Jacobs University
Fall 2018

November 19, 2018

Advanced Calculus

Homework 9

Due on November 26, 2018

Problem 1 [3 points]
Show that the functions

$$f_n(x) = (n+1)x^n(1-x)$$

converge to zero for all $x \in [0,1]$, but possess a maximum at $x = \frac{n}{1+n}$. What is the value of f_n at the maximum in the limit $n \to \infty$? Does f_n converge uniformly to zero?

Problem 2 [6 points]
Let's do a few more exercises about integration by substitution. Compute for $a, b > 0$

(a)

$$\int \frac{x^2}{\sqrt{a^2 - b^2x^2}}\,dx \quad \text{with a substitution using sine } \left(x < \frac{a^2}{b^2}\right),$$

(b)

$$\int \frac{1}{x^2\sqrt{a^2 + b^2x^2}}\,dx \quad \text{with a substitution using tangens,}$$

(c)

$$\int \sqrt{a^2 + b^2x^2}\,dx \quad \text{with a substitution using the hyperbolic sine.}$$

Problem 3 [3 points]
Find the solution to the logistic growth model

$$\frac{dy}{dx} = \lambda y \left(1 - \frac{y}{k}\right)$$

by separation of variables.

Problem 4 [3 points]
Imagine there is a hole in the earth, straight from the north to the south pole. Now you let a ball of mass m fall through that hole (with zero initial velocity). Then from Newton's gravitational law it follows that the potential energy for $r \leq R_E$ is

$$V(r) = -\frac{G_N m M_E}{2R_E^3}\left(3R_E^2 - r^2\right),$$

where r is the distance from the center of the earth, R_E the radius of the earth (we assume the earth is a perfect sphere), M_E the mass of the earth and G_N is Newton's gravitational constant. The corresponding gravitational force is $F_G(r) = -\frac{dV(r)}{dr}$. Newton's second law is $F = ma(t)$, where $a(t) = \frac{dv(t)}{dt}$ is the acceleration, and $v(t) = \frac{dr(t)}{dt}$ is the velocity.

(a) What differential equation follows from setting $F = F_G$?

(b) How long does it take the ball to go once from the north to the south pole?

Problem 5 [3 points]
The motion of a body in the earth's gravitational field is described by the differential equation

$$y'' = -\frac{G_N M}{y^2},$$

with constants G_N and M, and where y is the distance of the body to the center of earth. Determine the constants in the solution such that $y(0) = R$ (for some constant R, the radius of earth) and $y'(0) = v$ (for some constant v, the initial velocity). Then, find the escape velocity, i.e., the smallest velocity v for which the body will not return to earth. (Note that here it is enough to solve the ODE for $v(y)$ only. If v is always positive the body will never return to earth.)

Problem 6 [2 points]
Find the solution to

$$y'' + 5y' + 6y = 0$$

with initial conditions $y(0) = 2$ and $y'(0) = 3$.