Jacobs University
Fall 2018

December 6, 2018

Advanced Calculus

Some extra exercises for part II (not homework, not graded)

Problem 1 (Taylor Series)
Compute the Taylor series around $x = 0$ for $|x| \leq 1$ of

$$f(x) = \arctan(x).$$

Problem 2 (Maxima/Minima)
Find all maxima, minima, and points of inflection of

$$f(x) = x^3 - 3x + 3$$

and determine where the function is concave and where it is convex. Based on your results, qualitatively sketch the graph of f.

Problem 3 (Integration by Substitution)
Compute for any $a < b$

$$\int_a^b \Big((x-a)(b-x)\Big)^{-1/2}\,dx \quad \text{and} \quad \int_a^b \Big((x-a)(b-x)\Big)^{1/2}\,dx$$

using the substitution $x = a\cos^2 y + b\sin^2 y$.

Problem 4 (Integration by Parts)
Compute the integrals

$$\int x^2 e^{2x}\,dx, \qquad \int (\ln(x))^2\,dx.$$

Problem 5 (Improper Integrals)
Compute the following improper integrals, in case they exist.

$$\int_0^1 \frac{1}{x^2}\,dx, \qquad \int_1^\infty \frac{1}{x^2}\,dx, \qquad \int_0^1 \frac{x}{(1-x^2)^{1/2}}\,dx.$$

Problem 6 (Uniform Convergence and Exchange of Limits)
Consider the sequence of functions

$$f_n(x) = \begin{cases} n & \text{, for } 0 < x \leq \frac{1}{n} \\ 0 & \text{, otherwise.} \end{cases}$$

What is the pointwise limit $n \to \infty$? What is $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$? What is $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx$? Does f_n converge uniformly to some function f?

Problem 7 (ODEs: Separation of Variables)
Solve the ODE

$$\frac{dy}{dx} = x + xy$$

by separation of variables.

Problem 8 (Linear ODEs)
Give the general solution to the linear homogeneous ODE

$$y'' + y' - 2y = 0.$$

Then give the solution for the initial condition $y(0) = 2$ and $y'(0) = 5$. What is the behavior of the solution as $x \to \infty$? Also find one particular solution to the linear inhomogeneous ODE

$$y'' + y' - 2y = e^{-x}.$$

Finally, provide the general solution (i.e., involving two constants) to this inhomogeneous ODE.

Problem 9 (Fourier Series)
Consider the 2π-periodic function f which is $f(x) = \cosh(x - \pi)$ on the interval $[0, 2\pi]$. Does its Fourier series converge uniformly to f? Compute the Fourier series of f. Then, by evaluating f and its Fourier series at π, compute the value of the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{1 + k^2}.$$