Jacobs University
Fall 2017

November 20, 2017

Advanced Calculus (Elements of Analysis)

Homework 9

Due on November 27, 2017

Problem 1 [6 points]

Let's do a few more exercises about integration by substitution. Compute for $a, b > 0$

(a)
$$\int \frac{x^2}{\sqrt{a^2 - b^2x^2}}\, dx \quad \text{with a substitution using sine} \quad \left(x < \frac{a^2}{b^2}\right),$$

(b)
$$\int \frac{1}{x^2\sqrt{a^2 + b^2x^2}}\, dx \quad \text{with a substitution using tangens,}$$

(c)
$$\int \sqrt{a^2 + b^2x^2}\, dx \quad \text{with a substitution using the hyperbolic sine.}$$

Problem 2 [10 points]

Let's also do one more exercise about differential equations. We consider the harmonic oscillator again, but this time with friction and a driving force, i.e.,

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = f(t),$$

where $m, r, k > 0$, and $f(t)$ is an external driving force. We are after a solution $x(t)$ of this equation.

(a) Let us consider the homogeneous case first, i.e., $f = 0$.

(1) Find the general real solution to the homogeneous equation. (You will need to consider different cases depending on m, r, k.)

(2) How do the solutions behave for large t?

(3) Do you recover the solution of the harmonic oscillator without friction when $r \to 0$?

(b) Now consider the inhomogeneous case with periodic force.

(1) Find one particular complex solution to the inhomogeneous equation with $f(t) = e^{i\omega t}$.

(2) By using Euler's formula, use part (1) to find one particular real solution to the inhomogeneous equation with $f(t) = \sin(\omega t)$.

(3) For $f(t) = \sin(\omega t)$ and $r^2 < 2km$, which driving frequency ω makes the amplitude of the solution maximal? This is the so-called resonance frequency. What happens to the amplitude at resonance if $r \to 0$?

Problem 3 [4 points]

(a) Show that

$$\int_0^{2\pi} \sin(kx)\cos(\ell x)\,dx = 0 \quad \forall k, \ell \in \mathbb{N},$$

$$\int_0^{2\pi} \cos(kx)\cos(\ell x)\,dx = 0 = \int_0^{2\pi} \sin(kx)\sin(\ell x)\,dx \quad \forall k \neq \ell \in \mathbb{N},$$

$$\int_0^{2\pi} \cos^2(kx)\,dx = \pi = \int_0^{2\pi} \sin^2(kx)\,dx \quad \forall k \geq 1.$$

(b) Express the Fourier coefficients c_k and c_{-k} in terms of the a_k and b_k, and the other way around.

Bonus Problem [3 make-up points]

Note: The bonus problems go a bit beyond what is covered in class, and problems like that will not be posed in the exams. The total number of points for this homework sheet is still 20*, i.e., the bonus points can only be used to make up for point losses in the ordinary problems.*

We consider the ODE describing a population development with harvesting,

$$\frac{dx}{dt} = x(1-x) - c,$$

where $c > 0$ is the harvest quota. Find the zeros of the right-hand side of the equation and draw the direction field (arrows in the (t, x)-plane indicating the slope for different x) for the different cases of c. From that deduce how the solution will behave for large t depending on the initial condition for the different cases of c.