Jacobs University
Fall 2017

November 06, 2017

Advanced Calculus (Elements of Analysis)

Homework 7

Due on November 13, 2017

Problem 1 [3 points] Compute the integrals

(a)

$$\int \frac{\sin(2x)}{1+4\sin^2(x)}\,dx,$$

(b)

$$\int \frac{2x^3-4x^2+x-1}{x^3-4x^2+5x-2}\,dx.$$

Problem 2 [3 points] Compute the following improper integrals, in case they exist.

(a)

$$\int_0^\infty e^{-\lambda x}\,dx, \quad (\lambda \in \mathbb{R})$$

(b)

$$\int_{-\infty}^\infty \frac{x}{(x^2+\lambda^2)^2}\,dx, \quad (\lambda \in \mathbb{R})$$

(c)

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx.$$

Problem 3 [4 points] Let $R(x)$ be a rational function. Then integrals of the form $\int R(\sin x, \cos x, \tan x)dx$ can be solved by using substitution.

(a) One can start by replacing $\sin x = \frac{2y}{1+y^2}$. What is then the substitution for $\cos x$ and $\tan x$?

(b) Use this substitution to calculate

$$\int \frac{1}{2+\sin x}dx.$$

Problem 4 [4 points] It is known that the Fresnel integral

$$f(x) = \int_0^x \cos(t^2) dt$$

is not elementary.

(a) Express $f(x)$ as a power series.

(b) Show that the improper integral

$$\int_0^\infty \cos(t^2) dt$$

converges.

Problem 5 [6 points]

(a) The gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

Prove that $\Gamma(n) = (n-1)!$ for all natural numbers $n \geq 1$.

(b) In order to calculate integrals of the form $\int_a^b e^{nf(x)}\, dx$ one can use Laplace's method. Assume f has a unique maximum $x_m \in (a, b)$ and that f is twice (continuously) differentiable with $f''(x_m) < 0$. Then,

$$\lim_{n\to\infty} \frac{\int_a^b e^{nf(x)}\, dx}{\sqrt{\frac{2\pi}{n|f''(x_m)|}} e^{nf(x_m)}} = 1,$$

i.e., for very large n,

$$\int_a^b e^{nf(x)}\, dx \approx \sqrt{\frac{2\pi}{n|f''(x_m)|}} e^{nf(x_m)}.$$

Derive the latter formula in a non-rigorous way using a Taylor expansion to second order and just assuming that the remainder term behaves nicely. (You may use the fact that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.)

(c) Use the results from part a) and b) to derive (in a non-rigorous way) Stirling's approximation

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

for large n.

Bonus Problem [3 make-up points]

Note: The bonus problems go a bit beyond what is covered in class, and problems like that will not be posed in the exams. The total number of points for this homework sheet is still 20*, i.e., the bonus points can only be used to make up for point losses in the ordinary problems.*

In the last bonus problem, we studied convexity a bit closer and derived Jensen's inequality. Now, using convexity of $-\ln$, one can prove Young's inequality for products. It states that for positive real x, y and p, q with $\frac{1}{p} + \frac{1}{q} = 1$,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

With that inequality at hand, one can prove the following generalization of the Cauchy-Schwarz inequality. For positive p, q with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left| \int_a^b f(x)g(x)\,dx \right| \leq \left(\int_a^b |f(x)|^p\,dx \right)^{1/p} \left(\int_a^b |g(x)|^q\,dx \right)^{1/q}.$$

This is called Hölder's inequality. Prove these two inequalities.