Jacobs University
Fall 2018

October 15, 2018

Advanced Calculus

Some extra exercises for part I (not homework, not graded)

Problem 1 (Binomial Coefficients)
Compute

$$\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

for any n and $0 < p < 1$.

Problem 2 (Induction)
Prove by induction that

$$\sum_{k=0}^{n} k^3 = \frac{1}{4} n^2 (n+1)^2.$$

Problem 3 (Polynomials)
Factorize the polynomial $p(x) = x^3 - 3x^2 - 13x + 15$.

Problem 4 (Sequences and Convergence)
Show and carefully explain why the sequence

$$a_n = \frac{4n^3 + 3n}{(\sqrt{n+1} - \sqrt{n}) n^{7/2}}$$

converges, and what its limit is.

Problem 5 (Sequences and Convergence)
Determine $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ of the sequence

$$a_n = (-2)^n \left(2^{-n+1} + 10^{-n}\right).$$

Does $\lim_{n\to\infty} a_n$ exist?

Problem 6 (Infinite Series)
Compute

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+3)}$$

or show that the limit does not exist.

Problem 7 (Power Series)
Determine the radius of convergence ρ for the power series

$$P(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} x^k$$

and state whether it converges at $x = \pm\rho$ or not. What is the derivative $P'(x)$? Does it converge at $x = \pm\rho$ or not?

Problem 8 (Complex Numbers)
Find all roots of the equation

$$z^3 + 2 = 0.$$

Problem 9 (Complex Numbers)
Carefully derive the trigonometric identity

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

using Euler's formula.

Problem 10 (Derivatives)
Consider the function

$$f(x) = \frac{\ln(x)}{x - 3}.$$

What are the domain, image and derivative of f?

Problem 11 (Derivatives)
Compute the derivatives of

$$f(x) = \sin(x)\cos(x), \quad \text{and} \quad g(x) = \arcsin(x).$$