Jacobs University
Fall 2018

September 24, 2018

Advanced Calculus

Homework 3

Due on October 1, 2018

Problem 1 [**3 points**]

Prove that for $|x| < 1$ and any natural number $m \geq 1$, we have

$$(1-x)^{-m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k.$$

Hint: You could proceed by induction, using the Cauchy product formula and an identity from the first homework sheet.

Problem 2 [**4 points**]

Determine for each of the following series whether it converges conditionally, converges absolutely or diverges. In case of divergence, does it go to $\pm\infty$ or not?

(a)

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)},$$

(b)

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+5},$$

(c)

$$\sum_{k=0}^{\infty} \frac{e^k}{k},$$

(d)

$$\sum_{k=0}^{\infty} \frac{2^k + k}{k! + 1}.$$

Problem 3 [**3 points**]

For the following power series, determine the radius of convergence ρ and discuss the behavior at $\pm\rho$ (i.e., convergence or divergence (to $\pm\infty$ or not)).

(a)

$$\sum_{k=0}^{\infty} x^k,$$

(b)

$$\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!},$$

(c)

$$\sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} x^{2k+1} \quad \text{(here you do not need to discuss the endpoints!).}$$

Problem 4 [2 points]
State whether the following functions are injective (one-to-one) or not, and whether they are surjective (onto) or not.

(a)

$$f_1 : \mathbb{R} \to [0,1],\ x \mapsto \sin(x),$$

(b)

$$f_2 : \mathbb{R} \to \mathbb{R},\ x \mapsto e^x,$$

(c)

$$f_3 : \mathbb{R} \to [0,\infty),\ x \mapsto x^2,$$

(d)

$$f_4 : [0,\infty) \to [0,\infty),\ x \mapsto x^2.$$

Problem 5 [3 points]
The hyperbolic sine, hyperbolic cosine and hyperbolic tangent are defined as

$$\begin{aligned}
\sinh(x) &= \frac{e^x - e^{-x}}{2}, \\
\cosh(x) &= \frac{e^x + e^{-x}}{2}, \\
\tanh(x) &= \frac{\sinh(x)}{\cosh(x)}.
\end{aligned}$$

For each of these functions, determine their domain, range and their inverse.

Problem 6 [5 points]
Using the fact that $\ln(x)$ is the inverse function of e^x, prove the following calculation rules for the logarithm:

(a)

$$\ln(1) = 0,$$

(b)

$$\ln(xy) = \ln(x) + \ln(y), \quad \text{for } x, y > 0,$$

(c)
$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y), \quad \text{for } x, y > 0,$$

(d)
$$\ln(x^y) = y\ln(x), \quad \text{for } x > 0, y \in \mathbb{R},$$

(e)
$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)} \quad \text{for } a, b, x > 0.$$

Bonus Problem [2 extra points]

Note: The bonus problems go a bit beyond what is covered in class, and problems like that will not be posed in the exams.

In class we have learned how to expand $(x+y)^n$ when n is a positive integer (binomial theorem). In Problem 1 above, we have extended this result to negative powers, i.e., we have derived a series expansion for $(1-x)^{-n}$. Now one can do similar expansions for fractional powers, e.g., for $(1+x)^{1/2}$. On the one hand, we will learn soon how to do that in a very elegant way (Taylor expansion), but actually we can already guess the correct series with much simpler and rather heuristic means. Consider $(1+x)^{1/2}$ and make the ansatz
$$(1+x)^{1/2} = 1 + \delta_1 x + O(x^2)$$
for some δ_1. Here we assume ad hoc that the $O(x^2)$ terms are in some sense small. How do we find out what δ_1 is? We square both sides and get
$$1 + x = 1 + 2\delta_1 x + O(x^2),$$
where $O(x^2)$ now contains all terms of order x^2 (those are here different terms than above). Neglecting the $O(x^2)$ terms gives
$$\delta_1 = \frac{1}{2}.$$
One continues with the new ansatz $(1+x)^{1/2} = 1 + \frac{x}{2} + \delta_2 x^2 + O(x^3)$, and repeats the steps above to find the third order term. If you like to calculate, do this trick to find out all terms up to x^4, and try to find a pattern. The results can be found in Hairer/Wanner Theorem 2.2 (for arbitrary rational exponents). Looks familiar, right?

Now the problem that is worth the 2 bonus points is to find in a similar way an expansion for $\ln(1+x)$ for $x < 1$. What is the resulting series? What happens for $x = 1$? (This will answer a question that came up in class.)