Jacobs University

Fall 2018

October 29, 2018

Advanced Calculus

Homework 6

Due on November 5, 2018

Problem 1 [3 points]

Evaluate the following limits, e.g., by using L'Hospital's rule.

(a)

$$\lim_{x \to 1} \frac{\ln(x)}{x-1},$$

(b)

$$\lim_{x \to \infty} \left(1 - \frac{a}{x} + \frac{b}{x^2}\right)^x \quad \text{for any } a, b \in \mathbb{R},$$

(c)

$$\lim_{x \to 0} \frac{\tan(x) - x}{\sin(x) - x}.$$

Problem 2 [6 points]

Find the Taylor series around $x = 0$ (i.e., the Maclaurin series) of the following functions for $0 < x < 1$. Show that the functions are actually equal to their infinite power series by showing that the remainder term $R_n(x)$ of the Taylor expansion vanishes in the limit $n \to \infty$.

(a)

$$f_1(x) = \ln(1+x),$$

(b)

$$f_2(x) = (1+x)^a \quad \text{for any fixed } a \in \mathbb{R}.$$

Problem 3 [3 points]

Find all minima, maxima and points of inflection of $f(x) = \frac{x}{1+x^2}$. Where is the function convex, where is it concave? Based on your results, sketch the graphs of f, f' and f''.

Problem 4 [4 points]

Let $p(x)$ be a non-constant polynomial of degree less than m with $p(0) \neq 0$. Now consider the function $f(x) = x^m - p(x)$. We would like to approximate the solutions to $f(x) = 0$. Instead of Newton's method, consider the iteration scheme

$$x_{n+1} = [p(x_n)]^{1/m}.$$

(a) Show that if this scheme converges, then the convergence will in general only be of first order.

(b) Consider $m = 3$ and $p(x) = ax^2 + bx + c$. Give an example of non-zero coefficients a, b, c such that for such a $p(x)$ the convergence is actually second order.

Problem 5 [4 points]

We briefly mentioned convex functions in class. This concept can be defined also for functions which are not twice differentiable. We say that a function f is convex in an interval $[a, b]$ if for all $\lambda \in [0, 1]$ and for all $x_1, x_2 \in [a, b]$

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

(a) First, try to understand what this definition means for a convex function f. Draw some function, fix x_1 and x_2, and draw the line $\lambda f(x_1) + (1 - \lambda)f(x_2)$ that results from considering all possible $\lambda \in [0, 1]$.

(b) Then, consider a convex function f, some points $x_1, \ldots, x_n$, and some positive $\lambda_1, \ldots, \lambda_n$ with $\sum_{k=1}^{n} \lambda_k = 1$. Prove that

$$f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k f(x_k).$$

This is called Jensen's inequality. (You might see this again next year in your probability class.)