Jacobs University
Fall 2018

September 17, 2018

Advanced Calculus

Homework 2

Due on September 24, 2018

Problem 1 [8 points]

Do the following sequences converge or diverge as $n \to \infty$? In case a sequence converges, specify the limit. In case a sequence diverges, specify whether it goes to $-\infty$, $+\infty$ or neither. Explain your reasoning.

(a) $a_n = 2^n$,

(b) $b_n = \frac{(-1)^n}{\sqrt{n}}$,

(c) $c_n = x^{1/n}$ for some $x > 0$, (*Hint: One could use Bernoulli's inequality from the last homework sheet here.*)

(d) $d_n = \left(\sqrt{n+1} - \sqrt{n}\right)\sqrt{n + \frac{1}{3}}$,

(e) $e_n = \frac{10^n}{n!}$,

(f) $f_n = \frac{\sqrt{n^2+2\sqrt{n}}}{n}$.

Problem 2 [2 points]

Consider the sequence $(a_n)_{n\geq 0}$ with

$$a_n = \begin{cases} \frac{n+6}{n+4} & \text{, for } n \text{ even} \\ -\left(\frac{2}{n+3}\right) & \text{, for } n \text{ odd.} \end{cases}$$

What are $\liminf_{n\to\infty} a_n$, $\limsup_{n\to\infty} a_n$, $\inf\{a_n\}$ and $\sup\{a_n\}$?

Problem 3 [2 points]

Prove that the function

$$f(x) = \begin{cases} 0 & \text{, for } x = 0 \\ \sin\left(\frac{1}{x}\right) & \text{, for } x \neq 0 \end{cases}$$

is not continuous at $x = 0$ by constructing a sequence $(x_n)_{n\geq 0}$ that converges to 0, but for which $f(x_n)$ does not converge to $f(0)$.

Problem 4 [2 points]
Suppose that for some $x_0 \in \mathbb{R}$ the sequence

$$x_n = x_0 - \frac{1}{n}$$

is given. Now consider two functions $f(x)$ and $g(x)$ (from $\mathbb{R}$ to $\mathbb{R}$). Suppose that

$$\lim_{n\to\infty} f(x_n) = f(x_0),$$

but

$$\lim_{n\to\infty} g(x_n) \neq g(x_0).$$

What can be deduced about the continuity of f and g at x_0?

Problem 5 [2 points]
Using the definition of limits of functions via converging sequences, prove that for

$$f(x) = \begin{cases} 0 & \text{, for } x = 0 \\ \frac{x}{|x|} & \text{, for } x \neq 0 \end{cases}$$

we have $\lim_{x\to 0^+} f(x) = 1$, $\lim_{x\to 0^-} f(x) = -1$, and that $\lim_{x\to 0} f(x)$ does not exist.

Problem 6 [2 points]
Let

$$f(x) = \frac{x^2 + 3x - 10}{x - 2}.$$

What is the domain of f? What is $\lim_{x\to 2} f(x)$?

Problem 7 [2 points]
Compute

$$\sum_{k=1}^{n} \frac{1}{k(k+1)},$$

e.g., by using the difference method. *Hint: Decompose the summands into so-called partial fractions, i.e., find a, b such that*

$$\frac{1}{k(k+1)} = \frac{a}{k} + \frac{b}{k+1}.$$