Advanced Calculus

Session 1
Sep. 3, 2018

Prof. Sören Petrat

Office: 112, Res. I

Organization:

- see syllabus on campusnet
- website
- class: Mon, Wed: 11:15–12:30, LH Res. I
- weekly homework sheets (starting Mon, Sep. 10)
  - ↳ new sheet on Mondays, due Mondays <u>before</u> class
  - ↳ graded (two worst sheets disregarded for grading)
- weekly tutorial
- TAs: Maria Oprea, Benedikt Stock
- 2 exams: · midterm (Wed, Oct. 24)
  · final (in final exam period, Dec)

- grade: 20% HW (final > midterm
  30% midterm => midterm grade = final grade)
  50% final

Topics: 1. Introduction (function, limits, continuity, power series, complex numbers)

2. Differentiation

3. Integration

4. Differential Equations

5. Fourier series

6. Many variable (higher dimension)

1. Introduction

1.1 Polynomials

Definition: a polynomial of degree n ($n>0$) is

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

with some $a_0, \ldots, a_n \in \mathbb{R}$ ($\mathbb{R}$ = real numbers) ($a_n \neq 0$)

polynomial equation: $x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$

($a_n = 1$ here by convention)

↳ solutions are called roots of the eq.
= zeroes of the polynomial

n=1: linear eq.: $x + a_0 = 0 \Rightarrow$ sol.: $x = -a_0$

$\Rightarrow$ there is always exactly one solution

n=2: quadratic eq.: $x^2 + a_1 x + a_0 = 0$

solution: $x^2 + a_1 x + \left(\frac{a_1}{2}\right)^2 = \left(\frac{a_1}{2}\right)^2 - a_0$

$\Leftrightarrow \left(x + \frac{a_1}{2}\right)^2 = \left(\frac{a_1}{2}\right)^2 - a_0$

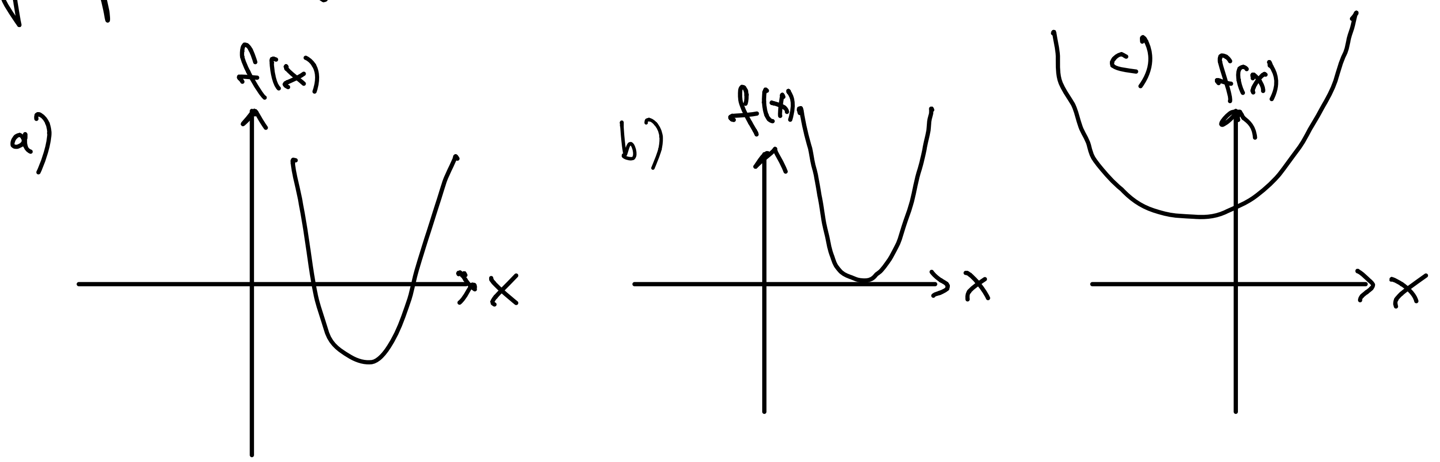
$\Rightarrow$ zeroes: $x_\pm = -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0}$

a) $\left(\frac{a_1}{2}\right)^2 > a_0 \Rightarrow$ two real roots

b) $\left(\frac{a_1}{2}\right)^2 = a_0 \Rightarrow$ one real root

c) $\left(\frac{a_1}{2}\right)^2 < a_0 \Rightarrow$ no real root (but two complex roots
↳ later)

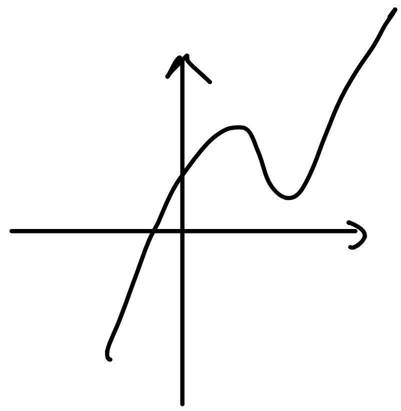
graphs of $f(x) = x^2 + a_1 x + a_0$



$\underline{n=3}$: cubic eq.: $x^3 + a_2 x^2 + a_1 x + a_0 = 0$

↳ no explicit formula, but recipes

↳ there is at least one real root



recipe: simplify eq. by setting $y = \frac{a_2}{3} + x$ $\left(x = y - \frac{a_2}{3}\right)$

$$\left(y - \frac{a_2}{3}\right)^3 + a_2\left(y - \frac{a_2}{3}\right)^2 + a_1\left(y - \frac{a_2}{3}\right) + a_0 = 0$$

note:
$(a+b)^2 = a^2 + 2ab + b^2$
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\Rightarrow$ compute ...

$$\Leftrightarrow y^3 + \underbrace{\left(-\frac{a_2^2}{3} + a_1\right)}_{=p} y + \underbrace{\left(\frac{2}{27} a_2^3 - \frac{a_1 a_2}{3} + a_0\right)}_{q} = 0$$

$\Rightarrow$ only need to solve $y^3 + py + q = 0$

solution trick (Vieta): set $y = z - \frac{p}{3z}$

$$\Rightarrow \left(z - \frac{p}{3z}\right)^3 + p\left(z - \frac{p}{3z}\right) + q = 0$$

$$\Rightarrow z^6 + qz^3 - \frac{p^3}{27} = 0$$

$$\Rightarrow (z^3)^2 + q(z^3) - \frac{p^3}{27} = 0$$

$$\Rightarrow \text{sol.: } z^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}}$$

$\Rightarrow$ depending on p and q there can be 1, 2, 3 real roots

(need complex numbers $\to$ later)

<u>$n>3$</u>: - for $n=4$ there is a solution recipe

- for $n>4$ no general recipe (Abel, Galois, 19th century)

  ↳ numerical methods necessary (later)

<u>Theorem:</u> any polynomial of degree n has exactly n possibly complex roots.

(no proof here)

↳ note: counted multiplicity, as in, e.g., $(x-1)^4 = 0$