Jacobs University
Fall 2017

October 23, 2017

Advanced Calculus (Elements of Analysis)

Homework 6

Due on November 06, 2017

Problem 1 [3 points]

Find all minima, maxima and points of inflection of $f(x) = \frac{x}{1+x^2}$. Where is the function convex, where is it concave? Based on your results, sketch the graphs of f, f' and f''.

Problem 2 [5 points]

Let $p(x)$ be a polynomial of degree less than m with $p(0) \neq 0$. Now consider the function $f(x) = x^m - p(x)$. We would like to approximate the solutions to $f(x) = 0$. Instead of Newton's method, consider the iteration scheme

$$x_{n+1} = [p(x_n)]^{1/m}.$$

(a) Show that if this scheme converges, then the convergence will in general only be of first order.

(b) Consider $m = 3$ and $p(x) = ax^2 + bx + c$. Give an example of non-zero coefficients a, b, c such that for such a $p(x)$ the convergence is actually second order.

Problem 3 [6 points]

Let $f : [1, \infty) \to [0, \infty)$ be nonincreasing (i.e., for $x \leq y$ we have $f(x) \geq f(y)$).

(a) Show that

$$\sum_{k=2}^{n} f(k) \leq \int_1^n f(x)\, dx \leq \sum_{k=1}^{n} f(k).$$

Here you can use the fact that nonincreasing functions are integrable.

(b) Show that $\sum_{k=1}^{n} \frac{1}{k}$ diverges logarithmically for large n.

(c) Show that $\sum_{k=1}^{\infty} \frac{1}{k^a}$ converges for all $a > 1$.

(d) Does $\sum_{k=2}^{\infty} \frac{1}{k \ln(k)}$ converge or diverge? What about $\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^b}$ for $b > 1$? (*Hint: substitution.*)

Problem 4 [3 points]

Compute the integrals

$$\int \frac{\cos(\ln(x))}{x}\,dx \quad \text{for } x>0, \qquad \int x^2 \sin(x)\,dx, \qquad \int \frac{1}{x^2+2x+6}\,dx.$$

Problem 5 [3 points]

Let f and g be integrable functions. Prove the Cauchy-Schwarz inequality

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq \sqrt{\int_a^b f(x)^2\,dx}\,\sqrt{\int_a^b g(x)^2\,dx}.$$

Hint: Start from the fact that the integral of $\big(f(x)-\lambda g(x)\big)^2$ is bigger or equal zero for all real λ.

Bonus Problem [4 make-up points]

Note: The bonus problems go a bit beyond what is covered in class, and problems like that will not be posed in the exams. The total number of points for this homework sheet is still 20, i.e., the bonus points can only be used to make up for point losses in the ordinary problems.

As briefly mentioned in class, convexity can be defined also for functions which are not twice differentiable. We say that a function f is convex in an interval $[a,b]$ if for all $\lambda \in [0,1]$ and for all $x_1, x_2 \in [a,b]$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

First, try to understand what this definition means for a convex function f. Draw some function, fix x_1 and x_2, and draw the line $\lambda f(x_1)+(1-\lambda)f(x_2)$ that results from considering all possible $\lambda \in [0,1]$.

Then, consider a convex function f, some points $x_1,\ldots,x_n$, and some positive $\lambda_1,\ldots,\lambda_n$ with $\sum_{k=1}^n \lambda_k = 1$. Prove that

$$f\left(\sum_{k=1}^n \lambda_k x_k\right) \leq \sum_{k=1}^n \lambda_k f(x_k).$$

This is called Jensen's inequality. (You might see this again next year in your probability class.)