Jacobs University November 13, 2017
Fall 2017

# Advanced Calculus (Elements of Analysis)

## Homework 8

Due on November 20, 2017

**Problem 1 [3 points]**
Show that the functions

$$f_n(x) = (n+1)x^n(1-x)$$

converge to zero for all $x \in [0,1]$, but possess a maximum at $x = \frac{n}{1+n}$. What is the value of $f_n$ at the maximum in the limit $n \to \infty$? Does $f_n$ converge uniformly to zero?

**Problem 2 [3 points]**
Find the solution to the logistic growth model

$$\frac{dy}{dx} = \lambda y \left(1 - \frac{y}{k}\right)$$

by separation of variables.

**Problem 3 [2 points]**
Imagine there is a hole in the earth, straight from the north to the south pole. Now you let a ball of mass $m$ fall through that hole (with zero initial velocity). Then from Newton's gravitational law it follows that the potential energy for $r \leq R_E$ is

$$V(r) = -\frac{G_N m M_E}{2R_E^3}\big(3R_E^2 - r^2\big),$$

where $r$ is the distance from the center of the earth, $R_E$ the radius of the earth (we assume the earth is a perfect sphere), $M_E$ the mass of the earth and $G_N$ is Newton's gravitational constant. The corresponding gravitational force is $F_G(r) = -\frac{dV(r)}{dr}$. Newton's second law is $F = ma(t)$, where $a(t) = \frac{dv(t)}{dt}$ is the acceleration, and $v(t) = \frac{dr(t)}{dt}$ is the velocity.

(a) What differential equation follows from setting $F = F_G$?

(b) How long does it take the ball to go once from the north to the south pole?

**Problem 4 [3 points]**

The motion of a body in the earth's gravitational field is described by the differential equation

$$y'' = -\frac{gR^2}{y^2},$$

with constants $g$ and $R$, and where $y$ is the distance of the body to the center of earth. Determine the constants in the solution such that $y(0) = R$ and $y'(0) = v$. Then, find the escape velocity, i.e., the smallest velocity $v$ for which the body will not return to earth.

**Problem 5 [3 points]**

Find the solution to

$$y'' + 5y' + 6y = 0$$

with initial conditions $y(0) = 2$ and $y'(0) = 3$.

**Problem 6 [3 points]**

Consider the linear ODE

$$y'' - 2\alpha y' + (\alpha^2 + \beta^2)y = 0$$

for some $\alpha, \beta \in \mathbb{R}$.

(a) Find the zeros of the characteristic polynomial and write down the general solution (involving complex exponentials).

(b) Find the general real solution.

**Problem 7 [3 points]**

Find the general real solution of

$$y_1'' + y_1' + y_1 = 0,$$

and of

$$y_2'' - 2y_2' + 10y_2 = 0.$$

How do the solutions $y_1(x)$ and $y_2(x)$ behave for large $x$?

**Bonus Problem [3 make-up points]**

*Note: The bonus problems go a bit beyond what is covered in class, and problems like that will not be posed in the exams. The total number of points for this homework sheet is still* 20*, i.e., the bonus points can only be used to make up for point losses in the ordinary problems.*

Show that for distinct $\lambda_1, \ldots, \lambda_n$, the functions $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ are linearly independent.

*Hint: There are several ways to do this. One is by induction. For the induction step, assume that* $e^{\lambda_{n+1} x}$ *can be written as a linear combination of the* $e^{\lambda_i x}$*. Then, let the differential operator* $D_{\lambda_{n+1}}$ *act on both sides of the equation.*