Session 2
Sep. 5, 2018

factorization:

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = (x - x_1)\cdot(x - x_2)\cdot \ldots \cdot(x - x_n)$$

where $x_1, \ldots, x_n$ are the roots

↳ helpful if one or several of the roots can be found by inspection (guessing)

Example: $x^3 - 3x^2 - 33x + 35 = 0$

check: $x = 1$: $1 - 3 - 33 + 35 = 0$ ✓

$$x^3 - 3x^2 - 33x + 35 = (x-1)(x^2 + ax + b)$$
$$= x^3 + (a-1)x^2 + (b-a)x - b$$

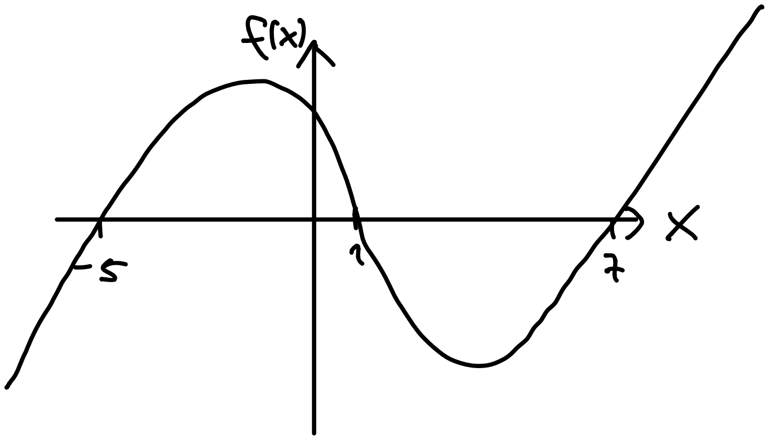
$\Rightarrow a - 1 = -3 \Rightarrow a = -2$

$b - a = -33 \Rightarrow b = -35$

roots of $x^2 - 2x - 35 = 0$?

$\Rightarrow x_{1/2} = 1 \pm \sqrt{1+35} = 1 \pm 6 \Rightarrow x_1 = 7, x_2 = -5$

$$x^3 - 3x^2 - 33x + 35 = (x-1)(x-7)(x+5) = f(x)$$

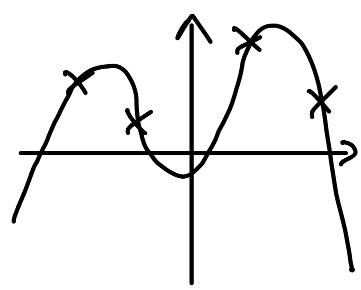


Polynomial Interpolation

find polynomial that passes through points $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$

with degree n or less

polynomial: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$



→ plug in points:

$$a_n x_0^n + a_{n-1} x_0^{n-1} + \dots + a_0 = y_0$$

$$a_n x_1^n + a_{n-1} x_1^{n-1} + \dots + a_0 = y_1$$

$$\vdots$$

$$a_n x_n^n + a_{n-1} x_n^{n-1} + \dots + a_0 = y_n$$

⇒ $n+1$ eq.s for the $n+1$ unknowns $a_0, \dots, a_n$

↳ can be solved, e.g., by Gaussian elimination

Theorem: The interpolating polynomial is unique.

Proof: Let $f(x)$ and $g(x)$ be polynomials of degree $\leq n$ with

$f(x_i) = y_i$ and $g(x_i) = y_i$, $i = 0, 1, \dots, n$.

define $d(x) = f(x) - g(x)$ $\Rightarrow$ this is pol. of degree $\leq n$

we have: $d(x_i) = f(x_i) - g(x_i) = y_i - y_i = 0$

$\Rightarrow$ x_i's are $n+1$ roots of $d(x) = 0$

but $d(x)$ has degree $\leq n$, so it can have at most n real roots

$\Rightarrow d(x) = 0$, i.e., $f(x) = g(x)$. $\square$

(qed)

proof finished ↗ (q.e.d.)

1.2 Binomial Expansion

we would like to expand $(a+b)^n$

Ex.:

$$(a+b)^0 = 1$$
$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

...

pattern: Pascal triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & \multicolumn{3}{c}{\underbrace{1 + 3}} & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$\downarrow n = 0, 1, 2, \dots$

$\rightarrow k = 0, 1, \dots$

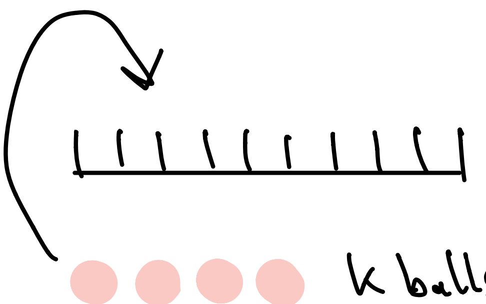
n^0

n^1

$\frac{n(n-1)}{2}$

in general: $\binom{n}{k} := \frac{n!}{(n-k)!\,k!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{1\cdot 2\cdot 3\cdot\ldots\cdot k}$

other notations: $\binom{n}{k} = C(n,k) = {}^{n}C_{k}$ "n choose k"



n boxes

k balls

Theorem: For positive integers n we have:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \ldots + \binom{n}{n} b^n$$

(sum notation: $\sum_{k=0}^{n} a_k = a_0 + a_1 + a_2 + \ldots + a_n$)

Connection to Pascal triangle:

- we define $0! := 1$, so $\binom{n}{0} = \frac{n!}{n!0!} = 1$, $\binom{n}{n} = \frac{n!}{(n-n)!n!} = 1$

- symmetry: $\binom{n}{k} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

- property of Pascal triangle:

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{(n-k)!\,k!} + \frac{n!}{(n-k+1)!\,(k-1)!}$$

$$= \frac{n!(n-k+1)}{(n-k+1)!\,k!} + \frac{n!\,k}{(n-k+1)!\,k!}$$

$$= \frac{n!(n-k+1+k)}{(n-k+1)!\,k!}$$

$$= \frac{(n+1)!}{(n+1-k)!\,k!}$$

$$= \binom{n+1}{k}$$

Proof of the Theorem:

we proceed by induction:

induction:
- allows to prove statement for all natural numbers
- <u>step 1:</u> show that statement holds for $n=0$ or $n=1$ (or some initial n)

- step 2: • assume statement holds for some n
  • then show that it holds for $n+1$

here: step 1: $(a+b)^1 = a+b = \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = a+b$ ✓

step 2: assume $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

then $(a+b)^{n+1} = (a+b)(a+b)^n$

$$= (a+b)\left(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

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now: "shift of indices" trick:

$$\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} = \sum_{\ell=1}^{n+1} \binom{n}{\ell-1} a^{n-\ell+1} b^{\ell} = \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n-k+1} b^k$$

set $\ell = k+1$, i.e., $k = \ell - 1$ rename ℓ back to k

$0 \le k \le n$

$1 \le k+1 \le n+1$

$1 \le \ell \le n+1$

(for example: $a_1 + a_2 + a_3 = \sum_{k=1}^{3} a_k = \sum_{k=0}^{2} a_{k+1}$)

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continue computation: $= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n+1} \binom{n}{k-1} a^{n-k+1} b^k$

$$= \underbrace{\binom{n}{0}}_{=\binom{n+1}{0}} a^{n+1} + \sum_{k=1}^{n} \underbrace{\left[\binom{n}{k} + \binom{n}{k-1}\right]}_{=\binom{n+1}{k}} a^{n+1-k} b^k + \underbrace{\binom{n}{n}}_{=\binom{n+1}{n+1}} b^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$$

$\Rightarrow$ formula holds for $n+1$. $\square$