Jacobs University
Fall 2018

November 26, 2018

Advanced Calculus

Homework 10

Due on December 3, 2018

Problem 1 [2 points]
Consider the linear ODE

$$y'' - 2\alpha y' + (\alpha^2 + \beta^2)y = 0$$

for some $\alpha, \beta \in \mathbb{R}$.

(a) Find the zeros of the characteristic polynomial and write down the general solution (involving complex exponentials).

(b) Find the general real solution.

Problem 2 [3 points]
Find the general real solution of

$$y_1'' + y_1' + y_1 = 0,$$

and of

$$y_2'' - 2y_2' + 10y_2 = 0.$$

How do the solutions $y_1(x)$ and $y_2(x)$ behave for large x?

Problem 3 [9 points]
Let's also do one more exercise about differential equations. We consider the harmonic oscillator again, but this time with friction and a driving force, i.e.,

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = f(t),$$

where $m, r, k > 0$, and $f(t)$ is an external driving force. We are after a solution $x(t)$ of this equation.

(a) Let us consider the homogeneous case first, i.e., $f = 0$.

(1) Find the general real solution to the homogeneous equation. (You will need to consider different cases depending on m, r, k.)

(2) How do the solutions behave for large t?

(3) Do you recover the solution of the harmonic oscillator without friction when $r \to 0$?

(b) Now consider the inhomogeneous case with periodic force.

(1) Find one particular complex solution to the inhomogeneous equation with $f(t) = e^{i\omega t}$.

(2) By using Euler's formula, use part (1) to find one particular real solution to the inhomogeneous equation with $f(t) = \sin(\omega t)$.

(3) For $f(t) = \sin(\omega t)$ and $r^2 < 2km$, which driving frequency ω makes the amplitude of the solution maximal? This is the so-called resonance frequency. What happens to the amplitude at resonance if $r \to 0$?

Problem 4 [3 points]
Show that

$$\int_0^{2\pi} \sin(kx)\cos(\ell x)\,dx = 0 \quad \forall k, \ell \in \mathbb{N},$$

$$\int_0^{2\pi} \cos(kx)\cos(\ell x)\,dx = 0 = \int_0^{2\pi} \sin(kx)\sin(\ell x)\,dx \quad \forall k \neq \ell \in \mathbb{N},$$

$$\int_0^{2\pi} \cos^2(kx)\,dx = \pi = \int_0^{2\pi} \sin^2(kx)\,dx \quad \forall k \geq 1.$$

Problem 5 [3 points]
Compute the Fourier series of the 2π-periodic function that is equal

$$f(x) = |x - \pi|$$

for $0 \leq x \leq 2\pi$.

Bonus Problem [3 extra points]
Note: The bonus problems go a bit beyond what is covered in class, and problems like that will not be posed in the exams.
We consider the ODE describing a population development with harvesting,

$$\frac{dx}{dt} = x(1 - x) - c,$$

where $c > 0$ is the harvest quota. Find the zeros of the right-hand side of the equation and draw the direction field (arrows in the (t, x)-plane indicating the slope for different x) for the different cases of c. From that deduce how the solution will behave for large t depending on the initial condition for the different cases of c.