > 0.$$

Bonus Problem [2 make-up points]

Note: The bonus problems go a bit beyond what is covered in class, and problems like that will not be posed in the exams. The total number of points for this homework sheet is still 20*, i.e., the bonus points can only be used to make up for point losses in the ordinary problems.*

In class we have learned how to expand $(x + y)^n$ when n is a positive integer (binomial theorem). On the last homework sheet (problem 7), we have extended this result to negative powers, i.e., we have derived a series expansion for $(1 - x)^{-n}$. Now one can do similar expansions for fractional powers, e.g., for $(1 + x)^{1/2}$. On the one hand, we will learn soon how to do that in a very elegant way (Taylor expansion), but actually we can already guess the correct series with much simpler and rather heuristic means. Consider $(1 + x)^{1/2}$ and make the ansatz

$$(1 + x)^{1/2} = 1 + \delta_1 x + O(x^2)$$

for some δ_1. Here we assume ad hoc that the $O(x^2)$ terms are in some sense small. How do we find out what δ_1 is? We square both sides and get

$$1 + x = 1 + 2\delta_1 x + O(x^2),$$

where $O(x^2)$ now contains all terms of order x^2 (those are here different terms than above). Neglecting the $O(x^2)$ terms gives

$$\delta_1 = \frac{1}{2}.$$

One continues with the new ansatz $(1 + x)^{1/2} = 1 + \frac{x}{2} + \delta_2 x^2 + O(x^3)$, and repeats the steps above to find the third order term. If you like to calculate, do this trick to find out all terms up to x^4, and try to find a pattern. The results can be found in Hairer/Wanner Theorem 2.2 (for arbitrary rational exponents). Looks familiar, right?

Now the problem that is worth the 2 bonus points is to find in a similar way an expansion for $\ln(1 + x)$ for $x < 1$. What is the resulting series? What happens for $x = 1$? (This will answer a question that came up in class.)