Jacobs University
Fall 2017

November 27, 2017

Advanced Calculus (Elements of Analysis)

Homework 10

Due on December 4, 2017

Problem 1 [**2 points**]

Generalize the definition of the Fourier series and the formula for the coefficients to L-periodic functions.

Problem 2 [**5 points**]

Show that

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \begin{cases} -\frac{(x-\pi)}{2} & \text{, for } x \in (0, 2\pi) \\ 0 & \text{, for } x = 0, x = 2\pi. \end{cases}$$

Hint: Write the finite sum as an integral like in class. Then use Problem 5 from Homework 4.

Problem 3 [**4 points**]

Compute the Fourier series of

$$f(x) = \frac{(\pi - x)^2}{4}.$$

With the help of your result, compute

$$\sum_{k=1}^{\infty} \frac{1}{k^4}.$$

Hint: Use what we established in class.

Problem 4 [**4 points**]

Compute the Fourier series of the 2π-periodic function that is equal

$$f(x) = |x - \pi|$$

for $0 \le x \le 2\pi$.

Problem 5 [5 points]

Consider the 2π periodic function

$$F : [0, 2\pi] \to \mathbb{C},\ F(x) = e^{i\frac{x^2}{2\pi}}.$$

(a) Let c_k be the Fourier coefficients of F. Show that $\sum_{k=-\infty}^{\infty} c_k = 1$ (without actually computing the c_k).

(b) Show that

$$c_k = \frac{1}{2\pi} \int_{-k\pi}^{(2-k)\pi} e^{i\frac{x^2}{2\pi}}\, dx \begin{cases} 1 & \text{, for } k \text{ even} \\ -i & \text{, for } k \text{ odd.} \end{cases}$$

(c) Combine part (a) and (b) to compute

$$\int_{-\infty}^{\infty} e^{i\frac{x^2}{2\pi}}\, dx.$$

Bonus Problem [4 make-up points]

Note: The bonus problems go a bit beyond what is covered in class, and problems like that will not be posed in the exams. The total number of points for this homework sheet is still 20*, i.e., the bonus points can only be used to make up for point losses in the ordinary problems.*

In class, we considered the simple ODE of exponential growth,

$$\frac{dx(t)}{dt} = \lambda x(t)$$

for some $\lambda > 0$. Now suppose that we only know that

$$\frac{dx(t)}{dt} \leq c(t)x(t)$$

for some given function $c(t)$. In that case one can show that

$$x(t) \leq x(0)\, e^{\int_0^t c(s)ds}.$$

This is called (a simple case of) Gronwall's inequality. Prove this inequality for continuous functions $c(t)$ and differentiable $x(t)$.