Jacobs University — October 09, 2017
Fall 2017

# Advanced Calculus (Element of Analysis)

## Homework 5

Due on October 16, 2017

**Problem 1 [4 points]**
Calculate the derivatives of the following functions:

(a)
$$f_1(x) = x\sin(x),$$

(b)
$$f_2(x) = \frac{x^3}{\sqrt{1-x}},$$

(c)
$$f_3(x) = \ln(a^x + a^{-x}) \quad \text{for any } a > 0,$$

(d)
$$f_4(x) = \arccos(x) \quad \text{(express the result without using trigonometric functions).}$$

**Problem 2 [2 points]**
Prove that the function $f(x) = e^{|x|}$ is not differentiable at $x = 0$.

**Problem 3 [2 points]**
In class we discussed the product rule $(fg)' = f'g + fg'$. Generalize this rule for the $n$-th derivative $(fg)^{(n)}$.

**Problem 4 [6 points]**
Consider the curve parametrized by $\varphi$ where $x = a\cos(3\varphi)$ and $y = a\cos(\varphi)$ for some given $a \in \mathbb{R}$.

1. Calculate $\frac{dy}{dx}$ using the given parametrization.
2. Show that the curve satisfies $4y^3 = a^2(x + 3y)$.
3. Calculate $\frac{dy}{dx}$ by implicit differentiation of the equation from (b). Does the result coincide with (a)?

**Problem 5 [2 points]**

Evaluate the following limits, e.g., by using L'Hospital's rule.

(a)

$$\lim_{x\to\infty}\left(1-\frac{a^2}{x^2}\right)^{x^2},$$

(b)

$$\lim_{x\to 0}\frac{\tan(x)-x}{\cos(x)-x}.$$

**Problem 6 [4 points]**

Find the Taylor series around $x = 0$ (i.e., the Maclaurin series) of the following functions for $0 < x < 1$. Show that the functions are actually equal to their infinite power series by showing that the remainder term $R_n(x)$ of the Taylor expansion vanishes in the limit $n \to \infty$.

(a)

$$f_1(x) = \ln(1+x),$$

(b)

$$f_2(x) = (1+x)^a \quad \text{for any fixed } a \in \mathbb{R}.$$